Dynamic Equilibrium Selection:
A General Uniqueness Result

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Abstract

This paper shows that in a dynamic context, under weak assumptions, the presence of payoff shocks can shrink the equilibrium set to a singleton. We study a model with a continuum of fully rational agents who switch between two actions or states over time (e.g., working in different sectors, employment vs. unemployment, etc.). An agent's incentive to pick a given action is greater if others do the same. Agents receive chances to change actions at random times and may influence the rate at which these chances arrive. Payoff shocks may follow any of a large class of stochastic processes that includes both seasonal and mean-reverting processes. In this general setting, payoff shocks give rise to a unique equilibrium. One implication is that the introduction of aggregate shocks leads to a unique equilibrium in two well-known macroeconomic search models with multiple equilibria (Diamond and Fudenberg, Howitt and McAfee).

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E pluribus unum. [Out of many, one.] – Motto on U.S. dollar.

1 Introduction

The problem of multiple equilibria is one of the most fundamental of modern economic theory. Models with multiple equilibria lead to ambiguous predictions and are harder to test and refute than models with unique outcomes. The assumption of equilibrium play is also harder to justify: one has to explain why all agents expect the same equilibrium. Finally, multiplicity distorts research: since the conclusion that "anything can happen" is seen as uninteresting and the use of equilibrium refinements is regarded with skepticism, ad-hoc assumptions are often made in order to guarantee a unique equilibrium.

Recently, economists have attacked this problem by looking for omitted features that have the potential to lead players to select one outcome in particular. One such factor is aggregate payoff shocks.[2] Payoff shocks have been shown to lead to uniqueness in models in which a continuum of fully rational players play over time a static game with strategic complementarities and frictions in changing actions. Without shocks, there are multiple equilibria. When payoff shocks are introduced, in the form of a payoff parameter that changes stochastically, a unique equilibrium emerges.[3]

This paper generalizes these results in two important ways. Prior work has assumed that the payoff parameter follows either a random walk or its continuous-time counterpart, a Brownian motion. Each process has a crucial property: the distribution of its future changes does not vary over time or depend on the current value of the process. This rules out features such as mean-reversion and seasonality, which characterize many real-world variables that affect economic activity.[4] Early work relied heavily on this

[2] Another factor that can lead to a unique equilibrium is asymmetric information: when agents receive noisy signals of a game's true payoffs (see Carlsson and van Damme [6], Morris, Rob, and Shin [16], Morris and Shin [17], and Frankel, Morris, and Pauzner [9]).

[3] See Burdzy, Frankel, and Pauzner [3, 5] and Frankel and Pauzner [10, 11].

[4] If a variable is mean-reverting, its future changes depend on its current value: when it is high, it

property; mean reversion and seasonality could be tolerated only in the limit as the shocks or frictions vanish (Burdzy, Frankel, and Pauzner [3, 5]; Frankel and Pauzner [10]).

We prove that a unique equilibrium is obtained for a general class of both mean-reverting and seasonal processes, *without taking any limits.* The argument has two parts. We first show the existence of a transformation of space and time that removes any mean reversion or seasonality. The resulting process still does not have to be a Brownian motion since its variance may fluctuate over time. But we show that the original uniqueness proof can be generalized in a natural way to cover processes with time-varying variances. Together, these arguments let us prove uniqueness for processes that are either mean-reverting or seasonal or both.

Prior uniqueness arguments have also relied on another strong assumption: that agents receive chances to switch actions at a common, fixed rate. This assumption is usually unrealistic. For example, a worker can lower her expected waiting time to change jobs, either through searching more intensively or being less selective about accepting job offers. A firm can lower the time needed to fill a vacancy by advertising more heavily or by lowering its job requirements. We show that this sort of "endogenous switching" does not upset the uniqueness result. As an application, we show that the introduction of aggregate shocks leads to a unique equilibrium in two well-known models that study the multiplicity of equilibria arising from costly search: Diamond [7] and Diamond and Fudenberg [8]; and Howitt and McAfee [13].

The rest of the paper is as follows. The model is described in section 2. Section 3 presents the main result. In section 4, we discuss the models of Diamond, Diamond and Fudenberg, and Howitt and McAfee. An intuition for the main result is then presented in section 5. Proofs are collected in an appendix.

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is more likely to fall. A seasonal variable changes in a way that depends on time: e.g., the maximum daily temperature tends to fall in the autumn months and to rise in the spring.

2 The Model

The game is played in continuous time $t \in [0, \infty)$. There is a continuum of players of measure 1. At any time t, each player is locked into one of two actions, R or L. Players change actions from time to time, according to independent Poisson processes.[5] A player who is playing the action $a = R, L$ and maintains the switching rate k^a during the period $[t, t + dt]$ will change actions with probability $k^a dt$ during that period. She also incurs the cost $c^a(k^a, X_t)dt \geq 0$ where X_t is the proportion of R players in the population. In addition to paying switching costs, a player also derives utility directly from playing a given action. This utility flow is $u(a, W_t, X_t)$, where $a = R, L$ is the player's action and W_t is an exogenous parameter that changes randomly over time.[6] A player's time-t continuation payoff is thus

$$E \int_{v=t}^{\infty} e^{-r(v-t)} \left[u(a_v, W_v, X_v) - c^{a_v}\left(k_v^{a_v}, X_v\right) \right] dv$$

where $a_v \in \{R, L\}$ is the action the player is locked into at time v, $k_v^{a_v}$ is her switching rate, and $r > 0$ is her pure rate of time preference.

Prior work in this area has assumed that the payoff parameter W follows a Brownian motion. In this paper we show that W can follow a more general stochastic process. We assume that

$$dW_t = (a_t W_t + b_t)dt + \sigma_t dB_t \tag{1}$$

where B is a Brownian motion with zero drift and unit variance.[7] This equation means that as $dt \to 0$, the increment $W_{t+dt} - W_t$ is asymptotically normal with mean $(a_t W_t + b_t)dt$ and variance $\sigma_t^2 dt$. If $a_t < 0$, for example, W is mean-reverting.

[5] The model has a continuum of independent Poisson processes. We assume that there is no aggregate uncertainty. See Judd [14], Boylan [2], and Gilboa and Matsui [12] for a discussion of the technical problems underlying this assumption, as well as some solutions.

[6] Pairwise random matching is a special case: if players are randomly matched to play a game with payoff function $v(a, a', W_t)$ (where a is the player's action and a' is her opponent's), then $u(a, W_t, X_t) = X_t v(a, R, W_t) + (1 - X_t) v(a, L, W_t)$.

[7] This means that for any $t > t'$, $B_t - B_{t'}$ is normally distributed with mean zero and variance $t - t'$.

Some technical assumptions on W are necessary: there are constants $0 < N_1 < N_2$ such that, for all t, $|a_t|, |b_t| < N_2$, $\int_{s=0}^{\infty} |a_s| \, ds < N_2$, $\sigma_t \in [N_1, N_2]$, and $\dot{\sigma}_t \leq N_2$. The assumption that $\int_{s=0}^{\infty} |a_s| \, ds < N_2$ means, e.g., that if W is mean reverting, the mean-reversion coefficient a_t must eventually converge to zero. Although this is somewhat restrictive, a_t can be arbitrarily large for an arbitrarily long time.

We also assume that the function c^a is weakly increasing and left-continuous in the switching rate k^a.[8] The switching rate k^a is constrained to come from some interval $[\underline{K}^a, \overline{K}^a]$ where $\infty > \overline{K}^a \geq \underline{K}^a \geq 0$. $\underline{K}^a$ and $\overline{K}^a$ may be Lipschitz functions of X_t; if so, we assume they are bounded by some constant K.[9]

It is important to note that this framework generalizes the earlier models of Burdzy, Frankel, and Pauzner [3, 5] and Frankel and Pauzner [10, 11], in which players costlessly receive chances to change actions at some fixed, exogenous rate δ. We can capture this in our model by constraining k^R and k^L to be in $[0, \delta]$ and letting $c^R = c^L = 0$ in this range. Choosing a switching rate of k in our model is then equivalent to switching actions with probability k/δ should an opportunity arise in the earlier models.

Let $\Delta(W_t, X_t, k_t^R, k_t^L)$ be the difference in static utilities from being locked into R and choosing switching rate k_t^R as opposed to being locked into L and choosing switching rate k_t^L:

$$\Delta(W_t, X_t, k_t^R, k_t^L) = \left[u(R, W_t, X_t) - c^R(k_t^R, X_t)\right] - \left[u(L, W_t, X_t) - c^L(k_t^L, X_t)\right] \qquad (2)$$

We assume that Δ is nondecreasing and Lipschitz in X_t: there is a $\beta > 0$ such that for any w and $x > x'$ and for any update rates k^R and k^L that are feasible at both states (w, x) and (w, x'),

$$\Delta(w, x, k^R, k^L) - \Delta(w, x', k^R, k^L) \in [0, \beta(x - x')] \qquad (3)$$

[8]Without left continuity an optimum might not exist. Suppose, e.g., that the cost is zero for switching rates below K and $c > 0$ for rates K and above. If the benefit of switching is only $c/2$, the agent has no optimal switching rate: any rate below K is too low while any rate greater than or equal to K is too high.

[9]It appears to be difficult to apply our approach to the case in which c^a, $\overline{K}^a$, and $\underline{K}^a$ are also functions of W_t. Whether this can be done is an interesting question.

This implies that there are strategic complementarities in the static game: the static relative payoff from playing a given action is increasing in the proportion who play that action.

Δ is also strictly increasing and Lipschitz in W_t: there is an $\overline{\alpha} > 0$ such that for any $w > w'$ and x and for any update rates k^R and k^L that are feasible at both states (w, x) and (w', x),

$$\Delta(w, x, k^R, k^L) - \Delta(w', x, k^R, k^L) \in (0, \overline{\alpha}(w - w')] \tag{4}$$

We assume the existence of *dominance regions*: for W_t sufficiently high, it is strictly dominant in the dynamic game for R players to pick their lowest feasible update rate and for L players to pick their highest feasible update rate; and analogously for low enough W_t. More precisely, there are constants $\overline{w} > \underline{w}$ such that if $W_t \geq \overline{w}$, then it is strictly dominant in the dynamic game to set $k_t^R = \overline{K}^R$ and $k_t^L = \underline{K}^L$; for $W_t < \underline{w}$, it is strictly dominant to choose $k_t^R = \underline{K}^R$ and $k_t^L = \overline{K}^L$.[10] We give some examples of how to check these conditions in the proofs of Propositions 1 and 2.

A player's information set at time t comprises the public history $(W_v, X_v)_{v \in [0,t)}$ and her private history (the actions she has played and the switching rates she has selected through time t). A (possibly mixed) strategy for a player specifies, at any information set, the distribution of switching rates that she will choose.

3 Showing Uniqueness

The game is solved using iterative conditional dominance. First we eliminate strategy profiles in which any player, after any history, plays a strictly dominated action; then we eliminated profiles in which any player, after any history, plays an action that is strictly dominated in the set of remaining strategy profiles; and so on. We show that the game has an *essentially unique* outcome: for almost any sequence of shocks (i.e., with probability one), iterative conditional dominance isolates a unique path of X. Importantly, every Nash equilibrium outcome survives the iterative procedure, so this

[10] As noted, these bounds may be functions of X_t.

path of X is also the unique *equilibrium* path.

THEOREM 1 *There is an essentially unique strategy profile that survives iterative conditional dominance: for almost any path* $(W_t)_{t \geq 0}$*, there is a unique path* $(X_t)_{t \geq 0}$ *that must occur.*

Proof: appendix (p. 18).

An intuition appears in section 5.

4 Application: Search Models

Some well-known search models with multiple equilibria are special cases of our model. Our results show that the indeterminacy in these models is sensitive to the introduction of payoff shocks.

4.1 Diamond [7] and Diamond and Fudenberg [8]

Diamond [7] presents a model in which unemployed agents receive production opportunities with random production costs. An agent who takes a production opportunity carries the resulting inventory until she finds a trading partner, at which time she trades, consumes, and becomes unemployed once again. The rate at which agents find trading partners is increasing in the proportion of agents with inventories because of increasing returns in the matching technology. Diamond and Fudenberg [8] show that this strategic complementarity gives rise to multiple equilibrium paths: the expectation that agents will have a low cost threshold, thus accepting only a few productive opportunities, is self-fulfilling since it implies that inventory will be hard to sell. They also use the model to study "endogenous business cycles": expectations-driven fluctuations in which agents alternate between optimistic expansions and pessimistic recessions.

We show, as an application of our findings, that when inventory costs are stochastic, the equilibrium is unique. Economic fluctuations are driven by cost shocks alone; agents' expectations no longer play an autonomous role.

Diamond's model has a continuum of identical agents who, at any given time, are either employed or unemployed. An unemployed agent receives production opportunities according to a Poisson process with arrival rate a. Each opportunity has a fixed output $y > 0$ and a random cost c, which is drawn from the continuous distribution $G(c)$ with support on $(\underline{c}, \overline{c})$ where $u > \overline{c} > \underline{c} > 0$. If the agent accepts the production opportunity, she becomes employed and begins to search for a trading partner. If the proportion of employed agents is e_t, then an agent meets trading partners according to a Poisson process with arrival rate $b(e_t)$ where $b' \in (0, \infty)$ and $b'' \leq 0$. When an agent trades, she consumes, getting the consumption utility u, and becomes unemployed once again. Agents cannot consume their own output.

We modify Diamond's model by assuming a stochastic cost of holding an inventory. Changes in inventory costs might be due, e.g., to fluctuations in warehouse rents or short term interest rates.[11] The cost of holding an inventory is assumed to be a function $h(W_t)$, where W_t is a stochastic process that satisfies (1). The function h is strictly increasing and Lipschitz.

The assumption of dominance regions requires that if inventory costs are low enough, it becomes strictly dominant to accept all productive opportunities. This raises a problem: since agents must trade in order to consume, if they expect all other agents to decline all productive opportunities, they will also choose not to produce, regardless of how low inventory costs are. To apply our results, the model must be modified so that this autarkic outcome is not an equilibrium for low enough inventory costs.

Wherever human beings have lived together, they have traded. One reason may be that people initially produced for their own consumption; since production and consumption opportunities did not perfectly coincide, they had to hold inventories. Trade began soon after, when people noticed that their inventories differed from those of their neighbors. We model this by assuming that agents sometimes receive opportunities to consumer their own production: $b(0) > 0$. (Diamond assumed $b(0) = 0$.) This

[11] One could also imagine production cost shocks - shocks to the distribution G. This is equivalent in our model to letting one of the switching cost functions (c^R or c^L) depend on the stochastic parameter W. We have not proved uniqueness in this case; whether it can be done is an interesting open question.

possibility of own-consumption eliminates the autarkic equilibrium for low enough inventory costs. The dominance region assumptions then become $\lim_{W_t \to +\infty} h(W_t) > b(1)u$ and $\lim_{W_t \to -\infty} h(W_t) < b(0)u - \overline{c}(r + b(0) + a)$. These conditions guarantee that for sufficiently high (low) values of W_t, it is strictly dominant to take no (all) productive opportunities. Finally, we assume that b is Lipschitz.

PROPOSITION 1 *The search model analyzed by Diamond [7] and Diamond and Fudenberg [8], with stochastic inventory costs and the aforementioned changes, has a unique equilibrium.*

Proof: appendix (p. 50).

4.2 Howitt and McAfee [13]

Howitt and McAfee [13] study a model with many identical firms that advertise to hire workers, who quit or "die" at a fixed rate. A firm's marginal revenue from hiring a worker is increasing in the aggregate employment rate since higher employment raises the number of customers. Moreover, the cost of attracting a job applicant with a given probability is decreasing in the employment rate since higher employment leads to a smaller pool of potential job applicants. These assumptions give rise to multiple equilibria: if firms expect high employment in the future, they will advertise intensely for workers in the present since they expect both marginal revenue and advertising costs to increase later. There are also "sunspot equilibria" in which economic fluctuations are driven by an extraneous variable (e.g., sunspots) that has no direct effect on agents' payoffs. Changes in this variable lead firms to alternate between optimism and pessimism, giving rise to expectations-driven business cycles or "animal spirits."

We show that when productivity shocks are introduced, the equilibrium becomes unique. Expectations do fluctuate, but in a unique way, as determined by the outcome of the productivity shocks. This implies that there can be no sunspot equilibria.

The model is as follows. Workers are either employed or unemployed. Unemployed workers search costlessly until they find a job. Employed workers become unemployed according to a Poisson process with arrival rate δ. While employed, a worker produces a flow of output at the rate $fG(n_t)$, where n_t is the proportion of employed workers, $G(n_t) \in [0,1]$, and $G' > 0$. $G(n_t)$ is the fraction of output f that is not spent on marketing. It is an increasing function of n_t because buyers are easier to find when employment is high. A worker's employer receives the fixed fraction $1-\omega$ of the worker's output, $fG(n_t)$.

In a period $[t, t+dt]$, a firm meets an unemployed worker with probability $\theta_t(1-n_t)dt$ and incurs an advertising cost of $c\theta_t dt$, where θ_t represents advertising intensity. Firms are restricted to choose $\theta_t \in [0,h]$, where $h > 0$; thus, there is an upper bound hdt on the percentage of workers (employed or unemployed) who can be reached in a period of length dt. The firms' discount rate is β; firms choose advertising intensities to maximize the integral of expected discounted future profits.

We introduce payoff shocks into the model by assuming that a worker's productivity is an increasing function $G(n_t, W_t)$ of a stochastic parameter W_t, which satisfies (1). We assume that G is increasing and Lipschitz in n_t and W_t, and that $\lim_{W_t \to -\infty} G(n_t, W_t) = 0$ and $\lim_{W_t \to +\infty} G(n_t, W_t) = \infty$ for all n_t.

The original model of Howitt and McAfee is set in discrete time. We have recast their model in continuous time in order to be able to apply our results. Our uniqueness argument relies on time being continuous or nearly so: this guarantees that while W remains at a particular value, only a negligible fraction of players (firms in the current case) can change their actions (vacancy status). If this were not so, one player's optimal switching (advertising) intensity might depend on the intensities chosen by others at the same value of W, and there could be multiple equilibria in parts of the state space. Burdzy, Frankel, and Pauzner [5] show, in a simpler model, that uniqueness is obtained in the limit of discrete models as the period length goes to zero.

We also assume a continuum of firms, while Howitt and McAfee specify only a "large number." With a finite number of firms, one firm's actions might be observed and punished by other firms. This would lead to multiple equilibria. Howitt and McAfee implicitly assume a continuum as they rule out equilibria that utilize punishment: firms do not think that their decisions will affect other firms' future behavior [13, equation

(1), p. 496]. Accordingly, the multiplicity of equilibria that they find is not due to having a finite number of firms (if that is indeed what they intended).

PROPOSITION 2 *The search model analyzed by Howitt and McAfee [13], with the above modifications, has a unique equilibrium.*

Proof: appendix (p. 52).

5 Intuition for Main Result

In the first part we will assume that the payoff parameter W follows a Brownian motion. Afterwards we will explain why the results still hold if W comes from a more general class of stochastic processes.

For simplicity, let us assume that players choose between only two positive update rates. The cost of choosing an update rate at or below $\kappa > 0$ is zero, the cost of choosing an update rate in $(\kappa, K]$ is some $c > 0$, and the cost of choosing an update rate above K is infinite. Clearly, if a player wants to change actions, it doesn't make sense to choose an update rate $k \in (0, \kappa)$ or $k \in (\kappa, K)$ since increasing the update rate would be costless. Thus, if a player strictly prefers one action over the other, she will choose among three update rates: 0 (if she is already playing her preferred action), κ (if she has a slight preference for switching), and K (if she strongly prefers the other action).

We iterate first by computing a declining sequence of upper bounds on $\dot{X}$ at any state (W, X). (Time subscripts are omitted.) Then we will compute an increasing sequence of lower bounds on $\dot{X}$. The final step is to explain why these two sequences must converge to each other, so that $\dot{X}$ is uniquely determined at any state.

Suppose all R players at some state choose the same switching rate k^R while all L players choose k^L. The rate of change of X must be $\dot{X} = k^L(1-X) - k^R X$, since there are $1 - X$ players of L and X players of R. This means that the fastest feasible rate of increase of X is $\dot{X} = K(1 - X)$: when $k^L = K$ and $k^R = 0$. Let $\dot{X}^{\mathbf{0}} = K(1 - X)$ be this highest upper bound.

We next compute another upper bound, $\dot{X}^1 \leq \dot{X}^{\mathbf{0}}$. This is the rate of change that results from optimal behavior if players expect that, in the future, $\dot{X}$ will always equal

$\dot{X}^0$. Since there are strategic complementarities, the belief that $\dot{X}$ will always equal $\dot{X}^0$ is the most favorable for R, so it maximizes the rate at which L players switch to R and minimizes the rate at which R players switch to L. Thus, $\dot{X}$ cannot exceed $\dot{X}^1$ if players know that $\dot{X}$ will never be higher than $\dot{X}^0$ and behave rationally.

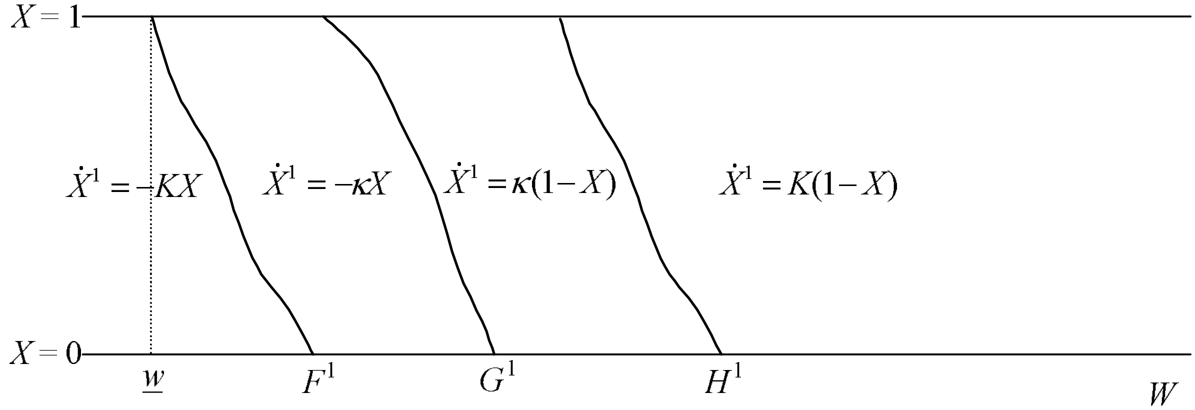

Figure 1: The Upper Bound $\dot{X}^1$.

Figure 1 depicts the bound $\dot{X}^1$. The payoff parameter W appears on the horizontal axis, while the proportion X_t of R players is on the vertical axis. In the region to the left of the curve F^1, the low W makes R so unappealing, even under the belief that $\dot{X}$ will equal $\dot{X}^0$, that R players must switch to L at the maximum possible rate, K, while L players will never switch to R. Thus, in this region, $\dot{X}^1 = -KX$. Such a region must exist by the assumption of dominance regions: if $W < \underline{w}$, these update rates are strictly dominant for any X. ($\underline{w}$ is shown in the figure.)

Between F^1 and G^1, players still prefer L under the belief that $\dot{X}$ will equal $\dot{X}^0$, but less so than in the first region since W is higher. Consequently, R players choose the lower switching rate κ: $\dot{X}^1 = -\kappa X$. In the region between G^1 and H^1, R is slightly preferred to L under the belief that $\dot{X}$ will equal $\dot{X}^0$. R players do not switch and L players choose the switching rate κ, so $\dot{X}^1 = \kappa(1-X)$. To the right of H^1, R is strongly preferred over L. L players now choose the highest switching rate, K, so $\dot{X}^1 = K(1-X)$.

We proceed to compute upper bounds $\dot{X}^2$, $\dot{X}^3$, ..., where $\dot{X}^n$ is the rate of change that results from optimizing behavior if players expect $\dot{X}$ to equal $\dot{X}^{n-1}$ in the future.

Strategic complementarities imply that if $\dot{X}^{n-1} \leq \dot{X}^{n-2}$, then $\dot{X}^{n} \leq \dot{X}^{n-1}$. Since $\dot{X}^{1} \leq \dot{X}^{0}$, this is a weakly decreasing sequence of upper bounds: $\dot{X}^{n} \leq \dot{X}^{n-1}$ for all n.

Let F^n, G^n, and H^n be the boundaries of the different regions of $\dot{X}^n$, in analogy to F^1, G^1, and H^1 in Figure 1. Since $\dot{X}^{n} \leq \dot{X}^{n-1}$, each curve F^n,G^n, and H^n lies weakly to the right of the corresponding curve F^{n-1},G^{n-1}, and H^{n-1}. Eventually we will reach a limiting upper bound on $\dot{X}$ with regions separated by curves F^{∞}, G^{∞},and H^{∞}. This limit is depicted in Figure 2.

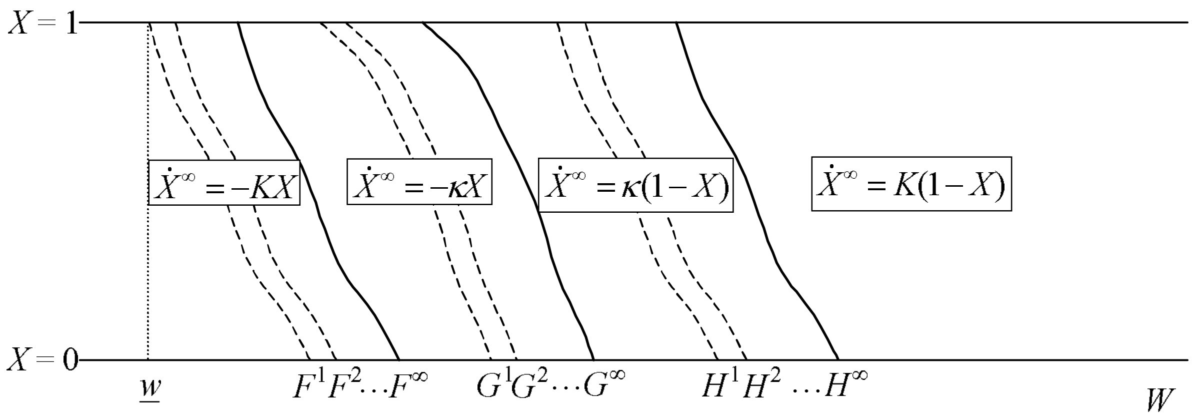

Figure 2: Limit of Iterations of Upper Bounds on $\dot{X}$.

By construction, for any n, if players expect $\dot{X}$ to equal $\dot{X}^{n-1}$, they will choose update rates that make $\dot{X}$ equal $\dot{X}^{n}$. Since one more iteration from $\dot{X}^{\infty}$ yields $\dot{X}^{\infty}$, if players expect $\dot{X}$ to equal $\dot{X}^{\infty}$, they must choose update rates that make this come true. That is, $\dot{X} = \dot{X}^{\infty}$ is an *equilibrium* in addition to being an upper bound on $\dot{X}$.

The next step is to find an increasing sequence of lower bounds on $\dot{X}$ at each state. However, the procedure is different: we iterate using *translations* of the limiting upper bound, $\dot{X}^{\infty}$. The reason will soon be apparent. We first shift the curves F^{∞}, G^{∞}, and H^{∞} in parallel far enough to the right that all three curves lie to the right of $\overline{w}$. Let this translation be $(\widehat{F}, \widehat{G}, \widehat{H})$ (Figure 3).

For any curves (F, G, H), let the $\dot{X}$ *given by* (F, G, H) be $-KX$ to the left of F, $-\kappa X$ between F and G, $\kappa(1-X)$ between G and H, and $K(1-X)$ to the right of H. The curves $(\widehat{F}, \widehat{G}, \widehat{H})$ give an $\dot{X}$ that is a lower bound on the rate of change that can

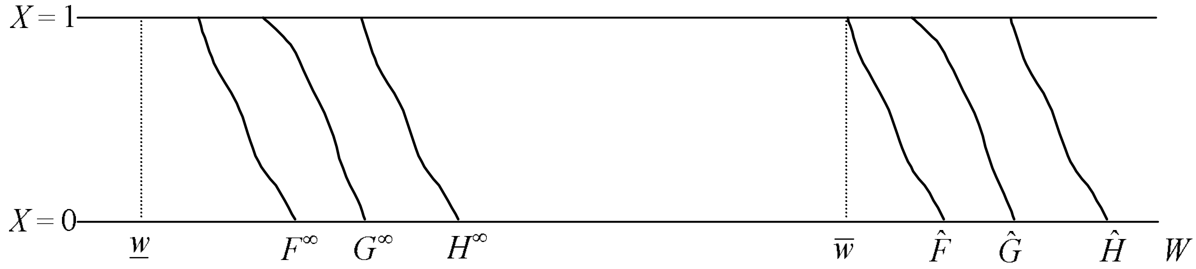

Figure 3: The Translation $(\widehat{F}, \widehat{G}, \widehat{H})$.

come from optimizing behavior. This is because $(\widehat{F}, \widehat{G}, \widehat{H})$ gives $\dot{X} = -KX$ to the left of $\overline{w}$, but no lower $\dot{X}$ is feasible; $(\widehat{F}, \widehat{G}, \widehat{H})$ gives higher values of to $\dot{X}$ the right of $\overline{w}$, but this is a region in which it is dominant for R players to stay in R and for L players to pick K, so the only optimal $\dot{X}$ is $K(1-X)$.

We now proceed to find an increasing sequence of lower bounds on $\dot{X}$ by iterating with translations of $(\widehat{F}, \widehat{G}, \widehat{H})$. If players expect the rate of change of X always to be that given by $(\widehat{F}, \widehat{G}, \widehat{H})$, their optimal switching rates imply, at every state, some "best response" $\dot{X}$; the next translation in the sequence is the leftmost translation of $(\widehat{F}, \widehat{G}, \widehat{H})$ that never gives an $\dot{X}$ above this best response. This means that the new translation is a new lower bound on $\dot{X}$. We iterate in this way ad infinitum. Let the limit be $(\widehat{F}^\infty, \widehat{G}^\infty, \widehat{H}^\infty)$. Let the $\dot{X}$ given by this limit be $\widehat{\dot{X}}^\infty$. Since $\widehat{\dot{X}}^\infty$ is a lower bound on $\dot{X}$ while $\dot{X}^\infty$ is an upper bound, it must be that $\widehat{\dot{X}}^\infty \leq \dot{X}^\infty$, and thus $\widehat{F}^\infty$, $\widehat{G}^\infty$, and $\widehat{H}^\infty$ each lies weakly to the right of F^∞, G^∞, and H^∞, respectively. We will show that in fact $\widehat{F}^\infty = F^\infty$, $\widehat{G}^\infty = G^\infty$, and $\widehat{H}^\infty = H^\infty$, so that $\widehat{\dot{X}}^\infty = \dot{X}^\infty$: the rate of change of X is uniquely determined by the current state (W, X).

The curves $(\widehat{F}^\infty, \widehat{G}^\infty, \widehat{H}^\infty)$ are depicted in Figure 4. The dotted curves are the boundaries for $\dot{X}^{BR}$, the rate of change that comes from optimizing behavior when players expect $\dot{X}$ to equal $\widehat{\dot{X}}^\infty$. By construction, $\dot{X}^{BR}$ is never less than $\widehat{\dot{X}}^\infty$. Moreover, since $(\widehat{F}^\infty, \widehat{G}^\infty, \widehat{H}^\infty)$ is the limit of the iterative process, any further translation must give an $\dot{X}$ that sometimes exceeds $\dot{X}^{BR}$. The only way this can be true is if one of the boundaries of the different regions of $\dot{X}^{BR}$ touches the corresponding boundary of

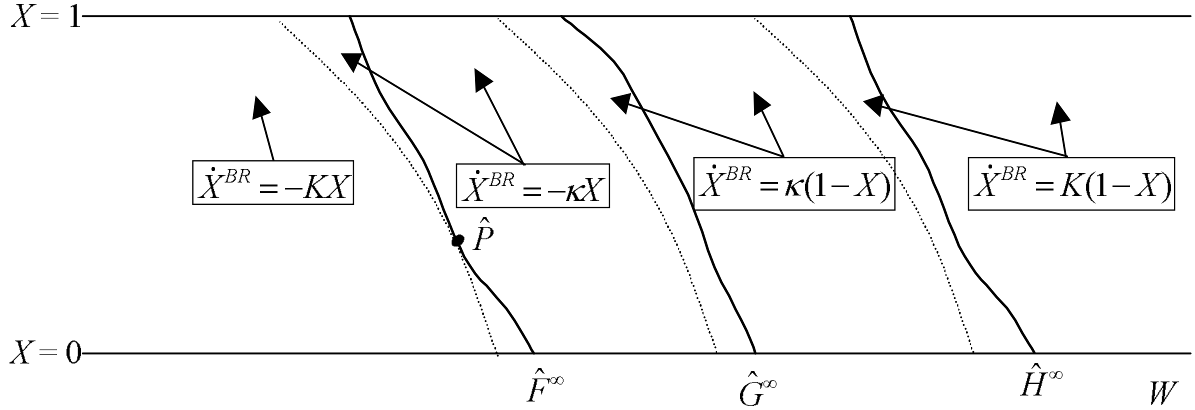

Figure 4: The Limit $(\widehat{F}^\infty, \widehat{G}^\infty, \widehat{H}^\infty)$.

$\dot{\widehat{X}}^\infty$. Let us suppose that the boundary $\widehat{F}^\infty$ of $\dot{\widehat{X}}^\infty$ is touched, as depicted in Figure 4. The point of contact is $\widehat{P}$. To the right of the dotted best response curve through $\widehat{P}$, R players prefer to switch to L at rate κ, while to the left they switch at rate K. Thus, at $\widehat{P}$, R players must be *indifferent* between the switching rates κ and K if they expect $\dot{X}$ always to equal $\dot{\widehat{X}}^\infty$ in the future.

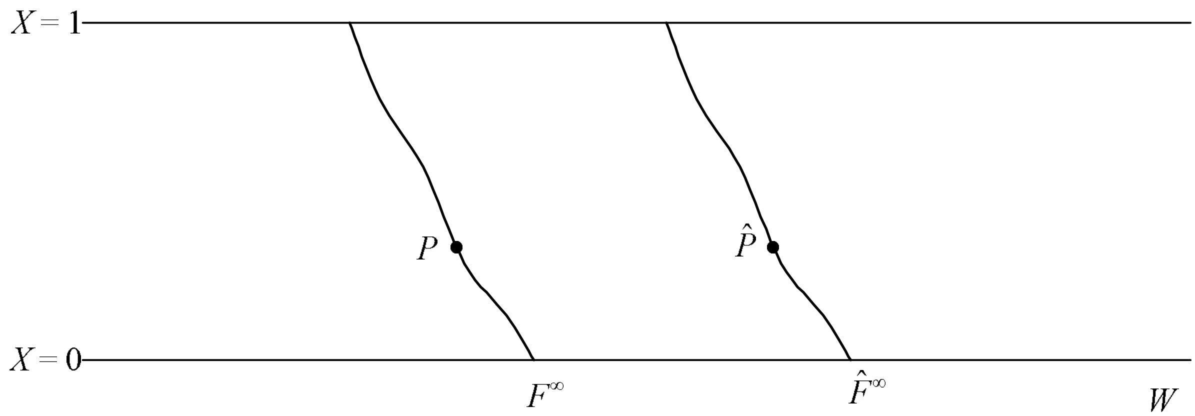

Figure 5: Comparison of P with $\widehat{P}$.

Let us now compare $\widehat{P}$ to the point P on F^∞ that is at the same vertical height as $\widehat{P}$ (Figure 5). Recall that $\dot{X}^\infty$, the $\dot{X}$ given by $(F^\infty, G^\infty, H^\infty)$, is an equilibrium. This means that R players are indifferent between the update rates κ and K at P if they expect $\dot{X}$ to equal $\dot{X}^{\infty}$. But at $\widehat{P}$, R players are indifferent between the update rates κ and K if they expect $\dot{X}$ to equal $\widehat{\dot{X}}^{\infty}$. Since $(\widehat{F}^{\infty}, \widehat{G}^{\infty}, \widehat{H}^{\infty})$ is a rightwards translation of $(F^{\infty}, G^{\infty}, H^{\infty})$ and P and $\widehat{P}$ are at the same vertical height on corresponding curves, a player at P who expects $\dot{X}$ to equal $\dot{X}^{\infty}$ anticipates the same joint distribution of changes in W and X as a player at $\widehat{P}$ who expects $\dot{X}$ to equal $\widehat{\dot{X}}^{\infty}$. That is, at any time t they expect the same distribution of *relative continuation paths*, $(W_v - W_t, X_v - X_t)_{v \geq t}$. Since X_t is the same at both points, the only difference is that W is expected to be uniformly higher at $\widehat{P}$ than at P. But then if R players are indifferent between the update rates κ and K at P, they must *strictly prefer* K at $\widehat{P}$ — unless $P = \widehat{P}$! This shows that $P = \widehat{P}$: F^{∞} coincides with $\widehat{F}^{\infty}$. Hence, the upper and lower bounds on $\dot{X}$ coincide: a unique strategy profile survives iterative conditional dominance.

We have glossed over a subtle point in the above argument. Players at P and $\widehat{P}$ might expect different distributions of relative continuation paths if the dynamical system at P or $\widehat{P}$ had more than one solution. In fact, the solution is unique, for the following reasons. First, our assumption that Δ, the relative static utility of playing R, is strictly increasing in W guarantees that in the compact region $W \in [\underline{w}, \overline{w}]$, the rate of change of Δ as a function of W is bounded below by a strictly positive constant. Since the rate of change of Δ as a function of X is bounded above, this guarantees that the curves F^{∞}, G^{∞}, and H^{∞}, where some players are indifferent between two update rates, have slopes that are bounded away from zero: these curves have no horizontal or near-horizontal segments. (Since each is a curve of indifference between two update rates, it could be horizontal only if the effect of W on the relative payoff from playing R vs. L were negligible relative to the effect of X; the above properties of Δ imply that this is not the case.) This implies that, if the state lies on one of these curves at some time t, its location at time $t + dt$ is governed by changes in W rather than by changes in X.[12] But from standard results on differential equations we know that the only time

[12]The reason is that over a short time interval $[t, t + dt]$, changes in X are of order dt while the standard deviation of changes in W are proportional to the square root of the time interval, $\sqrt{dt} \gg dt$. This is due to the central limit theorem: the variance of the sum of changes in W over many small time intervals of length dt equals the sum of the variances of these changes, so the variance of each change

at which there might be multiplicity is at a state where $\dot{X}$ is a non-Lipschitz function of X. This is true only on curves such as F^{∞}, where $\dot{X}$ changes discontinuously. Since changes in W govern the behavior of the system around these curves, multiplicity cannot creep in.

We have explained, for a simple cost structure, why a unique outcome survives the iterative procedure when W follows a Brownian motion. The same argument might appear to break down when W follows a more general stochastic process with such features as mean reversion and seasonality. The problem is in the last step: if P does not equal $\widehat{P}$, the distribution of changes in W will not, in general, be the same at P as at $\widehat{P}$, so the distribution of changes in X may differ as well. This means that a player might well be indifferent between two update rates at both points even if the points do not coincide. For example, suppose W reverts to a mean value that lies somewhere between the two curves F^{∞} and $\widehat{F}^{\infty}$. A player at P would expect W to trend upwards, making it likely that the state will move into the region where (under the beliefs corresponding to F^{∞}) $\dot{X}$ will equal $-\kappa X$. A player at $\widehat{P}$ would expect W to trend downwards, moving the state into the region where (under the beliefs corresponding to $\widehat{F}^{\infty}$) $\dot{X}$ will equal $-KX$. So a player at P would expect X to tend to be higher than would a player at $\widehat{P}$. On the other hand, in the near future W will tend to be lower for a player at P than for a player at $\widehat{P}$, since it starts lower. These two differences go in opposite directions: one makes R more appealing and one less. Thus, R players at the two points may well *both* be indifferent between the update rates κ and K.

How do we overcome this problem? We show the existence of a transformation of space and time that removes any mean reversion or seasonality from the process. The resulting process has no drift: its expected change is always zero. Its variance may change over time, but does not depend on the current value of the process. This means that the distribution of future changes in the transformed process does not depend on the process's current value. Thus, we can perform the iterations on the *transformed process*. One modification is necessary: since the transformed process has a time-

must remain proportional to dt as $dt \to 0$.

varying variance, players' optimal switching rates at a given state may change over time. This is addressed by letting the bounds on $\dot{X}$ depend on time as well as the current state. The rest of the intuition is essentially the same.

For example, suppose $dW_t = -aW_t dt + dB_t$, where a is a constant and B is a standard Brownian motion with no drift and unit variance. W has unit variance but is mean-reverting with a mean of zero. It has the same distribution of sample paths as the transformed process $\widehat{W}_t = e^{-at}B_{e^{2at}}$. Since B_t has no drift, neither does $B_{e^{2at}}$, but by multiplying by e^{-at} we create a drift towards zero - mimicking W. Since this requires us to progressively compress space, time must also be compressed to preserve the constant variance of the process. Hence, $\widehat{W}_t$ is based on the value of B at time e^{2at}, which grows exponentially. Together, these transformations of space and time give $\widehat{W}$ a unit variance and a drift towards zero.

The next step is to redefine the utility function so that we can work in the transformed space units: the instantaneous utility difference $\Delta(W_t, X_t, k_t^R, k_t^L)$ is rewritten as $\Delta_t(B_{e^{2at}}, X_t, k_t^R, k_t^L)$ where $\Delta_t(b, x, k^R, k^L) = \Delta(e^{-at}b, x, k^R, k^L)$. We then compute a declining sequence of upper bounds on $\dot{X}$ at each state and time, $(t, B_{e^{2at}}, X_t)$. Accordingly, players' switching rates can depend not only on the current state $(B_{e^{2at}}, X_t)$, but also on time, in response to the fluctuating variance of $B_{e^{2at}}$. The limiting upper bound $\dot{X}^\infty$ is then shifted over in the $B_{e^{2at}}$ dimension far enough that it becomes a lower bound on $\dot{X}$. We iterate using this translation, obtaining a limiting lower bound, and then show that the upper and lower bounds must coincide as in the prior argument. Here the stationarity argument does work: at any time t, the distribution of continuation paths $(B_{e^{2av}} - B_{e^{2at}})_{v \geq t}$ is independent of the current value of $B_{e^{2at}}$ since B is a Brownian motion and the time transformation is the same for all values of the parameter, $B_{e^{2at}}$.

Technically, we cannot actually prove uniqueness for processes that mean-revert forever such as $dW_t = -aW_t dt + dB_t$ since if $W_t = e^{-at}B_{e^{2at}}$, the nondominance region $W_t \in [\underline{w}, \overline{w}]$ corresponds to $B_{e^{2at}} \in [e^{at}\underline{w}, e^{at}\overline{w}]$, which continually expands. Thus, once we obtain the limiting upper bound $\dot{X}^\infty$, there may be no finite amount by which we can translate $\dot{X}^\infty$ in the $B_{e^{2at}}$ dimension so that at *all* times t, the shifted $\dot{X}^\infty$ equals

$-KX$ at every point in the nondominance region.[13] This means that the second round of iterations (in which translations are used) cannot get started: there may be no finite translation of $\dot{X}^\infty$ that is a lower bound on $\dot{X}$. To overcome this, we restrict to processes whose mean reversion coefficients a_t eventually die out, so that the analogue of e^{-at} does not go to zero as t grows.

A Proofs

Proof of THEOREM 1. We first show that we can write W in terms of a Brownian motion by simultaneously transforming space and time:

LEMMA 1 *Consider the diffusion given by $dW_t = (a_t W_t + b_t)dt + \sigma_t dB_t$ where B is a Brownian motion with zero drift and unit variance. Assume there are constants $0 < N_1 < N_2$ such that, for all t, $|a_t|, |b_t| < N_2$, $\int_{s=0}^{\infty} |a_s| ds < N_2$, $\sigma_t \in [N_1, N_2]$, and $\dot{\sigma}_t \leq N_2$. For the following functions g and h, the process $g(t, B_{h(t)})$ has the same distribution as the process W:*

$$g(t,z) = \exp\left(\int_{s=0}^{t} a_s ds\right) z + \int_{s=0}^{t} b_s \exp\left(\int_{v=s}^{t} a_v dv\right) ds \tag{5}$$

$$h(t) = \int_{s=0}^{t} \exp\left(-2\int_{v=0}^{s} a_v dv\right) \sigma_s^2 ds$$

where $B_0 = W_0$, h is strictly increasing and $h(0) = 0$. Moreover:

1. *There are constants $\overline{\gamma} \geq \underline{\gamma} > 0$ such that for all t and $z > z'$, $g(t,z) - g(t,z') \in \left[\underline{\gamma}(z-z'), \overline{\gamma}(z-z')\right]$ and $|g(t,z)| \leq \overline{\gamma}(|z| + t)$.*

2. *There are constants $\overline{\rho} \geq \underline{\rho} > 0$ such that for all $t > t'$, $h(t) - h(t') \in \left[\underline{\rho}(t-t'), \overline{\rho}(t-t')\right]$ and $|h'(t) - h'(t')| \leq \overline{\rho}|t - t'|$. ($h'$ is the derivative of h.)*

[13] More precisely, the translation by $\lambda > 0$ at state $(t, B_{e^{2at}}, X_t)$ equals the original $\dot{X}^\infty$ at state $(t, B_{e^{2at}} - \lambda, X_t)$; for large t, the nondominance region $B_{e^{2at}} \in [e^{at}\underline{w}, e^{at}\overline{w}]$ is large relative to any given λ, so we cannot pick λ large enough to guarantee that, at all times t, the translation equals $-KX$ (the absolute lower bound on $\dot{X}$) in the nondominance region.

As of time $t' \leq t$, W_t *is normal with mean* $\exp\left(\int_{s=t'}^{t} a_s ds\right) W_{t'} + \int_{s=t'}^{t} b_s \exp\left(\int_{v=s}^{t} a_v dv\right) ds$ *and variance* $\int_{s=t'}^{t} \exp\left(2\int_{v=s}^{t} a_v dv\right) \sigma_s^2 ds$.

Our assumptions imply that Δ has an important property. Since $\Delta(w, x, k^R, k^L)$ is strictly increasing in w, there must be a constant $\underline{\alpha} > 0$ such that if $w > w'$,

$$\Delta(w, x, k^R, k^L) - \Delta(w', x, k^R, k^L) \in [\underline{\alpha}(w - w'), \overline{\alpha}\,(w - w')] \tag{6}$$

for all

$$(w, x, k^R, k^L), (w', x, k^R, k^L) \in \left\{ (w, x, k^R, k^L) : \begin{array}{r} w \in [\underline{w}, \overline{w}], x \in [0, 1], \\ k^R \in [\underline{K}^R(x), \overline{K}^R(x)], \\ k^L \in [\underline{K}^L(x), \overline{K}^L(x)] \end{array} \right\}$$

(a compact set).

In the remainder of the proof, we normalize the cost of choosing the lowest possible switching rate to zero, by letting $\widehat{u}(a, w, x) = u(a, w, x) - c^a(\underline{K}^a(x), x)$ and $\widehat{c}^a(k, x) = c^a(k, x) - c^a(\underline{K}^a(x), x)$. We then relabel $\widehat{u}$ and $\widehat{c}$ to u and c^a, respectively.

Fix a state and a player's beliefs of how other players will respond to changes in W. Part 1 of the following Lemma gives an expression for the difference in the player's continuation payoffs from being locked into R vs. L. Part 2 proves some useful bounds on this difference. Part 3 gives the optimality condition that the update rates must satisfy.

LEMMA 2 *At state* (W_t, X_t)*, fix a player's beliefs over the path* $(X_v)_{v \geq t}$ *that will result from any path* $(W_v)_{v \geq t}$*. (These beliefs will be generated by her beliefs about other players' strategies, but that is not important for the lemma.) Let* $V_t^a = V^a(W_t, X_t)$ *be the player's continuation payoff if the player is locked into action* $a \in \{R, L\}$*. Let* k_v^a *be the player's optimal switching rate conditional on being locked into* a*. Then*

1.

$$V_t^R - V_t^L = E \int_{v=t}^{\infty} \exp\left(- \int_{s=t}^{v} (r + k_s^R + k_s^L) ds \right) \Delta(W_v, X_v, k_v^R, k_v^L) dv \tag{7}$$

2. *For all states* (W_t, X_t) *and for any beliefs over the path* $(X_v)_{v\geq t}$ *that will result from any path* $(W_v)_{v\geq t}$*,*

$$V_v^a - V_v^{a'} \leq E \int_{s=v}^{\infty} e^{-r(s-v)} \left(|u(a, W_s, X_s) - u(a', W_s, X_s)| + C\right) ds$$

for any $a, a' \in \{R, L\}$*. Moreover, there are positive constants* c_0 *and* c_1 *such that* $\left|V_t^R - V_t^L\right| \leq c_0 |W_t| + c_1$*.*

3. *For* $a \in \{R, L\}$*,* $k_v^a \in \text{argmax}_{k\geq 0} \left(k(V_v^{a'} - V_v^a) - c^a(k, X_v)\right)$*, where* $a' = R$ *if* $a = L$ *and vice-versa.*

Let $Z_t = B_{h(t)}$. We now redefine the state space to be the set of triplets (t, Z_t, X_t) rather than $(W_t, X_t) = (g(t, Z_t), X_t)$. Since players know t, by (5) they can invert $g(t, Z_t)$ to discover Z_t. Let $\Delta_v(Z_v, X_v, k_v^R, k_v^L)$ represent the static relative payoff to playing R vs. L at time v:

$$\Delta_v(Z_v, X_v, k_v^R, k_v^L) \triangleq \Delta(g(v, Z_v), X_v, k_v^R, k_v^L) = \Delta(W_v, X_v, k_v^R, k_v^L)$$

The iterative procedure begins by computing, at each state (t, Z_t, X_t), an upper bound $\Phi^0 = \Phi^0(t, Z_t, X_t)$ on $V_t^R - V_t^L$, the difference between the values of playing R and L. We compute this bound using a belief that maximizes the relative appeal of playing R: that all players will immediately switch to R and continue to play R forever.[14] By Lemma 2,

$$\Phi^0(t, Z_t, X_t) = E \int_{v=t}^{\infty} \exp\left(-\int_{s=t}^{v} (r + k_s^R + k_s^L) ds\right) \Delta_v(Z_v, 1, k_v^R, k_v^L) dv$$

where k_v^L, k_v^R, k_s^L, and k_s^R are optimal given these beliefs.

By part 3 of Lemma 2, $BR^R(y, x) = \text{argmax}_{k\geq 0}[-ky - c^R(k, x)]$ (resp., $BR^L(y, x) = \text{argmax}_{k\geq 0}[ky - c^L(k, x)]$) is the set of optimal switching rates for an R (resp., L) player when the relative payoff to R is y. These best response correspondences have the closed graph property and satisfy a single crossing property:

[14] The model restricts players to arrival rates below K. The belief that players will all immediately jump to R thus gives an (unattainable) upper bound on the difference between the expected payoffs from being in R and L.

LEMMA 3 *1. (Closed Graph.)* $BR^L(y,x)$ *and* $BR^R(y,x)$ *are upper hemicontinuous in* y.

2. (Single Crossing.) Suppose $y < y'$. *If* $k \in BR^L(y,x)$, *and* $k' \in BR^L(y',x)$, *then* $k \leq k'$. *If* $k \in BR^R(y,x)$, *and* $k' \in BR^R(y',x)$, *then* $k \geq k'$.

Since there are $1 - X_t$ L players, who switch to R at a rate no greater than the maximum of $BR^L(\Phi^0(t, Z_t, X_t))$, and X_t R players, who switch *out of* R at a rate no less than the minimum of $BR^R(\Phi^0(t, Z_t, X_t))$,[15]

$$\begin{aligned} \dot{X}_t &\leq \max BR^L(\Phi^0(t, Z_t, X_t), X_t) \cdot (1 - X_t) - \min BR^R(\Phi^0(t, Z_t, X_t), X_t) \cdot X_t \\ &\triangleq \pi(\Phi^0(t, Z_t, X_t), X_t) \end{aligned} \tag{8}$$

where we define $\pi(y,x)$ to be $\max BR^L(y,x) \cdot (1-x) - \min BR^R(y,x) \cdot x$, the highest rate of change of X that is consistent with rational behavior when $X_t = x$ and the relative payoff to R is y.

Equation (8) implies, for any state (t, Z_t, X_t), a new upper bound $\Phi^1(t, Z_t, X_t)$ on the relative payoff from being locked into R vs. L. Φ^1 is computed using the belief that is the most favorable for R: that for all $v \geq t$, $\dot{X}_v$ will *equal* its the old upper bound, $\pi(\Phi^0(v, Z_v, X_v), X_v)$. For all $n \geq 1$, let $\Phi^n(t, Z_t, X_t)$ be the relative payoff to R on the belief that, at all times $v \geq t$,

$$\dot{X}_v = \pi(\Phi^{n-1}(v, Z_v, X_v), X_v) \tag{9}$$

Let $\Phi^\infty(t, z, x) = \lim_{n \to \infty} \Phi^n(t, z, x)$.

A central fact used in our proof is that the dynamical system (9) has a unique solution for any n, including $n = \infty$. We prove this in a sequence of lemmas. For any t, t', and $v \geq t$, define $\phi(v, t, t')$ implicitly by

$$h(t' + \phi(v, t, t')) - h(t') = h(t + v) - h(t) \tag{10}$$

where h is defined in Lemma 1. Let

$$\tau(t, t') = \max_{v \geq 0} |t' + \phi(v, t, t') - t - v| \tag{11}$$

[15] The min and max of the respective sets exist since the cost functions are left continuous.

Lemma 4 proves four important properties of these functions.

LEMMA 4 *For any t and t', let $dt = t' - t$. For all v:*

1. $\tau(t,t') \in \left[|dt|, \frac{\overline{\rho}}{\underline{\rho}}|dt|\right]$.
2. $|\phi(v,t,t') - v| \leq 2\frac{\overline{\rho}}{\underline{\rho}}|dt|$ *and* $|dt + \phi(v,t,t') - v| \leq \frac{\overline{\rho}}{\underline{\rho}}|dt|$.
3. $|\phi_1(v,t,t') - 1| \leq \left(\frac{\overline{\rho}}{\underline{\rho}}\right)^2 |dt|$.
4. $\tau(t+v, t' + \phi(v,t,t')) \leq \tau(t,t')$.

For any y, let $f^L(y,x) = \max BR^L(y,x)$: the highest switching rate L players may choose if the relative payoff to R is y and $X_t = x$. Let $f^R(y,x) = -\min BR^R(y,x)$: the negative of the lowest switching rate R players may choose in the same situation. Equation (9) implies that

$$\dot{X}_v = f^L(\Phi^{n-1}(v,Z_v,X_v),X_v)(1-X_v) + f^R(\Phi^{n-1}(v,Z_v,X_v),X_v)X_v$$

The following lemma will be used to show that this system has a unique solution. In reading it, one should interpret the function $F^a(v,z,x)$ for $a = R, L$ as $f^a(\Phi^{n-1}(v,z,x),x)$. Later we will show that this function indeed has the properties assumed in Lemma 5. Equation (12), which appears in the lemma, is just the integral version of (9) for these functions F^R and F^L.

LEMMA 5 *Assume that $F^R(t,z,x)$ and $F^L(t,z,x)$ have the following properties:*

1. *They are weakly increasing in z.*
2. *There is a constant K such that $|F^a(t,z,x)| \leq K$ for $a = R, L$ and for all t, z, and x.*
3. *For $a = R, L$, there are constants c_2 and c_3 such that if*

$$z' - z > (c_2 + c_3[|z'| + |z|]) \cdot (|x' - x| + \tau(t,t'))$$

and $|x' - x| \geq \left(\frac{\overline{\rho}}{\underline{\rho}}\right)^2 \tau(t,t')$ *then* $F^a(t',z',x') \geq F^a(t,z,x)$.

Then for any $x_0 \in [0,1]$, $T > 0$, and almost every path $(Z_t)_{t\in[0,T]}$ there exists a unique Lipschitz path $(X_t)_{t\in[0,T]}$ such that[16]

$$X_t = x_0 + \int_{s=0}^{t} \left(F^L(s, Z_s, X_s)(1 - X_s) + F^R(s, Z_s, X_s)X_s\right) ds \qquad (12)$$

One implication of Lemma 5 is that there is a unique solution to (12) if F^R and F^L satisfy assumptions 1 and 2 and if z has a sufficiently large effect on these functions relative to x and t. To see this, suppose that if $z' - z > (c_2 + c_3 [|z'| + |z|]) \cdot (|x' - x| + \tau(t, t'))$, then $F^a(t', z', x') \geq F^a(t, z, x)$ for $a = R, L$. Then clearly, F^R and F^L also satisfy assumption 3 of the lemma, and so there is a unique solution to (12).

The following two lemmas prove important comparative statics properties of the solution to (12).

LEMMA 6 1. *Suppose that $(X_t^1)_{t\in[0,T]}$ and $(X_t^2)_{t\in[0,T]}$ are Lipschitz solutions to equation (12) corresponding to pairs of functions (F_1^R, F_1^L) and (F_2^R, F_2^L) that satisfy the properties of (F_1^R, F_1^L) in Lemma 5 and such that $F_1^a(t, z, x) \geq F_2^a(t, z, x)$ for $a = R, L$ and for all (t, z, x). Suppose the solutions $(X_t^1)_{t\in[0,T]}$ and $(X_t^2)_{t\in[0,T]}$ are defined relative to the same Brownian motion sample path, $(Z_t)_{t\in[0,T]}$. Assume also that $X_0^1 \geq X_0^2$. Then $X_t^1 \geq X_t^2$ for all $t \in [0, T]$ almost surely.*

2. *Suppose, in addition, that for any (t, x) and $a = R, L$, $F_1^a(t, z, x) = F_2^a(t, z, x)$ at all but a measure zero set of z's. If $X_0^1 = X_0^2$, then $X_t^1 = X_t^2$ for all $t \in [0, T]$, almost surely.*

LEMMA 7 *Suppose that $(X_t)_{t\in[0,T]}$ is the unique Lipschitz solution of (12), where F^R, F^L satisfy the assumptions of Lemma 5. Let $\widetilde{X}_t^{z,x}$ be the solution to (12) starting from $\widetilde{X}_0^{z,x} = x_0 + x$ and corresponding to $\widetilde{Z}_t = Z_t + z$. ($F^R, F^L$ remain the same in parts 1 and 2 of the lemma).*

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[16]One implication of this is that there is a unique solution if z has a sufficiently large effect on F relative to x and t. To see this, suppose that if $z' - z > (c_2 + c_3 [|z'| + |z|]) \cdot (|x' - x| + \tau(t, t'))$, then $F(t', z', x') \geq F(t, z, x)$. Then clearly, F satisfies assumption 3 of the lemma, and so there is a unique solution to (12) for such an F.

1. *If $z, x > 0$ then $\widetilde{X}_t^{z,\mathbf{0}} \geq X_t$ and $\widetilde{X}_t^{\mathbf{0},x} \geq X_t$ for all $t \in [0,T]$ almost surely.*

2. *As z and x go to 0, the processes $\widetilde{X}_t^{z,x}$ converge almost surely to X_t, uniformly on $[0,T]$.*

3. *Suppose that for $n = 1, 2, \ldots$, $\left(F_n^R, F_n^L\right)$ have the properties of $\left(F^R, F^L\right)$ in Lemma 5, for the same constants c_2 and c_3. Fix some x_0 and z_0. For each n, let $\widehat{X}_t^n$ be the solution to (12) on $t \in [0,T]$ with $\left(F_n^R, F_n^L\right)$ appearing in place of $\left(F^R, F^L\right)$. If $\lim_{n\to\infty} F_n^a = F^a$ for $a = R, L$, then the solutions $\widehat{X}_t^n$ converge to X_t, the solution of (12) corresponding to $\left(F^R, F^L\right)$.*

The following two lemmas imply that if $F^a(v,z,x) = f^a(\Phi^{n-1}(v,z,x),x)$ for $a = R, L$, then $\left(F^R, F^L\right)$ satisfies the assumptions of Lemma 5, so there is a unique solution to (9).

LEMMA 8 *The functions $f^R(y,x)$, $f^L(y,x)$, and $\pi(y,x)$ are weakly increasing in y and right-continuous in y.*

LEMMA 9 *For each $n \geq 0$, including $n = \infty$, and for all (t,z,x), and (t',z',x'),*

(i) *$\Phi^n(t,z,x)$ is strictly increasing in z;*

(ii) *there are constants c_2 and c_3, independent of n, such that if $g(t,z)$ and $g(t',z')$ are both in $[\underline{w}, \overline{w}]$ and $z' - z > (c_2 + c_3\left[|z'| + |z|\right]) \cdot (|x' - x| + \tau(t,t'))$ and $|x' - x| \geq \left(\frac{\overline{\rho}}{\underline{\rho}}\right)^2 \tau(t,t')$ then $\Phi^n(t',z',x') > \Phi^n(t,z,x)$;*

(iii) *$\Phi^n(t,z,x)$ is weakly decreasing in n;*

(iv) *for any $\lambda > 0$, $\Phi^n(t,z,x)$ is a uniformly continuous function of t, z, and x on the set $g(t,z) \in [\underline{w} - \lambda, \overline{w} + \lambda]$.*

By Lemma 8 and part (iii) of Lemma 9,

$$\dot{X}_v \leq \lim_{n\to\infty} \pi(\Phi^{n-1}(v,Z_v,X_v),X_v) = \pi(\Phi^{\infty}(v,Z_v,X_v),X_v) \tag{13}$$

Moreover, $\dot{X}_v = \pi(\Phi^\infty(v, Z_v, X_v), X_v)$ is an equilibrium: if $\dot{X}_v$ is expected to equal $\pi(\Phi^\infty(v, Z_v, X_v), X_v)$ for all $v \geq t$, it is a best response for $\dot{X}_t$ to equal $\pi(\Phi^\infty(t, Z_t, X_t), X_t)$. The reasoning is as follows. Let

$$\begin{aligned} \dot{X}_v^n &= \pi(\Phi^{n-1}(v, Z_v, X_v), X_v) \\ &= f^R(\Phi^{n-1}(v, Z_v, X_v))(1 - X_v) + f^L(\Phi^{n-1}(v, Z_v, X_v))X_v \end{aligned}$$

By Lemmas 8 and 9, for all n and for $a = R, L$, $f^a(\Phi^{n-1}(v, Z_v, X_v))$ has the properties of F^a assumed in Lemmas 5-7, so for any path $(Z_v)_{v \geq t}$ there is a unique Lipschitz solution $(X_v^n)_{v \geq t}$ to this dynamical system.[17] By Lemmas 8 and part (iii) of 9, $\lim_{n\to\infty} f^a(\Phi^n(v, Z_v, X_v)) = f^a(\Phi^\infty(v, Z_v, X_v))$ for $a = R, L$. Let $X_v^\infty = \lim_{n\to\infty} X_v^n$. By part 3 of Lemma 7, $(X_v^\infty)_{v \geq t}$ is the unique solution to $\dot{X}_v = \pi(\Phi^\infty(v, Z_v, X_v), X_v)$. This implies that $(X_v^\infty)_{v \geq t}$ is a best response when the relative payoff to R for any (v, Z_v, X_v) is $\Phi^\infty(v, Z_v, X_v)$. It remains to show that $\Phi^\infty(t, Z_t, X_t)$ is the relative payoff to R if for any $(Z_v)_{v \geq t}$ players expect $(X_v^\infty)_{v \geq t}$. By the envelope theorem and (**??**), the relative payoff to R is a continuous function of the path of X. But $X_v^\infty = \lim_{n\to\infty} X_v^n$, so $\Phi^\infty(t, Z_t, X_t) = \lim_{n\to\infty} \Phi^n(t, Z_t, X_t)$ must be the relative payoff to R when X follows $(X_v^\infty)_{v \geq t}$.

This proves that $\dot{X}_t = \pi(\Phi^\infty(t, Z_t, X_t), X_t)$ is both an upper bound on $\dot{X}_t$ and the equilibrium with the highest path of X for any path of Z. We now iterate from below: we construct a growing sequence of *lower* bounds on $\dot{X}_t$. Each lower bound in the sequence is now some *translation* of $\pi(\Phi^\infty(t, Z_t, X_t), X_t)$, the upper bound on $\dot{X}_t$. We will show that the limit of this sequence of lower bounds coincides with the upper bound. This will imply that the equilibrium $\dot{X}_t = \pi(\Phi^\infty(t, Z_t, X_t), X_t)$ is in fact the unique equilibrium of the model.

Since $\pi(y, x)$ is right continuous in y (Lemma 8) and $\Phi^\infty(t, z, x)$ is nondecreasing and continuous in z (Lemma 9), the upper bound on $\dot{X}_t$, $\pi(\Phi^\infty(t, Z_t, X_t), X_t)$, is right continuous in Z_t. Let $\widetilde{\pi}(y, x) = \lim_{\varepsilon \downarrow 0} \pi(y - \varepsilon, x)$ be the the left continuous (in y) version of π. By part (iv) of Lemma 9, $\widetilde{\pi}(\Phi^\infty(t, Z_t, X_t), X_t)$ is left continuous in Z_t; it is the

[17] Property 3 of lemma 5 holds by part (ii) of Lemma 9 since if $g(t, z) \notin [\underline{w}, \overline{w}]$, then $f^a(\Phi^{n-1}(t, z, x))$ is locally constant.

left continuous version of the upper bound on $\dot{X}_t$.

We iterate with translations of this $\widetilde{\pi}(\Phi^\infty(t, Z_t, X_t), X_t)$. Let $\lambda_0 > 0$ be large enough that regardless of their expectations for $(X_v)_{v \geq t}$, players at state (t, Z_t, X_t) must choose switching rates that yield a rate of change of X_t that is at least $\widetilde{\pi}(\Phi^\infty(t, Z_t - \lambda_0, X_t), X_t)$. There must be such a λ_0 by the existence of dominance regions and the assumption that the integral of the absolute drift terms is finite ($\int_{s=0}^{\infty} |a_s|\, ds < N_2$). To see this, consider the following three cases:

1. $W_t = g(t, Z_t) > \overline{w}$: then players must choose switching rates that yield the highest feasible $\dot{X}_t$ (which is $\overline{K}^L(X_t)(1 - X_t)$), so the result is trivial;

2. $W_t = g(t, Z_t) < \underline{w}$: then players must choose switching rates that yield the lowest feasible $\dot{X}_t$ (which is $-\overline{K}^R(X_t)X_t$); but $g(t, Z_t - \lambda_0) < \overline{w}$, so at $(t, Z_t - \lambda_0, X_t)$ they must also do so as well;

3. $W_t = g(t, Z_t) \in [\underline{w}, \overline{w}]$: then by equation (5) and since $\int_{s=0}^{\infty} |a_s|\, ds < N_2$, if $\lambda_0 > (\overline{w} - \underline{w})\, e^{N_2}$ then $g(t, Z_t - \lambda_0) < \underline{w}$, so players at $(t, Z_t - \lambda_0, X_t)$ must choose switching rates that yield the lowest feasible $\dot{X}_t$ $(= -\overline{K}^R(X_t)X_t)$; thus, the property holds here as well.

Let λ_n be the infimum of constants λ such that if players believe that $\dot{X}_v$ will be at least $\widetilde{\pi}(\Phi^\infty(v, Z_v - \lambda_{n-1}, X_v), X_v)$ for all $v \geq t$, they must choose switching rates that yield an $\dot{X}_t$ that is at least $\widetilde{\pi}(\Phi^\infty(t, Z_t - \lambda, X_t), X_t)$.

More precisely, let $\Phi_\lambda^\infty(t, Z_t, X_t)$ be the relative payoff to R on the belief that, for all $v \geq t$, $\dot{X}_v$ will equal the translation of the left continuous (LC) version of the upper bound on $\dot{X}_v$ by λ, $\widetilde{\pi}(\Phi^\infty(v, Z_v - \lambda, X_v), X_v)$. (Note that $\Phi_0^\infty(t, Z_t, X_t) = \Phi^\infty(t, Z_t, X_t)$.) Let $\underline{\pi}(y, x) = \min BR^L(y, x)(1 - x) - \max BR^R(y, x)x$: the lowest possible $\dot{X}_t$ when $X_t = x$ and the relative payoff to R is y. When the relative payoff to R is $\Phi_\lambda^\infty(t, Z_t, X_t)$, the rate of change $\dot{X}_t$ must be at least $\underline{\pi}(\Phi_\lambda^\infty(t, Z_t, X_t), X_t)$. For $n \geq 1$, let λ_n be the infimum of numbers λ such that, for all states (t, z, x), $\underline{\pi}(\Phi_{\lambda_{n-1}}^\infty(t, z, x), x)$ (the lowest possible rate of change when others are expected to play according to the translation of the LC version of the upper bound downward by λ_{n-1}) is at least $\widetilde{\pi}(\Phi^\infty(t, z - \lambda, x), x)$, the translation of the LC version of the upper bound downward by λ.

By construction, $\lambda_0 \geq \lambda_1$. By Lemma 6, for any path $(Z_v)_{v\geq t}$, the solution $(X_v)_{v\geq t}$ to the equation $\dot{X}_v = \widetilde{\pi}(\Phi^\infty(v, Z_v - \lambda, X_v), X_v)$ is weakly decreasing in λ; thus, by Lemma 2, $\lambda_1 \geq \lambda_2$. Continuing by induction, $\lambda_{n-1} \geq \lambda_n$ for all n. Let $\lambda_\infty = \lim_{n\to\infty} \lambda_n$. We know that $\dot{X}_t$ cannot lie above $\pi(\Phi^\infty(t, Z_t, X_t), X_t)$ nor below $\widetilde{\pi}(\Phi^\infty(t, Z_t - \lambda_\infty, X_t), X_t)$.

We now show that $\lambda_\infty = 0$. For any (t, z, x) and any λ, let $S^\lambda(t, z, x)$ stand for the situation in which players choose switching rates at state $(t, Z_t, X_t) = (t, z + \lambda, x)$ and believe that $\dot{X}_v$ will equal $\widetilde{\pi}(\Phi^\infty(v, Z_v - \lambda, X_v), X_v)$ for all $v \geq t$. The initial rate of change of X_t in situation $S^\lambda(t, z, x)$ is $\widetilde{\pi}(\Phi^\infty(t, z, x), x)$, independent of λ. The relative payoff to R in situation $S^\lambda(t, z, x)$ is just $\Phi^\infty_\lambda(t, z + \lambda, x)$.

For any $\lambda, \lambda' \in [0, \lambda_\infty]$, the distribution of continuation paths $(Z_v - Z_t)_{v\geq t}$ in situations $S^\lambda(t, z, x)$ and $S^{\lambda'}(t, z, x)$ is the same since $Z_v = B_{h(v)}$ where h is a fixed function and B is a Brownian motion. And given a continuation path of Z, the continuation path of X is determined by the same dynamical system: $X_t = x$ and $\dot{X}_v$ equals $\widetilde{\pi}(\Phi^\infty(v, Z_v - Z_t + z, X_v), X_v)$, independent of λ. By Lemmas 8 and parts (i) and (ii) of 9, for $a = R, L$, $F^a(v, Z_v, X_v) = f^a(\Phi^\infty(v, Z_v - Z_t + z, X_v))$ has the properties assumed in Lemma 5, so this dynamical system has a unique solution for each λ. So for any $\lambda, \lambda' \in [0, \lambda_\infty]$, players in situations $S^\lambda(t, z, x)$ and $S^{\lambda'}(t, z, x)$ expect the same distribution of continuation paths of the state, $(Z_v - Z_t, X_v - X_t)_{v\geq t}$. Fix any sample path $(z_v, x_v)_{v\geq t}$; since X_t is independent of λ, this sample path in situation λ has the same probability as the sample path $(z_v + \lambda' - \lambda, x_v)_{v\geq t}$ in situation $S^{\lambda'}(t, z, x)$. Hence, by Lemma 2 and the envelope theorem,

$$\frac{d\left[\Phi^\infty_\lambda(t, z + \lambda, x)\right]}{d\lambda} = E \int\limits_{v=t}^{\infty} \exp\left(-\int\limits_{s=t}^{v} (r + k_s^{R\lambda} + k_s^{L\lambda})ds\right) \frac{\partial \Delta_v(Z_v, X_v, k_v^R, k_v^L)}{\partial Z_v} dv \quad (14)$$

where $k_s^{R\lambda}$ and $k_s^{L\lambda}$ are the optimal update rates in situation $S^\lambda(t,z,x)$.[18] By Lemma 1 and equation (6), $\partial\Delta_v(Z_v,X_v,k_v^R,k_v^L)/\partial Z_v > \underline{\alpha}e^{-N_2}$ whenever $g(v,Z_v)\in[\underline{w},\overline{w}]$. Since $k_s^{R\lambda}+k_s^{L\lambda}\le 2K$,

$$\frac{d\left[\Phi_\lambda^\infty(t,z+\lambda,x)\right]}{d\lambda} \ge \underline{\alpha}e^{-N_2}E\int_{v=t}^{\infty} e^{-(r+2K)(v-t)}1\left(g(v,Z_v)\in[\underline{w},\overline{w}]\right)dv \ge \underline{\alpha}e^{-N_2}T(c) \quad (15)$$

where $c>0$ is is any constant such that $g(t,z+\lambda)\in[\underline{w}-c,\overline{w}+c]$ and $T(c)>0$ is the minimum expected discounted (at rate $r+2K$) amount of time $v>t$ that $g(v,Z_v)$ is expected to spend in the non-dominance region $[\underline{w},\overline{w}]$, given that $g(t,Z_t)$ is within c of this region (i.e., that $g(t,Z_t)\in[\underline{w}-c,\overline{w}+c]$). $T(c)$ is positive because the variance and drift of W are bounded in absolute value. Importantly, $\underline{\alpha}e^{-N_2}T(c)$ is independent of (t,λ,z,x), as long as $g(t,z+\lambda)\in[\underline{w}-c,\overline{w}+c]$.

By definition, λ_∞ is the infimum of numbers λ such that for all states (t,z,x), $\underline{\pi}(\Phi_{\lambda_\infty}^\infty(t,z,x),x)$ (the lowest possible rate of change at (t,z,x) when others are expected to play according to the translation of the left continuous version of the upper bound on $\dot{X}_t$ downward by λ_∞) is at least $\widetilde{\pi}(\Phi^\infty(t,z-\lambda,x),x)$ (the translation of the left continuous version of the upper bound on $\dot{X}_t$ downward by λ). Hence, for any $\varepsilon>0$ there must be a state $(t^\varepsilon,z^\varepsilon,x^\varepsilon)$ such that

$$\underline{\pi}(\Phi_{\lambda_\infty}^\infty(t^\varepsilon,z^\varepsilon,x^\varepsilon),x^\varepsilon) < \widetilde{\pi}(\Phi^\infty(t^\varepsilon,z^\varepsilon-\lambda_\infty+\varepsilon,x^\varepsilon),x^\varepsilon). \quad (16)$$

Otherwise, the infimum could be no greater than than $\lambda_\infty-\varepsilon$, a contradiction. Since (16) implies that players at $(t^\varepsilon,z^\varepsilon,x^\varepsilon)$ may choose switching rates that differ from those

[18] By the envelope theorem, equation (14) holds path-by-path (i.e., if $(Z_v-Z_t)_{v\ge t}$ is held constant as λ is varied); but the distribution of these paths is the same in all situations S_λ, so the equality holds in expectation as well. The envelope theorem applies even though $k_s^{R\lambda}$ and $k_s^{L\lambda}$ need not be continuous in λ. By construction, $k_s^{R\lambda}$ and $k_s^{L\lambda}$ are left continuous and monotonically increasing in λ; hence, either λ is a point of continuity of $k_s^{R\lambda}$ and $k_s^{L\lambda}$, in which case $d\lambda$ can be chosen small enough that $k_s^{R,\lambda+\varepsilon}$ and $k_s^{L,\lambda+\varepsilon}$ are close to $k_s^{R\lambda}$ and $k_s^{L\lambda}$ for $\varepsilon\in[0,d\lambda]$, or else λ is a point of right-discontinuity of either $k_s^{R\lambda}$ or $k_s^{L\lambda}$, in which case $d\lambda$ can be chosen small enough that $k_s^{R,\lambda+\varepsilon}$ and $k_s^{L,\lambda+\varepsilon}$ are close to $\lim_{\varepsilon\downarrow 0}k_s^{R,\lambda+\varepsilon}$ and $\lim_{\varepsilon\downarrow 0}k_s^{L,\lambda+\varepsilon}$, respectively, for $\varepsilon\in[0,d\lambda]$. Since the sample path $(Z)_{v\ge t}$ changes continuously as λ is varied, $\lim_{\varepsilon\downarrow 0}k_s^{R,\lambda+\varepsilon}$ and $\lim_{\varepsilon\downarrow 0}k_s^{L,\lambda+\varepsilon}$ must give the same payoffs to R and L as $k_s^{R\lambda}$ and $k_s^{L\lambda}$ do at λ. Thus, (14) holds at points of discontinuity in λ if we reinterpret $k_s^{R\lambda}$ and $k_s^{L\lambda}$ as $\lim_{\varepsilon\downarrow 0}k_s^{R,\lambda+\varepsilon}$ and $\lim_{\varepsilon\downarrow 0}k_s^{L,\lambda+\varepsilon}$, respectively, which suffices for equation (15).

chosen at $(t^\varepsilon, z^\varepsilon - \lambda_\infty + \varepsilon, x^\varepsilon)$, either z^ε or $z^\varepsilon - \lambda_\infty + \varepsilon$ must lie in the non-dominance region, so each can be no further than $\lambda_0 > \lambda_\infty - \varepsilon$ away from the non-dominance region.

We now show by contradiction (16) cannot hold for all $\varepsilon > 0$ unless $\lambda_\infty = 0$. By part (iv) of Lemma 9, there is a constant $c' > 0$ such that for all ε, $\Phi^\infty(t^\varepsilon, z^\varepsilon - \lambda_\infty + \varepsilon, x^\varepsilon) \leq \Phi^\infty(t^\varepsilon, z^\varepsilon - \lambda_\infty, x^\varepsilon) + c'\varepsilon$. By (15),

$$
\begin{aligned}
\Phi^\infty_{\lambda_\infty}(t^\varepsilon, z^\varepsilon, x^\varepsilon) &= \Phi^\infty_{\lambda_\infty}(t^\varepsilon, (z^\varepsilon - \lambda_\infty) + \lambda_\infty, x^\varepsilon) \\
&\geq \Phi^\infty_0(t^\varepsilon, (z^\varepsilon - \lambda_\infty) + 0, x^\varepsilon) + \underline{\alpha} e^{-N_2} T(\lambda_0)\lambda_\infty \\
&= \Phi^\infty(t^\varepsilon, z^\varepsilon - \lambda_\infty, x^\varepsilon) + \underline{\alpha} e^{-N_2} T(\lambda_0)\lambda_\infty \\
&\geq \Phi^\infty(t^\varepsilon, z^\varepsilon - \lambda_\infty + \varepsilon, x^\varepsilon) - c'\varepsilon + \underline{\alpha} e^{-N_2} T(\lambda_0)\lambda_\infty
\end{aligned}
$$

For $\varepsilon < \underline{\alpha} e^{-N_2} T(\lambda_0)\lambda_\infty$, $\Phi^\infty_{\lambda_\infty}(t^\varepsilon, z^\varepsilon, x^\varepsilon) > \Phi^\infty(t^\varepsilon, z^\varepsilon - \lambda_\infty + \varepsilon, x^\varepsilon)$. By part 2 of Lemma 3, this contradicts (16). This shows that $\lambda_\infty = 0$.

Consequently, the equilibrium is unique wherever $\pi(\Phi^\infty(t, Z_t, X_t), X_t)$ is continuous in Z_t. By Lemma 8, $\pi(\Phi^\infty(t, z, x), x)$ is weakly increasing in z and bounded, so for any (t, x), $\pi(\Phi^\infty(t, z, x), x)$ is almost everywhere continuous in z. Hence, by part 2 of Lemma 6, with probability one the path of X that results from any path of Z does not depend on whether players play according to the right or left continuous version of $\pi(\Phi^\infty(t, Z_t, X_t), X_t)$. (Intuitively, $\dot{X}_t$ is almost always the same and it is bounded, so $X_t = X_0 + \int_{v=0}^t \dot{X}_v dv$ is the same in the two cases.) Thus, for almost any path of Z, there is a unique equilibrium path of X. This completes the proof. Q.E.D.

Proof of LEMMA 1. We first verify that $g(t, B_{h(t)})$ has the same infinitesimal drift and variance as W_t. Since both processes have continuous paths a.s., this will imply that they are identically distributed. By definition,

$$
\begin{aligned}
g(t, B_{h(t)}) &= \exp\left(\int_{s=0}^t a_s ds\right) B_{h(t)} + \int_{s=0}^t b_s \exp\left(\int_{v=s}^t a_v dv\right) ds \\
&\implies d\left[g(t, B_{h(t)})\right] = \exp\left(\int_{s=0}^t a_s ds\right) dB_{h(t)} + \left[a_t g(t, B_{h(t)}) + b_t\right] dt
\end{aligned}
$$

so that $Ed\left[g(t, B_{h(t)})\right] = \left(a_t g(t, B_{h(t)}) + b_t\right) dt$ and

$$
E\left[dg(t, B_{h(t)})^2\right] = \exp\left(2\int_{s=0}^t a_s ds\right) E\left[\left(dB_{h(t)}\right)^2\right]
$$

$$
\begin{aligned}
&= \exp\left(2\int_{s=0}^{t} a_s ds\right)[h(t+dt)-h(t)] \\
&= \exp\left(2\int_{s=0}^{t} a_s ds\right)\exp\left(-2\int_{v=0}^{t} a_v dv\right)\sigma_t^2 dt = \sigma_t^2 dt
\end{aligned}
$$

proving that the two processes have the same distributions.

Since W is a Markov process,

$$
W_t \underset{\mathrm{L}}{=} \exp\left(\int_{s=t'}^{t} a_s ds\right)\widehat{B}_{\widehat{h}(t)} + \int_{s=t'}^{t} b_s \exp\left(\int_{v=s}^{t} a_v dv\right) ds
$$

where $\widehat{h}(t) = \int_{s=t'}^{t} \exp\left(-2\int_{v=t'}^{s} a_v dv\right)\sigma_s^2 ds$, "$\underset{\mathrm{L}}{=}$" denotes equality in law (distribution), and $\widehat{B}$ is another Brownian motion with zero drift and unit variance, satisfying $\widehat{B}_0 = W_{t'}$. The only stochastic term is $\widehat{B}_{\widehat{h}(t)}$, which is normal since $\widehat{B}$ is a Brownian motion. Hence, W_t is normal with mean $E_{t'}W_t = \exp\left(\int_{s=t'}^{t} a_s ds\right) W_{t'} + \int_{s=t'}^{t} b_s \exp\left(\int_{v=s}^{t} a_v dv\right) ds$ and variance

$$
\begin{aligned}
Var_{t'}W_t &= \exp\left(2\int_{s=t'}^{t} a_s ds\right) Var\left(\widehat{B}_{\widehat{h}(t)}\right) \\
&= \exp\left(2\int_{s=t'}^{t} a_s ds\right)\widehat{h}(t) = \int_{s=t'}^{t} \exp\left(2\int_{v=s}^{t} a_v dv\right)\sigma_s^2 ds
\end{aligned}
$$

For property 1, note that

$$
g(t,z) - g(t,z') = \exp\left(\int_{s=0}^{t} a_s ds\right)(z-z') \in \left[e^{-N_2}(z-z'), e^{N_2}(z-z')\right]
$$

Moreover,

$$
\begin{aligned}
|g(t,z)| &\leq \exp\left(\int_{s=0}^{t} |a_s|\, ds\right)|z| + \int_{s=0}^{t} |b_s| \exp\left(\int_{v=s}^{t} a_v dv\right) ds \\
&\leq N_2|z| + tN_2 e^{N_2} \leq N_2(1+e^{N_2})(|z|+t)
\end{aligned}
$$

For property 2,

$$
\begin{aligned}
h(t) - h(t') &= \int_{s=t'}^{t} \exp\left(-2\int_{v=0}^{s} a_v dv\right)\sigma_s^2 ds \\
&\in \left[N_1^2 e^{-2N_2}(t-t'), N_2^2 e^{2N_2}(t-t')\right]
\end{aligned}
$$

and

$$
|h'(t) - h'(t')| = \left|\exp\left(-2\int_{v=0}^{t} a_v dv\right)\sigma_t^2 - \exp\left(-2\int_{v=0}^{t'} a_v dv\right)\sigma_{t'}^2\right|
$$

$$\begin{aligned}
&\leq e^{2N_2}\left|\exp\left(-2\int_{v=t'}^{t} a_v dv\right)-1\right|\sigma_t^2+\left|\exp\left(-2\int_{v=0}^{t} a_v dv\right)\left(\sigma_t^2-\sigma_{t'}^2\right)\right| \\
&\leq e^{2N_2}N_2^2(e^{2N_2(t-t')}-1)+e^{2N_2}N_2(t-t') \\
&\leq e^{2N_2}N_2^2(3N_2(t-t'))+e^{2N_2}N_2(t-t')
\end{aligned}$$

for sufficiently small $t-t'$. By the triangle inequality, this generalizes to any $t-t'$, so $|h'(t)-h'(t')|\leq\overline{\rho}\,|t-t'|$ for any $\overline{\rho}\geq e^{2N_2}N_2(3N_2^2+1)$. This proves property 2. Q.E.D.$_{\text{Lemma 1}}$

Proof of LEMMA 2. For $a,a'\in\{R,L\}$, $a\neq a'$, the Bellman equation for V_v^a is

$$V_v^a\approx\begin{bmatrix}[u(a,W_v,X_v)-c^a(k_v^a,X_v)]\,dv\\ +k_v^a\cdot dv\cdot EV_{v+dv}^{a'}\\ +[1-k_v^a dv-rdv]\,EV_{v+dv}^a\end{bmatrix}\tag{17}$$

This becomes exact as $dv\to 0$, proving part 3. Rearranging (17), we obtain

$$EdV_v^a=\left[-u(a,W_v,X_v)+c^a(k_v^a,X_v)-k_v^aV_v^{a'}+(k_v^a+r)V_v^a\right]dv$$

where $dV_v^a=V_{v+dv}^a-V_v^a$. Therefore,

$$E\left(dV_v^R-dV_v^L\right)=\begin{bmatrix}-\Delta_v(Z_v,X_v,k_v^R,k_v^L)\\ +(k_v^R+k_v^L+r)\left(V_v^R-V_v^L\right)\end{bmatrix}dv\tag{18}$$

This expectation is as of time v. Now multiply both sides by $\exp\left[-\int_{s=t}^{v}(r+k_s^R+k_s^L)ds\right]$, integrate, and take the expectation as of time t, yielding (by iterated expectations)

$$\begin{aligned}
&E\int_{v=t}^{\infty}\exp\left(-\int_{s=t}^{v}(r+k_s^R+k_s^L)ds\right)\left[dV_v^R-dV_v^L\right]\\
=\;&E\int_{v=t}^{\infty}\exp\left(-\int_{s=t}^{v}(r+k_s^R+k_s^L)ds\right)\begin{bmatrix}-\Delta(W_v,X_v,k_v^R,k_v^L)\\ +(k_v^R+k_v^L+r)\left(V_v^R-V_v^L\right)\end{bmatrix}dv
\end{aligned}$$

Integrating by parts,

$$\begin{aligned}
&E\int_{v=t}^{\infty}\exp\left(-\int_{s=t}^{v}(r+k_s^R+k_s^L)ds\right)\left[dV_v^R-dV_v^L\right]\\
=\;&E\left(\exp\left(-\int_{s=t}^{v}(r+k_s^R+k_s^L)ds\right)\left(V_v^R-V_v^L\right)\right)_{v=t}^{\infty}\\
&+E\int_{v=t}^{\infty}\exp\left(-\int_{s=t}^{v}(r+k_s^R+k_s^L)ds\right)(k_v^R+k_v^L+r)\left(V_v^R-V_v^L\right)dv
\end{aligned}$$

But $\left|E\left(\lim_{v\to\infty}\exp\left(-\int_{s=t}^{v}(r+k_s^R+k_s^L)ds\right)\left(V_v^R-V_v^L\right)\right)\right| \leq \lim_{v\to\infty} e^{-r(v-t)}E\left|V_v^R-V_v^L\right|$. We will now show that

$$\lim_{v\to\infty} e^{-r(v-t)}E\left|V_v^R-V_v^L\right| = 0 \tag{19}$$

This will establish part 1.

For $a = R, L$, V_v^a is no greater than the payoff from always being in the "right" action and paying the lowest possible switching cost of zero:

$$V_v^a \leq E\int_{s=v}^{\infty} e^{-r(s-v)} \max\left\{u(R,W_s,X_s), u(L,W_s,X_s)\right\} ds$$

If one chooses the lowest switching cost, the worst that can happen is that one is always in the wrong action; hence, $V_v^a \geq E\int_{s=v}^{\infty} e^{-r(s-v)} \min\left\{u(R,W_s,X_s), u(L,W_s,X_s)\right\} ds$. Thus,

$$\begin{aligned}\left|V_v^R-V_v^L\right| &\leq E\int_{s=v}^{\infty} e^{-r(s-v)}\left|\Delta u(W_s,X_s)\right| ds \\ &\leq \frac{\max_{x\in[0,1]}\left|\Delta u(0,x)\right|}{r} \\ &\quad + E\int_{s=v}^{\infty} e^{-r(s-v)}\left|\Delta u(W_s,X_s)-\Delta u(0,X_s)\right| ds\end{aligned}$$

where $\Delta u(W_s,X_s) = u(R,W_s,X_s) - u(L,W_s,X_s)$. (This proves the first formula in part 2.) Since $c^a(\underline{K}^a(x),x) = 0$ for $a = R, L$ and for all x, (4) implies that $\Delta u(w,x) = \Delta(w,x,\underline{K}^R(x),\underline{K}^L(x))$ is Lipschitz in w with constant $\overline{\alpha}$, so $\left|\Delta u(W_s,X_s)-\Delta u(0,X_s)\right| \leq \overline{\alpha}\left|W_s\right|$. But

$$\begin{aligned}E\left|W_s\right| &= E\left[\sqrt{W_s^2}\right] \leq \sqrt{E\left[W_s^2\right]} = \sqrt{\left[EW_s\right]^2 + \mathrm{Var}\left(W_s\right)} \\ &\leq \sqrt{\left[EW_s\right]^2} + \sqrt{\mathrm{Var}\left(W_s\right)} = \left|EW_s\right| + \sqrt{\mathrm{Var}\left(W_s\right)}\end{aligned}$$

where all expectations are conditioned on W_v. Thus, by Lemma 1,

$$\begin{aligned}E\left|W_s\right| &\leq \exp\left(\int_{s'=v}^{s} a_{s'}ds'\right)\left|W_v\right| + \left|\int_{s'=v}^{s} b_{s'}\exp\left(\int_{v'=s'}^{s} a_{v'}dv'\right)ds'\right| \\ &\quad + \sqrt{\int_{s'=v}^{s}\exp\left(2\int_{v'=s'}^{s} a_{v'}dv'\right)\sigma_{s'}^2 ds'} \\ &\leq e^{N_2}\left|W_v\right| + (s-v)N_2e^{N_2} + e^{N_2}N_2\sqrt{s-v}\end{aligned} \tag{20}$$

Hence, there are positive constants c_0, c_1, and c_2 such that

$$\begin{aligned}\left|V_v^R - V_v^L\right| &\leq c_2 + \int_{s=v}^{\infty} e^{-r(s-v)}\overline{\alpha}\left(e^{N_2}\left|W_v\right| + (s-v)N_2 e^{N_2} + e^{N_2}N_2\sqrt{s-v}\right) ds \\ &= c_2 + \frac{\overline{\alpha}e^{N_2}\left|W_v\right|}{r} + \frac{\overline{\alpha}N_2 e^{N_2}}{r^2} + \frac{\overline{\alpha}N_2 e^{N_2}\sqrt{\pi}}{2r^{3/2}} = c_0\left|W_v\right| + c_1 \qquad (21)\end{aligned}$$

(establishing the second bound in part 2) so

$$\lim_{v\to\infty} e^{-r(v-t)}E\left|V_v^R - V_v^L\right| \leq c_6 \lim_{v\to\infty} e^{-r(v-t)}\left|W_v\right|$$

but by Chebyschev's inequality, for any $c_7 > 0$,

$$\Pr(e^{-r(v-t)}\left|W_v\right| > c_7) \leq \frac{E\left|W_v\right|}{c_7 e^{r(v-t)}} \leq e^{N_2 - r(v-t)}\left(\left|W_t\right| + (v-t)N_2 + \sqrt{v-t}N_2\right)$$

which goes to 0 as $v \to \infty$, establishing (19). Q.E.D.

Proof of LEMMA 3.

1. Fix x. Let $c(k)$ be shorthand for $c^R(k,x)$ or $c^L(k,x)$. We will show that if the function c is left-continuous, then $\zeta(y) = \text{argmax}_{k\geq 0}(ky - c(k))$ is upper hemicontinuous. A similar argument holds for the function $\text{argmax}_{k\geq 0}(-ky - c(k))$. Suppose there is a sequence $(y^n, k^n)_{n=1}^{\infty}$ such that $k^\infty = \lim_{n\to\infty} k^n$ and $y^\infty = \lim_{n\to\infty} y^n$ both exist and $k^n \in \zeta(y^n)$ for all n. Upper hemicontinuity means that $k^\infty \in \zeta(y^\infty)$ for all such sequences. We first show that $\lim_{n\to\infty} c(k^n) = c(k^\infty)$. This is trivial if c is continuous at k^∞. If not, we claim that there is an $I < \infty$ such that if $n > I$, then $k^n \leq k^\infty$. By assumption, c is left continuous, so it must not be right continuous at k^∞. So let $\lim_{k\downarrow k^\infty} c(k) = c(k^\infty) + \varepsilon$ where $\varepsilon > 0$. For any $k^n > k^\infty$, since c is weakly increasing, $k^n y^n - c(k^n) \leq k^n y^n - c(k^\infty) - \varepsilon$. Let I be large enough that $n > I$ implies $|k^n y^n - k^\infty y^n| < \varepsilon/2$. Then $k^n y^n - c(k^n) \leq k^\infty y^n - c(k^\infty) - \varepsilon/2$, so $k^n \notin \zeta(y^n)$ after all. Therefore, if $n > I$, then $k^n \leq k^\infty$. Since c is left continuous, $c(k^\infty) = \lim_{n\to\infty} c(k^n)$, so $\lim_{n\to\infty}\left[k^n y^n - c(k^n)\right] = k^\infty y^\infty - c(k^\infty)$.

Now suppose $k^\infty \notin \zeta(y^\infty)$. Then there is a k' and an $\varepsilon' > 0$ such that $k'y^\infty - c(k') > k^\infty y^\infty - c(k^\infty) + \varepsilon'$. We claim this implies $k^n \notin \zeta(y^n)$ for large enough n. Since $\lim_{n\to\infty}\left[k^n y^n - c(k^n)\right] = k^\infty y^\infty - c(k^\infty)$, for any $\varepsilon'' > 0$ there is an I' such that if $n > I'$, $|k^n y^n - c(k^n) - \left[k^\infty y^\infty - c(k^\infty)\right]| < \varepsilon''$. So $k'y^\infty - c(k') > k^n y^n - c(k^n) + \varepsilon' - \varepsilon''$ for all

$n > I'$. But there is also an I'' such that if $n > I''$, $|k'y^\infty - k'y^n| < \varepsilon''$ (as k' is bounded by K). So $k'y^n - c(k') > k^n y^n - c(k^n) + \varepsilon' - 2\varepsilon''$. So setting $\varepsilon'' = \varepsilon'/3$, we know that if $n > \max\{I', I''\}$, then $k'y^n - c(k') > k^n y^n - c(k^n)$, so $k^n \notin \zeta(y^n)$ - a contradiction.

2. Suppose $y < y'$, $k \in BR^L(y, x)$, and $k' \in BR^L(y', x)$. Then $k'y' - c^L(k', x) \geq ky' - c^L(k, x)$ while $ky - c^L(k, x) \geq k'y - c^L(k', x)$. Subtracting, we obtain $(k' - k)(y' - y) \geq 0$, so $k' \geq k$.[19] The proof for BR^R is analogous. Q.E.D.

Proof of LEMMA 4. By (10), $\phi(0, t, t') = 0$, so $\tau(t, t') \geq |dt|$. By equation (10),

$$|h(t + v + dt + [\phi(v, t, t') - v]) - h(t + v)| = |h(t + dt) - h(t)| \tag{22}$$

The left hand side of (22) is at least $\underline{\rho}\,|dt + \phi(v, t, t') - v|$ while the right hand side is no greater than $\overline{\rho}\,|dt|$ by assumption A3. So $|dt + \phi(v, t, t') - v| \leq \frac{\overline{\rho}}{\underline{\rho}}\,|dt|$, which shows parts 1 and 2. Differentiating (10) with respect to v,

$$\begin{aligned} |\phi_1(v, t, t') - 1| &= \left| \frac{h'(t' + \phi(v, t, t')) - h'(t + v)}{h'(t' + \phi(v, t, t'))} \right| \\ &\leq \frac{\overline{\rho}}{\underline{\rho}}\,|dt + \phi(v, t, t') - v| \leq \left(\frac{\overline{\rho}}{\underline{\rho}}\right)^2 |dt| \end{aligned} \tag{23}$$

by part 2 of Lemma 1 and the prior computation. This shows part 3.

For part 4, let $t'' = t + v$ and $t''' = t' + \phi(v, t, t')$. Suppose that

$$s_0 = \text{argmax}_{s \geq 0}\,|t''' + \phi(s, t'', t''') - t'' - s|$$

We will show that

$$t''' + \phi(s_0, t'', t''') - t'' - s_0 = t' + \phi(s_0 + v, t, t') - t - (s_0 + v) \tag{24}$$

implying

$$\begin{aligned} \tau(t'', t''') &= |t''' + \phi(s_0, t'', t''') - t'' - s_0| \\ &= |t' + \phi(s_0 + v, t, t') - t - (s_0 + v)| \leq \tau(t, t') \end{aligned}$$

Substituting,

$$\begin{aligned} & t''' + \phi(s_0, t'', t''') - t'' - s_0 \\ = \;& t' + \phi(v, t, t') + \phi(s_0, t + v, t' + \phi(v, t, t')) - t - v - s_0 \end{aligned}$$

[19] This relies on the fact that if $z' = z$, then $k' = k$.

This equals $t' + \phi(s_0 + v, t, t') - t - (s_0 + v)$ if

$$\phi(v, t, t') + \phi(s_0, t + v, t' + \phi(v, t, t')) = \phi(s_0 + v, t, t') \tag{25}$$

By repeatedly applying (10), we obtain

$$\begin{aligned} & h(t''' + \phi(s_0, t'', t''')) - h(t'' + s_0) = h(t''') - h(t'') \\ = \; & h(t' + \phi(v, t, t')) - h(t + v) = h(t') - h(t) \\ = \; & h(t' + \phi(s_0 + v, t, t')) - h(t + s_0 + v) \end{aligned}$$

But $t'' = t + v$. Thus, equating the first and last expressions,

$$h(t' + \phi(s_0 + v, t, t')) = h(t''' + \phi(s_0, t'', t'''))$$

Since h is strictly increasing by part 2 of Lemma 1,

$$\begin{aligned} t' + \phi(s_0 + v, t, t') &= t''' + \phi(s_0, t'', t''') \\ &= t' + \phi(v, t, t') + \phi(s_0, t + v, t' + \phi(v, t, t')) \end{aligned}$$

establishing (25). Q.E.D.$_{\text{Lemma 4}}$

Proof of LEMMA 5. For any $N > 0$ let T_N be the first time t at which $|Z_t| > N$. We will prove that almost surely, for $t \leq 1/2$ and for any N, there exists a unique solution to the version of (12) that is killed when $|Z|$ reaches N:

$$X_t = x_0 + \int_{s=0}^{t \wedge T_N} \Gamma(s, Z_s, X_s | F) ds \tag{26}$$

where $t \wedge T_N = \min\{t, T_N\}$, $F = (F^R, F^L)$, and

$$\Gamma(s, z, x | F) = F^L(s, z, x)(1 - x) + F^R(s, z, x)x \tag{27}$$

Since the same argument can be repeated for $t \in [1/2, 1]$ etc. and taking $N \to \infty$, this will prove the existence of a unique solution for all t. For brevity, we will write t in place of $t \wedge T_N$.

We first prove existence. For any $\delta > 0$, define $X_t^\delta = x_0 + \int_{s=0}^t \Gamma_s^\delta ds$, where $\Gamma_s^\delta = \frac{1}{\delta} \int_{v=s-\delta}^s \Gamma(v, Z_v, X_v^\delta | F) dv$. (For $v \in [-\delta, 0)$, let $Z_v = Z_0$ and $X_v^\delta = x_0$.) Note that

$\dot{X}_t^\delta = \frac{1}{\delta}\int_{v=t-\delta}^t \Gamma(v, Z_v, X_v^\delta|F)dv$; the right hand side is absolutely bounded by K and independent of X_t^δ, so this equation has a unique solution that is Lipschitz with constant K. Let $X_t = \limsup_{n\to\infty} Y_t^{1/n}$ where $Y_t^{1/n} = \sup_{m>n} X_t^{1/m}$. The supremum of an arbitrary family of Lipshitz functions with constant K is a Lipshitz function with the same constant, and the same is true for the limit of a sequence of such functions. Hence, for every n, the function $Y_t^n = \sup_{m>n} X_t^{1/m}$ is Lipschitz with constant K, and so is X_t. Moreover, for fixed t, there exists a subsequence $(m_j)_{j=1}^\infty$ such that $\lim_{j\to\infty} X_t^{1/m_j} = X_t$. By extracting further subsequences and then using the diagonal method we can obtain a subsequence $(m_j')_{j=1}^\infty$ of the original sequence $(m_j)_{j=1}^\infty$ such that $\lim_{j\to\infty} X_t^{1/m_j'} = X_t$ for every rational $t \geq 0$ (and hence for every $t \geq 0$). The convergence is uniform on compact intervals because all functions $X_t^{1/m_j'}$ are Lipschitz with constant K.

To finish the proof of existence, it remains only to show that (26) holds for $X_t = \lim_{j\to\infty} X_t^{1/m_j'}$. For any j,

$$\left| X_t - \left(x_0 + \int_{s=0}^t \Gamma(s, Z_s, X_s|F)ds \right) \right| \leq A_1^j + A_2^j + A_3^j \tag{28}$$

where

$$\begin{aligned} A_1^j &= \left| X_t - \left(x_0 + \int_{s=0}^t \Gamma_s^{1/m_j'} ds \right) \right| \\ A_2^j &= \left| \int_{s=0}^t \Gamma_s^{1/m_j'} ds - \int_{s=0}^t \Gamma(s, Z_s, X_s^{1/m_j'}|F)ds \right| \\ A_3^j &= \left| \int_{s=0}^t \Gamma(s, Z_s, X_s^{1/m_j'}|F)ds - \int_{s=0}^t \Gamma(s, Z_s, X_s|F)ds \right| \end{aligned}$$

Since $A_1^j = \left| X_t - X_t^{1/m_j'} \right|$, $\lim_{j\to\infty} A_1^j = 0$. Moreover,

$$\begin{aligned} \int_{s=0}^t \Gamma_s^{1/m_j'} ds &= m_j' \int_{s=0}^t \int_{v=s-1/m_j'}^s \Gamma(v, Z_v, X_v^{1/m_j'}|F)dvds \\ &= \int_{v=0}^t \Gamma(v, Z_v, X_v^{1/m_j'}|F)dv + o(1/m_j') \end{aligned}$$

(reversing the order of integration), so that $\lim_{j\to\infty} A_2^j = 0$.

We now prove that $\lim_{j\to\infty} A_3^j = 0$. For $a = R, L$, all $t \in \Re$, $y \in [0, K]$ and $x \in [0, 1]$, let $H^a(t, y, x) = \inf\{z \in [-N, N] : F^a(t, z, x) > y\}$; if this set is empty, define $H^a(t, y, x) = N$. Let $c_4 = 3\left(\frac{\overline{\rho}}{\underline{\rho}}\right)^4 (c_2 + 2c_3 N)$ and define $F^{RL}(s, z, x) = F^R(s, z, x) + F^L(s, z, x)$.

CLAIM 1 *For $a = R, L$:*

1. *For any t, t', z, x, x' such that* $\max\{|z|, |z + c_4(|t - t'| + |x - x'|)|\} \le N$,

   $F^a(t', z + c_4[|t' - t| + |x' - x|], x') \ge F^a(t, z, x)$.

2. $H^a(t, y, x)$ *is Lipschitz in t and x with constant c_4.*

3. *For any two states (t, z, x) and (t', z', x'), let $|dt| = |t - t'|$, $|dz| = |z - z'|$, and $|dx| = |x - x'|$ and $\alpha = |dz| + c_4(|dt| + |dx|)$. Then*

$$\begin{aligned} & \Gamma(t, z, x|F) - \Gamma(t', z', x'|F) \\ \le \; & \left|F^L(t, z, x) - F^L(t', z', x')\right| + \left|F^R(t, z, x) - F^R(t', z', x')\right| + 2K|x - x'| \qquad (29) \\ \le \; & \left|F^L(t, z + \alpha, x) - F^L(t, z - \alpha, x)\right| + \left|F^R(t, z + \alpha, x) - F^R(t, z - \alpha, x)\right| + 2K|x - x'| \end{aligned}$$

   Moreover, if $x \ge x'$, then

$$\begin{aligned} \Gamma(t, z, x|F) - \Gamma(t', z', x'|F) & \le F^{RL}(t, z, x) - F^{RL}(t', z', x') \\ & \le F^{RL}(t, z, x) - F^{RL}(t, z - \alpha, x) \end{aligned}$$

4. *For any processes $Y \ge 0$ and $X^1 \in [0, 1]$,*

$$\begin{aligned} & \int_{s=0}^{t} \left[F^a\left(s, Z_s + Y_s, X_s^1\right) - F^a\left(s, Z_s, X_s^1\right)\right] ds \\ \le \; & \int_{y=-K}^{K} \int_{s=0}^{t} \mathbf{1}(H^a(s, y, X_s^1) - Z_s \in [0, Y_s]) ds dy \end{aligned}$$

5. *Suppose that $\widehat{F}^a(t, z, x)$ satisfies the assumptions of Lemma 5 and for any (t, x), $\widehat{F}^a(t, z, x) = F^a(t, z, x)$ at all but a measure zero set of z's. Let $\widehat{H}^a(t, y, x) = \inf\{z \in [-N, N] : \widehat{F}^a(t, z, x) > y\}$. Then H^a and $\widehat{H}^a$ coincide everywhere.*

Proof of Claim. Part 1: Let $x'' = x + \left(\frac{\overline{\rho}}{\underline{\rho}}\right)^2 \tau(t, t')$, $z'' = z + (c_2 + 2c_3 N)\left[\left(\frac{\overline{\rho}}{\underline{\rho}}\right)^2 + 1\right]\tau(t, t')$, and $z' = z'' + (c_2 + 2c_3 N)\left[\left(\frac{\overline{\rho}}{\underline{\rho}}\right)^2 \tau(t, t') + |x' - x|\right] \ge z'' + (c_2 + 2c_3 N)[|x'' - x'|]$. (The inequality follows from $|x'' - x| + |x' - x| \geq |x'' - x'|$.) By assumption 3 of this lemma and part 1 of Lemma 4,

$$\begin{aligned} F^a(t,z,x) &\leq F^a(t',z'',x'') \leq F^a(t',z',x') \\ &= F^a\left(t', \left(\begin{array}{c} z + (c_2 + 2c_3N)\left[\left(\frac{\overline{\rho}}{\underline{\rho}}\right)^2 + 1\right]\tau(t,t') \\ +(c_2 + 2c_3N)\left[\left(\frac{\overline{\rho}}{\underline{\rho}}\right)^2 \tau(t,t') + |x' - x|\right] \end{array}\right), x'\right) \\ &\leq F^a\left(t', z + 3\left(\frac{\overline{\rho}}{\underline{\rho}}\right)^4 (c_2 + 2c_3N)\left[|t' - t| + |x' - x|\right], x'\right) \end{aligned}$$

as claimed.

Part 2: Consider any t, t', x, and x';

$$|H^a(t',y,x') - H^a(t,y,x)| = \left| \begin{array}{c} \inf\{z \in [-N,N] : F^a(t',z,x') > y\} \\ -\inf\{z \in [-N,N] : F^a(t,z,x) > y\} \end{array} \right|$$

By part 1,

$$\begin{aligned} \inf\{z &\in [-N,N] : F^a(t,z,x) > y\} \\ &\geq \inf\{z \in [-N,N] : F^a(t', z + c_4(|x - x'| + |t - t'|), x') > y\} \\ &\geq \inf\{z \in [-N,N] : F^a(t',z,x') > y\} - c_4(|x - x'| + |t - t'|) \end{aligned}$$

Hence, $|H^a(t',y,x') - H^a(t,y,x)| \leq c_4(|x - x'| + |t - t'|)$.

Part 3: For $a = R, L$, let $F^a = F^a(t,z,x)$ and $F^{a\prime} = F^{a\prime}(t,z,x)$. We have

$$\begin{aligned} &\Gamma(t,z,x|F) - \Gamma(t,z,x'|F) \\ = \;& F^L \cdot (1 - x) + F^R \cdot x - F^{L\prime} \cdot (1 - x') - F^{R\prime} \cdot x' \qquad (30) \\ = \;& (F^L - F^{L\prime})(1 - x) + F^{L\prime} \cdot (x' - x) + (F^R - F^{R\prime})x + F^{R\prime} \cdot (x - x') \end{aligned}$$

implying $|\Gamma(t,z,x|F) - \Gamma(t,z,x'|F)| \leq \left|F^L - F^{L\prime}\right| + \left|F^R - F^{R\prime}\right| + 2K\left|x - x'\right|$ as claimed. By part 1, for $a = R, L$, $F^{a\prime} \in [F^a(t, z - \alpha, x), F^a(t, z + \alpha, x)]$; clearly, F^a is in the same interval, so $|F^a - F^{a\prime}| \leq |F^a(t, z + \alpha, x) - F^a(t, z - \alpha, x)|$, which proves (29). If $x' \leq x$, then since $F^{L\prime} \geq 0$ and $F^{R\prime} \leq 0$, $F^{L\prime} \cdot (x' - x) + F^{R\prime} \cdot (x - x') = (F^{L\prime} - F^{R\prime})(x' - x) \leq 0$, so (30) implies

$$\Gamma(t,z,x|F) - \Gamma(t',z',x'|F) \;\leq\; (F^L - F^{L\prime})(1 - x) + (F^R - F^{R\prime})x$$

$$
\begin{aligned}
&\le (F^L - F^L(t, z-\alpha, x))(1-x) + (F^R - F^R(t, z-\alpha, x))x \\
&\le F^{RL}(t, z, x) - F^{RL}(t, z-\alpha, x)
\end{aligned}
$$

proving part 3.

Part 4: Since $F^a(s,z,x) = K - \int_{y=-K}^{K} \mathbf{1}(F^a(s,z,x) \le y)dy$,

$$
\begin{aligned}
&\int_{s=0}^{t} \left[F^a\left(s, Z_s + Y_s, X_s^1\right) - F^a\left(s, Z_s, X_s^1\right)\right] ds \\
= &\int_{y=-K}^{K} \int_{s=0}^{t} \left[\mathbf{1}(F^a(s, Z_s, X_s^1) \le y) - \mathbf{1}(F^a(s, Z_s + Y_s, X_s^1) \le y))\right] dsdy \\
= &\int_{y=-K}^{K} \int_{s=0}^{t} \mathbf{1}(Z_s \le H(s, y, X_s^1) < Z_s + Y_s) dsdy \\
= &\int_{y=-K}^{K} \int_{s=0}^{t} \mathbf{1}(H(s, y, X_s^1) - Z_s \in [0, Y_s)) dsdy
\end{aligned}
$$

Part 5: Since both F^a and $\widehat{F}^a$ are monotonic in z, the sets $\{z \in [-N,N] : F^a(t,z,x) > y\}$ and $\{z \in [-N,N] : \widehat{F}^a(t,z,x) > y\}$ are each intervals of the form $(\zeta, N]$ or $[\zeta, N]$. Since F^a and $\widehat{F}^a$ agree almost everywhere, these intervals must also agree almost everywhere; hence, their infima must coincide. Q.E.D.$_{\text{Claim 1}}$

By part 3 of Claim 1,

$$
A_3^j \le \int_{s=0}^{t} \left[\begin{array}{c} F^{RL}\left(s, Z_s + c_4 \left|X_s^{1/m_j'} - X_s\right|, \min\{X_s^{1/m_j'}, X_s\}\right) \\ -F^{RL}\left(s, Z_s, \min\{X_s^{1/m_j'}, X_s\}\right) \end{array} \right] ds + 2K \int_{s=0}^{t} \left|X_s^{1/m_j'} - X_s\right| ds
$$

By part 4,

$$
\begin{aligned}
A_3^j \;\le\; &\int_{y=-K}^{K} \int_{s=0}^{t} 1 \left(\begin{array}{c} H^R(s, y, \min\{X_s^{1/m_j'}, X_s\}) - Z_s \\ \in \left[0, c_4 \left|X_s^{1/m_j'} - X_s\right|\right) \end{array} \right) dsdy \\
&+ \int_{y=-K}^{K} \int_{s=0}^{t} 1 \left(\begin{array}{c} H^L(s, y, \min\{X_s^{1/m_j'}, X_s\}) - Z_s \\ \in \left[0, c_4 \left|X_s^{1/m_j'} - X_s\right|\right) \end{array} \right) dsdy + 2K \int_{s=0}^{t} \left|X_s^{1/m_j'} - X_s\right| ds
\end{aligned}
$$

Since $X_s^{1/m_j'}$ converges uniformly to X_s on $[0,t]$, $\lim_{j\to\infty} 2K \int_{s=0}^{t} \left|X_s^{1/m_j'} - X_s\right| ds = 0$. Since Brownian motion has a jointly continuous local time ([**?**, p. 310]),

$$
\lim_{j\to\infty} \int_{s=0}^{t} 1\left(-Z_s \in \left[0, c_4 \left|X_s^{1/m_j'} - X_s\right|\right)\right) ds = 0
$$

almost surely. But by part 2 of Claim 1, for $a = R, L$, $H^a(s, y, \min\{X_s^{1/m'_j}, X_s\})$ is Lipschitz in s with constant $c'_4 \triangleq c_4(1 + K)$. Thus, by the Girsanov Theorem [18], the law of $H^a(s, y, \min\{X_s^{1/m'_j}, X_s\}) - Z_s$ is mutually absolutely continuous with the law of $-Z_s$. Consequently, $\lim_{j \to \infty} A_3^j = 0$ almost surely. This proves existence.

We now prove uniqueness. Let X_t^+ and X_t^- be the maximal and minimal solutions to (26). Define $Y_t = X_t^+ - X_t^-$. By part 3 of Claim 1,

$$Y_t \leq \int_{s=0}^{t} \left[F^{RL}(s, Z_s + c_4 Y_s, X_s^-) - F^{RL}(s, Z_s, X_s^-) \right] ds \tag{31}$$

so that by part 4,

$$Y_t \leq \int_{y=-K}^{K} \int_{s=0}^{t} \left[\mathbf{1}(H^R(s, y, X_s^-) - Z_s \in [0, c_4 Y_s)) + \mathbf{1}(H^L(s, y, X_s^-) - Z_s \in [0, c_4 Y_s)) \right] dy$$

Since Z has zero drift, the probability distribution over $(H^a(s, y, X_s^-) - Z_s)_{s \geq 0}$ is the same as the probability distribution over $(Z_s + H^a(s, y, X_s^-))_{s \geq 0}$. Hence, if there is a positive probability that $Y_t > 0$, then this also occurs with positive probability if Y_t instead satisfies

$$Y_t \leq \int_{y=-K}^{K} \int_{s=0}^{t} \left[\mathbf{1}(Z_s + H^R(s, y, X_s^-) \in [0, c_4 Y_s)) + \mathbf{1}(Z_s + H^L(s, y, X_s^-) \in [0, c_4 Y_s)) \right] dy \tag{32}$$

We will show that if (32) holds, Y_t is identically zero for all $t \in [0, 1/2 \wedge T_N]$ almost surely.

Let $(\Omega, \mathcal{F}, P^z)$ be the probability space associated with Z when $Z_0 = z$, and let $(\mathcal{F}_u)_{u \geq 0}$ be the filtration generated by Z.[20] Since each $H^a(s, y, X_s^-)$ has paths that are Lipschitz-continuous in s with constant c'_4, by the Girsanov theorem (Øksendal [18]), for any $u \in [0, t]$, and for any positive A and α,

$$\begin{aligned} & E\left[\int_{s=u}^{t} \mathbf{1}(Z_s \in [0, A s^\alpha)) ds \mid \mathcal{F}_u \right] \\ = \; & E\left[\int_{s=u}^{t} \mathbf{1}(Z_s + H^a(s, y, X_s^-) \in [0, A s^\alpha)) \cdot M_s^y \, ds \mid \mathcal{F}_u \right] \end{aligned} \tag{33}$$

[20] Ω is the set of possible sample paths $(Z_t)_{t \geq 0}$; $\mathcal{F}$ is the σ-algebra of measurable subsets of Ω; for any $S \in \mathcal{F}$ and constant z, $P^z(S)$ is the probability, conditional on $Z_0 = z$, that the sample path will be in S. $\mathcal{F}_u$ is the σ-algebra that contains information about Z_v for $v \leq u$ but no information about Z_v for $v > u$.

where

$$M_s^y = \exp\begin{pmatrix} -\int_{v=u}^{s}\left[\frac{dH^a(v,y,X_v^-)}{dv}\right]dZ_v \\ -\frac{1}{2}\int_{v=u}^{s}\left[\frac{dH^a(v,y,X_v^-)}{dv}\right]^2 dv \end{pmatrix} \geq c_5 \exp\left(-\int_{v=u}^{s}\left[\frac{dH^a(v,y,X_v^-)}{dv}\right]dZ_v\right)$$

where $c_5 = \exp\left(-\frac{1}{4}\left[c_4'\right]^2\right)$ (as $s \leq \frac{1}{2}$ and $\left|\frac{dH^a(v,y,X_v^-)}{dv}\right| \leq c_4' = c_4(1+K)$). But

$$\begin{aligned}
\int_{v=u}^{s}\left[\frac{dH^a(v,y,X_v^-)}{dv}\right]dZ_v &= \int_{v=u}^{s}\left[\frac{dH^a(v,y,X_v^-)}{dv}\right]dB_{h(v)} \\
&= \int_{v=h(u)}^{h(s)}\left[\frac{dH^a\left(h^{-1}(v),y,X_{h^{-1}(v)}^-\right)}{dh^{-1}(v)}\right]dB_v
\end{aligned}$$

Hence, for any $\lambda > 0$,

$$\begin{aligned}
&\Pr\left(\min_{s\in[u,t]}\left\{-\int_{v=u}^{s}\left[\frac{dH^a(v,y,X_v^-)}{dv}\right]dZ_v\right\} < -\lambda \mid \mathcal{F}_u\right) \\
=\ &\Pr\left(\min_{s\in[u,t]}\left\{-\int_{v=h(u)}^{h(s)}\left[\frac{dH^a\left(h^{-1}(v),y,X_{h^{-1}(v)}^-\right)}{dh^{-1}(v)}\right]dB_v\right\} < -\lambda \mid \mathcal{F}_u\right) \\
\leq\ &\Pr\left(\max_{s\in[u,t]}\left|\int_{v=h(u)}^{h(s)}\left[\frac{dH^a\left(h^{-1}(v),y,X_{h^{-1}(v)}^-\right)}{dh^{-1}(v)}\right]dB_v\right| > \lambda \mid \mathcal{F}_u\right)
\end{aligned}$$

The integral in the last line is a martingale and B_v is a Brownian motion with zero drift and unit variance. Hence, by Doob's martingale inequality (Øksendal [18, p. 33]), the last line is no greater than

$$\frac{E\left[\int_{v=h(u)}^{h(s)}\left[\frac{dH^a\left(h^{-1}(v),y,X_{h^{-1}(v)}^-\right)}{dh^{-1}(v)}\right]^2 dv \mid \mathcal{F}_u\right]}{\lambda^2} \leq \frac{(c_4')^2\, h(s)}{\lambda^2} \leq \frac{c_6}{\lambda^2}$$

where $c_6 = c_3^2 h(1/2)$ (as $s \leq \frac{1}{2}$ and $h' > 0$). Thus, for any sufficiently small $m \in (0, c_5)$,

$$\begin{aligned}
\Pr(\min_{s\in[u,t]} M_s^y &< m \mid \mathcal{F}_u) \\
&\leq \Pr\left(\min_{s\in[u,t]}\left\{\exp\left(-\int_{v=u}^{s}\left[\frac{dH^a(v,y,X_v^-)}{dv}\right]dZ_v\right) < \frac{m}{c_5}\right\} \mid \mathcal{F}_u\right) \\
&\leq \Pr\left(\min_{s\in[u,t]}\left\{\int_{v=u}^{s}\left[-\frac{dH^a(v,y,X_v^-)}{dv}\right]dZ_v\right\} < -\ln\left(\frac{c_5}{m}\right) \mid \mathcal{F}_u\right) \\
&\leq \frac{c_6}{\ln\left(\frac{c_5}{m}\right)^2} \triangleq \rho(m)
\end{aligned}$$

where $\rho(m)$ is independent of y and u and $\lim_{m\to 0}\rho(m)=0$. Thus, for any $m>0$,

$$\begin{aligned} & E\left[\int_{y=-K}^{K}\int_{s=u}^{t}\mathbf{1}(Z_s+H^a(s,y,X_s^-)\in[0,As^\alpha))\,M_s^y\,ds\,dy \;\mid\; \mathcal{F}_u\right] \\ \geq\; & m(1-\rho(m))E\left[\int_{y=-K}^{K}\int_{s=u}^{t}\mathbf{1}(Z_s+H^a(s,y,X_s^-)\in[0,As^\alpha))\,ds\,dy \;\mid\; \mathcal{F}_u\right] \\ =\; & m(1-\rho(m))E\left[C_t-C_u \;\mid\; \mathcal{F}_u\right] \end{aligned}$$

where $C_t=\int_{y=-K}^{K}\int_{s=0}^{t}\mathbf{1}(Z_s+H^a(s,y,X_s^-)\in[0,As^\alpha))\,ds\,dy$. Hence, by (33), there is a positive constant c_7, independent of A, α, and u, such that

$$E\left[C_t-C_u \;\mid\; \mathcal{F}_u\right]\leq c_7\cdot E\left[\int_{s=u}^{t}\mathbf{1}(Z_s\in[0,As^\alpha))ds \;\mid\; \mathcal{F}_u\right]$$

Let $\left(\widehat{\mathcal{F}}_s\right)_{s\geq 0}$ be the filtration generated by B where $Z_t=B_{h(t)}$. Since B is a Brownian motion, there is a constant c_8 such that

$$\Pr(Z_t\in dy\mid\mathcal{F}_s)=\Pr(B_{h(t)}\in dy\mid\widehat{\mathcal{F}}_s)\leq\frac{c_8}{(h(t)-h(s))^{1/2}}dy\leq\frac{c_8/\underline{\rho}^{1/2}}{(t-s)^{1/2}}dy$$

Using this fact, the argument of Lemma 2.14 in Bass and Burdzy [1] implies that there exist constants c_9 and c_{10}, independent of A and α, such that

$$\Pr(C_t>\lambda)\leq c_9\exp(-c_{10}\lambda\alpha^{1/8}/(At^{\alpha+1/4}))$$

Using this fact, the argument of Lemma 2.15 of Bass and Burdzy implies that given $\zeta>0$ there exist constants c_{11} and c_{12} such that if $\alpha\geq 1$, $A,A_0>0$, $A_0/A>\zeta$, and $A_1=\alpha+1/8$, then $\Pr(C_t\geq A_0t^{A_1}$ for some $t\leq 1/2)\leq c_{11}\exp(-c_{12}A_0\alpha^{1/8}/A)$. Armed with this result, it is straightforward to adapt the argument in Lemma 2.17 of Bass and Burdzy to show that $\Pr(Y_t\neq 0$ for some $t\in[0,1/2\wedge T_N])=0$. By induction on t and letting $N\to\infty$, we then have $Y_t=0$ for all t almost surely. This proves uniqueness. Q.E.D.$_{\text{LEMMA 5}}$

Proof of LEMMA 6. Part 1: Let $Y_t=\max\{0,X_t^2-X_t^1\}$ and $F_1^{RL}(s,z,x)=F_1^R(s,z,x)+F_1^L(s,z,x)$. Then

$$\begin{aligned} \dot{Y}_t \;&=\; \left[\Gamma(t,Z_t,X_t^2|F_2)-\Gamma(t,Z_t,X_t^1|F_1)\right]\mathbf{1}(X_t^2\geq X_t^1) \\ &\leq\; \left[\Gamma(t,Z_t,X_t^2|F_1)-\Gamma(t,Z_t,X_t^1|F_1)\right]\mathbf{1}(X_t^2\geq X_t^1) \end{aligned}$$

$$
\begin{aligned}
&\le \left[F_1^{RL}\left(t, Z_t + c_4(X_t^2 - X_t^1), X_t^1\right) - F_1^{RL}\left(t, Z_t, X_t^1\right)\right] \mathbf{1}(X_t^2 \ge X_t^1) \\
&= \left[F_1^{RL}\left(t, Z_t + c_4 Y_t, X_t^1\right) - F_1^{RL}\left(t, Z_t, X_t^1\right)\right] \mathbf{1}(X_t^2 \ge X_t^1) \\
&= F_1^{RL}\left(t, Z_t + c_4 Y_t, X_t^1\right) - F_1^{RL}\left(t, Z_t, X_t^1\right)
\end{aligned}
$$

(The second inequality follows from part 3 of Claim 1.) This implies that equation (31) holds for this Y_t, with X_s^1 substituted for X_s^- and F_1^{RL} substituted for F^{RL}. The argument following equation (31) now applies verbatim to show that Y_t is identically zero.

Part 2: we will prove that

$$
\left| \int_{s=0}^{t} \Gamma(s, Z_s, X_s^1 | F_1) ds - \int_{s=0}^{t} \Gamma(s, Z_s, X_s^1 | F_2) ds \right| = 0 \tag{34}
$$

This implies that X_t^1 is a solution to (12) defined relative to F_2; by uniqueness, $X_t^1 = X_t^2$. To see (34), consider any (s, z, x); for $a = R, L$, and $n = 1, 2$, let $F_n^a(x)$ represent $F_n^a(s, z, x)$. We have

$$
\begin{aligned}
0 &\le \Gamma(s, z, x | F_1) - \Gamma(s, z, x | F_2) \\
&= F_1^L(x)(1 - x) + F_1^R(x)x - F_2^L(x)(1 - x) - F_2^R(x)x \\
&\le F_1^L(x) - F_2^L(x) + F_1^R(x) - F_2^R(x)
\end{aligned}
$$

Thus,

$$
\begin{aligned}
&0 \le \int_{s=0}^{t} \Gamma(s, Z_s, X_s^1 | F_1) ds - \int_{s=0}^{t} \Gamma(s, Z_s, X_s^1 | F_2) ds \\
\le &\int_{s=0}^{t} \left[F_1^L(s, Z_s, X_s^1) - F_2^L(s, Z_s, X_s^1)\right] ds + \int_{s=0}^{t} \left[F_1^R(s, Z_s, X_s^1) - F_2^R(s, Z_s, X_s^1)\right] ds
\end{aligned}
$$

As in the proof of part 4 of Claim 1, for $a = R, L$,

$$
\begin{aligned}
&\int_{s=0}^{t} \left[F_1^a\left(s, Z_s, X_s^1\right) - F_2^a\left(s, Z_s, X_s^1\right)\right] ds \\
= &\int_{s=0}^{t} \int_{y=-K}^{K} \left[\mathbf{1}(F_2^a(s, Z_s, X_s^1) \le y) - \mathbf{1}(F_1^a(s, Z_s, X_s^1) \le y)\right] dy ds
\end{aligned}
$$

Let $H_j^a(t, y, x) = \inf\{z \in [-N, N] : F_j^a(t, z, x) > y\}$ for $j = 1, 2$. By part 5 of Lemma **??**, H_1^a and H_2^a coincide everywhere. Moreover, $F_j^a(s, Z_s, X_s^1) \le y$ implies $H_j^a(s, y, X_s^1) \ge Z_s$ and $F_j^a(s, Z_s, X_s^1) > y$ implies $H_j^a(s, y, X_s^1) \le Z_s$. So $\mathbf{1}(F_2^a(s, Z_s, X_s^1) \le y) -$

$\mathbf{1}(F^a_1(s,Z_s,X^1_s) \le y) \ne 0$ only if $H^a_2(s,y,X^1_s) \ge Z_s \ge H^a_1(s,y,X^1_s)$. Since $H^a_1 = H^a_2$, this implies $Z_s = H^a_1(s,y,X^1_s)$. As $H^a_1(s,y,X^1_s)$ has paths that are Lipschitz-continuous in s with constant c'_4,

$$\int_{y=-K}^{K}\int_{s=0}^{t}\left[\mathbf{1}(F^a_2(s,Z_s,X^1_s) \le y) - \mathbf{1}(F^a_1(s,Z_s,X^1_s) \le y)\right] dsdy$$
$$\le \int_{y=-K}^{K}\int_{s=0}^{t}\mathbf{1}(Z_s = H^a_1(s,y,X^1_s))dsdy = \int_{y=-K}^{K} 0dy = 0 \text{ a.s.}$$

by the Girsanov theorem (Øksendal [18]). Q.E.D$_{\text{LEMMA 6}}$

Proof of Lemma 7. We will deduce part 1 from Lemma 6. For $a = R, L$, define $\widetilde{F}^a$ by $\widetilde{F}^a(t,Z_t,X_t) = F^a(t,Z_t + z, X_t) = F^a(t,\widetilde{Z}_t,X_t)$. Since $\widetilde{F}^a \ge F^a$, Lemma 6 implies that $\widetilde{X}^{z,\mathbf{0}}_t \ge X_t$. The assertion $\widetilde{X}^{\mathbf{0},x}_t \ge X_t$ follows directly from Lemma 6.

For part 2, take any sequence $\{(z_n,x_n)\}$ such that $z_n \to 0$ and $x_n \to 0$ as n goes to infinity. For a fixed t, there exists a subsequence $\{(z_{n_j},x_{n_j})\}$ such that $\widetilde{X}^{z_{n_j},x_{n_j}}_t$ converges. By extracting further subsequences and then using the diagonal method we can obtain a subsequence $\{(z'_n,x'_n)\}$ of the original sequence $\{(z_n,x_n)\}$ such that $\widetilde{X}^{z'_n,x'_n}_s$ converges to a limit X^*_s for every rational $s > 0$. The convergence is uniform on compact sets because all functions $\widetilde{X}^{z'_n,x'_n}_s$ are Lipschitz with constant K. We see that X^*_s must be a solution to (12) by the following argument. Let $\widetilde{F}^a_n(t,z+z_n,x) = F^a(t,z,x)$ and let $X^n_s = \widetilde{X}^{z'_n,x'_n}_s$. For any n,

$$\left|X^*_t - \left(x_0 + \int_{s=0}^{t}\Gamma(s,Z_s,X^*_s|F)ds\right)\right| \le A^n_1 + A^n_2 + A^n_3$$

where

$$\begin{aligned} A^n_1 &= \left|X^*_t - \left(x_0 + \int_{s=0}^{t}\Gamma(s,Z_s,X^n_s|F_n)ds\right)\right| \\ A^n_2 &= \left|\int_{s=0}^{t}\Gamma(s,Z_s,X^n_s|F_n)ds - \int_{s=0}^{t}\Gamma(s,Z_s,X^*_s|F)ds\right| \\ A^n_3 &= \left|\int_{s=0}^{t}\Gamma(s,Z_s,X^n_s|F)ds - \int_{s=0}^{t}\Gamma(s,Z_s,X^*_s|F)ds\right| \end{aligned}$$

Since $A^n_1 = |X^*_t - X^n_t|$, $\lim_{n\to\infty} A^n_1 = 0$. Since $F_n \to F$, $\lim_{n\to\infty} A^n_2 = 0$. One can prove that $\lim_{n\to\infty} A^n_3 = 0$ by the same argument used to prove that $A^j_3 \to 0$ in Lemma 5. By uniqueness, $X^*_s = X_s$ for all s. Since the same is true for any initial sequence

$\{(z_n, x_n)\}$, we conclude that $\widetilde{X}_t^{z,x}$ converges to X_t almost surely, uniformly on compact time intervals.

The proof of part 3 is completely analogous to that for part 2. One can show that for every subsequence of $\widehat{X}_t^n$, there is a further subsequence which converges and, moreover, it converges to a solution of (12). The argument is finished by invoking the uniqueness of the solution. Q.E.D$_{\text{LEMMA 7}}$

Proof of LEMMA 8. This is an easy consequence of parts 1 and 2 of Lemma 3.

Proof of LEMMA 9. We prove (i-iii) by induction. (iii) holds for $n = 0$ if we define Φ^n for $n = -1$ to be ∞. For given t and Z_t and any $N > 0$, let T_N be either N or the smallest v for which $g(t+v, Z_{t+v}) \notin (\underline{w} - N, \overline{w} + N)$, whichever is least. Let $\overline{g}(N) = \max\{|\underline{w} - N|, |\overline{w} + N|\}$. Then by Lemma 2, for any n, $\Phi^n(t,z,x) = \lim_{N\to\infty} \Phi_N^n(t,z,x)$, where

$$\Phi_N^n(t,z,x) = E \int_{v=0}^{T_N} \exp\left(- \int_{s=0}^{v} (r + k_{t+s}^R + k_{t+s}^L) ds \right) \left[\Delta_{t+v}(Z_{t+v}, X_{t+v}, k_{t+v}^R, k_{t+v}^L) \right] dv \tag{35}$$

conditioned on $Z_t = z$ and $X_t = x$.

Let $(b_v)_{v\geq 0}$ be a fixed Brownian sample path with $b_0 = 0$. We compare $\Phi_N^n(t', z', x')$ to $\Phi_N^n(t,z,x)$ path by path, so that the continuation path of Z from time t (t') on begins at z (z') and its changes are given by $(b_v)_{v\geq 0}$ with time suitably transformed. For the path starting at (t,z,x), let $Z_{t+v} = z + b_{h(t+v)-h(t)}$; for the path starting at (t', z', x'), let $Z'_{t'+v} = z' + b_{h(t'+v)-h(t')}$.

Let $dx = x' - x$, $dz = z' - z$, and $dt = t' - t$. Let

$$\Delta'_v = \Delta_{t'+\phi(v,t,t')} \begin{pmatrix} Z'_{t'+\phi(v,t,t')}, X'_{t'+\phi(v,t,t')}, \\ k^{R\prime}_{t'+\phi(v,t,t')}, k^{L\prime}_{t'+\phi(v,t,t')} \end{pmatrix}$$

Using the change of variables $v = \phi(\widehat{v}, t, t')$, and then replacing $\widehat{v}$ by v (noting $\phi(0,t,t') = 0$), we obtain

$$\Phi_N^n(t', z', x')$$

$$= E\int_{v=0}^{T_N'} \exp\left(-\int_{s=0}^{v}(r+k_{t'+s}^{R\prime}+k_{t'+s}^{L\prime})ds\right)\left[\Delta_{t'+v}(Z_{t'+v}',X_{t'+v}',k_{t'+v}^{R\prime},k_{t'+v}^{L\prime})\right]dv$$

$$= E\int_{\phi(\widehat{v},t,t')=0}^{T_N'} \exp\left(-\int_{s=0}^{\phi(\widehat{v},t,t')}(r+k_{t'+s}^{R\prime}+k_{t'+s}^{L\prime})ds\right)\Delta_{\widehat{v}}'\phi_1(\widehat{v},t,t')d\widehat{v}$$

$$= E\int_{v=0}^{\phi^{-1}(T_N',t,t')} \exp\left(-\int_{s=0}^{\phi(v,t,t')}(r+k_{t'+s}^{R\prime}+k_{t'+s}^{L\prime})ds\right)\Delta_{v}'\phi_1(v,t,t')dv$$

where we define ϕ^{-1} by $\phi(\phi^{-1}(T_N',t,t'),t,t')=T_N'$. By Lemma 4,

$$\left|T_N'-\phi^{-1}(T_N',t,t')\right|=\left|\phi(\phi^{-1}(T_N',t,t'),t,t')-\phi^{-1}(T_N',t,t')\right|\le\frac{\overline{\rho}}{\underline{\rho}}\left|dt\right|$$

and we can take N as high as we like, so we can approximate T_N' by T_N. Moreover, for small (dt,dz,dx), the choices $\left(k_{t'+s}^{R\prime},k_{t'+s}^{L\prime}\right)_{s\ge0}$ and $(k_{t'+\phi(s,t,t')}^{R\prime},k_{t'+\phi(s,t,t')}^{L\prime})_{s\ge0}$ must give approximately the same expected payoffs to playing R and L as $\left(k_{t+s}^{R},k_{t+s}^{L}\right)_{s\ge0}$ by the envelope theorem. Thus, letting $k_{t+v}=\left(k_{t+v}^{R},k_{t+v}^{L}\right)$,

$$\begin{aligned}&\Phi_N^n(t',z',x')-\Phi_N^n(t,z,x)\\ =\ & E\int_{v=0}^{T_N}\exp\left(-\int_{s=0}^{\phi(v,t,t')}(r+k_{t+s}^{R}+k_{t+s}^{L})ds\right)\Delta_{t'+\phi(v,t,t')}\begin{pmatrix}Z_{t'+\phi(v,t,t')}',\\ X_{t'+\phi(v,t,t')}',k_{t+v}\end{pmatrix}\phi_1(v,t,t')dv\\ &-E\int_{v=0}^{T_N}\exp\left(-\int_{s=0}^{v}(r+k_{t+s}^{R}+k_{t+s}^{L})ds\right)\Delta_{t+v}\left(Z_{t+v},X_{t+v},k_{t+v}\right)dv\end{aligned}$$

to first order, by (35). By definition,

$$Z_{t'+\phi(v,t,t')}'=z'+b_{h(t'+\phi(v,t,t'))-h(t')}=z'+b_{h(t+v)-h(t)}=Z_{t+v}+dz$$

Hence, $\Phi_N^n(t',z',x')-\Phi_N^n(t,z,x)=A_0+A_1+A_2$ where

$$A_0 = E\int_{v=0}^{T_N}\left[\begin{array}{c}\exp\left(-\int_{s=0}^{\phi(v,t,t')}\left(r+k_{t+s}^{R}+k_{t+s}^{L}\right)ds\right)\\ -\exp\left(-\int_{s=0}^{v}\left(r+k_{t+s}^{R}+k_{t+s}^{L}\right)ds\right)\end{array}\right]\Delta_{t'+\phi(v,t,t')}\begin{pmatrix}Z_{t+v}+dz,\\ X_{t'+\phi(v,t,t')}',\\ k_{t+v}\end{pmatrix}\phi_1(v,t,t')dv$$

$$A_1 = E\int_{v=0}^{T_N}\exp\left(-\int_{s=0}^{v}(r+k_{t+s}^{R}+k_{t+s}^{L})ds\right)\Delta_{t'+\phi(v,t,t')}\begin{pmatrix}Z_{t+v}+dz,\\ X_{t'+\phi(v,t,t')}',\\ k_{t+v}\end{pmatrix}\left[\phi_1(v,t,t')-1\right]dv$$

$$A_2 = E\int_{v=0}^{T_N} \exp\left(-\int_{s=0}^{v}(r+k_{t+s}^R+k_{t+s}^L)ds\right)[A_3+A_4+A_5]\,dv$$

and

$$\begin{aligned}
A_3 &= \Delta_{t+v}\left(Z_{t+v}+dz, X_{t+v}, k_{t+v}\right) - \Delta_{t+v}\left(Z_{t+v}, X_{t+v}, k_{t+v}\right)\\
A_4 &= \Delta_{t+v}\left(Z_{t+v}+dz, X'_{t'+\phi(v,t,t')}, k_{t+v}\right) - \Delta_{t+v}\left(Z_{t+v}+dz, X_{t+v}, k_{t+v}\right)\\
A_5 &= \Delta_{t'+\phi(v,t,t')}\left(Z_{t+v}+dz, X'_{t'+\phi(v,t,t')}, k_{t+v}\right) - \Delta_{t+v}\left(Z_{t+v}+dz, X'_{t'+\phi(v,t,t')}, k_{t+v}\right)
\end{aligned}$$

By Lemmas 1 and 4, part 2 of Lemma 2, and equations (3) and (6),

$$\begin{aligned}
|A_0| &\leq \left(1-e^{-2r\frac{\overline{\rho}}{\underline{\rho}}dt}\right)|\Phi_N^n(t',z',x')| < 4r\frac{\overline{\rho}}{\underline{\rho}}|\Phi_N^n(t',z',x')dt| \leq 4r\frac{\overline{\rho}}{\underline{\rho}}\left(c_0\overline{g}\left(N\right)+c_1\right)|dt|\\
|A_1| &\leq \left(\frac{\overline{\rho}}{\underline{\rho}}\right)^2\left(c_0\overline{g}\left(N\right)+c_1\right)|dt|\\
A_3 &\geq 0 \text{ and for } g(t+v,Z_{t+v})\in[\underline{w},\overline{w}],\ A_3 \geq \underline{\alpha\gamma}dz\\
A_4 &\geq \beta\min\left\{0, \min_{v\geq 0}\left(X'_{t'+\phi(v,t,t')}-X_{t+v}\right)\right\}\\
|A_5| &\leq \overline{\alpha\gamma}\frac{\overline{\rho}}{\underline{\rho}}|dt|
\end{aligned}$$

Let $T(N)$ be a (strictly positive) lower bound on $E\int_{v=0}^{\infty}e^{-(r+2K)v}1\left(g(t+v,Z_{t+v})\in[\underline{w},\overline{w}]\right)dv$ over all starting points $g(t,Z_t)\in[\underline{w}-N,\overline{w}+N]$. By the above inequalities,

$$\begin{aligned}
&\Phi_N^n(t',z',x')-\Phi_N^n(t,z,x) \geq \underline{\alpha\gamma}T(N)dz - \frac{\overline{\rho}}{\underline{\rho}}\left(\frac{\overline{\rho}}{\underline{\rho}}+4r\right)\left(c_0\overline{g}\left(N\right)+c_1\right)|dt|\\
&+\frac{1}{r}\left(\beta\min\left\{0,\min_{v\geq 0}\left(X'_{t'+\phi(v,t,t')}-X_{t+v}\right)\right\}-\overline{\alpha\gamma}\frac{\overline{\rho}}{\underline{\rho}}|dt|\right)
\end{aligned}$$

For $n=0$, X and X' are identically zero for any dx, so $\Phi^0(t,z,x)$ is strictly increasing in z and independent of x (part i). Moreover, since $|dt|\leq\tau(t,t')$,

$$\begin{aligned}
&\frac{1}{\underline{\alpha\gamma}T(N)}\left[\Phi_N^n(t',z',x')-\Phi_N^n(t,z,x)\right]\\
\geq\ & dz-\left(c_2'+c_3\overline{g}\left(N\right)\right)\cdot\tau(t,t')+\frac{\beta}{r\underline{\alpha\gamma}T(N)}\min\left\{0,\min_{v\geq 0}\left(X'_{t'+\phi(v,t,t')}-X_{t+v}\right)\right\} \quad (36)
\end{aligned}$$

where

$$\begin{aligned}
c_2' &= \frac{1}{\underline{\alpha\gamma}T(N)}\frac{\overline{\rho}}{\underline{\rho}}\left(\frac{\overline{\rho}}{\underline{\rho}}+4r\right)c_1+\frac{\overline{\alpha\gamma\rho}}{r\underline{\alpha\gamma\rho}T(N)}\\
c_3 &= \frac{1}{\underline{\alpha\gamma}T(N)}\frac{\overline{\rho}}{\underline{\rho}}\left(\frac{\overline{\rho}}{\underline{\rho}}+4r\right)c_0
\end{aligned}$$

Let $c_2 = c_2' + \frac{\beta}{r\underline{\alpha}\gamma T(N)}$. To prove (ii), it remains to show that if $dz \geq (c_2 + c_3\overline{g}(N)) \cdot (\tau(t,t') + |dx|)$ and $|dx| \geq \left(\frac{\overline{\rho}}{\underline{\rho}}\right)^2 \tau(t,t')$ then $\min\left\{0, \min_{v\geq 0}\left(X'_{t'+\phi(v,t,t')} - X_{t+v}\right)\right\}$ is not less than $-|dx|$ since then

$$\begin{aligned}
& \frac{1}{\underline{\alpha}\gamma T(N)}\left[\Phi_N^n(t',z',x') - \Phi_N^n(t,z,x)\right] \\
\geq\ & dz - (c_2' + c_3\overline{g}(N)) \cdot \tau(t,t') - \frac{\beta}{r\underline{\alpha}\gamma T(N)}|dx| \\
\geq\ & dz - (c_2 + c_3\overline{g}(N)) \cdot [\tau(t,t') + |dx|] \geq 0
\end{aligned}$$

This is trivial for the case $n = 0$ since X and X' are identically zero for any dx. Let $dX_{t+v} = X'_{t'+\phi(v,t,t')} - X_{t+v}$. For small enough $\varepsilon > 0$, we will show that $d(dX_{t+v})/dv \geq 0$ whenever $dX_{t+v} \in [-|dx| - \varepsilon, -|dx|]$. Since X has continuous paths, this will imply $dX_{t+v} \geq -|dx|$, so $dX_{t+v} \wedge 0 \geq -|dx|$, proving (ii).

To see why $d(dX_{t+v})/dv \geq 0$ in this range, recall that by Lemma 4, $\tau(t+v, t' + \phi(v,t,t')) \leq \tau(t,t')$, so $dz > (c_2 + c_3 N) \cdot (|dx| + \tau(t+v, t' + \phi(v,t,t')))$. Thus, if $dX_{t+v} \in [|dx| - \varepsilon, |dx|]$ for small enough ε, $dz > (c_2 + c_3 N) \cdot (dX_{t+v} + \tau(t+v, t' + \phi(v,t,t')))$ and $\left(\frac{\overline{\rho}}{\underline{\rho}}\right)^2 \tau(t+v, t'+\phi(v,t,t')) \leq \left(\frac{\overline{\rho}}{\underline{\rho}}\right)^2 \tau(t,t') \leq |dX_{t+v}|$. By the induction hypothesis,

$$0 < a_v \triangleq \Phi^{n-1}\left(t' + \phi(v,t,t'), Z_{t+v} + dz, X'_{t'+\phi(v,t,t')}\right) - \Phi^{n-1}\left(t+v, Z_{t+v}, X_{t+v}\right)$$

But

$$\begin{aligned}
& d(dX_{t+v})/dv \\
=\ & \pi(\Phi^{n-1}(t+v, Z_{t+v}, X_{t+v}) + a_v, X_{t+v} + dX_{t+v})\phi_1(v,t,t') - \pi(\Phi^{n-1}(t+v, Z_{t+v}, X_{t+v}), X_{t+v}) \\
=\ & \begin{pmatrix} \max BR^L(\Phi^{n-1}(t+v, Z_{t+v}, X_{t+v}) + a_v) \cdot (1 - X_{t+v} - dX_{t+v}) \\ -\min BR^R(\Phi^{n-1}(t+v, Z_{t+v}, X_{t+v}) + a_v) \cdot (X_{t+v} + dX_{t+v}) \end{pmatrix} (\phi_1(v,t,t') - 1) \\
& + \max BR^L(\Phi^{n-1}(t+v, Z_{t+v}, X_{t+v}) + a_v) \cdot (1 - X_{t+v} - dX_{t+v}) \\
& - \max BR^L(\Phi^{n-1}(t+v, Z_{t+v}, X_{t+v})) \cdot (1 - X_{t+v}) \\
& - \min BR^R(\Phi^{n-1}(t+v, Z_{t+v}, X_{t+v}) + a_v) \cdot (X_{t+v} + dX_{t+v}) \\
& + \min BR^R(\Phi^{n-1}(t+v, Z_{t+v}, X_{t+v})) \cdot X_{t+v} \\
\geq\ & \begin{pmatrix} \max BR^L(\Phi^{n-1}(t+v, Z_{t+v}, X_{t+v}) + a_v) \cdot (1 - X_{t+v} - dX_{t+v}) \\ -\min BR^R(\Phi^{n-1}(t+v, Z_{t+v}, X_{t+v}) + a_v) \cdot (X_{t+v} + dX_{t+v}) \end{pmatrix} (\phi_1(v,t,t') - 1)
\end{aligned}$$

$$-\left(\begin{array}{c}\max BR^L(\Phi^{n-1}(t+v,Z_{t+v},X_{t+v})+a_v)\\+\min BR^R(\Phi^{n-1}(t+v,Z_{t+v},X_{t+v})+a_v)\end{array}\right)dX_{t+v}$$

By Lemma 3, this is nonnegative if $|\phi_1(v,t,t')-1|\le|dx|$ since by hypothesis $dX_{t+v}\le -|dx|$. By Lemma 4, $|\phi_1(v,t,t')-1|\le\left(\frac{\overline{\rho}}{\underline{\rho}}\right)^2|dt|\le\left(\frac{\overline{\rho}}{\underline{\rho}}\right)^2\tau(t,t')\le|dx|$. This proves that $d\left(dX_{t+v}\right)/dv\ge 0$ and hence (ii) holds for finite $n>0$.

Now consider (i) for finite $n>0$. The relative payoff $\Phi_N^n(t,z,x)$ is computed assuming players believe that, in the future, $\dot{X}_v$ will equal $f^L(\Phi_N^{n-1}(v,Z_v,X_v))(1-X_v)+f^R(\Phi_N^{n-1}(v,Z_v,X_v))X_v$ (equation (9)). By induction and Lemma 8, for $a=R,L$, $f^a(\Phi_N^{n-1}(t,z,x))$ has the properties of $F^a(t,z,x)$ assumed in Lemma 7. Hence, if $dt=0$ and both dx and dz are nonnegative, then $X'_{t'+\phi(v,t,t')}=X'_{t+v}\ge X_{t+v}$. By (36) and the envelope theorem (and taking N large enough), $\Phi_N^n(t,z,x)$ is strictly increasing in z and weakly increasing in x, proving (i) for finite $n>0$.

For (iii) with finite $n>0$, we know by induction that $\Phi_N^{n-1}(t,z,x)\le\Phi_N^{n-2}(t,z,x)$ and that for $a=R,L$, both $f^a(\Phi_N^{n-1}(t,z,x))$ and $f^a(\Phi_N^{n-2}(t,z,x))$ satisfy the assumptions of $F^a(t,z,x)$ in Lemma 5. Hence, $\Phi_N^n(t,z,x)\le\Phi_N^{n-1}(t,z,x)$ by Lemma 6.[21]

For the case $n=\infty$, $f^a(\Phi_N^\infty(t,z,x))=f^a(\lim_{n\to\infty}\Phi_N^n(t,z,x))$ satisfies the properties of F^a in Lemma 5 as each $f^a(\Phi_N^n(t,z,x))$ does for $n<\infty$, and these properties clearly hold in the limit. In particular, c_2 and c_3 are independent of n, so if

$$z'-z>(c_2+c_3\left[|z'|+|z|\right])\cdot(|x'-x|+\tau(t,t'))$$

and $|x'-x|\ge\left(\frac{\overline{\rho}}{\underline{\rho}}\right)^2\tau(t,t')$ then $\Phi_N^\infty(t',z',x')\ge\Phi_N^\infty(t,z,x)$, whence $f^a(\Phi_N^\infty(t',z',x'))\ge f^a(\Phi_N^\infty(t,z,x))$. Thus, by Lemma 7, if $dt=0$ and both dx and dz are nonnegative, then $dX_{t+v}\ge 0$ for all $v\ge 0$. This shows (i). For (ii), if $z'-z>(c_2+c_3N)\cdot(|x'-x|+\tau(t,t'))$ and $|x'-x|\ge\left(\frac{\overline{\rho}}{\underline{\rho}}\right)^2\tau(t,t')$ then there is an $\varepsilon>0$ such that $(z'-\varepsilon)-z>(c_2+c_3N)\cdot(|x'-x|+\tau(t,t'))$, whence $\Phi_N^n(t',z'-\varepsilon,x')>\Phi_N^n(t,z,x)$ for all n, so $\Phi_N^\infty(t',z'-\varepsilon,x')\ge\Phi_N^\infty(t,z,x)$; by part (i), $\Phi_N^\infty(t',z',x')>\Phi_N^\infty(t,z,x)$.

We now show (iv) for $n=0,1,\ldots$. [STILL TO WRITE UP]. Q.E.D.$_{\text{Lemma 9}}$

[21] Evaluate them path-by-path in Z and use the envelope theorem to show that a lower X must lower the relative payoff to playing R; then apply Lemma 6.

Proof of PROPOSITION 1. For the proof, it is more convenient to let h be a declining function of W_t. (Let $\widehat{W_t} = -W_t$ and $\widehat{h}(\widehat{W_t}) = h(-\widehat{W_t}) = h(W_t)$; $\widehat{h}(\widehat{W_t})$ is a declining function. Now rename $\widehat{h}$ and $\widehat{W_t}$ to be h and W_t, respectively.) Let V_v^e and V_v^u be the values of being employed and unemployed at time v, respectively. Ignoring second order terms:

$$V_v^e = \begin{bmatrix} b(e_v)\left(u + EV_{v+dv}^u\right)dv - h(W_v)dv \\ +\left(1 - b(e_v)dv - rdv\right)EV_{v+dv}^e \end{bmatrix}$$

Let $dV_v^e = V_{v+dv}^e - V_v^e$; then

$$EdV_v^e = \begin{bmatrix} h(W_v) - b(e_v)(u + V_v^u) \\ +(b(e_v) + r)V_v^e \end{bmatrix} dv \tag{37}$$

where $E\left[dV_v^e\right]$ is the expectation of $V_{v+dv}^e - V_v^e$ as of time v.

If an agent accepts a proportion $k_t^u \in [0,1]$ of productive opportunities, her average cost for an accepted opportunity is $c^u(k_t^u) = \frac{1}{k_t^u}\int_{c=\underline{c}}^{c_t^*} cdG(c)$, where the threshold c_t^* is defined implicitly by $k_t^u = G(c_t^*)$. An agent chooses k_v^u to maximize the value of unemployment:

$$V_v^u = \left[ak_v^u dv(-c^u(k_v^u) + V_v^e) + \left[1 - ak_v^u dv - rdv\right]EV_{v+dv}^u\right] \tag{38}$$

Thus, k_v^u maximizes

$$k_v^u\left(\nabla_t - c^u(k_v^u)\right) = k_v^u\nabla_t - \int_{c=\underline{c}}^{c_v^*} cdG(c) \tag{39}$$

where where $\nabla_t = V_t^e - V_t^u$ is the relative value of being employed. Since $k_v^u = G(c_v^*)$, the derivative of the right hand side of (39) with respect to k_v^u is $\nabla_t - c_v^*$ so the optimal k_v^u satisfies

$$\text{optimal } k_v^u = \begin{cases} 0 & \text{if } \nabla_v < \underline{c} \\ G(\nabla_v) & \text{if } \nabla_v \in [\underline{c}, \overline{c}] \\ 1 & \text{if } \nabla_v > \overline{c} \end{cases} \tag{40}$$

By (38),

$$E\left[dV_v^u\right] = \left[ak_v^u\left(c^u(k_v^u) - \nabla_v\right) + rV_v^u\right]dv$$

Combining this with (37),

$$Ed\nabla_v = \begin{bmatrix} \left(b(e_v) + r + ak_v^u\right)\nabla_v \\ +h(W_v) - b(e_v)u - ak_v^u c^u(k_v^u) \end{bmatrix} dv$$

As in the proof of Lemma 2, this implies

$$\nabla_t = E \int_{v=t}^{\infty} \exp\left(- \int_{s=t}^{v} \left[r + b(e_v) + ak_s^u \right] ds \right) \left[\begin{array}{c} b(e_v)u - h(W_v) \\ +ak_v^u c^u(k_v^u) \end{array} \right] dv \tag{41}$$

and if all agents choose the same k_t^u, $\dot{e}_t = -b(e_t)e_t + ak_t^u(1-e_t)$. The comparable equations in our model are

$$V_t^R - V_t^L = E \int_{v=t}^{\infty} \exp\left(- \int_{s=t}^{v} \left[r + k_s^R + k_s^L \right] ds \right) \left[\begin{array}{c} u(R, W_v, X_v) - u(L, W_v, X_v) \\ +c^L(k_v^L, W_v, X_v) \\ -c^R(k_v^R, W_v, X_v) \end{array} \right] dv$$

(by Lemma 2) and $\dot{X}_t = -k_t^L X_t + k_t^R(1-X_t)$. To make these equivalent, we associate R with employment and L with unemployment and associate X_t with e_t, k_t^L with ak_t^u, k_t^R with $b(X_t)$; we let $\underline{K}^R(X_t) = \overline{K}^R(X_t) = b(X_t)$ and $[\underline{K}^L(X_t), \overline{K}^L(X_t)] = [0, a]$. Furthermore, let $u(R, W_t, X_t)$ equal $b(X_t)u - h(W_t)$ and $u(L, W_t, X_t) = 0$. Let $c^R(k_t^R, W_t, X_t) = 0$ as searching for a trading partner is costless; and $c^L(k_t^L, W_t, X_t) = k_t^L c^u(k_t^L/a)$.

We now verify that the assumptions of our model hold (section **??**). $\underline{K}^R$ and $\overline{K}^R$ are nondecreasing since $b' > 0$; $\underline{K}^L$ and $\overline{K}^L$ are constant. These functions are all bounded by $K = \max\{b(1), a\}$. The derivative of $c^L(k_t^L, W_t, X_t) = a \int_{c=\underline{c}}^{c_t^*} c dG(c)$ w.r.t. c_t^* is $c_t^* G'(c_t^*)$; since $k_t^L = aG(c_t^*)$, the derivative of c^L w.r.t. k_t^L is c_t^*/a, which proves that c^L is continuous in k^L. c^L clearly takes values in $[0, \infty]$, and satisfies $c^L(0, X_t) = 0$.

The static payoff difference $\Delta = b(X_t)u - h(W_t) + k_t^L c^u(k_t^L/a)$ is nondecreasing and Lipschitz in X_t since $b()$ has these properties. Δ is also increasing and Lipschitz in W_t as $-h(W_t)$ has these features.

We now show that $k_t^L = 0$ is dominant if W_t is low enough. By part 2 of Lemma 2,

$$\nabla_t \leq E \int_{v=t}^{\infty} e^{-r(v-t)} \left[b(X_v)u - h(W_v) \right] dv \tag{42}$$

Since $\lim_{W_t \to -\infty} h(W_t) > b(1)u$, for low enough W_t the right hand expression is less than $\underline{c}$, so $k_t^L = 0$ is dominant by (40).[22] By (41),

$$\frac{\partial \nabla_t}{\partial b(e_{\widehat{v}})} = \exp\left(- \int_{s=t}^{\widehat{v}} (r + b(e_s) + ak_s^u) ds \right) (u - \nabla_{\widehat{v}}) > 0$$

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[22]This uses the fact from lemma 1 that as of time t, W_v is normal with mean $\exp\left(\int_{s=t}^{v} a_s ds\right) W_t + \int_{s=t}^{v} b_s \exp\left(\int_{s'=s}^{v} a_{s'} ds'\right) ds$, which for $W_t < 0$ is no greater than $e^{-N_2} W_t + N_2 e^{-N_2}(v-t)$. The

since immediate consumption maximizes an employed agent's continuation payoff because of discounting and inventory costs. Thus,

$$\nabla_t \geq E \int_{v=t}^{\infty} e^{-(r+b(0)+a)(v-t)} \left[b(0)u - h(W_v)\right] dv$$

Since $\lim_{W_t \to +\infty} h(W_t) < b(0)u - \overline{c}(r + b(0) + a)$, the right hand side is strictly greater than $\overline{c}$ for high enough W_t, so $k_t^L = 1$ is dominant by (41).

Since this model is payoff-equivalent to a particular instance our model, Theorem 1 applies. Q.E.D.$_{\text{Proposition 1}}$

Proof of PROPOSITION 2. Let V_t^f and V_t^u be the value to a firm of a filled and unfilled vacancy, respectively. In the period $[t, t + dt]$ a filled vacancy produces profits $f(1 - \omega)G(n_t, W_t)dt$; with probability δdt it is vacated. Thus,

$$V_t^f = f(1 - \omega)G(n_t, W_t)dt + \delta V_t^u dt + (1 - \delta dt - \beta dt)EV_{t+dt}^f$$

implying that the expected change in this value must be

$$EdV_t^f \triangleq E(V_{t+dt}^f - V_t^f) = \left[-f(1 - \omega)G(n_t, W_t) + \delta\nabla_t + \beta V_t^f\right] dt$$

where $\nabla_t = V_t^f - V_t^u$ is the relative value of a filled vacancy. Define $\widehat{\theta}_t = \theta_t(1 - n_t)$; in the period $[t, t + dt]$, a firm with an unfilled vacancy pays the advertising cost $\frac{c\widehat{\theta}_t}{1-n_t}dt$ and fills its vacancy with probability $\widehat{\theta}_t dt$; thus,

$$V_t^u = -\frac{c\widehat{\theta}_t}{1 - n_t}dt + \widehat{\theta}_t V_t^f dt + \left(1 - \widehat{\theta}_t dt - \beta dt\right) EV_{t+dt}^u \qquad (43)$$

so that

$$EdV_t^u \triangleq E(V_{t+dt}^u - V_t^u) = \left[\frac{c\widehat{\theta}_t}{1 - n_t} - \widehat{\theta}_t\nabla_t + \beta V_t^u\right] dt$$

Hence,

$$Ed\nabla_t = \left[-f(1 - \omega)G(n_t, W_t) - \frac{c\widehat{\theta}_t}{1 - n_t} + \left(\delta + \beta + \widehat{\theta}_t\right)\nabla_t\right] dt$$

variance of W_v given W_t is $\int_{s=t}^{v} \exp\left(2\int_{s'=s}^{v} a_{s'}ds'\right)\sigma_s^2 ds$ which is no greater than $N_2^2 e^{N_2}(v - t)$. Thus, by choosing low enough W_t, we can guarantee that W_v will be below any given threshold with arbitrarily high probability for an arbitrarily long (finite) time.

As in the proof of Lemma 2, this implies

$$\nabla_t = E \int_{v=t}^{\infty} \exp\left(- \int_{s=t}^{v} \left[\delta + \beta + \widehat{\theta}_s \right] ds \right) \left[f(1-\omega)G(n_v, W_v) + \frac{c\widehat{\theta}_v}{1-n_v} \right] dv \tag{44}$$

and if all firms choose the same $\widehat{\theta}_t$, $\dot{n}_t = -\delta n_t + \widehat{\theta}_t(1-n_t)$. The comparable equations in our model are

$$V_t^R - V_t^L = E \int_{v=t}^{\infty} \exp\left(- \int_{s=t}^{v} \left[r + k_s^R + k_s^L \right] ds \right) \left[\begin{array}{c} u(R, W_v, X_v) - u(L, W_v, X_v) \\ + c^L(k_v^L, W_v, X_v) \\ - c^R(k_v^R, W_v, X_v) \end{array} \right] dv$$

(by Lemma 2) and $\dot{X}_t = -k_t^R X_t + k_t^L(1-X_t)$. To make these equivalent, we associate R with a filled vacancy, L with an unfilled vacancy, X_t with n_t, $u(R, W_t, X_t)$ with $f(1-\omega)G(n_t, W_t)$ (the profit flow from a filled vacancy), $u(L, W_t, X_t)$ with zero, k_t^R with δ, k_t^L with $\widehat{\theta}_t$, c^R with zero, and $c^L(k_t^L, W_t, X_t)$ with $\frac{ck_t^L}{1-X_t}\left(= \frac{c\widehat{\theta}_t}{1-n_t}\right)$. We let $\underline{K}^R(X_t) = \overline{K}^R(X_t) = \delta$ and $\underline{K}^L(X_t) = 0$, and $\overline{K}^L(X_t) = (1-X_t)h$.

We now verify that the assumptions of our model are satisfied. $\underline{K}^R$ and $\overline{K}^R$ are clearly nondecreasing functions while $\underline{K}^L$ and $\overline{K}^L$ are nonincreasing. They are all bounded by $K = \max\{h, \delta\}$. For $X_t < 1$, c^L is left-continuous in k_t^L and takes values in $[0, \infty]$, and satisfies $c^a(\underline{K}(X_t), W_t, X_t) = 0$. We must also address the case in which X_t is close to or equal to 1, which makes c^L non-Lipschitz in X_t and discontinuous at zero in k_t^L. Since $k_t^L \leq h$, $\dot{X}_t \leq -\delta X_t + h(1-X_t)$, so

$$X_t \leq \frac{h}{h+\delta} + \left(X_0 - \frac{h}{h+\delta} \right) e^{-(h+\delta)t} \leq e^{-(h+\delta)t} + \frac{h}{h+\delta}\left(1 - e^{-(h+\delta)t}\right)$$

The right hand side equals 1 when $t = 0$ and is strictly decreasing. Thus, for small $\varepsilon > 0$ consider the subgame from time ε onwards. In this subgame $X_t \leq \overline{x}$ for $\overline{x} < 0$. Our results will imply that the equilibrium is unique from time ε onwards. By letting $\varepsilon \to 0$ this implies uniqueness at all times $t > 0$.

The function $\Delta = f(1-\omega)G(X_t, W_t) + \frac{ck_t^L}{1-X_t}$ is increasing and Lipschitz in both W_t and X_t (for $t > \varepsilon$) as G has this property. By (43), a rational firm will choose k_t^L to maximize $k_t^L \left[\nabla_t - \frac{c}{1-X_t}\right]$, so

$$\text{optimal } k_t^L = \begin{cases} 0 & \text{if } \nabla_t < \frac{c}{1-X_t} \\ \text{anything in } [0,1] & \text{if } \nabla_t = \frac{c}{1-X_t} \\ 1 & \text{if } \nabla_t > \frac{c}{1-X_t} \end{cases} \tag{45}$$

We first show that $k_t^L = 0$ is dominant if W_t is low enough. By part 2 of Lemma 2,

$$\nabla_t \leq E \int_{v=t}^{\infty} e^{-r(v-t)} f(1-\omega) G(\overline{x}, W_v) dv \tag{46}$$

Since $\lim_{W_t \to -\infty} G(\overline{x}, W_t) = 0$, for low enough W_t the right hand expression is less than c, so $k_t^L = 0$ is dominant.[23] By (44),

$$\nabla_t \geq E \int_{v=t}^{\infty} e^{-(\delta+\beta+h)(v-t)} f(1-\omega) G(n_v, W_v) dv$$

Since $\lim_{W_t \to +\infty} G(\overline{x}, W_t) = \infty$, the right hand side is greater than $c/(1-\overline{x})$ for high enough W_t, so $k_t^L = 1$ is dominant.

Since this model is payoff-equivalent to a particular instance our model, Theorem 1 applies. Q.E.D.$_{\text{Proposition 2.}}$

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[23] See footnote 22.

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