Portfolio Delegation with Limited Liability*

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Abstract

We consider the portfolio delegation problem in a world with potentially incomplete contingent claim markets. A principal hires an agent to manage a portfolio. When the agent has limited liability (that is, there is a lower bound on the compensation contract), she may have an incentive to take on excessive risk. With complete markets, the precise nature of the risk the agent may take on is a large short position in the state with lowest probability, and a long position in every other state. Our main result is that, with limited liability and a large number of states, incentive compatibility alone restricts the feasible contract to be either a flat one or one with exactly two compensation levels (equal to the lower and upper bounds on compensation). We examine the effectiveness of Value at Risk compensation schemes in this context. An appropriately set VaR scheme can be effective at controlling the size of the maximum loss suffered by the portfolio. However, in general, we do not expect it to attain the same outcome as the optimal contract.

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1 Introduction

The portfolio delegation problem studies the agency problem that arises when an investor (or "principal") contracts with a money manager (or "agent") to invest funds. An important real-world feature of this agency relationship is the limited liability of the agent: the principal cannot force the agent to share losses on the investment. For example, the agent may be a fund manager or a trader at an investment bank. In the former case, the contract usually has a fixed component that represents a lower bound. In the latter, the agent may be fired if the portfolio performs poorly, but cannot be forced to share in large losses.

In this paper, we study this principal-agent problem, and characterize the optimal incentive scheme with limited liability. A unique feature of our analysis is that, unlike existing literature on the delegated portfolio management problem (e.g., Bhattacharyya and Pfleiderer, 1985, and Stoughton, 1993), we describe the solution in a contingent claims model. Our main result is that, if the limited liability constraint binds at all and markets are complete, then the optimal contract takes a very special form: the agent gets a flat fee and gets a bonus if a large enough return on the investment is obtained. We call this a "bang-bang" contract.

The contingent claims model highlights the role of limited liability in encouraging the agent to take on excessive risk. With complete markets, a feasible strategy for the agent is to hold a portfolio that has a large negative payoff in the lowest probability state, and a positive payoff in every other state. We show that one implication of this is that, if the lowest probability state has small enough probability, then the only feasible (i.e., incentive compatible) contracts are close to either the flat contract, in which the compensation does not depend at all on the outcome, or the bang-bang one.

One of the main roles played by the principal in our model is that he offers the agent a chance to gamble. Limited liability for the agent is equivalent to the principal underwriting all losses. It is, therefore, interesting to examine the relationship between our work and recent regulation designed to control excessive risk taking by financial institutions. Banking regulation now requires banks to report the Value at Risk (VaR) of their portfolios.[1] This, and related measures of risk, are intended to reduce excessive risk-taking. One reason given for the imposition of such risk measures

[1] Value at Risk is defined in Section 4 below.

is the limited liability of most financial institutions and money managers.

However, a limit on some risk measure such as VaR is only one feature of a contract, and its implications must be studied in the context of the complete contract, which includes a description of the compensation scheme. As a benchmark, we compare our results to a linear compensation scheme with a VaR limit. We call this a VaR scheme. We show that, if there are only two securities, the two schemes are identical. However, with three or more securities, the VaR scheme is not optimal.

We argue that the contingent claims framework is a natural one in which to study risk-taking behavior by an agent. It highlights the nature of the risks an agent will take under limited liability: large short positions in low probability states to finance long positions in high probability ones. While we do not characterize the optimal contract in the general case, the model points to the strong implications of incentive compatibility when there are a large number of states: the only incentive compatible contracts are essentially flat, in the sense that the payment to the agent, when positive, is constant across realized wealth levels. This is consistent, for example, with the flat fee contracts often observed in the mutual fund industry.

The contingent claims framework also highlights one of the downsides to financial innovation. We study the problem in an incomplete markets setting in general, with complete markets as a special case. Financial innovation leads to a greater degree of market completeness. To the extent that this increases the ability of the agent to bet on finer partitions of the state space, it exacerbates the agency problem inherent in portfolio delegation. If the incentive compatibility constraint is binding in the presence of innovation, the contract will have to be continually revised to remain incentive compatible.

Previous work in this area includes Bhattacharyya and Pfleiderer (1985), Dybvig and Spatt (1986), Grinblatt and Titman (1989) and Stoughton (1993). Bhattacharyya and Pfleiderer (1985) consider a problem in which agents have different types, and are required to invest. They point out that quadratic contracts are incentive compatible in this context, and lead agents to reveal their true information. As with Stoughton below, they restrict attention to exponential utilities and a normal distribution for the risky asset. Dybvig and Spatt (1986) show that risk-sharing is efficient if and only if the investor and manager have "similar" preferences.

Stoughton (1993) considers a moral hazard problem in this context. Set Θ, the set of types, to be a singleton. There is one risky and one riskless asset. The agent expends effort e which leads to a signal, that gives a posterior distribution H over the return of the risky asset. Stoughton considers two cases: contracts that are linear and quadratic in w.

Under linear contracting, the agent chooses α_x, the weight on the risky asset, since it is not incentive compatible for the agent to simply reveal the signal truthfully and let the principal invest. Since H is assumed known only to the agent, the principal must compute VaR based only on F, the prior distribution. A VaR rule then prohibits certain choices of α_x.

If the revelation principle is applied, one can think of the principal directly choosing α_x, after the agent has revealed his signal to the principal. In this setting, a VaR-rule is clearly redundant. Bhattacharyya and Pfleiderer (1985) show that a quadratic contract is incentive compatible for the agent, and leads to truthful revelation of signal. Stoughton (1993) shows that, as the principal's risk aversion coefficient approaches zero, the portfolio induced by the optimal quadratic contract approaches first best.

Grinblatt and Titman (1989) show that, if there is limited liability, the agent has an incentive to take on a riskier portfolio than otherwise. The solution they propose is that the loss (to the agent) of underperformance outweigh the gain from overperformance. In our model, this solution is infeasible due to limited liability; instead, we focus on the implications of limited liability for the set of feasible contracts.

Some of the work in this area, including Grinblatt and Titman (1989) under partial equilibrium, Admati and Pfleiderer (1997), Lynch and Musto (1997) and Das and Sundaram (1998) considers contracts based on a performance benchmark. Dybvig, Farnsworth and Carpenter (2000) show that the optimal contract may involve the use of a benchmark, over and above portfolio performance. In our framework, there is no natural benchmark to use. In any case, even if contracts were conditioned on some benchmark, limited liability restricts the set of feasible contracts. Ou-Yang (1998) considers the optimal contract in a continuous time setting, but with unlimited liability.

Palomino and Prat (1999) also consider a moral hazard problem in this context. They consider a setting in which portfolios are defined by their mean and risk measure (that is unspecified), and in which a unique efficient portfolio exists. This appears to rule out the standard mean-variance case of a minimum variance frontier that gives rise to a continuum of efficient portfolios. They characterize the first-best single period contract in their setting, and show that the first-best outcome cannot be achieved in two periods.

Our paper shares some common features with Palomino and Prat (1999), most notably the lower bound on the agent's compensation function. However, unlike their paper, we do not presuppose the existence of a risk measure; indeed, such a measure should depend on the preferences of the principal and agent, and should therefore be endogenous to a model. Another point of departure is that there is a unique efficient asset in the setting of Palomino and Prat (1999), whereas our framework allows for a continuum of efficient assets.

The notion of bounds on compensation contracts has been explored in contract theory by, among others, Innes (1990). Innes considers a risk-neutral principal (entrepreneur) and agents (investors), with moral hazard on the part of the principal. In our model, the moral hazard problem relates directly to the agent (the party taking the contract), rather than the principal (the party designing the contract). Hence, limited liability can lead to the agent taking on excessive risk, a notion absent in Innes' model.

Recent work on VaR based portfolio management includes Basak and Shapiro (1999). In a continuous time framework, they find that a VaR rule leads portfolio managers to be uninsured in the worst states of the world, and insure against intermediate-loss states. This corresponds to our bang-bang solution. In our framework, since every state of the world has positive probability, an appropriately chosen VaR rule can limit the worst outcome. Basak and Shapiro further find that an expected loss based rule is more effective at controlling the size of the maximum loss. This will necessarily hold in any context in which the probability in a VaR rule is set higher than that of the lowest probability state. A discussion between desirable properties of various of a risk measure and the degree to which various popular risk measures satisfy these is contained in Artzner, et. al. (1998).

While moral hazard is certainly present in our model (since the agent cannot be forced to choose a particular portfolio, but must be induced to do so), there is no asymmetric information or costly effort. Jiang (2000) considers a general model which involves both traditional moral hazard (costly effort) and the form present in our model, and shows that the optimal contract can involve convex and concave regions. Garcia (2000), in a setting with CARA preferences and normally distributed asset returns, finds that the optimal contract may be a fixed wage. In our setting, with a large number of assets and complete markets, a fixed wage contract is the only incentive compatible one.

We describe our model in Section 2 below. The importance of limited liability and the bang-bang solution are studied in Section 3. Section 4 compares the optimal contract to a VaR scheme, and is followed by some concluding remarks in Section 5.

2 Model

A principal hires an agent to manage his investments. Since the agent possesses no superior information or skill, we assume that the principal lacks the time to invest on his own. The agent constructs a portfolio of financial assets at time 0. There is a finite set of states, $S = \{1, \ldots, S\}$, one of which will be revealed at time 1. π_s represents the probability of state s, so that $\pi_s > 0$ for all s, and $\sum_{s=1}^{S} \pi_s = 1$. Let $\pi = (\pi_1, \ldots, \pi_S)$. There are J securities that can be traded at time 0, where $2 \leq J \leq S$. Security j pays off an amount $a_{js} \geq 0$ in state s.[2] The J securities are linearly independent, so there are no redundant securities. The $S \times J$ matrix of security payoffs is denoted by A, and the price of security j is represented by q_j.

A special case of this model, of course, is that of complete markets; that is, $J = S$. In this case, lack of arbitrage implies a unique set of state prices that can be used to price securities. Further, when markets are complete, we can construct portfolios that are Arrow–Debreu securities, where security s pays off 1 unit in state s and 0 in every other state. These securities are priced via the unique state prices vector. To sharpen intuition about the problem, when $J = S$, we will assume (without loss of generality) that the securities are Arrow–Debreu securities.

At time 0, the value of the initial portfolio is w^0. This is interpreted as the amount of money the investor turns over to the agent to manage. We assume that the wealth of the principal is unbounded, so that the portfolio can sustain any finite loss.

The goal of the investor is to maximize his expected utility at time 1. The investor's utility is a function of time 1 wealth alone, and is represented by $u(\cdot)$. The investor is assumed to be risk-averse, so that $u''(w) < 0$ for all w.[3] Further, $u(\cdot)$ is defined for all wealth levels in $\mathbf{R}$.[4]

[2] Disallowing negative payoffs leads to a cleaner definition of a short position in a security.

[3] If the investor were risk-neutral, her utility-maximizing problem is potentially unbounded.

[4] This rules out certain functional forms, such as $u(w) = w^\gamma$, but does permit the CARA form

The investor chooses a compensation function, $I(\cdot)$, which is a function of realized wealth alone. That is, the investor does not observe the realized state, and can make compensation contingent only on the realized value of the portfolio, w.[5]

The manager chooses a portfolio taking this compensation function as given. The utility function of the manager is denoted by $v(\cdot)$. The manager is weakly risk averse, with $v''(w) \leq 0$ for all w. In particular, this permits the risk neutral case.

We assume that the compensation function is bounded below, so that there is a maximum penalty that the principal can impose if the outcome is not satisfactory. Without loss of generality, this lower bound is taken to be 0. Therefore, we have $I(w) \geq 0$ for all w. Further, $v(0)$ is normalized to 0, and $v'(0) < \infty$.[6] Both $u(\cdot)$ and $v(\cdot)$ are assumed to be twice differentiable and strictly increasing at all w.

Let x_j be the number of units purchased of the j^{th} asset, with $x = (x_1, \ldots, x_J)'$ representing the entire portfolio. The realized income of the portfolio at time 1 is $w_1 = (w_{11}, \ldots, w_{1S})'$, where $w_{1s} = \sum_{j=1}^{J} a_{js} x_j$ for each $s = 1, \ldots, S$. With a slight abuse of notation, we define $I(x) = (I(w_{11}), \ldots, I(w_{1S}))'$ to be the income of the agent across states when portfolio x is chosen. $I_s(x)$ represents the s^{th} element of the vector $I(x)$.

The problem faced by the agent can therefore be described as

$$\text{Problem (A)} \qquad \max_x \sum_{s=1}^{S} \pi(s) v(I_s(x))$$
$$\text{subject to:} \quad \sum_{j=1}^{J} q_j x_j \leq w^0.$$

The problem faced by the principal is:

$$\text{Problem (P)} \qquad \max_{x, I(\cdot)} \sum_{s=1}^{S} \pi(s) u\left(w_{1s} - I_s(x)\right)$$
$$\text{subject to:} \quad x \in \arg\max \text{Problem (A)}, \tag{1}$$
$$\sum_{s=1}^{S} \pi_s v(I_s(x)) \geq v(I^o), \tag{2}$$

$u(w) = -\frac{1}{\gamma} e^{-\gamma w}$.

[5]This is the easiest method of introducing an agency problem, and a need to control risk-taking by the agent, in this context. If either the state or the portfolio were directly observed by the principal, the principal could induce the first-best outcome.

[6]The requirement that $I(w) \geq 0$ is meaningless if $v'(c) \to \infty$ as $c \to 0$. In such a situation, there would be no limit on the ability to penalize the agent.

$$I(w) \geq 0 \text{ for all } w \in (-\infty, \infty) \tag{3}$$

Above, (1) is the incentive constraint on the agent, (2) the participation constraint, and (3) the restriction on $I(\cdot)$, reflecting the limited liability of the agent. Note that the participation constraint is specified in terms of a certain income level from an outside offer, I^o. This allows us to examine the effects of changing $v(\cdot)$.

The gambles that an agent can make will be defined in terms of self-financing portfolios. These are portfolios that cost zero in aggregate, and so are easily replicated to achieve unbounded security positions. For convenience, we compare across self-financing portfolios for which the Euclidean norm of the payoffs is 1.

Definition 1 A *self-financing portfolio* is a portfolio x with $\sum_{j=1}^{J} q_j x_j = 0$ and $\|Ax\| = 1$.

Let F be the set of self-financing portfolios. This set is non-empty for any subset of securities, $\hat{J}$, with $|\hat{J}| \geq 2$. Corresponding to every portfolio $f \in F$ are a set of states in which the portfolio has a negative payoff and a corresponding set in which it has a positive payoff. Let $S_f = \{s \in S | \sum_{j \in J} a_{js} f_j < 0\}$ denote the negative payoff states, and let $S^f = \{s \in S | \sum_{j \in J} a_{js} f_j > 0\}$ denote the positive payoff ones. These sets, of course, are mutually exclusive, with $S_f \cup S^f \subseteq S$. Further, asset prices are assumed to satisfy no arbitrage, so both S_f and S^f are non-empty.

To build some intuition about our problem, we first show that the compensation function must be bounded. Given a compensation scheme $I(\cdot)$, let $x^I = (x_1^I, \ldots, x_J^I)'$ be the portfolio chosen by the agent; that is, x^I is a solution to Problem (A). x^I is referred to as the portfolio induced by $I(\cdot)$.

Proposition 1 *If $v \circ I$ is unbounded, then x^I is unbounded.*

Proofs of all results are relegated to the Appendix (Section 6).

This proposition makes precise the notion of excessive risk-taking with limited liability. If $v \circ I$ is unbounded, the agent sells short an infinite amount of some security $\hat{j}$, and assumes an infinite long position in another security $\tilde{j}$. Limited liability ensures that $I_s(x) \geq 0$ for all s, and, if $v \circ I$ is unbounded, $v(I_s(x)) \to \infty$ for some subset of states (intuitively, states which the portfolio of securities $\hat{j}$ and $\tilde{j}$ has a positive payoff).

Since the principal is risk-averse, an unbounded portfolio x^I leads to a utility to the principal of negative infinity, so can never be optimal.

Proposition 2 *If $I(\cdot)$ is an optimal contract, then x^I is bounded.*

An immediate consequence of the proposition is that, if $v(\cdot)$ is unbounded (for example, if the agent is risk neutral), $I(\cdot)$ must be bounded. Conversely, an unbounded $I(\cdot)$ (such as a linear, increasing function) can be optimal only if $v(\cdot)$ is bounded.

Corollary 2.1 *If $I(\cdot)$ is an optimal contract, then*
(i) $I(x^I)$ is bounded.
(ii) x^I can be implemented by an equivalent contract bounded above by $\max_{s\in S} I(x^I)$.

Therefore, without loss of generality, we can restrict attention to compensation schemes that are bounded above.

As is usual in principal-agent models, we assume that, when indifferent over portfolios, the agent chooses the portfolio most beneficial to the principal. Then, given that $I(\cdot)$ is bounded, the agent picks a bounded portfolio. Note that this assumption is less innocuous here than in many other moral hazard models. In a model where the action set is finite, for example, the principal can often strictly induce the desired action by a slight strengthening of the binding incentive compatibility constraints to strict inequalities. In our model, strict incentives to choose the desired portfolio cannot be achieved without a contract that is decreasing in some range of wealth, as demonstrated in example 1 below.

Example 1

Suppose that $J = S = 2$. The states have unequal probability, and the securities are Arrow–Debreu securities. Let h, ℓ denote the states with high and low probability, respectively. Consider a non-decreasing salary plus bonus scheme, which can be implemented with one target level, $\hat{t}$, and an associated bonus $\hat{I}$. That is, the agent gets 0 if realized wealth $w(s)$ satisfies $w(s) < \hat{t}$, and $\hat{I}$ if $w(s) \geq \hat{t}$. Suppose further that $\hat{t} > \frac{w_0}{q_h+q_\ell}$, so that it is not feasible to attain wealth $\hat{t}$ in both states.

Then, any portfolio in the following class is utility-maximizing for the agent: choose $x_h \geq \hat{t}$ in the high probability state, and let $x_\ell = \frac{w_0 - q_h x_h}{q_\ell}$. In particular, a portfolio with a large long position in security h and a corresponding large short position in security ℓ is utility-maximizing.

Suppose that the principal is highly risk-averse, and, of this set of portfolios, prefers the one with $x_h = \hat{t}$, rather than $x_h > \hat{t}$. Given the compensation scheme described above, this portfolio is only weakly implementable. A strict incentive to choose $x_h = \hat{t}$ must necessarily involve a compensation scheme that has $I(w) < I(\hat{t})$ when $w > \hat{t}$.

Compensation schemes decreasing over some range of wealth are not ruled out by our assumption that the realized compensation is increasing in wealth. Such schemes correspond to penalizing the agent for doing too well. In our framework, there are no arbitrage opportunities. The only way for an agent to earn excessive wealth in one state is by being exposed to an excessive loss in some other state. A penalty for superior performance in our model could correspond to, for example, a costly audit if the return exceeds some threshold. Retirement funds, for example, routinely scrutinize the performance of managers who earn returns above some threshold, on the principle that they must be taking on too much risk.

Any portfolio x that the principal can implement can be implemented with a target scheme, defined below.

Definition 2 (i) A *targeted bonus scheme* is defined by a pair $(t, I) \in \mathbf{R}^S \times \mathbf{R}^S_+$, with $t_s \geq t_{s-1}$. The agent receives I_k if $w_{1s} = t_k$ for some k, and zero otherwise.
(ii) A *step scheme* is defined by a pair $(t, I) \in \mathbf{R}^S \times \mathbf{R}^S_+$, with $t_s \geq t_{s-1}$ and $I_s \geq I_{s-1}$ for $s = 2, \ldots, S$. The agent receives zero if $w_{1s} < t_1$, I_k if $w_{1s} \in [t_k, t_{k+1})$ for some k, and I_S if $w_{1s} \geq t_S$.

A targeted bonus scheme, therefore, need not be monotonic. By definition, a step scheme is monotonic.

The S income levels $I_1, \ldots, I_S$ can all be taken to be equal, so that this definition encompasses the flat fee contract. It also includes the null contract (e.g. set $I_s = 0$ for all s; since $I^o > 0$, the agent will refuse this contract), and a bonus scheme whereby the agent gets a bonus b if a target level $\bar{t}$ is hit (set $t_1 = \bar{t}, I_1 = b$, and $(t_s, I_s) = (t_1, I_1)$ for all $s > 1$).

Any portfolio that can be induced by any compensation scheme (increasing or not) can be induced by an appropriate targeted bonus scheme. If the original compensation scheme is non-decreasing, then the target scheme can, further, be a step scheme.

Proposition 3 *Consider any compensation scheme $I(\cdot)$, and the portfolio induced by it, x^I. The same portfolio is induced by an appropriate targeted bonus scheme, (t, I^t). Further, if $I(\cdot)$ is non-decreasing in w, then x^I can be induced by a step scheme.*

We therefore restrict attention to targeted bonus schemes, where both t and I are bounded.

The problem of the principal can now be stated as:

$$\text{Problem (P'):} \qquad \max_{\{x_j\}_{j=1}^J,\,\{I_s\}_{s=1}^S} \sum_{s=1}^S \pi_s u\left(\sum_{j=1}^J a_{js}x_j - I_s\right)$$

$$\text{subject to:} \qquad \sum_{s=1}^S \pi_s v(I_s) \geq v(I^o), \tag{4}$$

$$\sum_{j=1}^J q_j x_j \leq w^0, \tag{5}$$

$$I_s \geq 0 \quad \text{for all } s = 1, \ldots, S, \tag{6}$$

$$x \text{ is Incentive Compatible for the agent, given } \{I_s\}_{s=1}^S. \tag{7}$$

The solution (x, I) to this problem is converted to a targeted bonus scheme (t, I^t) by a suitable permutation of states. The only goal of the permutation is to ensure that t is non-decreasing, that is, to satisfy $t_s \geq t_{s-1}$ for $s = 2, \ldots, S$.

The contract, therefore, is defined by (x, I). Since x must solve the agent's problem, we sometimes refer to $I(\cdot)$ directly as the contract, and to x as the portfolio induced by the contract I.

3 Role of Limited Liability

Consider any bounded targeted bonus scheme, (t, I). Let $\bar{t}$ and $\bar{I}$ be the maxima of t and I, respectively. We define a limited liability portfolio to be one that achieves zero compensation in states S_f for some self-financing portfolio f, and compensation $\bar{I}$ in states S_f. By replicating a self-financing portfolio often enough, the agent ensures that $w_{1s} \geq \bar{t}$ for all $s \in S^f$. Conversely, $w_{1s} < t_1$ for all $s \in S_f$, so that $I_s(x) = 0$ for these states.

The limited liability portfolio that maximizes the agent's utility is termed a bang-bang portfolio. This latter portfolio describes the optimal gamble an agent would like to take, and therefore characterizes an important implication of the limited liability constraint.

Definition 3 (i) Given a bounded targeted bonus scheme (t, I) and a wealth level w_0, a limited liability portfolio is a portfolio x such that (a) $\sum_{j=1}^{J} q_j x_j - w^0$, and (b) for some self-financing portfolio f, $I_s(x) = 0$ for all $s \in S_f$ and $I_s(x) = \bar{I}$ for all $s \in S^f$.
(ii) A *bang-bang* portfolio is a portfolio $\bar{x}$ that maximizes the agent's utility, over the set of limited liability portfolios.

A limited liability portfolio, therefore, can be formed by choosing some portfolio x that satisfies the budget constraint with equality, some self-financing portfolio f, and replicating f often enough to reach the highest target level in states S^f and fall below the lowest positive income target level in states S_f. The set of limited liability portfolios will therefore vary with the security structure and compensation scheme.

A bang-bang portfolio further depends on the agent's utility function, $v(\cdot)$. The agent earns zero in states S_f, $\bar{I}$ in states S^f, and an intermediate compensation level in states $S \backslash (S_f \cup S^f)$. The following example illustrates this dependence on $v(\cdot)$.

Example 2

Suppose $S = 4$ and $J = 3$. The payoffs of the three securities across the four states are, respectively, $(0, 1, 1, 0)'$, $(1, 0, 1, 1)'$, and $(1, 1, 0, 0)'$. $\pi = (0.4, 0.2, 0.2, 0.2)'$, and $q_j = q$ for all j. Consider some bounded targeted bonus scheme, (t, I).

Clearly, the bang-bang portfolio will involve replications of one of the following two self-financing portfolios: $f_1 = (-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, 0)'$, or $f_2 = (-\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}})'$. Every other self-financing portfolio leads to a lower payoff.

For these two portfolios, we have

	f_1	f_2
S^f	$\{1, 4\}$	$\{1\}$
S_f	$\{2\}$	$\{3\}$

The utility of the agent from the optimal limited liability portfolio that includes replications of f_1 is

$$v_1 = 0.6v(\bar{I}) + 0.2v(\frac{w^0}{q}).$$

From the optimal limited liability portfolio that includes replications of f_2, the agent obtains

$$v_2 = 0.4v(\bar{I}) + 0.4v(\frac{w^0}{2q}).$$

The bang-bang portfolio will be the portfolio that results in utility $\max\{v_1, v_2\}$. Clearly, this depends on $v(\cdot)$.

In the special case of complete markets (that is, $J = S$), the bang-bang portfolio can be characterized more fully. Let $\underline{S} = \arg\min_{s \in S} \pi_s$ be the set of states that have (weakly) lower probability than all other states, with $\underline{s}$ denoting a specific element of the set. $\underline{S}$ is clearly non-empty. Let $\pi_{\underline{s}}$ denote the probability of any single state in $\underline{S}$.[7]

In this case, if $I(\cdot)$ is *any* compensation scheme that satisfies the limited liability constraint, the bang-bang portfolio consists of a short position in some state $\underline{s} \in \underline{S}$ and a long position in all other states.

Proposition 4 *Suppose $J = S$, $I(w) \geq 0$ for all w, and $I(w) > 0$ for some w. Then, the bang-bang portfolio earns the agent $I_{\underline{s}}(x) = 0$ for some $\underline{s} \in \underline{S}$, and $I_s(x) = \bar{I}$ for all $s \neq \underline{s}$.*

There is a continuum of bang-bang portfolios. Given any bang-bang portfolio, adding another unit of portfolio f, the underlying self-financing portfolio, creates a new bang-bang portfolio. Where necessary, we will assume that, of this set, the agent chooses the portfolio that is best for the principal. We note again that implementation is, obviously, a serious issue in this context.

The bang-bang portfolio describes one of the implications of limited liability. Given any targeted bonus scheme (t, I), the agent has the option of choosing to take an income of zero in states S_f, where f is the self-financing portfolio associated with some bang-bang portfolio $\bar{x}$, and obtaining $\bar{I}$ in states S^f. That is, the target in states S_f is simply ignored. Incentive compatibility, therefore, must rule out the agent deviating to the bang-bang portfolio. While this does not characterize all implications of incentive compatibility, this intuition itself is strong enough to characterize the optimal contract in some cases.

[7] All states in $\underline{S}$ have equal probability, by definition.

One such case is if the optimal contract is designed to induce a portfolio that leads to the agent achieving an income zero in some set of states that can be reached by a self-financing portfolio. In this case, the induced portfolio x^I must be a bang-bang portfolio.

Proposition 5 *Let $I(\cdot)$ be an optimal contract. Suppose there exists a limited liability portfolio f such that $I(x^I_s) = 0$ for all $s \in S_f$. Then x^I is a bang-bang portfolio.*

This proposition contains the main implication of limited liability. If the limited liability constraint binds in any states that are reachable by a self-financing portfolio, then the only achievable portfolio is a bang-bang portfolio.

If markets are complete (that is, $J = S$), then, if the limited liability constraint is hit in any state, the induced portfolio leads to zero compensation in some state $\underline{s} \in \underline{S}$ and a constant positive compensation $\bar{I}$ in every other state.

Corollary 5.1 *Suppose that $J = S$, $I(\cdot)$ is an optimal contract, and $I(x^I_s) = 0$ for some $s \in S$. Then $I(x^I_{\underline{s}}) = 0$ for some $\underline{s} \in \underline{S}$, and $I(x^I_s) = \bar{I} > 0$ for all $s \neq \underline{s}$. Further, $x_{\underline{s}} < x_s$ for all $s \neq \underline{s}$.*

The size of the position in state $\underline{s}$ depends on the target wealth $\bar{t}$ required to achieve the income $\bar{I}$. Since the portfolio actually chosen is assumed to maximize the principal's utility, the size of this position also depends on the portfolio the principal wishes to induce. That is, the risk inherent in the portfolio is controlled through $\bar{t}$.

We call the compensation scheme in Proposition 3.3 a bang-bang scheme. If markets are complete, this contract is interpretable as a salary plus bonus contract. The agent gets a salary of zero in all states. In every state except $\underline{s}$, she also gets a bonus $\bar{I}$.

With a bang-bang portfolio, the expected utility of the agent is at least as high as $\pi(S^f)v(\bar{I})$. When $\pi(S^f)$ is close to 1, the agent has a strong incentive to deviate from the induced portfolio to the bang-bang portfolio. In this case, this implication of incentive compatibility (that the portfolio suggested by the principal offer higher utility to the agent than a bang-bang portfolio) restricts the set of portfolios that the principal can induce.

Given a targeted bonus scheme, let f denote the self-financing portfolio associated with a bang-bang portfolio, $\bar{x}$. As before, S^f denotes the states in which f has a positive payoff.[8] Finally, let $\underline{I} = \min_{s \in S^f} \{I(x_s) > 0\}$ be the minimum positive income obtained by the agent across all positive payoff states, and $\bar{I} = \max_{s \in S^f} \{I(x_s) > 0\}$ be the corresponding maximum positive payoff.

Proposition 6 *Consider a sequence $(q^k, \pi^k) \to (q, \pi)$, and an associated sequence of self-financing portfolios, f^k, and incentive compatible compensation schemes, I^k. If $\pi^k(S_{f^k}) \to 1$ then $(\bar{I}^k - \underline{I}^k) \to 0$.*

That is, if the set of states in which the agent can earn a positive payoff has a high probability, the compensation earned in all states $s \in S^f$ is either zero or approximately equal to $\bar{I}$.

With complete markets, we have

Corollary 6.1 *Suppose $J = S$. Consider a sequence $(q^k, \pi^k) \to (q, \pi)$, and an associated sequence of incentive compatible compensation schemes, I^k. If $\pi^k_{\underline{s}} \to 0$ then $(\bar{I}^k - \underline{I}^k) \to 0$.*

In other words, as the probability of the lowest probability state becomes small, the lowest positive compensation level (hence all compensation levels less than the highest one) converges to the highest one.

One implication of Proposition 3.5 is that completing markets can exacerbate the agency problem. Suppose that there are a countably infinite number of states, and the number of securities is increased. As the number of securities becomes large, the agent is able to gamble on finer sets of states, sharpening the incentive compatibility constraint.

Proposition 7 *Suppose the set of states is countably infinite. As $J \to \infty$, $\underline{I}^J \to \bar{I}^J$.*

That is, completing markets restricts the set of incentive compatible contracts, and hence adversely affects risk-sharing between the principal and agent, especially when the latter is the less risk averse party.

As Elul (1995) shows, in an economy with no delegation, adding new securities can fail to be Pareto improving (in fact, can reduce all agents welfare) if markets are

[8] Note that the sets S^f and S_f are common across all bang-bang portfolios, since they do not depend on the number of replications of f required to achieve the upper bound $\bar{I}$.

sufficiently incomplete. Complete markets, however, imply perfect risk-sharing. However, in an economy with delegated portfolio management, while complete markets do facilitate risk-sharing between principals, they intensify the agency problem.

In a situation with financial innovation, the contract offered to the agent may need to be continually revised, if the incentive compatibility constraint is binding. If there is a cost to such revision, or a lag in the process for any reason, financial innovation (which lets the agent bet on finer partitions of the state space) may lead to incentive compatibility being violated, and the agent taking on excessive risk.

Two incentive compatible schemes of particular interest are the the flat scheme and the bang-bang one. The flat scheme is defined by $I(w) = \bar{I}$ for all w, and the bang-bang one by $I(w) = 0$ if $w < \bar{t}$ and $I(w) = \bar{I}$ if $w \geq \bar{t}$.

Hence, if the number of states and securities is large, all incentive compatible schemes approximate either the flat scheme or the bang-bang one. Therefore, in this case, case, limited liability strongly restricts the feasible set of contracts. The choice between a flat scheme and a bang-bang one will depend on risk-sharing considerations between the principal and agent.

The problem of multiple optimal portfolios with a bang-bang scheme was discussed at the end of the previous section. Note that it applies just as strongly with the flat scheme. When compensation does not depend on the portfolio at all, the principal can weakly implement any portfolio. However, since there is no reason for the agent to choose any portfolio over any other one, it is difficult to characterize the principal's portfolio as the outcome we can expect. We can only make the weaker statement, that the agent has no incentive to choose a large short position in any security.

4 Value at Risk

A general definition of Value at Risk is provided by Jorion (1997). Let $F(\cdot)$ be the probability distribution (over wealth at time 1, w_1) induced by a specific portfolio, x. Then, the value of risk (VaR) of the portfolio, given a probability p, is defined as a loss amount,

$$L(p, x) \quad = \quad \max \{ \; \tilde{L} \mid F(w^0 - \tilde{L} | x) = p \}.$$

If F is strictly increasing, this can be written more simply as $L(p, x) = F^{-1}(p) - w^0$.

That is, the VaR indicates the maximum amount that the portfolio will lose with probability p. Given p, portfolios with a higher VaR are considered risky. In applied portfolio delegation contexts, the agent is often given a target VaR, $\bar{L}$, in addition to a probability p, and is required to ensure that the chosen portfolio, x, satisfy $L(p,x) \leq \bar{L}$.

In our model, $w_{1s} = \sum_{j=1}^{J} a_{js}x_s$. Hence, the loss in any state s is just $w^0 - \sum_{j=1}^{J} a_{js}x_j$. To define the VaR of a portfolio x, order the states and associated probabilities in increasing order of w_{1s} (that is, in increasing order of wealth at time 1). Using these ordered states, define state $\tilde{s}$ as the state for which

$$\sum_{s=1}^{\tilde{s}-1} \pi_s < p,$$
$$\text{and } \sum_{s=1}^{\tilde{s}} \pi_s \geq p.$$

The VaR of the portfolio, $L(p,x)$, is then defined simply as $L(p,x) = w^0 - \sum_{j=1}^{J} a_{j\tilde{s}}x_j$.

Since the portfolio x is not directly observed in our setting, a VaR rule is not enforceable. In some applied contexts, the VaR of the portfolio is self-reported by the agent to the principal in a repeated game. The principal can then estimate the empirical frequency of a loss exceeding the reported VaR in any given time period, and compare this to p. For example, the Federal Reserve Board uses such a mechanism to adjust the capital requirements of banks. Banks whose losses exceed their self-reported VaR with a frequency significantly higher than p have their capital requirements raised.

We abstract away from the enforcement and observability issue in our static setting. The results of this section can therefore be interpreted as a best case scenario for VaR: if VaR were perfectly observed and enforceable, would a VaR rule be as effective in controlling excessive risk, or lead to similar outcomes, as the optimal contract?

Since a VaR restriction by itself does not help define the size of the position taken by the agent in states in which the VaR restriction is irrelevant, we define a VaR compensation scheme to include an associated compensation scheme, I^r.

Definition 4 A VaR compensation scheme consists of a triple (L, p, I^r), where (L, p) is a VaR restriction and $I^r(\cdot)$ is a compensation scheme.

Now, we consider the following question. Suppose $I(\cdot)$ is a contract that solves the principal's problem; that is, $I(\cdot)$ is incentive compatible for the agent and optimal for the principal in this class of contracts. Does there exist a corresponding VaR compensation scheme, (L, p, I^r), that implements the same induced portfolio, x^I?

Suppose, first, that $I^r = I$ for all w. Then, a VaR rule is redundant at best. If the extra constraint added by the VaR restriction is not binding, then it is redundant. If it is binding, then it induces a portfolio different from x^I, and so must lead to lower utility for the principal. That is, it is sub-optimal if binding, and redundant if not.

Therefore, a VaR compensation scheme (L, p, I^r) is interesting only if $I^r(x_s^I) \neq I(x_s^I)$ for some s. In terms of excessive risk-taking, a natural comparison is to compensation schemes I^r that are unbounded above. Following Proposition 2.2 above, we consider the case in which the agent's problem is potentially unbounded. If, for example, the agent were extremely risk averse, there may be no need for a VaR scheme; even a linear, increasing contract will induce an interior solution and a limit on the risk taken by the agent.

Definition 5 Given a compensation scheme $I(\cdot)$, the agent's problem is *potentially unbounded* if $v \circ I$ is unbounded.

One example in which the agent's problem is potentially unbounded is when the agent is risk neutral, and $I(\cdot)$ is unbounded. This includes, but is not limited to, the case of a linear $I(\cdot)$.

Consider the optimal action of the agent when presented with a VaR scheme $(L, p, I^r(\cdot))$. The problem of the agent is expressed as

$$\text{Problem (A')} \qquad \max_x \sum_{s=1}^{S} \pi(s) v(I_s^r(x))$$

$$\text{subject to:} \quad \sum_{s=1}^{S} q_s x_s \leq w^0$$

$$VaR(x, p) \leq L.$$

Let x^r be the portfolio chosen by the agent when faced with the VaR compensation scheme (L, p, I^r). For any portfolio x, define the maximum loss the portfolio can suffer as $\mathcal{L}(x) = w^0 - \min_s \left\{ \sum_{j=1}^{J} a_{js} x_j \right\}$. If a VaR scheme $L(p, I^r)$ results in a maximum loss that is finite, we term the scheme effective.

Definition 6 A VaR scheme (L, p, I^r) is *effective* if x^r satisfies the VaR restriction (L, p), and $\mathcal{L}(x^r) < \infty$.

That is, a VaR scheme is effective if the portfolio chosen by the agent is finite (i.e., has a finite position in each security). The scheme is ineffective if the agent takes on an unbounded gamble, which leads to an infinite short position in some state. An ineffective scheme, therefore, results in an unbounded loss for the principal with positive probability. The requirement that x^r satisfy the VaR rule is to prevent vacuous rules that no portfolio can satisfy.

To motivate the rest of this section, we present an example on the effectiveness of VaR in this setting.

Example 3

Suppose $S = 3$, and $J = 2$. The probabilities of the three states are $(0.2,\ 0.1,\ 0.7)'$. The payoffs of the securities are $(1,\ 2y,\ 0)'$ for security 1, and $(0,\ y,\ 0.5)'$ for security 2, where $y > 0$. Further, $q_1 = 0.6$, $q_2 = 0.3$, and $w_0 = 6$.

With two securities, there are exactly two self-financing portfolios, with $f_1 = (\frac{1}{\sqrt{2}},\ -\sqrt{2})$ and $f_2 = -f_1 = (-\frac{1}{\sqrt{2}},\ \sqrt{2})$.

Suppose the agent is risk-neutral (so that $v(c) = c$), and $I^r(w) = \max\{0, \alpha w\}$, for some $\alpha > 0$. That is, $I^r(\cdot)$ satisfies limited liability, and is linear for positive w.

Consider the following VaR scheme: (L, p, I^r), where $L = 4$, and $p = 0.25$.

Suppose, first, that $y = 0.1$. Then, every portfolio $x = (x_1, x_2)$ with $q_1x_1 + q_2x_2 = 6$ suffers a loss of 4 in state 2. Therefore, to satisfy the VaR rule, the chosen portfolio x^r must have a loss strictly less than 4 in every other state. Therefore, it must be that $x_1^r > -4$, and $x_2^r > -8$. Alternatively, the chosen portfolio can include at most $4\sqrt{2}$ replications of f_1 or f_2. In this case, the VaR scheme is equivalent to a short sale constraint on either self-financing portfolio, and is effective, since $\mathcal{L}(x^r) = L = 4$.

Next, suppose that $y = 0.2$. Then, every portfolio that costs 6 has a payoff of 4 (and hence a loss of 2) in state 2. In this case, the VaR rule is ineffective. Consider any portfolio $\tilde{x}$ that satisfies the budget constraint, and let $x = \tilde{x} + zf_2$, where $z \in \mathbf{R}$ represents the number of replications of f_2. Since the agent's problem is potentially unbounded, the agent maximizes utility by letting $z \to \infty$. This leads to an infinite loss in state 1, but with a probability $0.2 < p$. Hence, this portfolio satisfies the VaR constraint.

The implication of this example is that, if p is too high, the VaR scheme will be ineffective at controlling the size of the maximum loss. It is clear that, if $p \leq \pi_{\underline{s}}$, the probability of the lowest probability state, such a scheme will result in $\mathcal{L}(x^r) \leq L$. However, as the example indicates, depending on the security structure, p can be higher than this and yet be effective.

Let $\underline{f} = \arg\min_{f \in F} \ \pi(S_f)$, and $\hat{f} = \arg\max_{f \in F} \ \pi(S^f)$. That is, $\underline{f}$ is the portfolio in the set of self-financing portfolios that has the smallest probability over states of negative payoff. In terms of the utility of the agent, this is the cheapest gamble the agent can take, in the set of self-financing portfolios. It minimizes the probability of states over which the agent's limited liability constraint is binding. Conversely, $\hat{f}$ maximizes the probability over states with positive payoff. This is the most rewarding gamble the agent can take. Note that the portfolios $\underline{f}$ and $\hat{f}$ need not be the same, but it must be that $\pi(S_{\underline{f}}) \leq 1 - \pi(S^{\hat{f}})$.

The portfolio $\hat{f}$ helps define a necessary condition for a VaR scheme to be effective, and $\underline{f}$ defines a sufficient condition.

Proposition 8 *Consider a VaR compensation scheme, (L, p, I^r). Suppose the agent's problem is potentially unbounded under I^r. Then, the VaR scheme is effective if $p \leq \pi(S_{\underline{f}})$, and only if $p \leq 1 - \pi(S^{\hat{f}})$.*

With complete markets, $\underline{f} = \hat{f}$. This portfolio has a negative payoff in state $\underline{s}$ and a positive one in every other state. Hence,

Corollary 8.1 *Suppose $J = S$. Consider a VaR compensation scheme, (L, p, I^r). Suppose the agent's problem is potentially unbounded under I^r. Then, the VaR scheme is effective if and only if $p \leq \pi_{\underline{s}}$.*

Consider a situation in which there are a countably infinite number of states, and $J \to \infty$, that is, the setting of Proposition 3.7. A static VaR rule will eventually be in this situation as well. For any finite p, the agent will eventually be able to gamble on losses that have smaller probability.

Proposition 9 *Consider a fixed VaR compensation scheme, (L, p, I^r). Suppose there are a countably infinite number of states, and the agent's problem is potentially unbounded under I^r. As $J \to \infty$, there exists a $\hat{J}$ such that the VaR scheme is ineffective for $J \geq \hat{J}$.*

This Proposition provides some insight into the intuition behind the Basak and Shapiro (1999) result that, in their framework, VaR rules are completely ineffective at controlling risk-taking behavior by the agent. In their continuous time framework with dynamically complete markets, there is no discrete analog to $\underline{s}$ or $\pi_{\underline{s}}$. However low p is set in a VaR rule, any finite p is too high, since the agent can take on excessive risk with a smaller probability.

The optimal contract, $I(\cdot)$, induces a maximum loss $\mathcal{L}(x^I)$. To achieve a maximum loss no higher than this, a VaR scheme requires a tighter sufficient condition than in Proposition 4.4. Suppose that $S_{\underline{f}} = \{s_1, s_2\}$, with $\pi(s_1) < \pi(s_2)$, and $\sum_{j=1}^{J} a_{j\,s_1}\ \underline{f}_j < \sum_{j=1}^{J} a_{j\,s_2}\ \underline{f}_j$. That is, the self-financing portfolio $\underline{f}$ has a smaller payoff in state s_1 than s_2. Then, a VaR restriction (L, p) with $p = \pi(S_{\underline{f}})$ is sufficient to ensure that $w_{1s_2} \geq w_0 - L$, but it is possible that $w_{1s_1} > w_0 - L$: the loss in state s_1 may well be higher than L. This is not a violation of the VaR restriction, since this state has probability less than p.

To account for this, let $\hat{s} = \arg\min_s \pi(s \mid s \in S_f \text{ for some } f \in F)$. Then, $\hat{s}$ represents the lowest probability state in which a negative payoff can be obtained. If $p \leq \tilde{\pi}_{\hat{s}}$, the VaR scheme ensures that $\mathcal{L}(x^r) \leq L$. The necessary condition remains the same as earlier.

Proposition 10 *Consider any optimal contract, $I(\cdot)$, and suppose that the agent's problem is potentially unbounded under $I^r(\cdot)$. Then, a VaR scheme (L, p, I^r) with $L = \mathcal{L}(x^I)$ results in $\mathcal{L}(x^r) \leq L$ if $p \leq \pi_{\hat{s}}$ and only if $p \leq 1 - \pi(S^{\hat{f}})$.*

Hence, an appropriately defined VaR rule is effective at controlling the maximum loss of a portfolio. In this setting, with $p \leq \pi_{\hat{s}}$, the VaR rule is equivalent to a short sale constraint on all self-financing portfolios with a negative payoff in state $\hat{s}$. However, a VaR scheme with p set too high is completely ineffective.

As before, with complete markets, the necessary and sufficient conditions collapse to a single condition, $p \leq \pi_{\underline{s}}$.

Corollary 10.1 *Suppose $J = S$. Consider any optimal contract, $I(\cdot)$, and suppose that the agent's problem is potentially unbounded under $I^r(\cdot)$. Then, a VaR scheme (L, p, I^r) with $L = \mathcal{L}(x^I)$ results in $\mathcal{L}(x^r) \leq L$ if and only if $p \leq \pi_{\underline{s}}$.*

The intuition is clearest in the case of complete markets: if p is too high, there is no effective restriction on $x_{\underline{s}}$, since the probability of losing $w^0 - x_{\underline{s}}$ is only $\pi_{\underline{s}} < p$, which is permissible under the VaR rule as specified. Hence, for example, when there are a large number of states, even a VaR rule with a low p may not be effective in controlling the size of the maximum loss.

Finally, through an example, we show that a VaR scheme is not a good replacement for an upper bound on compensation unless the number of securities is exactly 2.

Example 4

Suppose markets are complete ($J = S$), and the agent is risk-neutral. Let $I(\cdot)$ be an optimal contract that achieves the bang-bang outcome, with $x_s^I > 0$ for at least two states $s \neq \underline{s}$. Let $\bar{I}$ be the upper bound on $I(\cdot)$, and $\bar{w}$ the wealth level that achieves compensation $\bar{I}$. Define $I^r(\cdot) = I(\cdot)$ for $w \leq \bar{w}$, and $I^r(w) = \bar{I} + \alpha(w - \bar{w})$ for $w > \bar{w}$, where $\alpha > 0$.

Consider a VaR scheme with $p = \pi_{\underline{s}}$, and $L = w^0 - \sum_{j=1}^{J} a_{j\underline{s}} x_j^I$. That is, L is set to the loss suffered by x^I in state $\underline{s}$.

Proposition 11 *Under the conditions of Example 4, (i) $x^r = x^I$ if and only if $S = 2$. (ii) the VaR scheme leads to the same utility for the principal as the optimal contract $I(\cdot)$ if and only if $S = 2$.*

That is, the VaR rule (L, p) can be thought of as equivalent to the optimal contract (in terms of utility achieved by the principal) if and only if $S = 2$. However, in this case, it must also be that the portfolio chosen by the agent, x^{I^r}, and the realized income of the agent, $x^r I^r$, match. That is, $I^r = I$ at the two points of interest, x_1 and x_2. In this case, as argued above, the VaR rule is redundant if I^r is incentive compatible. Hence, one interpretation of the use of a VaR scheme in this context is that it is a substitute for the incentive compatibility constraints.

In the setting of Example 4, when faced with a VaR scheme with $p = \underline{s}$, the agent's optimal portfolio consists of a short position $w^0 - L$ in security $\underline{s}$, and a corresponding long position in the "cheapest" of the remaining states (that is, the state with lowest $\frac{\pi_s}{q_s}$. In all other securities, the agent invests zero. By contrast, the bang-bang solution induces the same short position in security $\underline{s}$, but an equal long position in every other state. Hence, the overall riskiness of the VaR scheme is higher, even though it leads to the same maximum loss.

5 Conclusion

We examine the effects of portfolio delegation with limited liability, in a two-period contingent claims framework. As has been observed by, among other, Grinblatt and Titman (1989), limited liability provides the agent with an incentive to take on too much risk. The innovation in our paper is that we incorporate this into the incentive compatibility constraints for the principal's problem, and examine the resultant set of incentive compatible contracts.

Limited liability in our model is interpreted simply as a lower bound on the agent's payoff. We first show that, in any solution to the principal's problem, the agent must earn at most a finite payoff in any state. More generally, this implies that any outcome that can be achieved can be achieved with a bounded contract.

The contingent claims framework enables characterization of the optimal gamble the agent would like to take, in terms of the bang-bang portfolio. We find that the impact of limited liability on incentive compatibility depends critically on the set of states in which the agent is exposed to getting his worst outcome. If the probability of this set is low, incentive compatibility is extremely restrictive. The contract must be close to a flat contract over all states in which the agent does not get this lower bound. In the limiting case, as the probability of this set goes to zero, the contract approaches the flat contract over the remaining states. Depending on risk-sharing needs, the optimal contract here may be either the flat contract or the bang-bang one.

We find that a VaR compensation scheme, appropriately set, can be effective in controlling the maximum exposure of the portfolio chosen by the agent. In a situation of financial innovation, however, the probability inherent in a VaR scheme may need to be continually adjusted (i.e., lowered). Financial innovation allows an agent to gamble on finer partitions of the state space. This has the dual effect of sharpening the incentive compatibility constraints (i.e., fewer compensation schemes are now incentive compatible) and weakening the impact of an existing VaR scheme with a fixed p.

While we study the static delegation problem, the results generalize in a straightforward manner to any finite horizon economy. This is easy to see with complete markets, where we can re-interpret the states as time-event pairs. Then, an agent's optimal deviation would be to take on a large short position in the time-event pair with lowest probability, and all our results would follow. The infinite horizon case remains to be studied.

6 Appendix: Proofs

Proposition 2.2

Suppose $v \circ I$ is unbounded. Note that, if x^I is finite, the resultant utility of the agent is also finite.

Now, consider the following strategy for the agent. Choose any two securities $\hat{j}$ and $\tilde{j}$. Consider a self-financing portfolio f, that consists of these two securities alone (that is, $\sum_{j=\hat{j},\tilde{j}} q_j x_j = 0$. Let x consist of any security positions such that $\sum_{j=1}^{J} q_j x_j = w^0$, plus n replications of the portfolio f.

Now, as $n \to \infty$, $I_s(x))$ is bounded below by 0, so that $v(I_s(x))$ is bounded below by 0. However, since $v \circ I$ is unbounded, as $n \to \infty$, $v(I_s(x))) \to \infty$ for $s \in S^f$. Hence, $\sum_{s=1}^{S} \pi_s v(I_s(x)) \to \infty$.

Any portfolio chosen by the agent must offer at least as high a utility as the above portfolio, and therefore must be unbounded. ∎

Proposition 2.3

Suppose $I(\cdot)$ is unbounded. As argued in Proposition (2.2), the agent picks an unbounded portfolio, x. In particular, x contains infinite replications of some self-financing portfolio, $\hat{f}$. Hence, $w(s) - I_s \to -\infty$ for $s \in S_{\hat{f}}$. For $s \in S^{\hat{f}}$, we could have any of $w_{1s} - I_s \to -\infty$, $w_{1s} - I_s \in (-\infty, \infty)$, or $w_{1s} - I_s \to \infty$. In the first two cases, it is obvious that $\sum_{s=1}^{S} \pi_s u(w_{1s} - I_s) \to -\infty$. In the third, since $u(\cdot)$ is strictly concave, again $\sum_{s=1}^{S} \pi_s u(w_{1s} - I_s) \to -\infty$.

Finally, note that when $I(\cdot)$ is bounded, the set of utility maximizing portfolios for the agents includes bounded portfolios. For any such bounded portfolio, the utility of the principal is also bounded. One feasible, bounded $I(\cdot)$ that leads to bounded utility is the flat contract; $I(w) = I^o$ for all w. This satisfies the participation constraint and all incentive constraints, and can induce a finite portfolio. ∎

Proposition 2.6

Let x^I denote the portfolio chosen by the agent when offered the contract $I(\cdot)$. Re-order w, the realized wealth level under x, in ascending order; let w^a indicate this re-ordered vector.

Consider the targeted bonus scheme $(t, I^t) = (w^a, I(w^a))$. That is, if $w_s = w_k^a$ for some k, then the agent's compensation is $I(w_k^a)$, otherwise the agent is paid zero.

Let $B(q) = \{x | \sum_{s=1}^S q_s x_s \leq w^0\}$ denote the budget set faced by the agent when choosing the portfolio. Then, if x^I is chosen under compensation scheme $I(\cdot)$,

$$\sum_{s=1}^S \pi_s v(I_s(x^I)) \quad \geq \quad \sum_{s=1}^S \pi_s v(I_s(x)) \text{ for all } x \in B(q).$$

Now, consider the agent's choice when offered the targeted bonus scheme (t, I^t) defined above. Clearly, $\sum_{s=1}^S \pi_s v(I_s^t(x^I)) = \sum_{s=1}^S \pi_s v(I_s(x^I))$, and

$$\sum_{s=1}^S \pi_s v(I_s^t(x^I)) \quad < \quad \sum_{s=1}^S \pi_s v(I_s(x^I)) \text{ for all } x \in B(q),\ x \neq x^I.$$

Hence, $\sum_{s=1}^S \pi_s v(I^t(x_s^I)) \geq \sum_{s=1}^S \pi_s v(I^t(x_s))$ for all $x \in B(q)$. That is, portfolio x^I is optimal for the agent under the targeted bonus scheme (t, I^t).

Next, suppose $I(\cdot)$ is non-decreasing. Then, the above arguments follow through completely for a step scheme, with (t, I^t) as defined above. Hence, in this case, x^I can be induced by a step scheme. ■

Proposition 3.2

First, choose any portfolio $\hat{x}$ using the wealth w_0. Then, consider a portfolio that has a long position y in states $s \neq \underline{s}$, and $x_{\underline{s}} = -\frac{y \sum_{s \neq \underline{s}} q_s}{q_{\underline{s}}} < 0$. The portfolio so defined is a self-financing portfolio. Let f denote this portfolio.

Now, replicate this portfolio often enough to ensure that $I_s \geq \bar{I}$ for all $s \neq \underline{s}$, and $I_{\underline{s}} = 0$. Let n be the number of replications required, where n must be finite, since (t, I) is bounded.

Consider the portfolio $\tilde{x} = \hat{x} + nf$. Then, the expected utility from this portfolio is

$$\pi_{\underline{s}} v(0) + (1 - \pi_{\underline{s}}) v(\bar{I}) \quad = \quad (1 - \underline{s}) v(\bar{I}).$$

Now, every other limited liability portfolio must have $I_s = 0$ for some s. Therefore, the portfolio $\tilde{x}$ yields expected utility at least as high as the utility derived from any other limited liability portfolio. ■

Proposition 3.3

For some limited liability portfolio f, $I(x_s^I) = 0$ for all $s \in S_f$. Now, the agent can replicated f often enough to earn $\bar{I}$ in states $s \in S^f$, without affecting payoffs in states $s \in S_f$ or $s \in S \backslash \{S_f \cup S^f\}$. Hence, x^I must be a limited liability portfolio.

However, in the set of limited liability portfolios, the bang-bang portfolios are optimal for the agent. Hence, any limited liability portfolio that is not a bang-bang portfolio is not incentive compatible, and x^I must be a bang-bang portfolio. ■

Corollary 3.4

Follows immediately from Propositions 3.3 and 3.2. ■

Proposition 3.5

Suppose $I(\cdot)$ is an incentive compatible contract. The expected utility the agent obtains from the bang-bang strategy is

$$\pi(S_f)v(0) + \pi(S^f)v(\bar{I}) + \sum_{s \in S \backslash \{S^f \cup S_f\}} \pi_s v(I_s) \quad = \quad \pi(S^f)v(\bar{I}) + \sum_{s \in S \backslash \{S^f \cup S_f\}} \pi_s v(I_s).$$

Incentive compatibility implies that this must be no higher than $\sum_{s=1}^{S} \pi_s v(I_s)$, the expected utility from the portfolio prescribed by the principal. This implies that

$$\pi(S^f)v(\bar{I}) \quad \leq \quad \sum_{s \in \{S^f \cup S_f\}} \pi_s v(I_s).$$

As $\pi(S^f)^k \to 1$, $\sum_{s \in S_f} \pi_s v(I_s)$ converges to zero (since $I(\cdot)$ is bounded). Further, by definition, $I_s \leq \bar{I}$ for all $s \in S$. it must be that each term I_s on the right hand side also converges to $\bar{I}$. Hence, $(\bar{I}^k - \underline{I}^k) \to 0$. ■

Corollary 3.6

When $J = S$, $S^f = S \backslash \{\underline{s}\}$. Hence, as $\pi_{\underline{s}} \to 0$, $\pi(S^f) \to 1$. ■

Proposition 3.7

In the limit, with $J = S$, there exists a self-replicating portfolio that has a negative payoff in the state with minimum probability. $\underline{s}$, and positive payoffs in every other state. Hence, $S^f \to S \backslash \underline{s}$ as $J \to \infty$. Since S is infinite, $\pi(S^f) \to 1$ as $J \to \infty$. From Proposition 3.5, $\underline{I}^J \to \bar{I}^J$. ■

Proposition 4.4

Suppose first that $p > 1 - \pi(S^{\hat{f}})$. Consider any portfolio x that satisfies the VaR rule (L, p), and let $\tilde{x} = x + z\hat{f}$, where $z \in \mathbf{R}_+$. Since the agent's problem is potentially unbounded, the agent achieves a higher utility from letting $z \to \infty$ than any from choosing any finite z. But then any portfolio chosen by the agent must involve infinite gain in some states, and infinite losses in others. Hence, the VaR rule is ineffective, proving the "only if" part of part (i) of the Proposition.

Next, consider the "if" part. Let x^r be the portfolio chosen by the agent. Suppose the VaR scheme is ineffective. Then, there must exist some self-financing portfolio f such that x^r includes infinite replications of f. Therefore, $\text{Prob}(w_0 - w_{1s} \leq L) \geq \pi(S_f) \geq \pi(S_{\underline{f}})$, where the last inequality follows by definition of $\underline{f}$. Now, $p \leq \pi(S_{\underline{f}})$. Hence, the VaR of x^r at the probability level p exceeds L, which is a contradiction. This proves part (i). ■

Corollary 4.5

With $J = S$, $\underline{f} = \hat{f}$. Further, $S_{\underline{f}} = \{\underline{s}\}$, and $S^{\hat{f}} = S \backslash \{\underline{s}\}$. ■

Proposition 4.6

Consider $\pi^J(S_{\underline{f}})$. As $J \to \infty$, $\pi^J(S_{\underline{f}}) \to 0$. Hence, for any fixed p, and any sequence of securities $J \to \infty$, there must exist some $\hat{J}$ such that $\pi^J(S_{\underline{f}}) \leq p$ for $J \geq \hat{J}$. The statement of the proposition now follows from Proposition 4.4. ■

Proposition 4.7

Consider any VaR scheme (L, p, I^r), with $L \leq \mathcal{L}(x^I)$, where x^I is the portfolio chosen under the optimal contract $I(\cdot)$. Suppose first that $p \leq \pi_{\hat{s}}$. Then, the portfolio chosen by the agent, x^r, must satisfy $\text{Prob}(w \leq w^0 - \mathcal{L}(x^I)) \leq \pi_{\hat{s}} + \pi(\hat{S})$. Let $\hat{S} = \cup_{f \in F} S_f$ be the union of all states over which negative payoffs can be obtained. By definition of $\pi(\hat{s})$, it follows that $w_{1s} \geq w^0 - \mathcal{L}(x^I)$ for all $s \in \hat{S}$, with strict equality for $s \in \hat{S}$ such that $\pi(s) > \pi_{\hat{s}}$.

Now, if $\hat{S} = S$, then $\mathcal{L}(x^r) \leq L$. Suppose $\hat{S} \neq S$. Then, states in $S \backslash \hat{S}$ cannot be reached by any security, and any portfolio must lose w^0 in these states. But this must be true of portfolio x^I as well, so that $\mathcal{L}(x^I) \geq w^0$. Therefore, $L \leq \mathcal{L}(x^I)$, together with the feasibility requirement of a VaR scheme, ensures that $\mathcal{L}(x^r) \leq \mathcal{L}(x^I)$.

The "if" part follows directly from Proposition 4.7: if a VaR scheme is ineffective, its maximum loss is infinite, necessarily greater than any finite L. ■

Corollary 4.8

When $J = S$, $\hat{s} = \underline{s}$, and $S^{\hat{f}} = S \backslash \{\underline{s}\}$. ■

Proposition 4.9

Suppose first that $S = 2$. Then, L restricts $x^r_{\underline{s}} \leq x^I_{\underline{s}}$. Since $x^I_{\underline{s}}$ was optimal under $I(\cdot)$, it must remain optimal under $I^r(\cdot)$. Given the budget constraint, part (i) of the Proposition now follows. Since $I^r(w) = I(w)$ over this range of w, part (ii) also follows.

Suppose, next that $S > 2$. Under the VaR scheme, the agent chooses $x^r_{\underline{s}} = w^0 - L$. The net wealth obtained is $(w^0 + q_{\underline{s}} x^r_{\underline{s}})$. Since I^r is linear in w, the optimal investment for the agent is to invest this entire net wealth in security $\hat{s}$, where $\hat{s} = \arg\min_{s \in S \backslash \underline{s}} \frac{\pi_s}{q_s}$, and choose $x^r_s = 0$ for all $s \neq \underline{s}, \hat{s}$. Clearly, this is a riskier portfolio than x^I. Parts (i) and (ii) of the proposition now follow. ■

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