Sequentially Optimal Mechanisms

Vasiliki Skreta*

Department of Economics, University of Pittsburgh, Pittsburgh, PA 15260

email: vasst10+@pitt.edu

01. 30. 2000

Abstract

We characterize the revenue maximizing mechanism in a two-period model. A risk neutral seller owns one unit of a durable good and faces a risk neutral buyer whose valuation is private information. The seller has all the bargaining power; she designs an institution to sell the object at t=0 but she cannot commit not to change the institution at t=1 if trade does not occur at t=0. The seller's objective is to pick the revenue maximizing mechanism. We show that if the probability density function of the buyer's valuation satisfies certain conditions, the optimal mechanism is to post a price in each period. Previous work has examined price dynamics when the seller behaves sequentially rationally. We provide a reason for the seller's choice to post a price even though she can use infinitely many other possible institutions: posted price selling is the optimal strategy in the sense that it maximizes the seller's revenues. *Keywords: mechanism design, optimal auctions, sequential rationality. JEL Classification Codes: C72, D44, D82.*

1 Introduction

The main goal of a seller is to maximize profits. Theorists (see Myerson (1981) and Riley and Samuelson (1981)) have provided an answer to the following question. When the seller has incomplete information

*I am indebted to Professor Reny and Professor Aoyagi for their guidance and support. P. Reny made numerous constructive suggestions on the model used in this paper. All errors are of course mine.

about the buyers' willingness to pay, what is form of the institution that maximizes a seller's revenues out of all possible arrangements? The answer is provided under the assumption that the seller can tie her hands and is able to commit to the institution she chose, even though ex post there may exist more profitable arrangements. In other words, the revenue maximizing institution is characterized under the restriction that the seller will behave in a non-credible way. If one relaxes this assumption, in other words, if the seller behaves sequentially rationally, what is the form of the revenue maximizing mechanism? This is the question we aim to answer.

Why ? It seems a more natural scenario. In many instances the assumption that the seller can tie her hands, even though ex-post it may turn out that she has more attractive possibilities, is far fetched. In practice people are often tempted not to commit as the following examples demonstrate. Christies in Chicago auctions the same bottles of wine that failed to sell in earlier auctions. The government reauctions properties that fail to sell: lumber tracts, oil tracts and real estate are put up in a new auction if no bidder bids above the reserve price[1]. McAfee and Vincent (1997), note that "either implicitly or by explicit policy, auctioneers are acknowledging the impossibility of resisting a temptation to try to resell an object who failed to meet a reserve price at an earlier auction."

To illustrate the situation let us look at the following scenario. Consider a risk neutral seller who owns one unit of a durable good that is of no value to her. She decides to auction the item using a revenue maximizing institution. She faces just one risk neutral buyer whose valuation is unknown but is distributed according to a continuous distribution $f(.)$ on a closed interval, which is common knowledge. The seller who is aware of the work of Myerson (1981) knows that in this situation the optimal, in terms of revenue, institution is to post a price. Suppose that the there are T=2 periods, t=0,1. The buyer announces his valuation to the seller and if it is above the seller's posted price, he obtains the object and pays the price. Suppose that at date 0 the seller posts the price that maximizes ex-ante revenues but the buyer announces a valuation below the price. No trade takes place. It is well known that in order for the seller to maximize her ex-ante expected revenues she should commit to tie her hands and not try to sell the item again using

[1] These examples are also mentioned in McAfee and Vincent (1997).

a different institution in a subsequent period. After the buyer declines the initial sale contract, it is not sequentially rational for the seller to tie her hands. At date 1 the seller knows that there exist gains from trade but they were not realized because the price she posted was above the buyer's valuation. If the seller behaves sequentially rationally, then if no trade takes place at t=0, the seller will try to sell the item at t=1 using a different institution that maximizes revenues from that point on, which clearly changes the buyer's strategic considerations at t=0. What does the revenue maximizing mechanism look like in this case? Does the seller offer a set of lotteries at period t=0 or does she simply post a price? Does the seller use a first period mechanism that allows her to infer with precision the type of the buyer in the case the buyer rejects this mechanism, hoping that she can use her sharper estimate to extract the buyer's surplus in the second period? Or is it too costly in terms of expected revenues to do so?

In this paper we characterize the revenue maximizing mechanism when the seller behaves sequentially rationally. We show that the seller will maximize her expected revenues by posting a price in each period. The revenue maximizing mechanism is derived out of a very general class of mechanisms. This work extends the works of Myerson (1981), Riley and Samuelson (1981) and Hart and Tirole (1988). We show that for any possible history of the game, at the beginning of the period t=1 the seller will maximize expected revenues by posting a price. This result is derived restricting attention to direct revelation mechanisms since the seller's problem at the beginning of t=1 is isomorphic to the problem with full commitment. Subsequently, we derive the revenue maximizing set of contracts for t=0, denoted by M_0. This is a more difficult task since one cannot appeal to the revelation principle. M_0 is derived under some conditions imposed on the probability density function of the buyer's valuation. We do not impose any restrictions on the potential form of M_0. Under the assumptions made, M_0 is equivalent to a posted price. Previous work has assumed that the seller's strategy is to post a price and the problem of the seller is to find what price to post. We provide a reason for the seller's choice to post a price even though she can use infinitely many other possible institutions: posted price selling is the optimal strategy in the sense that it maximizes the seller's revenues.

1.1 Related Literature

This question has not been addressed previously in the economics literature but is related to the optimal auctions literature, to the durable goods monopolist literature and to the incentives literature that studies repeated relationships.

The Optimal Auction Literature (Myerson (1981), Riley and Samuelson (1981)), characterizes the revenue maximizing mechanism for a risk neutral seller facing a fixed number of risk neutral buyers. The optimal auction form is derived assuming that the seller can tie her hands in the case that the item failed to sell in the auction. When the seller is not able to commit to this static optimal institution, and she offers a new setup, in case the static one failed to realize any gains from trade, she scarifies ex-ante expected revenues.

The Durable Goods Monopolist literature (Bulow (1982), Stockey (1981), Gul-Sonnenschein-Wilson (1986)) examines the situation where a monopolist who owns a unit of a durable good faces a continuum of consumers. The seller is not able to commit to post the same price at each period. In other words the seller is assumed to behave sequentially rationally. These papers examine whether the equilibrium sequence of price offers satisfies Coase's Conjecture, which states that when offers take place quickly the seller opens the market with a price close to 0 (the lowest possible buyer's valuation), and all potential gains from trade are realized instantaneously. These papers verify the Coase's Conjecture under certain assumptions, among others stationarity of the buyer's strategy in the no-gap case.[2] The catalytic effect of introducing sequential rationality into the monopolist's problem must be considered with caution. Ausubel and Deneckere (1989a) show that in the no-gap case, without the stationarity assumption about the buyer's strategy, a durable good monopolist's profits can be arbitrarily close to the static monopoly profits.

The durable goods monopoly literature characterizes the revenue maximizing sequence of prices when the seller behaves sequentially rationally. In this paper we characterize the revenue maximizing mechanism when the seller behaves sequentially rationally in a finite horizon framework.

The literature of incentives has analyzed repeated principal - agent relationships under three assump-

[2]In the "no-gap" case the lowest possible valuation of the buyer, v, is less or equal to the seller's cost.

tions regarding commitment. In the first approach, "full commitment", the principal commits to the optimal static incentive scheme at each period, which can be enforced by a third party if either of the parties refuses to do so. See for instance Baron and Bensako (1984a). The principal also commits never to renegotiate this incentive scheme even though a renegotiation may lead to a pareto improving incentive scheme at some point in time.

In the "non-commitment" approach the principal offers at the beginning of each period a contract that will be implemented in the current period. The principal is restricted to offering only short term contracts. The analysis of dynamic incentives run by short term contracts is complex (see Freixas, Guisnerie, Tirole (1985), Laffont and Tirole (1988), Laffont and Tirole (1993) ch 9, among others).

The last approach, "commitment with renegotiation" assumes that a principal involved in a repeated relationship can offer a long term incentive scheme that can be renegotiated at each date. All transfer payments take place in the last period. The principal (the uninformed party) can offer at each date a new incentive scheme for the remainder of the relationship. Renegotiation proof contracts are ex-post pareto efficient.

The literature of non-commitment and commitment and renegotiation studies repeated relationships. In our model, trade will take place once, if at all. This is similar to the durable good sale model. As noted earlier, the durable good monopoly literature has looked so far at sale contracts that consist of a price at each period. We will allow for any possible arrangement. We should note that the optimal incentive scheme under non-commitment and under commitment and renegotiation is not known for the case that the agent's possible types are a continuum.

The papers most closely related to this work are the papers by Hart and Tirole (1988) and McAfee and Vincent (1997).

Hart and Tirole (1988) analyze the sale and the rental model of a durable good in the case that the buyer has two possible types under non-commitment, and under commitment and renegotiation. In their analysis, the seller's strategy under non-commitment, is a price for each period and the result for the sale model is the one of the durable good monopoly. The equilibrium of the rental model under non-commitment converges to a non-discrimination equilibrium where the seller charges the lowest valuation each period as the horizon tends to infinity. In their analysis of the rental and the sale model with commitment and renegotiation, they allow for any possible contract form. The optimal contract form exhibits coasian dynamics and is the same for the sale and the rental model. The optimal long term renegotiation-proof contract is not known for the case that the buyer's possible types are a continuum.

McAfee and Vincent in Sequentially Optimal Auctions (1997) examine sequentially optimal auctions under the assumption that the seller's strategy consists of posting a reserve price, and the buyers follow a stationary strategy. In their model the seller faces a fixed number of buyers and the optimal institution is derived out of a restricted class: the auction format is fixed and the seller posts at each date a reserve price. They show that when the time period between offers is short, the reserve price in the first period is close to the lowest possible valuation and the seller's revenues converge to the revenues with no reserve price.

In short, most of the literature[3] has fixed the form of the mechanism (a price in the durable goods literature and a reserve price in the sequentially optimal auctions literature) and searches for the optimal price (reserve price) path when the seller is unable to commit intertemporally to an institution. In this paper we study the sale model of a durable good under non-commitment and we *derive* the optimal mechanism.

2 The Environment

A seller owns one unit of an item that is durable for 2 periods, t=0,1. The seller's valuation for the item is normalized to zero. She faces just one buyer whose valuation is unknown. Let $f : [0, 1] \rightarrow \Re_+$ continuous, denote the probability density function of the buyer's valuation. We assume that $f(.)$ is common knowledge.

Both the seller and the buyer are risk neutral. The seller can chose any institution to sell her object but cannot commit not to change it in case it fails to sell the object. Her goal is to maximize revenues. Let

[3] With the exception of Hart and Tirole's (1988) two-type model, where they characterize the optimal renegotiation proof contract out of a general class. Note though, that when they derive the optimal sale contract under no commitment, the seller's strategy is just to post a price at each period.

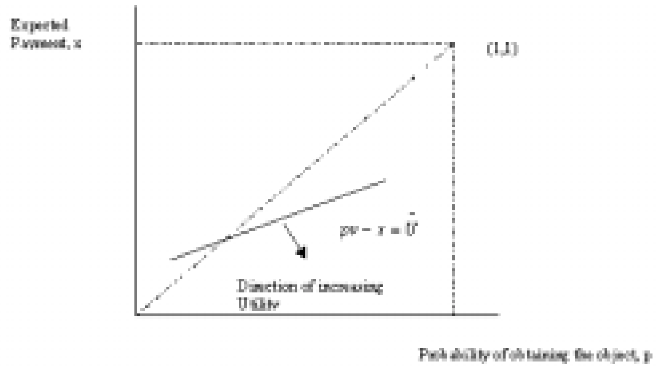

Figure 1: The space of contracts.

δ denote the common discount factor.

Definition 1 *A mechanism consists of 2 sets of points in* $\Re^2$ *one for each period, denoted by* M_0 *and* M_1*.* M_i*,* $i = 1, 2$ *is the set of contracts available at period i. A point* $(p, x) \in M_i$*,* $i = 1, 2$*, denotes a contract. The first element of each pair denotes the probability of obtaining the object and the second element of each pair denotes the expected payment.*

A pair (p, x) is the reduced outcome of some potentially very complicated institution. We refer to it as contract for simplicity. If a buyer with valuation v accepts contract (p, x) his expected payoff is given by

$$pv - x.$$

See Figure 1 for a typical indifference curve in the space of contracts.

In our setting the buyer and the seller will trade only once, if at all. The timing is as follows. Nature moves first and draws the valuation of the buyer.

At the beginning of period zero the seller offers a set of contracts M_0 that are realized at period t=0. The buyer can choose a contract out of M_0, choose the contract $(0, 0)$ at t=0, which is the 'exit' option, or wait until period t=1. The contract $(0, 0)$ is taken to be the legal status quo. If the buyer chooses a contract out of M_0, or the status quo contract, the game ends at t=0.

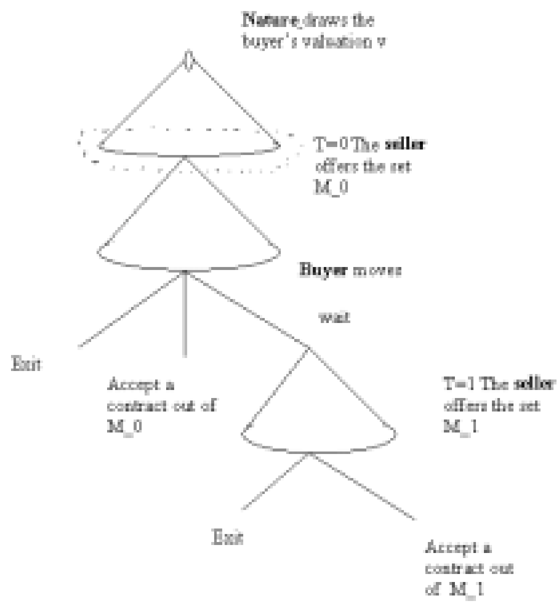

Figure 2: The game

If the buyer, after seeing M_0, decides to wait we move on to period t=1. The seller offers a set of contracts M_1 that are realized in period 1. At t=1 the buyer can choose a contract out of M_1 or contract $(0,0)$. A contract (p,x) in M_1 corresponds to $(\delta p, \delta x)$ at t=0. We assume that the buyer prefers to accept the status quo contract $(0,0)$ at t=0 than at t=1.

2.1 Equilibrium

The seller's strategy, σ_S, is a mechanism (M_0, $M_1(M_0, \sigma_{B_0})$). The set of contracts offered at t=1, M_1, is a function of the seller's information at date t=1, i.e. the buyer's action at t=0 given that M_0 was offered. The buyer's strategy, σ_B is a sequence of choices$(\sigma_{B_0}(v, M_0), \sigma_{B_1}(v, M_0, \sigma_{B_0}, M_1))$. The buyer at t=0 can accept a contract out of M_0, accept the status quo contract, $(0,0)$, or wait (choose $\emptyset$). The buyer's action at t=0 depends on M_0, i.e. the set of contracts offered at t=0, and on his valuation. The buyer at t=1 can choose a contract out of M_1 or the status quo contract. Again the buyer's action depends on the history of the game at t=1 given by (M_0, σ_{B_0}, M_1) and on his valuation. In a Perfect Bayesian Equilibrium of the game the following conditions must hold.

1.For all $v \in [0,1]$ σ_{B_1} maximizes the buyer's payoff at t=1.

2.M_1 maximizes expected revenues given the seller's updated beliefs and the buyer's strategy at t=1. Posterior beliefs f_1 depend on the history (M_0, σ_{B_0}). Let f_1 denote the posterior PDF of the buyer's valuation.

3.For all $v \in [0,1]$ σ_{B_0} maximizes the buyer's payoff given t=1 strategies.

4.M_0 maximizes the seller's expected revenues given subsequent strategies.

5. f_1 is Bayes' consistent with f, the buyer's strategy, and the buyer's observed actions at t=0.

For reasons of tractability we will look at equilibria of the following reduced game. At the beginning of period zero the seller announces her strategy (M_0, M_1).[4] M_0 contains the contract $(0,0)$ which is the legal status quo. The legal status quo contract is equivalent to the buyer rejecting all the contracts in $M_0 \backslash (0,0)$ and in M_1. The buyer can do so in period t=0. This contract should be interpreted as an outside option,

[4]For brevity we omit the arguments from the players' strategies.

as noted earlier. The seller's strategy, σ_S, is a mechanism (M_0, M_1) and the buyer's strategy, σ_B is a choice of a point of $\Re^2$ that belongs in $M_0 \cup M_1$. If the buyer chooses a contract out of M_0, it is realized at period t=0 and the game ends. Contracts in M_1 are realized at t=1. A contract (p, x) in M_1 corresponds to $(\delta p, \delta x)$ at t=0.

The difference of the reduced game is that the seller announces his strategy at the beginning of t=0 and the fact that the outside option is a contract in M_0.

We are looking for a Perfect Bayesian Equilibrium of the reduced game.

1. For all $v \in [0, 1]$, σ_B maximizes the buyer's payoff. In other words, a buyer with valuation v chooses

$$(p^*, x^*) \in \arg \max_{(p,v) \in M_0 \cup M_1} (pv - x). \quad (1)$$

2. M_1 must be optimal at the beginning of t=1 given the seller's posterior beliefs about the buyer's valuation given by $f_1(.)$, and the buyer's strategy at t=1.

3. M_0 must maximize expected revenues for the seller given subsequent strategies.

4. f_1 is Bayes' consistent with f, the buyer's strategy and the buyer's observed actions at t=0.

Note that in a Perfect Bayesian Equilibrium of the original game when the seller announces M_0 the buyer can figure out what the seller will do in the history $(M_0, wait)$. In other words, the buyer after seeing M_0 can figure out the optimal M_1. Given that when the buyer sees M_0 he can induce what the seller will do, if he decides to wait, it follows that in the case the buyer prefers to exit and not choose a contract out of M_0 or out of M_1, he will do so at t=0. The reduced game allows the buyer to exit at t=0 after seeing M_0, if he wishes to do so. From the above observations it follows that an equilibrium of the reduced game is an equilibrium of the original game. The opposite is not necessarily true.

2.2 Methodology

Before we move on to characterize the optimal mechanism we would like to make a couple of remarks regarding the methodology we will use to derive the equilibrium. Recall that t=1 is the final period of the game. This operates as a device for commitment which implies that the seller's problem at t=1 is isomorphic to the static problem. For this reason the mechanism designer can appeal to the revelation principle and choose M_1 among the class of direct revelation mechanisms. For a definition of a direct revelation auction game see Myerson (1981).

At t=0 the situation is different. In the absence of commitment, the seller cannot appeal to the revelation principle when she designs the sale contracts in M_0. Suppose that M_0 is a direct revelation mechanism, the buyer has claimed to have valuation v, and according to M_0 no trade takes place. If the seller behaves sequentially rationally, she will try to sell the object at t=1 using a different mechanism. And in the case that the buyer has revealed his valuation at t=0, the seller will use this information to extract all his surplus. In this situation the buyer has an incentive to manipulate the seller's beliefs at t=0. One would expect that the buyer will not reveal his valuation truthfully at the beginning of the relationship. This is the reason why, when the seller behaves sequentially rationally, restricting attention to direct revelation mechanisms *is* with loss of generality. The seller, since she does not have commitment power, cannot play the role of the "machine" that exogenously specifies the direct revelation game that implements an equilibrium of some general game. This is the reason why M_0 is determined among arbitrary sets of contracts.

•EXAMPLE 1: When choosing M_0 the seller cannot appeal to the revelation principle:

Consider a seller who owns an item of zero value to her facing one buyer whose valuation is drawn from the uniform distribution on [0,1]. The seller decides to employ the mechanism described in Myerson in (1981). If the buyer claims to have valuation above 0.5 he obtains the object and pays 0.5. Otherwise, no transaction is made. In the case that the seller can commit not to resell the item, this mechanism is incentive compatible, individually rational and maximizes expected revenues for the seller. Suppose that the seller cannot commit, and that the buyer claimed that his valuation is 0.48. Under the original arrangement no transaction is made. In the absence of commitment, the seller will make another offer at the second stage of, say 0.4799. In such an environment the seller will use the information revealed by the buyer in the first stage to extract all his surplus; it is no longer incentive compatible for the buyer to reveal his valuation truthfully. In any situation where the seller cannot commit herself to an institution (and choosing the right one in each information set is part of her strategy) one is no longer able to appeal to the revelation principle. This is because the seller is a player in the game and her role is not restricted to just pick the right institution once.

3 The seller's problem at t=1

3.1 Characterization of continuation equilibria

In this section we characterize the revenue maximizing set of contracts at t=1, M_1, for any history of the game. A given M_0 characterizes the history $h_1 = (M_0,\ \text{wait})$. In a Perfect Bayesian Equilibrium M_1 must be an equilibrium of the continuation game that starts after a history h_1. At t=1 the seller, given her posterior beliefs about the buyer's valuation, which are determined by the history of the game h_1, seeks M_1 that maximizes her expected revenues from that point on. Recall that f_1 stands for the posterior probability distribution of the buyer's valuation. In the case that f_1 is generic, in other words the buyer's valuation is fully revealed after some history $h_1 = (M_0, wait)$, the seller's problem at t=1 is trivial. The seller names a price equal to the buyer's valuation and extracts all his surplus. In what follows we assume that f_1 is non-generic, which will in fact be the case in equilibrium.

Let

$$V_0 = \left\{ \begin{array}{c} v \in [0,1] \text{ such that the buyer with valuation } v \text{ accepts} \\ \text{a contract out of } M_0 \end{array} \right\} \tag{2}$$

$$V_1 = \left\{ \begin{array}{c} v \in [0,1] \text{ such that the buyer with valuation } v \\ \text{accepts a contract out of } M_1 \end{array} \right\}. \tag{3}$$

We choose M_1 among the class of direct revelation mechanisms. It is shown, that for any history $h_1 = (M_0, wait)$, the revenue maximizing mechanism at t=1 is a posted price. We also show a way to find the optimal price. The buyer's valuation is drawn from a posterior distribution $f_1(.)$ that is not necessarily continuous or strictly positive on $[0,1]$. We will assume that f_1 is measurable. The seller's expected revenues are given by[5],[6]

$$\int_0^1 p(v)\left[vf_1(v) - [1 - F_1(v)]\right] dv - U(0). \tag{5}$$

The optimal mechanism should extract all the surplus from the lowest valuation buyer, $U(0) = 0$. So the seller's revenues can be rewritten as

$$\int_0^1 p(v)\left[vf_1(v) - [1 - F_1(v)]\right] dv. \tag{6}$$

Let

$$\phi(v) = vf_1(v) - [1 - F_1(v)]. \tag{7}$$

The seller wants to maximize

$$\max_{p \in \Im} \int_0^1 p(v)\phi(v)dv \tag{8}$$

where

$$\Im = \left\{ \begin{array}{c} p : [0,1] \to [0,1] \text{ such that } p \text{ is measurable,} \\ \text{continuous everywhere form above, increasing} \\ \text{and } p(0) = 0 \text{ and } p(1) = 1 \end{array} \right\} \tag{9}$$

and $\phi(.)$ is measurable.

We first establish that the seller's maximization problem is well defined[7].

Proposition 1 [8]*(i)The maximization problem defined in (8) has a maximum.*

(ii)The maximizer of (8) is of the form

$$\begin{aligned} p_v(t) &= 1 \textit{ if } t > v \\ &= 0 \textit{ if } t \le v. \end{aligned} \tag{10}$$

[5]See Myerson (1981) for more details.

[6]Note that in the standard case where $f_1(v) > 0$ for every $v \in [0,1]$ the seller's revenues are usually written as

$$\int_0^1 p(v)\left[v - \frac{[1 - F_1(v)]}{f_1(v)}\right] f_1(v)dv \tag{4}$$

but since this quantity is not always well defined when $f_1(t) = 0$ for some $t \in [0,1]$ we will use the form given by (5).

[7]Proofs of the results not provided in the main text, are in the Appendix.

[8]The proof of Proposotion (1) has been outlined by Phil Reny.

Proof. See Appendix.

The following proposition is closely related to Proposition (1). It not only shows that the maximizer is one of the extreme points of $\Im$, it also describes a way to find the maximizer. It states that, for any possible history of the game, the seller at t=1 will maximize expected revenues from that point on by posting a price. Since t=1 is the last period of the game, the seller's problem is the same as in the case of commitment. Myerson (1981) characterizes the optimal auction under commitment. We extend his analysis a step further by not requiring f_1 to be continuous nor strictly positive.

Proposition 2 *(i) The maximizer of the problem defined in (8) is of the following form*

$$\begin{aligned} p_{v^*}(t) &= 1 \text{ if } t > v^* \\ &= 0 \text{ if } t \leq v^* \end{aligned} \tag{11}$$

where

$$v^* \equiv \inf \left\{ v \in [0,1] \text{ such that } \int_v^{\tilde{v}} \phi(t)dt \geq 0, \text{ for all } \tilde{v} \in [v,1] \right\}. \tag{12}$$

(ii) Let $x = v^*$. The following mechanism

$$\begin{aligned} p(v) &= 1 \text{ if } v > x \\ &= 0 \text{ if } v \leq x \end{aligned} \tag{13}$$

and

$$\begin{aligned} x(v) &= x \text{ if } v > x \\ &= 0 \text{ if } v \leq x \end{aligned} \tag{14}$$

is the optimal feasible mechanism for t=1.

Proof. See Appendix.

The optimal mechanism specifies that if the buyer claims to have valuation above x obtains the object with probability one and pays x. In other words the seller will maximize her expected revenues at t=1 by posting a price equal to x.

3.2 Implications

So far we have proved that no matter how the seller's objective function at t=1 looks like, the optimal mechanism for the seller in period 1 is to post a price. Proposition (2) shows a way to find this optimal price. Recall that V_1 is the set of $v's$ such that a buyer with valuation v, prefers M_1 to M_0. The fact that M_1 is a posted price implies the following result.

Proposition 3 *Suppose that the set of possible buyer's valuations is convex. In an equilibrium the set V_1 is convex.*

Proof. See Appendix.

Proposition (3) says that if the buyer's possible type is a convex set, say an interval, V_1 will be an interval as well (it may be possible that V_1 contains only one element; if this is the case if the buyer after seeing M_0 decides to wait, the sellers knows the buyer's valuation at the beginning of t=1).

Suppose that the buyer seeing M_0 decides to wait. We know from Proposition (3) that V_1 is convex. Assume that f_1 is non-generic, that is assume that V_1 contains more than one element. If the buyer after seeing M_0 prefers to wait, the seller believes at t=1 that she is facing a buyer with valuation in $[\underline{v}, \bar{v}]$. M_1 must be chosen optimally given posterior beliefs f_1 and beliefs must be fulfilled along the equilibrium path. We designate with x the price posted at t=1 which is given by

$$x = \inf \left\{ v \in [0,1] \text{ such that } \int_v^{\tilde{v}} (t f_1(t) - [1 - F_1(t)])\, dt \geq 0, \text{ for all } \tilde{v} \in [v, 1] \right\}. \tag{15}$$

where

$$F_1(x) = \int_{\underline{v}}^{x} \frac{f(t)}{F(\bar{v}) - F(\underline{v})} dt = \frac{F(x) - F(\underline{v})}{F(\bar{v}) - F(\underline{v})} \text{ and } f_1(x) = \frac{f(x)}{F(\bar{v}) - F(\underline{v})}. \tag{16}$$

Let $v_L = \inf V_1$ and $v_H = \sup V_1$.[9]

In an equilibrium beliefs must be fulfilled so it must hold that

$$\underline{v} = v_L \text{ and } \bar{v} = v_H$$

[9] Note that $v_H > v_L$, since we assumed that V_1 contains more than one element.

We assumed that $f(.)$ is continuous which implies that $f_1(.)$ defined in (16) is continuous as well. Now we argue that x defined in (15) will be a root of the equation

$$xf_1(x) - [1 - F_1(x)] = 0.$$

This follows from the following arguments. If x is such that this expression is negative, i.e.

$$xf_1(x) - [1 - F_1(x)] < 0$$

then, by continuity there exists $\varepsilon > 0$ small enough such that

$$\int_x^{x+\varepsilon} [xf_1(x) - [1 - F_1(x)]]\, dt < 0$$

but this contradicts the definition of x given in (15). In the case that x is such that

$$xf_1(x) - [1 - F_1(x)] > 0$$

then by continuity, there exists $\varepsilon' > 0$ small enough such that

$$\int_{x-\varepsilon'}^{x} [xf_1(x) - [1 - F_1(x)]]\, dt > 0$$

which contradicts the definition of x.

Hence, given that the seller is facing a buyer with valuation in $[v_L, v_H]$ the price posted at t=1 will be one of the solutions of the following equation

$$xf_1(x) - [1 - F_1(x)] = 0 \qquad (17)$$

or

$$\text{or } x\left(\frac{f(x)}{F(v_H) - F(v_L)}\right) - \left[\frac{F(v_H) - F(x)}{F(v_H) - F(v_L)}\right] = 0$$

which is equivalent to

$$xf(x) + F(x) - F(v_H) = 0. \qquad (18)$$

Proposition 4 *The price posted at t=1, x, is a non-decreasing function of v_H, and hence it is differentiable almost everywhere.*

Proof. Recall that x is a root of $yf(y) + F(y) - F(v_H) = 0$ (the solution of this equation, x, depends on v_H). Let x' be a root of $yf(y) + F(y) - F(v_H + \varepsilon) = 0$, where $\varepsilon > 0$. Note that x' is a root of the function $yf(y) + F(y)$ shifted down by $F(v_H + \varepsilon)$ and x is a root of the same function shifted down by $F(v_H)$, where of course $F(v_H + \varepsilon) > F(v_H)$. From this observation and from the definition of x and x'it holds that $x \leq x'$. ■

We conclude that the price posted at t=1 will be an increasing function of v_H, which put simply means that the price that the seller charges in the second period is higher, the higher the upper bound of the support of the posterior PDF of the buyer's valuation is.

4 The behavior of the Buyer- Relevant Contracts

At t=0 the seller announces her strategy. The buyer is faced with M_0 and M_1 and wants to pick the contract that maximizes his welfare. Recall that the buyer solves

$$(p^*, x^*) \in \arg \max_{(p,v) \in M_0 \cup M_1} (pv - x)$$

Definition 2 *A contract C in $\Re^2_+$ is irrelevant with respect to $M \subset \Re^2_+$ if there does not exist $v \in [0, 1]$ such that, given all the other available contracts in M, a buyer with valuation v finds weakly optimal to choose C.*

Definition 3 *M is a relevant set of contracts if for all $C \in M$, C is not irrelevant with respect to M. A contract that is not irrelevant is called relevant.*

The seller is interested only in offering relevant sets of contracts, since irrelevant contracts will not affect the buyer's behavior no matter what his valuation is.

Remark 1 *It is without loss of generality that the seller restricts attention to relevant sets of contracts.*

5 Equilibrium when M_0 contains a finite number of contracts.

5.1 The structure of M_0 when it contains K relevant contracts.

We examine the situation where the seller offers M_0 that is of the following form

$$M_0 = \cup_{i=0}^{K-1} \{(p_i, x_i)\} \tag{19}$$

where K is finite and fixed but potentially very large. These contracts are taken to be distinct. The seller is interested in offering points that can be potentially chosen by the buyer. In this section we examine the structure of a set that contains a fixed and finite number of relevant contracts.

The elements of M_0 are taken to be distinct since offering the same point twice will not change the allocation. Consider the set M_0 defined in (19). We order this finite set of points according to the first coordinate, that is, the point which has the lowest p is called point 0 and

$$p_0 \leq p_1 \leq p_2 \leq .... \leq p_{K-1}.$$

Now consider two (distinct) points in M_0, (p_i, x_i) and (p_j, x_j) that have the same first coordinate, that is $p_i = p_j$. Since these points are distinct one of the two must have a higher second coordinate say without loss $x_j > x_i$. For all v in [0,1]

$$p_i v - x_i > p_i v - x_j$$

therefore no buyer will choose (p_j, x_j) when (p_i, x_i) is available, which implies that (p_j, x_j) is irrelevant. Hence when the seller offers M_0 that contains K relevant points it must be the case that

$$0 = p_0 < p_1 < p_2 < ..... < p_{K-1}.$$

We assumed that $(0, 0)$, which is the status quo contract, is an element of M_0 so we take $p_0 = 0$. Note also in order that (p_0, x_0) be relevant it must be the case that $x_0 = 0$, since if it were positive no buyer would find individually rational to pick a contract that assigns zero probability of obtaining the object and positive expected payment.

Now consider two adjacent contracts, say (p_1, x_1) and (p_2, x_2). These points define a line with some slope v_1 which is a real number and is given by

$$v_1 = \frac{x_2 - x_1}{p_2 - p_1}.$$

Note that v_1 denotes the valuation of the buyer who is indifferent between these two contracts. A buyer with valuation $v > v_1$ prefers (p_2, x_2) to (p_1, x_1) and a buyer with valuation $v < v_1$ prefers (p_1, x_1) to (p_2, x_2).

If $v_1 > 1$ no buyer with valuation in [0,1] picks point (p_2, x_2). If $v_1 < 0$ no buyer with valuation v in [0,1] picks contract (p_1, x_1). Hence in order that (p_1, x_1) and (p_2, x_2) be relevant, v_1 must be in [0,1]. This fact implies that since $p_1 < p_2$ and

$$p_1 v_1 - x_1 = p_2 v_1 - x_2$$

that

$$(p_2 - p_1)v_1 = x_2 - x_1 \geq 0.$$

In other words, when (p_1, x_1) and (p_2, x_2) are relevant, then it holds that $x_1 \leq x_2$. So if M_0 contains K relevant points it holds that

$$0 = x_0 \leq x_1 \leq x_2 \leq ...... \leq x_{K-1}.$$

Let $v_2 = \frac{x_3 - x_2}{p_3 - p_2}$ be the valuation of the buyer who is indifferent between (p_2, x_2) and (p_3, x_3). Now we move on to show that if $v_1 > v_2$, then contract (p_2, x_2) will be irrelevant. Recall that $v_1 = \frac{x_2 - x_1}{p_2 - p_1}$. If $v_1 > v_2$ then the line segment that connects contract (p_1, x_1) and contract (p_3, x_3) has slope v'. So given that (p_3, x_3) is available a buyer with valuation v less than $v' = \frac{x_3 - x_1}{p_3 - p_1}$ prefers (p_1, x_1) to (p_3, x_3) and a buyer with valuation greater than v'prefers (p_3, x_3) to (p_1, x_1). Note that $v' \in (v_2, v_1)$. Contract (p_2, x_2) is irrelevant. By assuming that the seller offers only relevant contracts we get that

$$\frac{x_1}{p_1} \leq v_1 \leq v_2 \leq ...... \leq v_{K-2}.$$

A typical set of contracts that contains a finite number of elements is depicted in Figure 3.

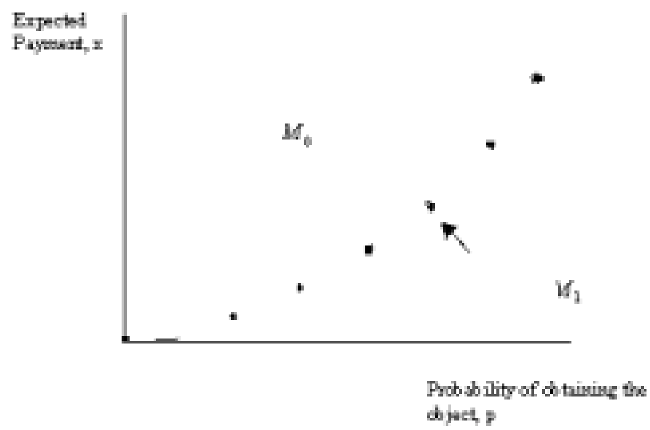

Figure 3: A relevant set of contracts with a finite number of elements.

5.2 Relevant mechanisms (M_0, M_1) when M_0 contains K contracts.

We proceed to examine the relationship of M_1 with the K contracts in M_0 when the set $M_0 \cup M_1$ is relevant. We require M_1, the price posted at t=1, to be optimal given beliefs after the history $(M_0, wait)$. It has been shown that no matter what posterior beliefs are at the beginning of t=1, M_1 will be a posted price.

M_1 is a posted price at t=1 so at t=0 it corresponds to a point $(\delta, \delta x)$. Now suppose that the seller offers M_0 that consists of K contracts, $(p_0, x_0), \ldots\ldots, (p_{K-1}, x_{K-1})$. If $M_0 \cup M_1$ is relevant, then M_1 relates to M_0 in the following way

$$0 = p_0 < p_1 < \ldots\ldots < p_{l-1} < \delta < p_l < \ldots\ldots < p_{K-1} \tag{20}$$

$$x_0 \leq x_1 \leq \ldots \leq x_{l-1} \leq \delta x \leq x_l \leq \ldots\ldots \leq x_{K-1}. \tag{21}$$

Note that it is very well possible that M_0 contains a contract (p_m, x_m) such that $p_m = \delta$ but then either (p_m, x_m) or M_1 will be irrelevant. In an equilibrium where the set $M_0 \cup M_1$ is relevant, M_0 will not contain contracts like (p_m, x_m).

Label M_1 by a name that indicates its ordering relative to contracts in M_0, say (p_l, x_l) and relabel the contracts to the right of (p_l, x_l) accordingly. Now let (p_{l-1}, x_{l-1}) be its neighboring point to the left and

(p_{l+1}, x_{l+1}) be its neighboring point to the right. After renaming we get that

$$0 = p_0 < p_1 < ..... < p_{l-1} < p_l < p_{l+1} < ..... < p_K \tag{22}$$

$$x_0 \leq x_1 \leq ... \leq x_{l-1} \leq x_l \leq x_{l+1} \leq .... \leq x_K \tag{23}$$

where $(p_l, x_l) = (\delta, \delta x) = M_1$.

Let v_{l-1} be the valuation of the buyer who is indifferent between (p_{l-1}, x_{l-1}) and $M_1 = \{(p_l, x_l)\}$ and let v_l be the valuation of the buyer who is indifferent between M_1 and (p_{l+1}, x_{l+1}). When $M_1 = \{(p_l, x_l)\}$ a buyer with valuation in $[v_{l-1}, v_l]$ prefers M_1 to accepting a contract out of M_0.

By the assumption that $M_0 \cup M_1$ is a relevant set of contracts we have that

$$\frac{x_1}{p_1} \leq v_1 \leq v_2 \leq v_3 \leq .... \leq v_{l-1} \leq v_l \leq ..... \leq v_{K-1} \tag{24}$$

where $v_1 = \frac{x_2 - x_1}{p_2 - p_1}$, $v_2 = \frac{x_3 - x_2}{p_3 - p_2}$,..., $v_{l-1} = \frac{\delta x - x_{l-1}}{\delta - p_{l-1}}$, $v_l = \frac{x_{l+1} - \delta x}{p_{l+1} - \delta}$,...., $v_{K-1} = \frac{x_K - x_{K-1}}{p_K - p_{K-1}}$.

5.3 Characterization of Equilibrium M_0

In this section we characterize the optimal M_0. In equilibrium M_0 must maximize the seller's expected revenues. We restricted the seller's strategy by assuming that in equilibrium $M_0 \cup M_1$ must be a relevant set of contracts. This is without loss of generality because contracts that are irrelevant are never chosen in equilibrium nor affect the buyer's behavior.

Given that $M_0 \cup M_1$ is a relevant set, expected revenues are given by[10]

$$R = \int_{\frac{x_1}{p_1}}^{v_1} p_1 x_1 f(t)dt + \int_{v_1}^{v_2} p_2 x_2 f(t)dt + ... +$$
$$\int_{v_{l-1}}^{v_l} \delta x f(t)dt + \int_{v_l}^{v_{l+1}} p_{l+1} x_{l+1} f(t)dt + ... + \int_{v_{K-1}}^{1} p_K x_K f(t)dt.$$

The goal of the seller is, taking $p_i's$ as given, to chose corresponding expected payments $x_i's$ that maximize her revenues taking into account the fact that $x_i's$ affect $v_i's$ and M_1 is chosen optimally given updated

[10] Recall that (p_0, x_0) is the status quo contract $(0, 0)$.

beliefs. We require that $M_0 \cup M_1$ be relevant which implies the constraints

$$\frac{x_1}{p_1} \leq v_1 \leq v_2 \leq v_3 \leq .... \leq v_{l-1} \leq v_l \leq ..... \leq v_{K-1}.$$

We also require that M_1 be a continuation equilibrium. In the event that the buyer, after seeing M_0 prefers to wait, the seller believes at t=1 that she is facing a buyer with valuation in $[v_{l-1}, v_l]$. M_1 must be chosen optimally given posterior beliefs f_1 and beliefs must be fulfilled along the equilibrium path.

Assumption D: Fix arbitrary x and y in $[0, 1]$, such that $y > x$. We assume that

$$F(y) - F(x) \geq f(x) - f(y).$$ [11]

Lemma 1 *Assume that the valuation of the buyer is distributed according to f that satisfies Assumption D. Suppose that M_0 is chosen among sets that contain finitely many contracts. Then in equilibrium $x_i's$ in M_0 are chosen such that*

$$\frac{x_1}{p_1} = v_1 = ... = v_{l-1} = x \tag{25}$$

and

$$v_l = v_{l+1} = v_{l+2} = ... = v_{K-1} = \frac{x_K - \delta x}{p_K - \delta} \tag{26}$$

Contracts $(p_1, x_1), ..., (p_{l-1}, x_{l-1})$ and $(p_{l+1}, x_{l+1}), ..., (p_{K-1}, x_{K-1})$ are chosen with probability zero; they are essentially *irrelevant. Contract (p_K, x_K) is the only contract in M_0, other than status quo contract $(0, 0)$, which is chosen with strictly positive probability in equilibrium.*

Proof. See Appendix.

The above lemma states that in equilibrium $x_i's$ are chosen such that contracts $(p_1, x_1), ..., (p_{l-1}, x_{l-1})$ lie on the line segment through $(0, 0)$ and $(\delta, \delta x)$[12] which has slope x. Any of theses contracts may be chosen only in the event that the buyer has valuation x, which occurs with probability zero.

Similarly, in equilibrium $(p_{l+1}, x_{l+1}), ..., (p_{K-1}, x_{K-1})$ lie on the line segment that goes through $(\delta, \delta x)$ and (p_K, x_K) and has slope v_{K-1}. These contracts are chosen with probability zero. Contract (p_K, x_K) is the

[11] From the mean value theorem for integrals, this inequality can be rewritten as $f(c)(y-x) \geq f(x)-f(y)$ for some $c \in (x, y)$.

[12] Recall that contract $(\delta, \delta x)$ has been renamed as (p_l, x_l).

only contract in M_0, other than status quo contract $(0,0)$, which is chosen with strictly positive probability in equilibrium.

We proceed to show that in equilibrium $p_K = 1$.

Lemma 2 *In equilibrium when M_0 is chosen among sets that contain finitely many contracts, the seller sets $p_K = 1$.*

Proof. See Appendix.

Summarizing, in equilibrium, if the buyer has valuation less than x[13], he will choose the status quo contract at t=0, if his valuation is between x and v_{K-1}, where $x < v_{K-1}$, he will choose to wait and pay x at t=1, whereas if his valuation is greater than v_{K-1} he chooses to pay x_K at t=0.

Theorem 1 *When the seller chooses M_0 among sets that contain finitely many contracts, the seller maximizes expected revenues by posting a price in each period.*

Proof. The result follows from Lemma 1 and Lemma 2.

Our result implies that M_0 equivalent to a posted price is weakly better than M_0 containing just the exit option. The reason is that x_K can be chosen such that all trade takes place at t=1.

6 The revenue maximizing mechanism (M_0, M_1) in the general case.

In this section we will examine the revenue maximizing mechanism in the case that M_0, the set of contracts available at t=0, takes any possible form in the space (p, x). We will, without loss of generality, restrict attention to sets that contain relevant contracts.

Lemma 3 *Consider an arbitrary set of contracts M. Then the boundary of the convex hull of M is equivalent to the set of contracts in M that are relevant.*

[13] Recall that we renamed $(\delta, \delta x)$ as (p_l, x_l).

Proof. Let M be an arbitrary set of contracts. Let $\mathbf{v}$ be the vector $\begin{pmatrix} v \\ -1 \end{pmatrix}$ and $\mathbf{x}$ be the vector $\begin{pmatrix} p \\ x \end{pmatrix}$. A buyer with valuation $v \in [0,1]$ faced with M will choose the contract that maximizes his expected payoff; call it (p_v, x_v). The buyer's maximized payoff is given by

$$U_M(v) = \sup \left\{ \mathbf{v}^T \mathbf{x} : \mathbf{x} \in M \right\}.$$

Then the convex hull of the set M is given by

$$C_M = \left\{ x \in \Re^2 : \mathbf{v}^T \mathbf{x} \leq U_M(v) \text{ for } v \in \Re^2 \right\}.$$

Consider the set of contracts in M that are optimal choices of a buyer with valuation v. This set is a subset of the boundary of the convex hull of M. Let $M_{relevant}$ denote the boundary of the convex hull of M which is given by

$$M_{relevant} = \left\{ \mathbf{x} \in \Re^2 \text{ such that } U_M(v) = \sup \left\{ \mathbf{v}^T \mathbf{x} : \mathbf{x} \in M \right\} \text{ for } v \in [0,1] \right\}.$$

Note that if M is not convex than the set $M_{relevant}$ will contain some points that are not elements of M. Strictly speaking only contracts in $M \cap M_{relevant}$ are available to the buyer. The contracts in $M_{relevant}$ that are not elements of M are on straight lines with slope $v \in [0,1]$. Consider such a line and let v be its slope . The contracts on this line will be chosen in the event that the buyer has valuation v. Since this is a probability zero event, the sets $M_{relevant}$ and $M \cap M_{relevant}$ generate the same expected revenues. We therefore, consider these sets as equivalent. In other words, the boundary of the convex hull of M is equivalent to the set of contracts in M that are relevant with respect to M. ■

Lemma 4 *Any set M_K that consists of K relevant contracts can be equivalently described by a piecewise linear continuous function $\bar{M}_K(p)$ which is constructed by connecting two adjacent contracts of M_K by straight lines.*

Proof. Straightforward.

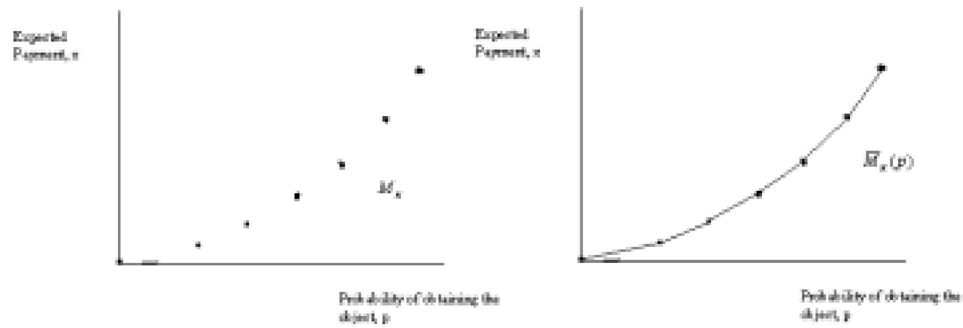

Figure 4: Two equivalent sets of contracts.

Recall that a relevant set of contracts does not contain two contracts with the same first coordinate. Hence, $M_{relevant}$ can be represented by a function of p, denoted by $m(p)$. This function is convex since it represents contracts that are on the boundary of the convex hull of M. It follows from the definition of $m(p)$ that its slope will be between zero and one, since $v \in [0,1]$. The function $m : [0,1] \to [0,1]$[14] is continuous on $[0,1]$ (see Rockafellar (1970) Theorem 10.1 p. 82). In what follows we will use $m(p)$ to represent the set of contracts $M \cap M_{relevant}$. Figure (5) provides an illustration of the above analysis.

Lemma 5 *An arbitrary set of relevant contracts can be uniformly approximated by a piecewise linear function which is equivalent to a set that contains a finite number of contracts.*

Proof. Let M be an arbitrary set of relevant contracts. Since M is relevant we can represent it by a continuous and convex function on $[0,1]$; call this function $m(p)$. Since $m(p)$ is continuous on a compact subset of the real line it is uniformly continuous. Divide $[0,1]$ into cells adding intermediate points $0 \leq p_1 \leq p_2 \leq p_3 \leq .. \leq p_l \leq .. \leq p_n = 1$, so that $p_i - p_{i-1} < \Delta(\frac{1}{n})$. Let $p_l =\delta$. Connect the points $(p_i, m(p_i))$ and define the resulting piecewise continuous function $m_{\frac{1}{n}}(p)$. By Lemma (4) this function is equivalent to a set that contains contracts $\{(0,0), (p_1, x_1), ..., (\delta, \delta x), ...(p_n, x_n)\}$. These contracts are relevant since they belong to M which is by assumption relevant.

[14] A contract with expected payment strictly greater than 1 is not individually rational for a buyer with valuation in $[0,1]$.

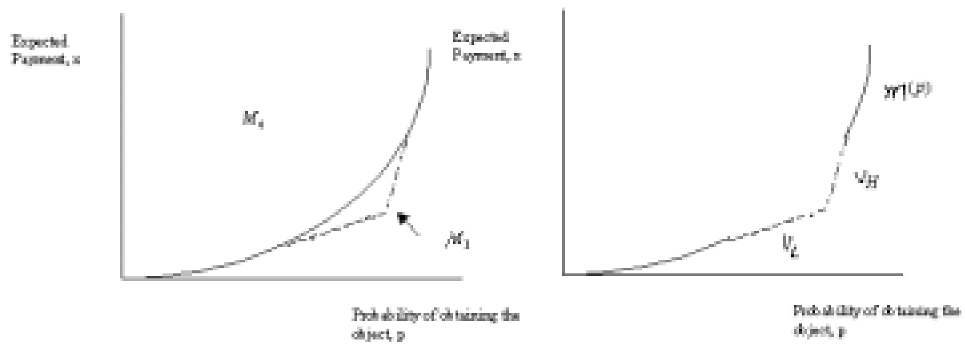

Figure 5: The left figure depicts a mechanism (M_0, M_1). The right figure depicts the relevant contracts of the mechanism (M_0, M_1) represented by the function $m(p)$.

We claim that $m_{\frac{1}{n}}(p)$ approximates $m(p)$ uniformly on $[0,1]$ within $\frac{1}{n}$. We want to show that

$$\left| m_{\frac{1}{n}}(p) - m(p) \right| < \frac{1}{n}. \tag{27}$$

Consider an arbitrary p. Then it must belong in an interval; call it $I_j = [p_{j-1}, p_j]$. The function $m_{\frac{1}{n}}$ on this interval is given by the line segment that connects $(p_j, m(p_j))$ and $(p_{j-1}, m(p_{j-1}))$. Using the linearity of $m_{\frac{1}{n}}$ on I_j we get that

$$\begin{aligned}
\left| m_{\frac{1}{n}}(p) - m(p) \right| &\leq \left| m_{\frac{1}{n}}(p_{j-1}) + \frac{\left(m_{\frac{1}{n}}(p_j) - m_{\frac{1}{n}}(p_{j-1}) \right)}{p_j - p_{j-1}} (p - p_{j-1}) - m(p) \right| \\
&\leq \left| m_{\frac{1}{n}}(p_{j-1}) + \frac{\left(m_{\frac{1}{n}}(p_j) - m_{\frac{1}{n}}(p_{j-1}) \right)}{p_j - p_{j-1}} (p_j - p_{j-1}) - m(p) \right| \\
&\leq \left| m_{\frac{1}{n}}(p_{j-1}) + m_{\frac{1}{n}}(p_j) - m_{\frac{1}{n}}(p_{j-1}) - m(p) \right| \\
&= |m(p_{j-1}) + m(p_j) - m(p_{j-1}) - m(p)| \\
&= |m(p_j) - m(p)| \leq \frac{1}{n}.
\end{aligned}$$

The first inequality follows the fact that $p_j \geq p$, and the rest follows after replacing $m_{\frac{1}{n}}$ by its definition. It follows that we can approximate a set that contains a continuum of relevant contracts uniformly by piecewise linear and continuous functions $m_{\frac{1}{n}}(p)$. ■

We proceed to show that even when the seller has the option to choose M_0 out of a class of sets that contain a continuum of contracts, in equilibrium M_0 will be equivalent to a posted price. This is the main result of the paper.

Theorem 2 *When M is chosen among arbitrary sets and f satisfies Assumption D, the revenue maximizing mechanism is equivalent to posting a price in each period.*

Proof. The proof follows from the following steps.

Step 1 : In this step we show that the behavior of the buyer faced with two different sets of contracts that are arbitrarily close essentially remains the same.

Consider a mechanism (M_0, M_1) and let $m(p)$ represent the relevant contracts of $M_0 \cup M_1$. Now define a sequence of piecewise linear continuous functions $m_{\frac{1}{n}}(p)$ by connecting two nearby points of , $m(p)$ $(p, m(p))$ and $(p + \frac{1}{n}, m(p + \frac{1}{n}))$. Then, by Lemma (5), we know that the sequence $\left(m_{\frac{1}{n}}\right) n \in N$ uniformly approximates $m(p)$. Consider an equilibrium where the seller offers $m(p)$. Suppose that the buyer's valuation is $v \in [0, 1]$ and given the available choices the buyer chooses a contract out of $m(p)$ call it $(p_v, m(p_v))$. Consider the situation where the seller offers $m_{\frac{1}{n}}(p)$ instead of $m(p)$. We need to examine whether the buyer's behavior when faced with $m_{\frac{1}{n}}(p)$,will be 'close' to his behavior when he is faced with $m(p)$.

Suppose that the buyer, when faced with $m_{\frac{1}{n}}$ chooses the contract $(p'_{v(\frac{1}{n})}, m_{\frac{1}{n}}(p'_{v(\frac{1}{n})})$ which is such that such that $p'_{v(\frac{1}{n})} \neq p_{v(\frac{1}{n})}$. The buyer's behavior in this case implies that

$$p'_{v(\frac{1}{n})}v - m_{\frac{1}{n}}(p'_{v(\frac{1}{n})}) > p_{v(\frac{1}{n})}v - m_{\frac{1}{n}}(p_{v(\frac{1}{n})}). \tag{28}$$

Given the continuity of the buyers payoff, taking limit as $n \to \infty$ of the above expression we get that[15]

$$p'_v v - m(p'_v) > p_v v - m(p_v) \tag{29}$$

which contradicts the fact that the buyer chose $(p_v, m(p_v))$ when contract $(p'_v, m(p'_v))$ was available.

[15]Since $m_{\frac{1}{n}}(p)$ uniformly approximates $m(p)$, there is a contract $(p_{v(\frac{1}{n})}, m_{\frac{1}{n}}(p_{v(\frac{1}{n})})$ within $\frac{1}{n}$ distance from $(p_v, m(p_v))$.

From the above analysis it follows that for n large enough, the buyer with valuation v will find optimal to choose the contract $(p_{v(\frac{1}{n})}, x_{\frac{1}{n}}(p_{v(\frac{1}{n})})$ which is close to $(p_v, m(p_v))$.

The behavior of the buyer remains essentially unchanged when the seller instead of offering $m(p)$ offers $m_{\frac{1}{n}}(p)$ that uniformly approximates $m(p)$.

Step 2: Consider a set of period 0 contracts that consists of the status quo contract and n other relevant contracts, call it $M_{0(\frac{1}{n})}$ and let M_1 denote the set of period 1 contracts. We require that the set $M_{0(\frac{1}{n})} \cup M_1$ be relevant. In this step we argue that, holding M_1 fixed, the seller's revenues weakly increase when $M_{0(\frac{1}{n})}$ is chosen to be equivalent to the set $\{(0,0),(p_n,x_n)\}$. This set contains a singleton contract and the outside option. The proof of the argument follows closely the proof of Lemma 1, with the only difference being that $M_1 = (\delta, \delta x)$ is now being held fixed, so we omit it. Holding M_1[16] fixed, the seller will maximize revenues by setting $M_{0(\frac{1}{n})} = \{(0,0),(p_n,x_n)\}$. It follows from Lemma (2) that $p_n = 1$. In equilibrium the set $M_{0(\frac{1}{n})}$ is equivalent to a posted price.

In step 1 we saw that the buyer's behavior when faced with $m(p)$ and $m_{\frac{1}{n}}$ (p),that uniformly approximates $m(p)$, remains essentially unchanged. In step 2 we proved that in equilibrium the set $M_{0(\frac{1}{n})}(p)$ will be set equivalent to a posted price. From Lemma (4), we know that $M_{0(\frac{1}{n})} \cup M_1$ can be equivalently described by a piecewise linear function; call this function $m_{\frac{1}{n}}$. Combining the above results we see that, in the case that the seller chooses M_0 out of a general class of sets, the optimal mechanism is equivalent to posting a price in each period. ■

7 Commitment and Non-Commitment: Revenue comparisons.

Now that we have characterized the revenue maximizing mechanism in a two-period environment with no commitment, we turn to compare the seller's revenues under commitment and under non-commitment.

Expected revenues under commitment in our model are characterized in Myerson (1981), who derives the optimal static auction. Recall that under commitment, the seller will never change the rules of the institution she initially chose, even if it turns out that ex-post it failed to sell the object. In our environment

[16]Recall that $M_1 = \{(\delta, \delta x)\}$.

the seller faces the same buyer at the beginning of period 0 and at the beginning of period 1. Under commitment the seller offers the same institution at the beginning period 0 and at the beginning of period 1. This implies that the final allocation will be determined at t=0.

From Myerson's analysis it follows that, under commitment and when the seller faces one buyer, the revenue maximizing institution is a posted price. The seller's maximized expected revenues are given by

$$\Pi_C = \int_x^1 x f(t) dt.$$

In the case that the buyer behaves sequentially rationally, it was shown in this paper that the revenue maximizing mechanism is to post a price in each period. The seller's expected maximized revenues are given by

$$\Pi_{NC}(\delta) = \int_{x_1}^{v_1} \delta x_1 f(t) dt + \int_{x_0}^1 x_0 f(t) dt.$$

Observe that the seller can replicate the situation under non-commitment in the commitment case by offering contracts $(\delta, \delta x_1)$ and $(1, x_0)$ instead of offering contract $(1, x)$. By doing so, she can obtain the same expected revenues as in the non-commitment case. From this observation it follows that in general

$$\Pi_C \geq \Pi_{NC}(\delta).$$

Claim: There exists some $\bar{\delta}$ high enough such that when $\delta \geq \bar{\delta}$, then the seller will post such a price at t=0, such that all trade takes place in the final period of the game where the seller has commitment power.

When the buyer and the seller are very patient, (in this model the buyer and the seller have the same discount factor), the seller will find it beneficial to shift all trade at the last period of the game. In the last period of the game she has commitment power. If $\delta = 1$ by shifting all trade at t=1 she obtains expected revenues equal to Π_C, which is the best she can hope for. When $\delta = 1$ the seller will post a price at t=1 such that $v_1 = 1$. Revenues when $\delta = 1$ are given by

$$\Pi_{NC}(1) = \int_x^1 x f(t) dt = \Pi_C. \tag{30}$$

Claim: There exists $\varepsilon > 0$ such that when $\delta < \varepsilon$, the seller will post such a price at t=1 such that, if trade takes place, it takes place at t=0.

For δ very small in equilibrium all buyer types that accept a sales contract will accept a sales contract offered at t=0. This follows from the fact that when δ is very small the value of the object at t=1 is almost zero to the buyer no matter what his valuation is, so there is not much surplus for the seller to extract. When the seller and the buyer are very impatient the situation is almost equivalent to the full commitment case. A small discount factor implies that the future is relatively unimportant, so the gain from behaving sequentially rationally is minimal. When $\delta \to 0$ the seller posts at t=0 the revenue maximizing price as in the environment with commitment. Her expected revenues are $\Pi_{NC}(0) = \Pi_C$.

From the above analysis it follows that for extreme values of the discount factor the seller can achieve the expected revenues under commitment even when she behaves sequentially rationally. For intermediate values of the discount factor it holds that $\Pi_{NC} < \Pi_C$. To get some idea about the magnitude of the difference in expected revenues, the reader is referred to the examples presented in the next section.

8 Examples

To illustrate the ideas developed in the previous sections we present two examples in some detail.

We first look at the case where the seller in period t=0 makes the following offer to the buyer: "you will obtain the object with probability 1 if you pay x_0 ". Formally this offer means that M_0 contains two contracts ($K = 2$): the exit option $(0, 0)$ and contract $(1, x_0)$. If the buyer decides to wait, the seller makes the following offer to the buyer: "you will obtain the object with probability 1 if you pay x_1". More formally, the seller offers at t=1 the contract $(1, x_1)$. Basically, the seller posts the price x_0 at period t=0 and x_1 at t=1. We have shown, that under some circumstances, this in fact the optimal thing to do.

In the second example the seller at period zero makes the following offer: "you will obtain the object with probability p_1 if you pay x_1 and with probability 1 if you pay x_2 ". Formally the seller offers the contracts $(0, 0)$, (p_1, x_1) and $(1, x_2)$ at t=0. At t=1 she posts a price of x. We demonstrate that by offering the extra contract (p_1, x_1) at t=0 the seller can do no better than by just offering $(1, x_2)$.

These simple examples demonstrate the essence of our results: even by offering infinitely many contracts at period t=0 the seller will do no better than by posting a price: choosing the appropriate price to post is all the seller needs to do in order to maximize expected revenues. Calculating the optimal price is a relatively straightforward task and it is best the seller can do!

8.1 I. Posting a Price at each Period.

Assume that the buyer's valuation is uniformly distributed on the interval [0,1]. There are two periods t=0,1. As already described, $M_0 = \{(0,0),(1,x_0)\}$ and $M_1 = \{(1,x_1)\}$. Let

$$v_1 = \frac{x_0 - \delta x_1}{1-\delta}$$

denote the valuation of the buyer who is indifferent between $(1,x_0)$ and $(1,x_1)$. We assume that all contracts are relevant which implies that $x_0 \leq x_1$, then

- a buyer with valuation $[0,x_1)$ chooses $(0,0)$ at t=0
- a buyer with valuation $[x_1,v_1]$ chooses contract M_1
- a buyer with valuation $(v_1,1]$ chooses contract $(1,x_0)$.

Suppose that the buyer after seeing M_0 decides to wait. The seller believes that she faces a buyer with valuation in $[\underline{v},\bar{v}]$. In an equilibrium M_1 must be optimal given beliefs and beliefs must be consistent with players' actions. Posterior beliefs are given by $F_1(t) = \frac{F(t)-F(\underline{v})}{F(\bar{v})-F(\underline{v})}$. The price posted at t=1, x_1, must satisfy

$$x_1 = \frac{\bar{v}}{2}.$$

In order that this is an equilibrium it must hold that

$$\underline{v} = x_1 \text{ and } \bar{v} = v_1.$$

Substituting this expression of x_1 into v_1 we get that

$$v_1 = \frac{x_0}{1-0.5\delta}.$$

Given the above relationship between x_1, v_1 and x_0 the seller will pick x_0 that maximizes revenues, given equilibrium beliefs, i.e. we are looking for

$$x_0 \in \arg\max \left\{ \int_{v_1}^{1} x_0 f(t)dt + \delta \int_{x_1}^{v_1} x_1 f(t)dt \right\}.$$

The maximizer is given by

$$x_0 = \frac{(1-0.5\delta)^2}{2-1.5\delta}.$$

Discount Factor δ	**Price at t=0, x_0**	**Price at t=1, x_1**	$v_1 = \frac{x_0}{1-0.5\delta}$	Π_{NC}
0.0001	0.49999	0.25001	0.50002	0.24999
0.3	0.46612	0.27419	0. 54839	0. 23306
0.4	0.45714	0.28571	0. 57143	0. 22857
0.45	0.45330	0.29245	0.58491	0. 22665
0.5	0. 45	0.3	0. 6	0. 225
0.7	0. 44474	0. 34211	0. 68422	0. 22237
0.9	0. 46538	0. 42308	0.84615	0. 23269
0.9999	0. 49995	0. 4999	0.9998	0. 24998
1	0.5	0.5	1	0.25

For this example the seller's expected revenue when she is able to commit is $\Pi_C = 0.25$.

8.2 II. "Type II Equilibria":Offering More options at t=0.

Suppose that the seller offers at t=0 the choice between three sale contracts $M_0 = \{(0,0), (p_1, x_1)$ and $(1, x_2)\}$ and at period t=1 she posts a price x, i.e. $M_1 = \{(\delta, \delta x)\}$. Again, we assume that the buyer's valuation is uniformly distributed on the interval [0,1].

8.2.1 Case 1:$p_1 < \delta < 1$

Fix p_1 and δ. Since we require that in equilibrium all contracts in M_0 and in M_1 be relevant, it holds that

$$p_1 < \delta < 1$$

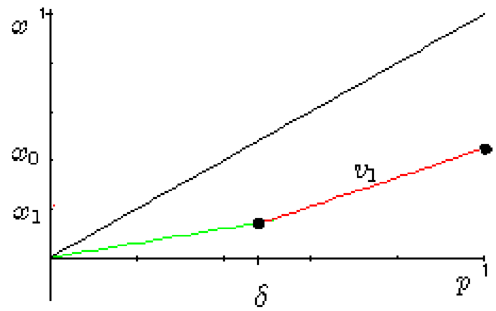

Figure 6: The sequentially optimal mechanism

$$x_1 \le \delta x \le x_2$$
$$\frac{x_1}{p_1} \le v_1 \le v_2.$$

Given (p_1, x_1), $(1, x_2)$ and $(\delta, \delta x)$ a buyer with valuation

$$\tilde{v}_1 = \frac{\delta x - x_1}{\delta - p_1}$$

is indifferent between (p_1, x_1) and paying price x at period 1. A buyer with valuation $v < \tilde{v}_1$ prefers (p_1, x_1) to $(\delta, \delta x)$ and a buyer with valuation $v > \tilde{v}_1$ prefers $(\delta, \delta x)$ to (p_1, x_1). Now, a buyer who has valuation

$$\tilde{v}_2 = \frac{x_2 - \delta x}{1 - \delta}$$

is indifferent between paying x_2 t=0 and paying x at t=1. A buyer with valuation $v > \tilde{v}_2$ prefers $(1, x_2)$ to $(\delta, \delta x)$ and a buyer with valuation $v < \tilde{v}_2$ prefers $(\delta, \delta x)$ to $(1, x_2)$.

Suppose that the buyer after seeing M_0 decides to wait. The seller believes that she is facing a buyer with valuation in $[\underline{v}, \bar{v}]$. M_1 must be optimal given beliefs and in order that this is an equilibrium it must hold that

$$\underline{v} = \tilde{v}_1 \text{ and } \bar{v} = \tilde{v}_2.$$

At t=1 the conditional CDF for the buyer's valuation is given by $\frac{F(x)-F(\tilde{v}_1)}{F(\tilde{v}_2)-F(\tilde{v}_1)}$. At t=1 given posterior distribution, $F_1(.)$,which is determined by $\tilde{v}_1$ and $\tilde{v}_2$, the seller will pick the price that maximizes her expected revenues from t=1 onwards. The price posted at t=1, x, is a root of the following equation

$$f(x)x + F(x) - F(\tilde{v}_2) = 0.$$

The optimal mechanism for t=1 in a type II equilibrium will be of the form $x(\tilde{v}_2)$. We can substitute this expression in the $\tilde{v}_1$ and $\tilde{v}_2$ and solve for x_1 and x_2 respectively. In the uniform case

$$x = \frac{\tilde{v}_2}{2}.$$

This expresses the revenue maximizing M_1 given posterior beliefs. Substituting this equation into $\tilde{v}_1$ and $\tilde{v}_2$ and after rearranging we get that

$$\begin{aligned} x_1 &= 0.5\delta\tilde{v}_2 - (\delta - p_1)\tilde{v}_1 \\ x_2 &= (1 - 0.5\delta)\tilde{v}_2. \end{aligned}$$

Given our assumption that all contracts in M_0 and in M_1 are relevant, the seller's objective function is given by

$$\max_{\{x_1, x_2\}} \int_{\frac{x_1}{p_1}}^{\tilde{v}_1} x_1 p_1 f(t)dt + \int_{\tilde{v}_1}^{\tilde{v}_2} \delta x f(t)dt + \int_{\tilde{v}_2}^{1} x_2 f(t)dt.$$

Substituting $x(\tilde{v}_2)$, $x_1(\delta, \tilde{v}_1, \tilde{v}_2)$ and $x_2(\delta, \tilde{v}_1, \tilde{v}_2)$ into revenues we get an expression that depends only on δ, p_0, (which are fixed parameters) and $\tilde{v}_1$, $\tilde{v}_2$ which are the choice variables. In the uniform case the seller seeks

$$(\tilde{v}_1, \tilde{v}_2) \in \arg \max_{v_i \in [0,1], i=1,2} \left\{ \begin{array}{c} \left(\tilde{v}_1 - \frac{[\delta 0.5\tilde{v}_2 - (\delta - p_1)\tilde{v}_1]}{p_1}\right) [0.5\delta\tilde{v}_2 - (\delta - p_1)\tilde{v}_1] \, p_1 \\ +\delta(\tilde{v}_2 - \tilde{v}_1)\frac{\tilde{v}_2}{2} + (1 - \tilde{v}_2) \left[(1 - 0.5\delta)\tilde{v}_2\right] \end{array} \right\}.$$

We solved the problem imposing the restriction that all contracts offered are relevant in equilibrium. Recall that δ and p_1 are fixed parameters in the problem.

Results for case 1: $p_1 < \delta < 1$

δ	p_1	x_1	x	x_2	$\frac{x_1}{p_1}$	$\tilde{v}_1$	$\tilde{v}_2$	Π_{NC}
0.2	0.1	0.0265	0.2647	0.4765	0.2647	0.2647	0.5294	0.2382
0.3	0.1	0.0274	0.2742	0.4661	0.2742	0.2742	0.5484	0.2331
0.4	0.1	0.0286	0.2857	0.4571	0.2857	0.2857	0.5714	0.2286
0.5	0.1	0.03	0.3	0.45	0.3	0.3	0.6	0.225
0.7	0.1	0.0342	0.3421	0.4447	0.3421	0.3421	0.6842	0.2224
0.9	0.1	0.0423	0.4231	0.4654	0.4231	0.4231	0.8462	0.2327
0.9999	0.1	0.05	0.4999	0.4999	0.4999	0.4999	0.9998	0.25
0.9999	0.5	0.2499	0.4999	0.4999	0.4999	0.4999	0.9998	0.25
0.7	0.5	0.1711	0.3421	0.4447	0.3421	0.3421	0.6842	0.2224

The solution was obtained numerically. Note that for any discount factor we get that $x = \frac{x_1}{p_1} = \tilde{v}_1$ which implies that no buyer picks (p_1, x_1). A buyer with valuation in $[0, \tilde{v}_1 = \frac{x_1}{p_1} = x)$ chooses $(0,0)$. A buyer with valuation in $[x = \tilde{v}_1, \tilde{v}_2]$ chooses $M_1 = \{(\delta, \delta x)\}$, and a buyer with valuation in $[\tilde{v}_2, 1]$ chooses $(1, x_2)$. Hence contract (p_1, x_1) is superfluous. Recall that p_1 is a fixed parameter and notice that the results do not depend on p_1; as an illustration we provide the result for $\delta = 0.9999$ and for $\delta = 0.7$ in the case that $p_1 = 0.1$ and $p_1 = 0.5$. We see that the results obtained in this example are identical to the case that the seller posts a price at t=0. They generate the same revenues (up to some rounding) and $v_1 = \tilde{v}_2$.

8.2.2 Case 2: $\delta < p_1 < 1$.

Fix δ and p_1. Since we require that the set $M_0 \cup M_1$ be relevant, it must hold that

$$\delta < p_1 < 1 \tag{31}$$

$$\delta x \leq x_1 \leq x_2 \tag{32}$$

and

$$x \leq v_1 \leq v_2. \tag{33}$$

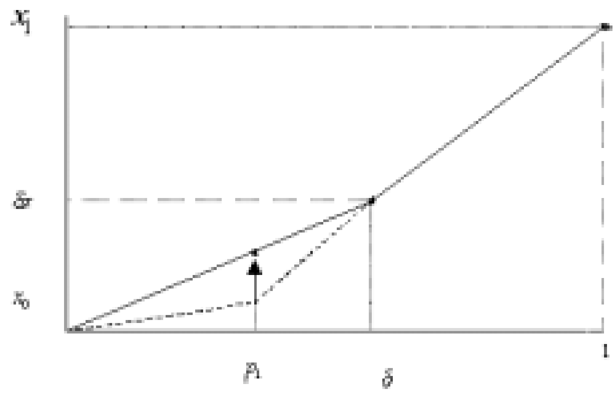

Figure 7: Type II Equilibrium. Case 1: $p_1 < \delta < 1$.

Let $\bar{v}_1$ and $\bar{v}_2$ be defined by

$$\bar{v}_1 = \frac{x_1 - \delta x}{p_1 - \delta} \text{ and } \bar{v}_2 = \frac{x_2 - x_1}{1 - p_1}. \tag{34}$$

A buyer with valuation $\bar{v}_1$ is indifferent between (p_1, x_1) and $(1, x_2)$. A buyer with valuation $v > \bar{v}_1$ prefers (p_1, x_1) to $(\delta, \delta x)$ and a buyer with valuation $v < \bar{v}_1$ prefers $(\delta, \delta x)$ to (p_1, x_1). Similarly a buyer with valuation $\bar{v}_2$ is indifferent between (p_1, x_1) and $(1, x_2)$. A buyer with valuation $v < \bar{v}_2$ prefers (p_1, x_1) to $(1, x_2)$ and a buyer with valuation $v > \bar{v}_2$ prefers $(1, x_2)$ to (p_1, x_1).

In summary a buyer with valuation in $[0, x)$ chooses $(0, 0)$ at t=0, with valuation $[x, \bar{v}_1)$ chooses $M_1 = \{(\delta, \delta x)\}$, a buyer with valuation in $[\bar{v}_1, \bar{v}_2)$ chooses (p_1, x_1) at t=0 and finally a buyer with valuation in $[\bar{v}_2, 1]$ chooses $(1, x_2)$ at t=0.

Following the same procedure as in case 1, we find that, in the uniform case, at t=1 the seller posts

$$x = \frac{\bar{v}_1}{2}. \tag{35}$$

Substituting this expression into $\bar{v}_1$ and $\bar{v}_2$ and after rearranging we get that

$$x_2 = (1 - p_1)\bar{v}_2 + (p_1 - 0.5\delta)\bar{v}_1 \tag{36}$$

$$x_1 = (p_1 - 0.5\delta)\bar{v}_1. \tag{37}$$

Given the assumption that all contracts in M_0 and in M_1 are relevant the seller's revenues are given by

$$R = \delta \int_x^{\bar{v}_1} x f(t)dt + \int_{\bar{v}_1}^{\bar{v}_2} p_1 x_1 f(t)dt + \int_{\bar{v}_2}^{1} x_2 f(t)dt. \tag{38}$$

In the uniform case this reduces to

$$R = \delta \left(\frac{\bar{v}_1}{2}\right)^2 + (\bar{v}_2 - \bar{v}_1)\, p_1(p_1 - 0.5\delta)\bar{v}_1 + (1 - \bar{v}_2)\left[(1 - p_1)\bar{v}_2 + (p_1 - 0.5\delta)\bar{v}_1\right]. \tag{39}$$

We derived the solution assuming that the contracts in M_0 and in M_1 must be relevant.

Results for case 2: $\delta < p_1 < 1$

$\boldsymbol{\delta}$	$\mathbf{p}_1$	$\mathbf{x}_1$	$\mathbf{x}$	$\mathbf{x}_2$	$\frac{x_1}{p_1}$	$\bar{\mathbf{v}}_1$	$\bar{\mathbf{v}}_2$	$\boldsymbol{\Pi}_{NC}$
0.3	0.5	0.1919	0.2742	0.4661	0.3839	0.5484	0.5484	0.2331
0.4	0.5	0.1714	0.2857	0.4571	0.3429	0.5714	0.5714	0.2286
0.45	0.5	0.1608	0.2925	0.4533	0.3217	0.5849	0.5849	0.2267

The solution was obtained numerically. Note that for any discount factor we get that $\bar{v}_1 = \bar{v}_2$ that is, (p_1, x_1) will be chosen only if the buyer has valuation $\bar{v}_1$, which is a probability zero event. A buyer with valuation $v \in [x, \bar{v}_2)$ chooses $M_1 = \{(\delta, \delta x)\}$ and a buyer with valuation in $[\bar{v}_2, 1]$ chooses $(1, x_2)$. The solution is identical to the one we derived for the case that the seller just posts a price at t=0! Contract (p_1, x_1) is superfluous and expected revenues are the same as in the case that the seller posts a price at t=0 (along with the status quo contract).

9 Conclusion

In this paper we characterize the revenue maximizing mechanism when the seller behaves sequentially rationally. We show that the revenue maximizing mechanism in an environment with no commitment is to post a price in each period. In particular, for any possible history of the game, at the beginning of the period t=1 the seller will maximize expected revenues by posting a price. In deriving his result, we restrict attention to direct revelation mechanisms. We are able to do so, since the seller's problem at the beginning of t=1 is isomorphic to her problem with full commitment. Subsequently, we derive the revenue maximizing set of contracts for t=0; this is a more difficult task since one cannot appeal to the revelation principle. We do not impose any restrictions on the potential form of the mechanism. Under some conditions on the probability density function of the buyer's valuation, the revenue maximizing mechanism at period t=0 is to post a price.

This work extends the works of Myerson (1981), Riley and Samuelson (1981) and Hart and Tirole (1988). The most closely related paper is the one by Hart and Tirole (1988). In a T-period framework where the buyer's valuation is either high or low, Hart and Tirole examine, among others, the situation where the seller and the buyer sign one-period contracts, known as the non-commitment case. They assume, that under non-commitment the seller's strategy is to post a price. In this paper we derive the revenue maximizing mechanism under non-commitment for the case that the buyer's valuation is drawn from a continuum. The methodology developed in this paper may be useful in deriving the optimal incentive scheme in other asymmetric information environments where the principle is unable to commit and the agent's type is drawn from a continuum. This has been done so far only for the case that the agent has two possible types.

Previous work has assumed that the seller's strategy is to post a price and the problem of the seller is to find what price to post. We provide a reason for the seller's choice to post a price, even though she can use infinitely many other possible institutions: posted price selling is the optimal strategy in the sense that it maximizes the seller's revenues. In the future we plan to work on eliminating the somewhat restrictive assumptions we made on the probability density function of the buyer's valuation. We also plan to study the problem in a T-period framework and in the case that the seller faces more than one buyer. Another important question related to this research is how the seller's inability to commit affects efficiency. This question is relevant when the seller faces more then one buyer.

If in reality individuals in charge of designing institutions are unable to commit, then the appropriate framework to study a mechanism design problem is the one where the principal behaves sequentially rationally.

10 Appendix

10.1 Mathematical Preliminaries

Definition 4 *Given a point x of [0,1] and an open set U of space [0,1] let*

$$S(x,U) = \left\{ p \mid p \in [0,1]^{[0,1]} \text{ and } p(x) \in U \right\}$$

The sets $S(x,U)$ are a subbasis for a topology on $[0,1]^{[0,1]}$ which is called the topology of pointwise convergence. The typical basis element about a function p consists of all functions g that are close to p at finitely many points.

Theorem 3 *A sequence p_n of functions converges to a function p in the topology of pointwise convergence if and only if for each $x \in X(= [0,1]$ in our problem), the sequence $p_n(x)$ of points of Y $(= [0,1]$ in our problem) converges to the point $p(x)$.*

Proof. See Munkres "Topology: A first Course" page 281.

Definition 5 *A topological space X is said to be sequentially compact if every infinite sequence from X has a convergent subsequence. A subspace Y of X is sequentially compact if every sequence from Y has a convergent subsequence that converges to an element of Y.*

Theorem 4 *Let X be a metrizable topological space . Then the following are equivalent. (1) X is compact (2) X is limit point compact (3) X is sequentially compact*

Proof. See Munkres "Topology: A first Course" p. 181.

Theorem 5 *(Helly) Let $\{p_n\}$ be a sequence of functions in $\Im$. Then there exists a function $p \in \Im$ and a subsequence of $\{p_n\}$ that converges weakly to p.*

This is a slightly adapted version of Helly's Theorem discussed in Stockey &Lucas: 'Recursive Methods in Economic Dynamics' pages 371-373.

Remark 2 *From Theorem (3) and Helly's Theorem it follows that the sequence p_n converges to p in the topology of pointwise convergence.*

Theorem 6 *(Lebesque's Dominated Convergence Theorem) Let g be a measurable function over a measurable set E, and suppose that $\{p_n\}$ is a sequence of measurable functions on E such that*

$$|p_n(x)| \leq g(x)$$

and for almost all $x \in E$ we have $p_n(x) \to p(x)$. Then

$$\int_E p = \lim \int_E p_n$$

Proof. See Royden (1962) p.76.

Remark 3 *The above theorem shows that $\int_E p$ is lower and upper semicontinuous and hence continuous.*

10.2 Proofs of the results.

Proof of Proposition (1)

(i) We start by proving the existence result.

Step 1: (Compactness) The set $\Im$ is compact in the topology of pointwise convergence. This follows from Helly's Theorem together with Theorems (3) and (4).

Step2: (Continuity) We want to show that the objective function is continuous on [0,1] in the topology of pointwise convergence. Take E=[0,1] which is a measurable set, and g is given by

$$g(t) = 1 \; \forall t \in [0,1].$$

Note that g is measurable since it is a constant function. Our space $\Im$ consists of measurable functions $p(.)$. Continuity of the objective function follows from Lebesque's Dominated Convergence Theorem.

We are done since a continuous function on a compact set assumes its minimum and its maximum ∎

So far we have established that the maximization problem given by (8) has a maximum. We will now proceed to show that the maximizer is of the form where $v \in [0,1]$. Note that the functions defined above are in $\Im$ since they are measurable, continuous everywhere form above, increasing and $p(0) = 0$ and $p(1) = 1$.

(ii).The objective function is linear in the choice variable so the maximizer will be an extreme point of the set of maximizers.

The set of extreme points of $\Im$ is

$$K = \cup_{v \in [0,1]} p_v(.)$$

where p_v is defined in (10) .

Every increasing, non-negative function F with $F(1) = 1$ can be written as a convex combination of functions as defined in (10)

$$F(s) = \int_0^1 p_v(s) dF(v).$$

Let p^* be a maximizer of the problem defined in (8). Let M^* denote the maximum value of the objective function. Then using the above representation and Fubini's theorem we have

$$\begin{aligned} \int_0^1 p^*(s)\phi(s)ds &= \int_0^1 \left\{ \int_0^1 p_v(s) dp^*(v) \right\} \phi(s) ds = \\ &= \int_0^1 \left\{ \int_0^1 p_v(s)\phi(s)ds \right\} dp^*(v) \geq M^*. \end{aligned}$$

This is a convex combination of functions of the form given in (10). Hence one of these functions is a maximizer. ■

Proof of Proposition(2)

(i) First note that v^* is well-defined because the set

$$\left\{ v \in [0,1] \text{ such that } \int_v^{\tilde{v}} \phi(t)dt \geq 0, \text{ for all } \tilde{v} \in [v,1] \right\}$$

is non-empty since it contains 1.

Fix arbitrary $v_1 \in [0,1]$ such that $v_1 < v^*$. Suppose that $p(v_1) > 0$, this implies that $\forall v' \geq v_1$

$$p(v') \geq p(v_1)$$

by the requirement that $p(.)$ be increasing. By the definition of v^* and since $v_1 < v^*$, there must exist some $\bar{v}_1$ where $\bar{v}_1 \in [0,1]$ such that

$$\int_{v_1}^{\bar{v}_1} \phi(t)dt < 0.$$

Lemma. We can take, without loss of generality, $\bar{v}_1 \leq v^*$. In other words if there exists $\bar{v}_{\ 1} > v^*$ such that

$$\int_{v_1}^{\bar{v}_1} \phi(t)dt < 0$$

then there exists $\tilde{v}_1$ such that

$$\int_{v}^{\tilde{v}_1} \phi(t)dt \text{ and } \tilde{v} \leq v^*.$$

Proof. Suppose $\bar{v}_1 > v^*$ is such that

$$\int_{v_1}^{\bar{v}_1} \phi(t)dt < 0$$

$$\int_{v_1}^{\bar{v}_1} \phi(t)dt = \int_{v_1}^{v^*} \phi(t)dt + \int_{v^*}^{\bar{v}_1} \phi(t)dt, \text{ since } \bar{v}_1 > v^*.$$

But

$$\int_{v^*}^{\bar{v}_1} \phi(t)dt \geq 0$$

this implies that

$$\int_{v_1}^{v^*} \phi(t)dt < 0.$$

So there must exist $\tilde{v}_1 \leq v^*$ such that

$$\int_{v_1}^{\tilde{v}_1} \phi(t)dt < 0.$$

for every $v_1 \in [0, v^*]$. Call this $\tilde{v}_1 = \bar{v}_1$. ■

Back to the proof of Proposition (2). We proceed to demonstrate why it can not be optimal to set $p(t) > 0$ for $t < v^*$.

Suppose that $v^* > 0$.Fix $v < v^*$.For any $v < v^*$ there exists $v' \leq v^*$such that

$$\int_{v}^{v'} \phi(t)dt < 0. \tag{40}$$

Case 1: Suppose that there exists $v'' \in [v, v')$ such that

$$\int_{v}^{v''} \phi(t)dt > 0. \tag{41}$$

Let

$$\bar{v} = \sup\left\{v'' \in [v, v') \text{ such that } \int_{v}^{v''} \phi(t)dt > 0\right\}.$$

It follows that

$$\left|\int_{\bar{v}}^{v'} \phi(t)dt\right| > \left|\int_{v}^{\bar{v}} \phi(t)dt\right| \tag{42}$$

since

$$\int_{v}^{\bar{v}} \phi(t)dt + \int_{\bar{v}}^{v'} \phi(t)dt < 0.$$

Suppose that $p(t) > 0$ for some $t \in [v, \bar{v}]$. By the requirement that $p(.)$ be non-decreasing it must hold that

$$p(s) \geq p(\bar{v}) \geq p(t) \text{ for all } t \in [\bar{v}, v'].$$

Suppose

$$p(s) = p(\bar{v}) \text{ for all } s \in [\bar{v}, v']$$

then

$$\int_{v}^{\bar{v}} p(t)\phi(t)dt + \int_{\bar{v}}^{v'} p(\bar{v})\phi(t)dt < \int_{v}^{\bar{v}} 0\phi(t)dt + \int_{\bar{v}}^{v'} 0\phi(t)dt.$$

The last inequality follows from (40), (41) and (42). Hence $p(t) > 0$ for $t \in [v, v']$ cannot be optimal.

Case 2: Suppose that there does NOT exist $v'' \in [v, v')$ such that

$$\int_{v}^{v''} \phi(t)dt > 0$$

then setting $p(t) > 0$ for some $t \in [v, v']$ clearly cannot be optimal.

Now we'd like to show that setting $p(t) < 1$ for $t \in [v^*, 1]$ cannot be optimal. We will argue by contradiction. Suppose that for some $t \geq v^*$ it holds that $p(t) < 1$. This implies that for every $s < t$, $p(s) \leq p(t)$ by the requirement that $p(.)$ is increasing in t; but

$$\int_{v^*}^{t} \phi(s)ds \geq 0$$

by the definition of v^*. Such a $p(.)$ cannot be a maximizer since

$$\int_{v^*}^{t} p(t)\phi(t)dt + \int_{t}^{1} p(t)\phi(t)dt < \int_{v^*}^{t} \phi(t)dt + \int_{t}^{1} p(t)\phi(t)dt.$$

Contradiction.

(ii) This is straightforward. ■

Proof of Proposition (3)

If V_1 is empty or it contains one element we are done. Suppose that the cardinality V_1 is at least 2. Let v_1 and v_2 be elements of V_1. This implies that a buyer with valuation v_1 and a buyer with valuation v_2 prefer $M_1 = (\delta, \delta v^*)$ to their most preferred point of M_0. Let (p_{01}, x_{01}) be such that

$$(p_{01}, x_{01}) \in \arg \max_{(p,x)\in M_0} (pv_1 - x).$$

Then it must hold that

$$\delta v_1 - \delta v^* \geq p_{01}v_1 - x_{01}. \tag{43}$$

Similarly, let (p_{02}, x_{02}) be such that

$$(p_{02}, x_{02}) \in \arg \max_{(p,x)\in M_0} (pv_2 - x).$$

Then it must hold that

$$\delta v_2 - \delta v^* \geq p_{02}v_2 - x_{02}. \tag{44}$$

Consider a buyer with type $\alpha v_1 + (1-\alpha)v_2$; a buyer with this type exists since we assumed that the support of the buyer's valuation is convex. Let

$$(\bar{p}, \bar{x}) \in \arg \max_{(p,x)\in M_0} p(\alpha v_1 + (1-\alpha)v_2) - x.$$

We want to show that

$$\delta(\alpha v_1 + (1-\alpha)v_2) - \delta v^* \geq \bar{p}(\alpha v_1 + (1-\alpha)v_2) - \bar{x}.$$

We will argue by contradiction. Suppose

$$\delta(\alpha v_1 + (1-\alpha)v_2) - \delta v^* < \bar{p}(\alpha v_1 + (1-\alpha)v_2) - \bar{x}$$

which can be rewritten as

$$\alpha \left[(\bar{p}v_1 - \bar{x}) - (\delta v_1 - \delta v^*) \right] + (1-\alpha) \left[(\bar{p}v_2 - \bar{x}) - (\delta v_2 - \delta v^*) \right] > 0.$$

This inequality implies that at least one of the two summands of the LHS is strictly greater than zero, which in turn implies that

$$\bar{p}v_1 - \bar{x} > \delta v_1 - \delta v^*$$

or

$$\bar{p}v_2 - \bar{x} > \delta v_2 - \delta v^*$$

or both hold. The last statement contradicts the fact that if the buyers has valuation v_i, $i = 1, 2$, he prefers $(\delta, \delta v^*)$ to every point of M_0. ■

Proof of Lemma 1.

In an equilibrium, where M_0 is chosen among the sets of contracts that are relevant and contain K contracts (including the status quo contract), the seller's equilibrium strategy is a vector $(x_1, x_2, ..., x_{k-1})$ that maximizes expected revenues taking $(p_1, ..., p_{k-1})$ as given.

We will look at the partial derivatives of expected revenues, R, with respect to each x_i in M_0.[17] Differentiating R with respect to x_1 we get

$$\begin{aligned} \frac{\partial R}{\partial x_1} &= \left(F(v_1) - F(\frac{x_1}{p_1}) \right) p_1 + \left(f(v_1) \left(\frac{-1}{p_2 - p_1} \right) - f(\frac{x_1}{p_1}) \frac{1}{p_1} \right) x_1 p_1 \\ &\quad + f(v_1) \frac{x_2 p_2}{p_2 - p_1}. \end{aligned}$$

Adding and substracting $f(v_1)x_1$ this can be rewritten as

$$\frac{\partial R}{\partial x_1} = \left(F(v_1) - F\left(\frac{x_1}{p_1}\right) \right) p_1 + f(v_1) \left(\frac{p_2 x_2 - p_2 x_1}{p_2 - p_1} \right) - \left(f(\frac{x_1}{p_1}) - f(v_1) \right) x_1.$$

[17] Essentially the seller determines p too, since by picking appropriate x, she can make a point (p, x) essentially irrelevant (this simply means that in equilibrium, given all the other available contracts, contract (p,x) will be chosen with probability zero).

If $f(v_1) \geq f(\frac{x_1}{p_1})$ revenues are increasing in x_1. Otherwise, revenues will be increasing in x_1 if

$$\left(F(v_1) - F\left(\frac{x_1}{p_1}\right)\right) p_1 - \left(f(\frac{x_1}{p_1}) - f(v_1)\right) x_1 \geq 0$$

or since $p_1 > x_1$

$$\left(F(v_1) - F\left(\frac{x_1}{p_1}\right)\right) p_1 - \left(f(\frac{x_1}{p_1}) - f(v_1)\right) p_1 \geq 0$$

which reduces to

$$\left(F(v_1) - F\left(\frac{x_1}{p_1}\right)\right) \geq \left(f(\frac{x_1}{p_1}) - f(v_1)\right). \tag{45}$$

This inequality holds under Assumption D. Under our assumptions revenues are non-decreasing in x_1. The constraint $\frac{x_1}{p_1} \leq v_1$ is binding so the seller will set x_1 as large as possible. The seller sets x_1 such that $\frac{x_1}{p_1} = v_1$. Notice that $\frac{x_1}{p_1} = v_1$ implies that $v_1 = \frac{x_2}{p_2}$ and when $\frac{x_1}{p_1} = v_1 = \frac{x_2}{p_2}$ holds, expected revenues are given by

$$\begin{aligned} R &= \left(F(v_2) - F(\frac{x_2}{p_2})\right) x_2 p_2 + (F(v_3) - F(v_2)) p_3 x_3 \\ &+ (F(v_4) - F(v_3)) x_4 p_4 + (F(v_5) - F(v_4)) x_5 p_5 \\ &+ (F(v_6) - F(v_5)) p_6 x_6 ..... + (1 - F(v_{K-1})) p_K x_K \end{aligned}$$

Now given that at the optimum it holds that $\frac{x_1}{p_1} = v_1 = \frac{x_2}{p_2}$, let us examine the effect of x_2 on revenues. Differentiating R with respect to x_2 we get that

$$\frac{\partial R}{\partial x_2} = \left(F(v_2) - F(\frac{x_2}{p_2})\right) p_2 + f(v_2) \left(\frac{p_3 x_3 - x_2 p_2}{p_3 - p_2}\right) - f(\frac{x_2}{p_2}) x_2.$$

Using parallel arguments to the ones used above it can be shown that, under our assumptions, revenues are increasing in x_2 which implies that the seller will set x_2 as large as possible: $\frac{x_2}{p_2} = v_2$ which implies that $\frac{x_1}{p_1} = v_1 = \frac{x_2}{p_2} = v_2 = \frac{x_3}{p_3}$. Continuing in the same way we see that revenues are increasing in x_i, $i = 1, ...l - 2$.

Now we examine whether revenues are increasing in x_{l-1}. This is the payment associated with the contract adjacent to M_1. When $\frac{x_1}{p_1} = v_1 = \frac{x_2}{p_2} = v_2 = .. = \frac{x_{l-2}}{p_{l-2}} = v_{l-2}$ expected revenues are given by

$$\begin{aligned} R &= \left(F(v_{l-1}) - F(\frac{x_{l-1}}{p_{l-1}})\right) x_{l-1}p_{l-1} + \left(F(v_l) - F(v_{l-1})\right)\delta x \\ &\quad + \left(F(v_{l+1}) - F(v_l)\right) x_{l+1}p_{l+1} + ... + \left(1 - F(v_{K-1})\right) p_K x_K \\ \frac{\partial R}{\partial x_{l-1}} &= \left(F(v_{l-1}) - F(\frac{x_{l-1}}{p_{l-1}})\right) p_{l-1} + f(v_{l-1})\frac{\delta x - x_{l-1}p_{l-1}}{\delta - p_{l-1}} - f(\frac{x_{l-1}}{p_{l-1}})x_{l-1} \end{aligned}$$

which can be shown to be non-negative under the assumptions made so far. Hence at the optimum the seller will set x_{l-1} as large as possible that is

$$\frac{x_{l-1}}{p_{l-1}} = v_{l-1} \tag{46}$$

which implies $\frac{x_{l-1}}{p_{l-1}} = v_{l-1} = x$.

Now we look at the effect of a change in x_{l+1} on revenues, when $\frac{x_1}{p_1} = v_1 = \frac{x_2}{p_2} = v_2 = .. = \frac{x_{l-2}}{p_{l-2}} = v_{l-2} = \frac{x_{l-1}}{p_{l-1}} = v_{l-1}$. Note that x_{l+1} affects v_l, which affects in turn x. Hence

$$\frac{\partial v_l}{\partial x_{l+1}} = \frac{1 - \delta\frac{\partial x}{\partial v_l}\frac{\partial v_l}{\partial x_{l+1}}}{p_{l+1} - \delta}$$

which reduces to

$$\frac{\partial v_l}{\partial x_{l+1}} = \frac{1}{p_{l+1} - \delta + \delta\frac{\partial x}{\partial v_l}}$$

Recall from Proposition (4) that x is non-decreasing and differentiable almost everywhere, hence $\frac{\partial x}{\partial v_l} \geq 0$. This says that as the upper bound of the support of the posterior distribution of the buyer's valuation increases the price posted at t=1 will not decrease. When $\frac{x_1}{p_1} = v_1 = \frac{x_2}{p_2} = v_2 = .. = \frac{x_{l-2}}{p_{l-2}} = v_{l-2} = \frac{x_{l-1}}{p_{l-1}} = v_{l-1}$ holds, then expected revenues are given by

$$\begin{aligned} R &= \left(F(v_l) - F(x)\right)\delta x + \left(F(v_{l+1}) - F(v_l)\right) p_{l+1}x_{l+1} + \\ &\quad + \left(F(v_{l+2}) - F(v_{l+1})\right) p_{l+2}x_{l+2} + ... + \left(1 - F(v_{K-1})\right) p_K x_K. \end{aligned}$$

Differentiating with respect to x_{l+1} we get that

$$\begin{aligned}
\frac{\partial R}{\partial x_{l+1}} &= \delta \frac{\partial x}{\partial v_l} \frac{1}{p_{l+1-\delta+\delta\frac{\partial x}{\partial v_l}}} \left(F(v_l) - F(x) - f(x)x\right) + \\
&+ \left(F(v_{l+1}) - F(v_l)\right) p_{l+1} + f(v_{l+1}) \frac{p_{l+2}x_{l+2} - p_{l+1}x_{l+1}}{p_{l+2} - p_{l+1}} \\
&+ f(v_l) \left(\frac{\delta x - p_{l+1}x_{l+1}}{p_{l+1} - \delta + \delta \frac{\partial x}{\partial v_l}} \right).
\end{aligned}$$

Notice that by (18), for $v_H = v_l$, the first term of the above expression is always equal to zero. From the assumption that the seller offers relevant contracts it holds that $v_{l+1} \geq v_l$ and $p_{l+2}x_{l+2} \geq p_{l+1}x_{l+1}$. If $\delta x \geq p_{l+1}x_{l+1}$ it follows immediately that expected revenues are non-decreasing in x_{l+1}. In the case that $\delta x \leq p_{l+1}x_{l+1}$ we need to do a little bit more work.

We start by adding and substracting $f(v_{l+1}) \left(\frac{\delta x - x_{l+1}p_{l+1}}{p_{l+1} - \delta + \delta \frac{\partial x}{\partial v_l}} \right)$ which gives that

$$\begin{aligned}
& \frac{\partial R}{\partial x_{l+1}} && (47) \\
\geq\ & \left(F(v_{l+1}) - F(v_l)\right) p_{l+1} \\
& + f(v_{l+1}) \left(\frac{p_{l+2}x_{l+2}\left(p_{l+1} - \delta\right) + \delta x \left(p_{l+2} - p_{l+1}\right) - p_{l+1}x_{l+1}\left(p_{l+2} - \delta\right)}{\left(p_{l+2} - p_{l+1}\right)\left(p_{l+1} - \delta + \delta \frac{\partial x}{\partial v_l}\right)} \right) && (48) \\
& - \left(f(v_l) - f(v_{l+1})\right) \left(\frac{p_{l+1}x_{l+1} - \delta x}{p_{l+1} - \delta + \delta \frac{\partial x}{\partial v_l}} \right) && (49) \\
& + f(v_{l+1}) \left(\frac{\left(p_{l+2}x_{l+2} - p_{l+1}x_{l+1}\right) \delta \frac{\partial x}{\partial v_l}}{\left(p_{l+2} - p_{l+1}\right)\left(p_{l+1} - \delta + \delta \frac{\partial x}{\partial v_l}\right)} \right) && (50) \\
\geq\ & 0.
\end{aligned}$$

The first term of the above expression is non-negative given that $v_{l+1} \geq v_l$. The second term is also non-negative for the following reasons

$$\begin{aligned}
v_{l+1} &\geq v_l \text{ reduces to} \\
x_{l+2}\left(p_{l+1} - \delta\right) + \delta x \left(p_{l+2} - p_{l+1}\right) &\geq x_{l+1}\left(p_{l+2} - \delta\right)
\end{aligned}$$

multiplying both sides of the above inequality by $p_{l+2} > 0$ we get that

$$p_{l+2}\left[x_{l+2}\left(p_{l+1}-\delta\right)+\delta x\left(p_{l+2}-p_{l+1}\right)\right] \geq p_{l+2}\left[x_{l+1}\left(p_{l+2}-\delta\right)\right]$$

but since $p_{l+1} < p_{l+2} \leq 1$ we get that

$$p_{l+2}x_{l+2}\left(p_{l+1}-\delta\right)+\delta x\left(p_{l+2}-p_{l+1}\right) \geq p_{l+1}x_{l+1}\left(p_{l+2}-\delta\right).$$

The third term of (47) is non-negative since, by the assumption that the seller offers relevant contracts, we have that $p_{l+2}x_{l+2} > p_{l+1}x_{l+1}$. Hence for expected revenues to be non-decreasing in x_{l+1} it will suffice if

$$\left(F(v_{l+1})-F(v_l)\right)p_{l+1}+\left(f(v_l)-f(v_{l+1})\right)\left(\frac{p_{l+1}x_{l+1}-\delta x}{p_{l+1}-\delta+\delta\frac{\partial x}{sdx_l}}\right) \geq 0$$

holds. This follows from Assumption D. Given the hypotheses made, the seller will pick x_{l+1} as large as possible, which implies that at the optimum it will hold

$$v_l = v_{l+1}.$$

Now replace v_l by v_{l+1}. Taking derivative with respect to x_{l+2}, which is just a renaming of our previous step, we get that v_{l+1} must be set equal to v_{l+2}. Continue analogously. So in the end we get that

$$v_l = v_{l+1} = ... = v_{K-1}.$$

The revenue maximizing vector $(x_1, ...x_K)$ is such that all contracts in M_0, but contract[18] (p_K, x_K) are essentially irrelevant. In equilibrium when M_0 is taken among the class of sets that contain K relevant contracts, then it, (M_0), is chosen to be equivalent to a singleton contract (p_K, x_K). ■

Proof of Lemma 2.

Proof. We have shown so far that in an equilibrium, when M_0 is chosen among the class of sets that contain K relevant contracts, x_i, i=1, ...K-1 will be chosen such that the following holds

$$\frac{x_1}{p_1} = v_1 = ... = v_{l-1} = x$$

[18]Contract (p_0, x_0) is treated as an outside option.

and

$$v_l = v_{l+1} = v_{l+2} = ... = v_{K-1}.$$

In equilibrium, only one contract in M_0 is chosen with positive probability, contract (p_K, x_K). All the other contracts in M_0 are essentially irrelevant, since they are chosen with zero probability. In such an equilibrium revenues are given by

$$R = \left[F\left(\frac{x_K - \delta x}{p_K - \delta}\right) - F(x)\right]\delta x + \left[1 - F\left(\frac{x_K - \delta x}{p_K - \delta}\right)\right] p_K x_K.$$

We proceed to show that in equilibrium M_0 is equivalent to a posted price, in other words p_K is set equal to 1.

Taking partial derivative of revenues with respect to p_K we get that

$$\begin{aligned}\frac{\partial R}{\partial p_K} &= f\left(\frac{x_K - \delta x}{p_K - \delta}\right)\left(\frac{x_K - \delta x}{(p_K - \delta)^2}\right)(p_K x_K - \delta x) \\ +\left[1 - F\left(\frac{x_K - \delta x}{p_K - \delta}\right)\right] x_K &\geq 0.\end{aligned}$$

The above inequality holds since x_K is relevant, which implies that $F\left(\frac{x_K - \delta x}{p_K - \delta}\right) \leq 1$, and $\delta x \leq x_K$. Since expected revenues are non-decreasing in p_K, the seller will chose p_K to be as large as possible. Hence $p_K = 1$. ■

References

[1] Ausubel, L., and R. Deneckere. (1989). Reputation in Bargaining and Durable Goods Monopoly. *Econometrica*, 57, 511-531.

[2] Baron, D., and D. Bensako. (1984). Regulation and Information in a Continuing Relationship. *Information Economics and Policy*, 1, 447-470.

[3] Bulow, J. (1982). Durable Goods Monopolists. *Journal of Political Economy*, 90, 314-322.

[4] Coase, R. (1972). Durability and Monopoly. *Journal of Law and Economics*, 15, 143-149.

[5] Dewatripont, M. (1986). Renegotiation and Information Revelation over Time in Optimal Labor Contracts. Chap.1 in On the Theory of Commitment, with Applications to the Labor Market, Ph.D. dissertation, Harvard University. Also in *Quarterly Journal of Economics,* 104, (1989), 589-620.

[6] Freixas, X., R. Guesnerie, and Tirole. (1985). Planning under Incomplete Information and the Ratchet Effect. *Review of Economic Studies,* 52, 173-192.

[7] Fudenberg, D., D. Levine, and J. Tirole. (1985). Infinite-horizon Models of Bargaining with One-sided Incomplete Information. In *'Game Theoretic Models of Bargaining'*, ed. A. Roth. Cambridge University Press.

[8] Fudenberg, D. and J. Tirole. (1991). *Game Theory.* Cambridge, Mass. : MIT Press.

[9] Gul, F., Sonnenschein, H., and Wilson, R. (1986). Foundations of Dynamic Monopoly and the Coase Conjecture, *J. Econ. Theory* 39, 155-190.

[10] Hart, O., and J. Tirole. (1988). Contract Renegotiation and Coasian Dynamics. *Review of Economic Studies*, 55, 509-540.

[11] Laffont,J.J., and J. Tirole. (1988). The Dynamics of Incentive Contracts. *Econometrica*, 56, 1153-1175.

[12] Laffont,J.J., and J. Tirole. (1990). Adverse Selection and Renegotiation in Procurement. *Review of Economic Studies*, 57, 597-625.

[13] Laffont, J.-J. and Tirole J.(1993). *A Theory of Incentives in Procurement and Regulation.* Cambridge, Mass. MIT Press.

[14] Maskin, E., and J. Tirole. (1992). The Principal-Agent Relationship with an Informed Principal. II: Common Values. *Econometrica*, 60, 1-42.

[15] McAfee, R. P. and Vincent, D. (1997). Sequentially Optimal Auction, *Games and Economic Behavior*, 18, 246-276.

[16] Munkres, I. (1975). *Topology: A First Course.* Prentice Hall.

[17] Myerson, R. (1981) Optimal Auction Design, *Math. Oper. Res.*, 6, 58-73.

[18] Rey, P. and B. Salanie.(1996). On the Value of Commitment with Asymmetric Information. *Econometrica*, 64, 1395-1414.

[19] Riley, J. G. and Samuelson, W. F.(1981) Optimal Auctions, *American Economic Review, 71, 381-392.*

[20] Rockafellar, R. T. (1970). *Convex Analysis.* Princeton University Press.

[21] Royden, H. (1962). *Real Analysis*, 2nd Edition, New York: Macmillan.

[22] Sobel, J. and I. Takahashi. (1983). A multi-stage Model of Bargaining. *Review of Economic Studies*, 50, 411-426.

[23] Stokey, N. (1981). Rational Expectations and Durable Goods Pricing. *Bell Journal of Economics*, 12, 112-128.

[24] Stokey, N. and R. Lucas with E. Prescott. (1989). *Recursive Methods in Economic Dynamics.* Harvard University Press.