Why Are Asset Returns More Volatile during Recessions? A Theoretical Explanation

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Abstract

During recession, many macroeconomic variables display higher levels of volatility. We show how introducing an AR(1)-ARCH(1) driving process into the canonical Lucas consumption CAPM framework can account for the empirically observed greater volatilty of asset returns during recessions. In particular, agents' joint forecasting of levels and time-varying second moments transforms symmetric-volatility driving processes into asymmetric-volatility endogenous variables. Moreover, numerical examples show that the model can indeed account for the degree of cyclical variation in both bond and equity returns in the U.S. data. Finally, we argue that the underlying mechanism is not specific to financial markets, and has the potential to explain cyclical variation in the volatilities of a wide variety of macroeconomic variables.

1 Introduction

When it rains, it pours. In times of recession, not only are many macroeconomic variables in bad shape with respect to their levels, but they are also plagued by higher levels of volatility. But what is it about recessions that seems to exacerbate volatility? In order to address this question, a framework must allow not only the first but also the second moments of relevant processes to change over time. In such a framework, agents need to base their optimal decisions upon forecasts both of future levels and of future variances. The main contribution of this paper is to show that agents' optimal joint forecasting of levels and variances leads to volatility which is asymmetric over the business cycle. In more technical terms, symmetric-volatility forcing processes are transformed into asymmetric-volatility endogenous variables. High-volatility recessions and low-volatility expansions may emerge endogenously due to agents' optimal behavior when forecasting levels and variances jointly.

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What is it about forecasting variances and levels jointly that produces asymmetries? In the ARCH-forecasting framework developed in this paper, agents use observations on the innovations u_t to forecast the level of some process, while using u_t^2 to forecast its variance. This means that each realization of the shock carries two messages: one on the level and another on the variance. These two pieces of news may reinforce one another, but they can also contradict one another. For example, a large negative shock will hold two pieces of bad news: bad news on the level due to $u_t < 0$ and bad news on the variance due to u_t^2 large. Thus, any dismay about the bad news of a large negative shock will be amplified: *when it rains, it pours.* A large and positive shock, on the other hand, will carry both good news and bad news. Any exuberance about the good news on the level $u_t > 0$ will be dampened by the bad news about the variance due to u_t^2 large. Thus, agents' reactions to large positive shocks and large negative shocks will be asymmetric, which in turn generates asymmetries in endogenous variables. These asymmetries turn out to be more important for the volatilities than for the levels of the variables.

The greater part of the paper will be devoted to studying the *when it rains it pours* mechanism in the context of a consumption CAPM model, of the kind first introduced by Lucas (1978). Despite its well known failings, the consumption CAPM has one important advantage: simplicity. Although the CCAPM is a dynamic general equilibrium model, it is possible to find closed form solutions for some types of asset returns, namely bond returns. By studying these closed form solutions it will be possible to gain some insight into how the *when it rains it pours* mechanism works. Moreover, the CCAPM is flexible enough to generate a wide variety of asset returns, providing an opportunity to compare the workings of the mechanism in bond and equity returns. The final part of the paper is then devoted to a numerical exercise, in order to determine whether the volatility asymmetries generated by the *when it rains it pours* mechanism are empirically relevant and quantitatively significant for reasonable parameter values. It turns out that the degree of countercyclical heteroscedasticity in both bond and equity returns is indeed quantitatively significant and quite similar to that found in the data.

1.1 GARCH Processes

Clearly, the *when it rains it pours* mechanism depends crucially on the use of squared residuals u_t^2 in variance forecasting. The most direct way to induce agents to use squared residuals in their optimal forecasts is to assume that innovations are governed by ARCH or GARCH processes. The ARCH specification, introduced by Engle (1982) and generalized by Bollerslev (1986) to GARCH, have been some of the most popular approaches to modelling time-varying second moments. In an ARCH(q) process, next period's conditional variance σ_{t+1}^2 is a linear and stochastic function of q lagged squared residuals $\left(u_t^2, u_{t-1}^2, ...u_{t-q}^2\right)$. The GARCH(p,q) specification adds linear dependence on p previous variances $\left(\sigma_t^2, \sigma_{t-1}^2, ...\sigma_{t-p}^2\right)$. Such a specification is said to include p GARCH terms and q ARCH terms.

Literally hundreds of papers have documented the empirical success of GARCH specifications, especially in modelling volatility in financial markets. (For a survey see Bollerslev, Chou and Kroner (1992).) GARCH has also been employed quite widely in modeling the variance of non-financial variables. For example, Gallant and Tauchen (1989) find evidence of GARCH-type volatility in the aggregate consumption process. This latter finding motivates the assumption in this paper that variance of consumption growth follows a (G)ARCH process.[1] That is, since (G)ARCH specifications do very well at representing the *empirical* properties of variance processes, it seems natural to integrate them into *theoretical* models as well.

In contrast to the voluminous body of empirical literature, relatively few theoretical models take time-varying second-moments into account. Notable exceptions are Kandel and Stambaugh (1990), Canova and Marrinan (1991,1993), and Bollerslev, Engle and Wooldridge (1988). Canova and Marrinan introduce time varying-volatility by means of GARCH innovations to the money supply and government expenditure functions into an ICCAP model, which they then use to study exchange rate volatility and the term structure of interest rates. Bollerslev, Engle and Wooldridge (1988) introduce time-varying covariances into a CAPM model. Closest in approach to the present model is the work of Kandel and Stambaugh (1990): they introduce second moments of consumption growth which follow a simple autoregressive process into a consumption CAPM framework, and examine the ability of such models to generate large equity premia. In contrast, here we concentrate on variances rather than levels, and upon (G)ARCH second moments.

1.2 Asymmetric Volatility

The main objective of this paper is to demonstrate the ability of ARCH-forecasting to explain the presence of asymmetric volatility in asset returns. The greater volatility of asset returns during recessions was first noted by Officer (1973). Schwert (1989) presents further evidence that equity and short-term bond returns are more volatile during recessions. In particular, Schwert (1989) reports estimates that monthly equity returns were 68% more volatile during recessions than during expansions in the post-war U.S. data (1953-1987). Over the same period, monthly short-term bond returns were estimated to be 134% more volatile. Such countercyclical heteroscedasticity also seems to be a property of other kinds of economic variables: Schwert (1989) also presents evidence that production growth rates are more volatile during recessions. Further, Heaton and Lucas (1996), using data from the PSID, find that income shocks are more volatile during recessions than during expansions. To our knowledge, no theoretical explanation has been proposed for CCH in any of these variables, with the exception of equity returns.

[1] The theoretical part of the paper will assume, for the sake of tractability, that consumption growth follows an ARCH(1) process. We have, however, also examined a numerical example with a GARCH(1,1) consumption process, which gives qualitatively similar (and somewhat stronger) results. Details are available upon request.

For equity returns, two explanations for CCH have been advanced. The most prominent explanation is the "leverage effect", originally due to Black (1976), for which Schwert (1989) provides some empirical support. During economic contractions, an asset's total value declines, so that the proportion of its value which is levered increases. More highly levered assets are riskier, so the leverage effect leads to equity returns which are more volatile during recessions. However, leverage is not of much help for fully-levered assets, most notably bonds, whose returns also display countercyclical heteroscedasticity. Leverage is of even less help in explaining CCH in more general macroeconomic variables, such as production growth. Thus, it seems that a deeper mechanism is needed, one which is capable of generating asymmetries in volatility over the business cycle in a wider range of variables.

The first objective of this paper is to describe such a deeper mechanism, one which is based upon agents' joint forecasting of levels and variances of relevant driving processes in a dynamic general equilibrium framework. The mechanism is based upon the idea that it is the *sign* of an innovation which determines whether it carries good or bad news on the level, but the *magnitude* which determines whether news on the variance is good or bad. Since sign and magnitude need not coincide, we obtain a richer set of implications for the equilibrium dynamics of endogenous variables. Among these implications is asymmetric volatility in endogenous variables over the business cycle.

To our knowledge, the only other formal model analyzing a similar mechanism is that of Campbell and Hentschel (1992). They develop a *volatility feedback* mechanism which is similar to the *when it rains it pours* mechanism presented in this paper. In the *volatility feedback* mechanism, time-varying second moments also serve to amplify equity returns' reactions to negative innovations in dividends, helping to account for the empirically observed correlation between negative innovations and volatility of equity returns. As the name suggests, it is a feedback machanism: its focus is upon the effects of current innovations to dividends on current equity return volatility. Further, *volatility feedback* operates within an empirical (non-equilibrium) framework. In particular, it is based upon a log-linear approximation to the ex definition relationship between returns, prices and dividends, better known as the present-value dividend model of Campbell and Shiller (1988a,b). The present-value dividend approach is based only upon the definition of asset returns $R_{t+1} = \frac{P_{t+1}+D_{t+1}}{P_t}$, where P_{t+1} is the ex dividend asset price and D_{t+1} is the dividend, both at date $t+1$.

In contrast, we are interested in forecasting in dynamic general equilibrium. General equilibrium places stronger restrictions on the relationship between returns, prices and dividends. Not only must the present-value dividend relationship of the *volatility feedback* mechanism continue to hold, but the returns must also be consistent with agents' risk preferences, in conjunction with their expectations on the stochastic dividend process. Given the fundamental importance of agents' risk preferences in determining their reactions to volatility, there is reason to believe that these restrictions may indeed be important, both theoretically and quantitatively. Moreover, *when it rains it pours* differs from *volatility feedback* in its timing. While Campbell and Hentschel (1992) stress the simultaneous effects of innovations, the focus in this paper is on the effects of innovations on agents' *forecasting*, and thus upon future asset return volatility. In this sense, the two approaches are complementary, and will turn out to deliver complementary results.

Our approach is to incorporate forecasting of (G)ARCH variances into a dynamic general equilibrium model with arbitrary risk aversion, which allows for general types of assets. Both the precise nature of the underlying asset, the degree of risk aversion, and the timing turn out to be quite important in determining whether returns will react more strongly to negative or to positive innovations. We show that equilibrium bond returns display countercyclical heteroscedasticity at *all* levels of risk aversion under quite general conditions[2]. Furthermore, CCH in bond returns generated by the *when it rains it pours* mechanism turn out to be quantitatively significant and empirically relevant. If agents have low to moderate levels of relative risk aversion, then the model is able to match reasonably well the 29% by which short-term bond returns were estimated by Schwert (1989) to be more volatile during recessions over the last century.

Moreover, our results are complementary to those of Campbell and Hentschel (1992) for equity returns. In our framework, next-period equity returns turn out to be more volatile during recession, but only at low to moderate degrees of risk aversion. When parameters are chosen to match U.S. monthly consumption data[3], simulated equity returns turn out to be more than twice as volatile during recessions as during expansions. However, the degree of countercyclical heteroscedasticity in equity returns is decreasing in risk aversion. The behavior of *when it rains it pours* in conjunction with risk aversion turns out to very important in interpreting the results, and will be discussed at length in Section 5.

The remainder of the paper is organized as follows: Section 2 presents the general framework and derives equilibrium asset returns with ARCH(1) variance forecasting. Section 3 derives closed form solutions for ARCH(1) bond returns and discusses some general properties of bond returns when variances are time-varying. In Section 4 the focus is upon ARCH(1) bond return *volatility*: we first link volatility to innovations, and then innovations to recessions, in order to examine the relationship between volatility and the business cycle more closely. Section 5 presents simulation results on bond and equity returns from both ARCH and constant variance models calibrated to U.S. data. Section 6 discusses extensions of the results to non-financial variables, and concludes.

[2]Briefly, the conditions sufficient for CCH in bond returns are that consumption growth is growing sufficiently and that volatilities are time-varying and positively serially correlated.

[3]See Section 5 for a discussion of the use of monthly versus quarterly data. All simulations have also been performed for the calibration to quarterly data presented in Appendix B.1. Quarterly results do *not* vary in any significant way, and are available upon request.

2 Consumption CAPM with ARCH(1) Variance

In this section, we introduce symmetric heteroscedasticity in the driving process into a consumption-CAPM model of the type first described in Lucas (1978). This provides a simple dynamic general equilibrium framework in which to test the ability of joint variance and level forecasting to generate asymmetries in the volatility of endogenous variables. In consumption-CAPM models, these endogenous variables are the returns on claims to aggregate consumption (equity returns) and the returns to one-period bonds. We will describe how equilibrium asset returns are related to the symmetrically heteroscedastic consumption growth process. This relationship will provide a precise basis for the discussion on asymmetric heteroscedasticity in endogenous variables in the sections to follow.

2.1 Asset Returns in an Exchange Economy

Consider a simple dynamic general equilibrium model of the type first introduced by Lucas (1978). Agents choose streams of consumption and asset holdings $\{c_t, z_t\}_t$ to maximize the discounted sum of future utilities, given the stochastic process for endowments $\{y_t\}_t$. Formally, they solve

$$\max \sum_{t=0}^{\infty} \beta^t u(c_t) \tag{1}$$

subject to the resource constraint

$$\begin{array}{cccc} y_t + R_{t+1} z_t & = & c_t + z_{t+1} & t = 0, 1, ... \\ & \{y_t\}_t & c_{-1}, z_{-1} & \text{given} \end{array} \tag{2}$$

where R_{t+1} is the gross return on the asset z_t. The asset is in zero net supply, so that in equilibrium $z_t = 0$. The equilibrium solution to this optimization problem takes the form of an Euler equation, which may be written as[4]

$$1 = \beta E_t \left\{ \frac{u'(c_{t+1})}{u'(c_t)} R_{t+1} \right\} \tag{3}$$

Assuming power utility $u(c_t) = \frac{1}{1-\gamma} c_t^{1-\gamma}$, the Euler equation may be expressed in terms of the consumption growth rate $x_{t+1} \equiv \frac{c_{t+1}}{c_t}$ as:

$$1 = \beta E_t \left\{ x_{t+1}^{-\gamma} R_{t+1} \right\} \tag{4}$$

where γ represents the coefficient of relative risk aversion.

One can now apply the Euler equation (4) for general asset returns in an Lucas exchange economy to two types of assets. We follow the tradition in the Macrofinance literature, and focus on the returns to equity and to one-period bonds.

[4] This Euler equation for general assets was first derived by Grossman and Shiller (1981).

2.1.1 Equity Returns

In the CCAPM, equity is defined as a claim to consumption and saving is ruled out, hence in equilibrium the dividend d_t is equal to aggregate (per capita) consumption c_t. Its gross return may be expressed as $R_{t+1} = \frac{p_{t+1}+d_{t+1}}{p_t}$. Substituting into equation (4) yields

$$p_t = \beta E_t \left\{ x_{t+1}^{-\gamma} \left\{ p_{t+1} + d_{t+1} \right\} \right\}$$

In a growing economy, dividends d_t and prices p_t are non-stationary. Under balanced growth, however, these variables grow at the same average rate x_t, so that the price-dividend ratio $\frac{p_t}{d_t}$ is stationary. Thus, it is helpful to write the Euler equation for a claim to consumption in terms of stationary variables as

$$\frac{p_t}{d_t} = \beta E_t \left\{ x_{t+1}^{1-\gamma} \left(\frac{p_{t+1}}{d_{t+1}} + 1 \right) \right\} \tag{5}$$

The sequence of price-dividend ratios $\left\{ \frac{p_t}{d_t} \right\}_t$ satisfying equation (5) may be approximated using the parameterized expectations approach of Marcet and Marshall (1994) [See Appendix B.3 for details.]. Once one has obtained the equilibrium price-dividend sequence $\left\{ \frac{p_t}{d_t} \right\}_t$, equilibrium returns may be recovered as

$$R_{t+1} = x_{t+1} \frac{p_{t+1}/d_{t+1} + 1}{p_t/d_t} \tag{6}$$

2.1.2 Bond Returns

One-period bonds may be represented as claims to an asset paying a dividend of one unit of the consumption good ($d_t = 1$) which mature at $t+1$ ($p_{t+1} = 0$). Substituting into equation (4) yields the following expression for the price of the one-period bond

$$q_t = \beta E_t \left\{ x_{t+1}^{-\gamma} \right\} \tag{7}$$

Under power utility and log-normally distributed consumption growth rates x_{t+1}, Hansen and Singleton (1983) show that it is possible to find analytical solutions for prices and returns on one-period bonds. Since the gross return on a one-period bond is $R_{t+1} = \frac{1}{q_t}$, and thus $r_{t+1}^f = \log R_{t+1} = -\log q_t$, one may use equation (7) above and write:

$$r_{t+1}^f = -\log\beta + \underbrace{\gamma E_t \log x_{t+1}}_{\text{smoothing term}} - \underbrace{\frac{\gamma^2}{2} var_t \log x_{t+1}}_{\text{precautionary term}} \tag{8}$$

The smoothing term reflects the fact that agents wish to borrow against future consumption growth, placing upward pressure on interest rates. The precautionary term, on the other hand, captures the effects of consumption volatility.

The more volatile the future growth rate, the more agents wish to insure themselves by means of precautionary savings. Increased demand for savings will place upward pressure on the interest rate.

Equations (5) and (8) define a relationship between the expected growth rate of consumption x_{t+1} and the equilibrium returns of equity and one-period bonds. Thus, the precise nature of the process governing the consumption growth rate plays a crucial role in determining the properties of the equilibrium returns. The next section describes the driving process for $\log x_{t+1}$ assumed here, which incorporates time-varying second moments.

2.2 Incorporating AR(1)-ARCH(1) Consumption Growth

It is at this point that our framework diverges from the canonical Lucas/Mehra-Prescott model. The differing element is the introduction of time-varying second moments in the driving process. In particular, we incorporate symmetric heteroscedasticity in the innovation u_t by means of an ARCH(1) specification.

To see how this works, suppose that the consumption growth rate $\log x_{t+1} \equiv \log\left(\frac{c_{t+1}}{c_t}\right)$ follows an AR(1) process

$$\log x_{t+1} = c + \rho \log x_t + u_{t+1} \qquad |\rho| < 1 \tag{9}$$

This is a standard approach to modeling consumption growth in the consumption-based asset pricing literature. Now we incorporate heteroscedasticity in the innovation u_t by means of an ARCH(1) process. Innovations which are $u_t \sim ARCH(1)$ may be described as:

$$\begin{aligned} u_{t+1} &\sim N\left(0, \sigma_{t+1}^2\right) \\ \sigma_{t+1} &= \sqrt{\xi + \alpha u_t^2} \cdot v_{t+1} \qquad \text{where } v_{t+1} \sim i.i.d.N(0,1) \end{aligned}$$

The ARCH(1) specification has the convenient property that the conditional expectation of the variance is linear in the lagged squared innovation u_t^2:

$$E_t \sigma_{t+1}^2 = \xi + \alpha u_t^2 \tag{10}$$

Heteroscedasticity in the innovations induces heteroscedasticity in the consumption growth rates $\log x_{t+1}$. Moreover, this heteroscedasticity is *symmetric* over the business cycle: consumption growth is just as volatile in recessions as in expansions. In particular, the consumption growth rate is symmetrically and conditionally log-normally distributed with moments:

$$E_t \log x_{t+1} = c + \rho \log x_t \tag{11}$$

$$var_t \log x_{t+1} = \xi + \alpha u_t^2 \tag{12}$$

Thus, each innovation u_t has an impact on both the expectation of $\log x_{t+1}$ (via $\log x_t$), and upon its variance. A large negative innovation will cause agents to expect future consumption growth to be low and volatile: *when it rains it* *pours*. Agents observing a large positive innovation, on the other hand, will expect future consumption growth to be high but volatile. In this latter case, any exuberance about high future growth will be dampened by worries about the economy "overheating" due to increased volatility.

3 Bond Returns with and without ARCH

Now we wish to analyze more carefully the effects of time-varying volatility and "when it rains it pours" variance forecasting. Since bond returns' closed form solutions lend themselves to such careful analysis, we begin by examining bond returns in some detail. The most natural place to start is by comparing the equilibrium bond returns with and without ARCH. For the AR(1)-ARCH(1) framework introduced above, the return to a one-period bond may be obtained by substituting the conditional moments (11) and (12) into the general equation for the bond return (8) to obtain:

$$r^f_{t+1} = -\log\beta + \underbrace{\gamma\left[c + \rho\log x_t\right]}_{\text{smoothing term}} - \underbrace{\frac{\gamma^2}{2}\left[\xi + \alpha u_t^2\right]}_{\text{precautionary term}} \qquad (r^f_{t+1}\text{–ARCH})$$

Similarly, when consumption growth variance is constant and equal to σ_x^2 the bond return may be written as:

$$r^f_{t+1} = -\log\beta + \underbrace{\gamma\left[c + \rho\log x_t\right]}_{\text{smoothing term}} - \underbrace{\frac{\gamma^2}{2}\sigma_x^2}_{\text{precautionary term}} \qquad (r^f_{t+1}\text{–no ARCH})$$

Clearly, the smoothing effect will be identical in both the ARCH and no-ARCH cases. Thus, any differences in the properties of ARCH and no-ARCH bond returns must be due to their differing precautionary terms, and thus to their variances.

Indeed, the precautionary effects will most likely differ. This is due to the fact that the ARCH variance forecasts are varying over time. At times, the ARCH variance forecast $\xi + \alpha u_t^2$ will be greater than its constant unconditional mean σ_x^2, placing additional downward pressure on the riskfree rate via a stronger precautionary effect. At other times, however, the ARCH variance forecast will be lower than average: relatively calm times will weaken the precautionary motive to save, placing additional upward pressure on the riskfree rate.

Figure (1) illustrates this variance-based difference between ARCH and no-ARCH bond returns. The solid line plots ARCH bond returns as a function of u_t, while the dashed line represents bond returns when variance is constant. For large magnitude innovations, the news on the variance is bad, placing downward pressure on the risk-free rate via the precautionary motive, and the ARCH bond returns curve lies below the corresponding constant variance line. For small magnitude innovations, however, the good news on the variance puts upward pressure on the risk-free rate. As a result, the ARCH bond return is greater than its constant-variance counterpart, and the ARCH curve lies above the no-ARCH line.

The quantitatively significant effects of time-varying second moments are not, however, on the bond returns themselves nor on their overall volatility. In fact, overall unconditional moments of bond returns in the ARCH and no ARCH cases are almost indistinguishable. Table 3.1 reports sample mean bond returns for the ARCH and no ARCH calibrations: the greatest difference is one basis point (one-hundredth of a percentage point). Similarly, Table 3.2 below shows that overall bond return volatility is not affected to any great extent by variance-forecasting either. Thus, introducing time-varying second moments to the driving process leads to bond returns whose overall moments are nearly indistinguishable from their constant variance counterparts.

Table 3.1: Average Bond Returns

γ	1	2	4
ARCH	1.26 %	1.34 %	1.68 %
no ARCH	1.26 %	1.34 %	1.67 %

Table 3.2: Bond Return Volatilities

γ	1	2	4
ARCH	0.10%	0.15 %	0.29 %
no ARCH	0.12 %	0.16 %	0.33 %

4 Asymmetric Volatility of ARCH Bond Returns

Although the unconditional moments are scarcely affected by variance-forecasting, variance forecasting *does* have significant and interesting effects on the volatility of bond returns over the business cycle. To see this, take a second look at Figure (1): it illustrates the fact that the ARCH bond return may be expressed as a *quadratic* function of the date t innovation to consumption growth u_t as:

$$r^f_{t+1}(u_t) = k_A + \gamma\rho u_t - \frac{\gamma^2}{2}\alpha u_t^2 \qquad (r^f_{t+1} \text{ - ARCH})$$

$$\frac{dr^f_{t+1}(u_t)}{du_t} = \gamma\rho - \gamma^2\alpha u_t \qquad \left(\frac{dr^f_{t+1}}{du_t} \text{ - ARCH}\right)$$

where all elements of $k_A \equiv -\log\beta + \gamma\left[c(1+\rho) + \rho^2 \log x_{t-1}\right] - \frac{\gamma^2}{2}\xi$ are parameters or constant at date $t-1$.

In contrast, when variance is constant and equal to σ_x^2, as it is in the canonical model, the bond return may be written as a *linear* function of u_t:

$$r^f_{t+1} = k_{nA} + \gamma\rho u_t \qquad (r^f_{t+1} \text{ - no ARCH})$$

$$\frac{dr^f_{t+1}}{du_t} = \gamma\rho \qquad (r^f_{t+1} \text{ - no ARCH})$$

where $k_{nA} = -\log\beta + \gamma\left[c(1+\rho) + \rho^2 \log x_{t-1}\right] - \frac{\gamma^2}{2}\sigma_x^2$.

What does this mean for the reaction of bond returns to positive and negative innovations? Note that for the concave bond return function (13), the slope of the bond return is decreasing in u_t, while for the linear no-ARCH bond return line, the slope is constant. This is illustrated in Figure (2): in the no-ARCH case, the constant slope translates into bond return which react symmetrically to positive and negative innovations. In the ARCH case, however, the decreasing slope translates into stronger reactions to negative innovations than to positive ones of equal magnitude. This is reflected by the fact that the ARCH plot of $\frac{dr_{t+1}^f}{du_t}$ lies above the horizontal no-ARCH line for all negative innovations, while it lies below the horizontal no-ARCH line whenever $u_t > 0$. *Thus, variance-forecasting tends to amplify the effects of negative shocks, while dampening the effects of positive shocks.* This clearly induces a negative correlation between innovations to the consumption growth rate process and bond return volatility, which is considered in greater detail in what follows.

4.1 Negatively Skewed Heteroscedasticity

Negatively Skewed Heteroscedasticity (NSH) captures the idea that negative innovations are associated with greater volatility. Formally, NSH may be expressed by means of bond return variance which is greater conditional on innovations being negative:

$$\underbrace{E\left\{\left(r_{t+1}^f(u_t) - Er^f\right)^2 | u_t < 0\right\}}_{var\left[r_{t+1}^f | u_t < 0\right]} > \underbrace{E\left\{\left(r_{t+1}^f(u_t) - Er^f\right)^2 | u_t \geq 0\right\}}_{var\left[r_{t+1}^f | u_t \geq 0\right]} \tag{13}$$

Equivalently, one may write the above equation in terms of absolute deviations from the unconditional mean as:

$$E\left\{\left|r_{t+1}^f(u_t) - Er^f\right| | u_t < 0\right\} > E\left\{\left|r_{t+1}^f(u_t) - Er^f\right| | u_t \geq 0\right\} \tag{14}$$

4.1.1 No ARCH, no NSH

First, it can be shown that constant-variance bond returns do *not* exhibit skewed heteroscedasticity. That is, in the no-ARCH case symmetrically time-varying second moments to the driving process lead to symmetrically time-varying second moments in the endogenous variable.

Begin by noting that, on average, the innovation needed to induce the unconditional mean bond return is $\widetilde{u}_t = 0$.[5] For constant-variance bond returns, then, the absolute deviation in the bond return from its mean due to an innovation

[5] That is, on average, $\widetilde{u}_t$ is the innovation which satisfies $Er^f = r_{t+1}^f(\widetilde{u}_t)$. See Appendix A.1 for details. Of course, $\widetilde{u}_t = 0$ always induces the conditional mean bond return in the constant variance model.

u_t may be written as

$$s^{NA}(u_t) \equiv \left| r^f_{t+1}(u_t) - Er^f \right| = \left| r^f_{t+1}(u_t) - r^f_{t+1}(0) \right|$$

Thus, $s^{NA}(u_t)$ may also be expressed as the integral:

$$s^{NA}(u_t) = \left| \int_0^{u_t} \frac{dr^f_{t+1}(u)}{du} du \right| \tag{15}$$

The integral (15) yields a particularly convenient graphical representation of $s^{NA}(u_t)$. In Figure (3), one can see that (15) corresponds to the shaded area under the horizontal no-ARCH line between u_t and 0.

To show that bond return volatility is symmetric, we need to show that volatility is equally large conditional on innovations being negative or positive. Thus, we need to show that the average size of the shaded area taken only over negative innovations $u_t < 0$ is just as large as the average size of the shaded area taken only over positive innovations. This is clearly the case. To see this, note that for each pair of positive and negative innovations of equal magnitude (u^+, u^-), the absolute deviations are also equal, as illustrated in Figure 3(a). Formally:

$$s^{nA}(u+) = s^{nA}(u-) \qquad \text{for} \begin{array}{c} u^+ > 0 \\ u^- = -u^+ \end{array} \tag{16}$$

Since u_t is symmetrically distributed about zero, the equal magnitude positive and negative innovations u^+ and u^- are equally likely. Thus, it is easy to see that for $\tilde{u}_t = 0$

$$E\left\{s^{nA}(u_t) | u_t < 0\right\} = E\left\{s^{nA}(u_t) | u_t \geq 0\right\} \tag{17}$$

Finally, the fact that $\tilde{u}_t$ is also symmetrically distributed about zero implies that (17) holds also when expectations are taken over $\tilde{u}_t$. Thus, in the no-ARCH case, positive and negative innovations have *symmetric* effects not only on the bond returns themselves, but also upon their deviations from Er^f, and thus on their volatility.

4.1.2 ARCH and NSH

The ARCH case is somewhat more complex. First consider something close to the absolute deviation of the bond return from its mean, namely the deviation from its $u_t = 0$ value:

$$s^A_0(u_t) = \left| r^f_{t+1}(u_t) - r^f_{t+1}(0) \right|$$

It is easy to see that $s^A_0(u_t)$ corresponds to the vertically striped region under the ARCH line between u_t and 0 in Figure (4). It is also clear that the ARCH deviation will be of greater magnitude for negative innovations. The area under the ARCH line between 0 and u^- is clearly greater than that between 0 and u^+ in Figure 4(a), so that

$$s_0^A\left(u^-\right) > s_0^A\left(u+\right) \qquad \text{for} \quad \begin{array}{c} u^+ > 0 \\ u^- = -u^+ \end{array}$$

Now symmetry of the distribution of u_t about zero imply that the average size of the shaded area under the ARCH line will be greater over all negative shocks:

$$E\left\{s_0^A\left(u_t\right)|u_t < 0\right\} > E\left\{s_0^A\left(u_t\right)|u_t \geq 0\right\} \tag{18}$$

Thus, in the ARCH case, negative innovations have greater effects on s_0^A than do positive innovations, inducing an asymmetry in the reaction of bond returns to positive and negative innovations *when* $\widetilde{u}_t = 0$.

Unfortunately, however, (18) is not necessarily equivalent to asymmetric volatility for the ARCH case. This is due to the fact that in the ARCH case, $u_t = 0$ is *not* the innovation required to induce the unconditional mean bond return Er^f. In fact, when $\rho > 0$, the innovation needed to induce the mean bond return is negative on average, so that $E\widetilde{u}_t < 0$ [see Appendix A.1.2 for a detailed discussion]. Negativity of $E\widetilde{u}_t$ makes it more difficult for NSH to hold. The reason is that any given negative innovation will be closer to a negative $\widetilde{u}_t$. This works *against* the asymmetry due to the negatively sloped ARCH $\frac{dr^f}{du}$ line.

To see this more clearly, recall that $s^A\left(u_t\right)$ may be expressed as the integral:

$$s^A\left(u_t\right) = \left|\int_{\widetilde{u}_t}^{u_t} \frac{dr_{t+1}^f\left(u\right)}{du} du\right| \tag{19}$$

For $\widetilde{u}_t = 0$, the integrals for positive and negative shocks u^+ and u^- are taken over equally sized ranges. Thus, the fact that the integrand is always greater for u^- is sufficient for the u^- -integral to be greater, so that NSH holds. For $\widetilde{u}_t > 0$, negative innovations must travel further to reach $\widetilde{u}_t$, so that the u^-- integral is taken over a greater range than the corresponding u^+-integral, reinforcing NSH. For $\widetilde{u}_t < 0$, however, negative innovations do not have to "travel" as far to reach $\widetilde{u}_t$ as do positive ones. Thus, the range of the u^- integral is smaller, while its integrand is greater, so that it is not certain which of these countervailing influences on $s^A\left(u_t\right)$ will prevail. As long as $\widetilde{u}_t$ is symmetrically distributed about zero, the positive and negative range effects cancel one another out, as was the case with the no-ARCH bond return. When, however, $\widetilde{u}_t$ is more likely to be negative, then it is no longer certain that NSH will prevail.

Table 4.1: Mean-Inducing Innovations and Asymmetric Absolute Deviations

Case	Range Effect	Integrand Effect	Total Effect
$\widetilde{u}_t < 0$	$\lvert\widetilde{u}_t - u^-\rvert < \lvert\widetilde{u}_t - u^+\rvert$	$\frac{dr_{t+1}^f\left(u^-\right)}{du} > \frac{dr_{t+1}^f\left(u^+\right)}{du}$	ambiguous
$\widetilde{u}_t = 0$	$\lvert\widetilde{u}_t - u^-\rvert = \lvert\widetilde{u}_t - u^+\rvert$	$\frac{dr_{t+1}^f\left(u^-\right)}{du} > \frac{dr_{t+1}^f\left(u^+\right)}{du}$	$s^A\left(u^-\right) > s^A\left(u^+\right)$
$\widetilde{u}_t > 0$	$\lvert\widetilde{u}_t - u^-\rvert > \lvert\widetilde{u}_t - u^+\rvert$	$\frac{dr_{t+1}^f\left(u^-\right)}{du} > \frac{dr_{t+1}^f\left(u^+\right)}{du}$	$s^A\left(u^-\right) > s^A\left(u^+\right)$

Loosely speaking, the greater is the slope of the ARCH-line in Figure 4(a), the more often the greater integrand will prevail over the smaller range, and the more likely will NSH hold. Further, the smaller is the magnitude $|E\widetilde{u}_t|$, the smaller will be the differences in the ranges, and thus the easier it is for the greater integrand to prevail and NSH to hold.

4.2 Greater Volatility during Recessions

What we are really interested in, however, is the relationship between *recessions* and bond return volatility. Bond returns which are conditionally counter-cyclically heteroscedastic (CCH) display conditional variances which are greater whenever the innovation u_t is recessionary. Formally, CCH is said to hold whenever

$$E\left\{s^A\left(u_t\right)|u_t \text{ recessionary}\right\} > E\left\{s^A\left(u_t\right)|u_t \text{ expansionary}\right\}$$

To the extent that negative innovations to consumption growth are linked to recessions, one would expect NSH to be linked with CCH. It turns out, however, that in growing economies, CCH is more likely to hold than NSH. That is, as long as the economy is growing at a sufficient rate, returns may be more volatile during recessions than during expansions, even if NSH does not hold. In order to examine this more closely, we must first define more precisely what we mean by recession and expansion.

4.2.1 Recession and Expansion

An innovation u_t is called *recessionary* whenever it causes the growth rate $\log x_t$ to be negative. More precisely: u_t is recessionary whenever $\log x_t = c + \rho \log x_{t-1} + u_t \leq 0$, which translates into a condition on u_t as

$$u_t \in \{U_t^{rec}\} \quad \text{whenever} \quad u_t \leq -c - \rho \log x_{t-1} \equiv \overline{u}_t \tag{20}$$

$$u_t \in \{U_t^{\exp}\} \quad \text{whenever} \quad u_t > -c - \rho \log x_{t-1} \equiv \overline{u}_t \tag{21}$$

Thus, for any given $\log x_{t-1}$ and any AR(1) parameters, the recessionary threshold $\overline{u}_t$ divides the support of innovations into two disjoint subsets. This is illustrated in Figure (5). The subset of recessionary innovations U_t^{rec} is that subset of the real numbers which is lies below the recessionary threshold, while the subset of expansionary innovations $U_t^{\exp}$ is its complement.

Note that the recessionary threshold $\overline{u}_t$ will be shifting over time depending upon last period's growth rate $\log x_{t-1}$. If last period's growth rate was large and positive, then it will take a relatively large negative innovation to throw the economy into recession. If, however, last period's growth rate $\log x_{t-1}$ was already recessionary, then it is possible that even small positive innovations will be sufficient to keep the economy in recession. From equation (20) one obtains that the recessionary threshold will be symmetrically and normally distributed with mean $-\mu_x$ and variance $\rho^2\sigma_x^2$. Thus, the greater the growth trend in the economy, the more strongly negative will the recessionary threshold tend to be.

Mean consumption growth μ_x turns out to play a crucial role for CCH, both in the formal analysis in the next subsection, and in the simulation results of Section 5.

4.2.2 Countercyclical Heteroscedasticity

Now we can make more precise the idea that bond returns are more volatile during recessions than expansions. Formally, bond returns satisfy CCH whenever:

$$E\left\{\left|r_{t+1}^{f}\left(u_{t}\right)-Er^{f}\right|\left|u_{t}\in U_{t}^{rec}\right.\right\}>E\left\{\left|r^{f}\left(u_{t}\right)-Er_{t+1}^{f}\right|\left|u_{t}\in U_{t}^{rec}\right.\right\} \tag{22}$$

The absolute deviation in the bond return at date $t+1$ may be written as a function of the innovation at date t as

$$s_{t+1}^{A}\left(u_{t}\right)=\left|r_{t+1}^{f}\left(u_{t}\right)-r_{t+1}^{f}\left(\widetilde{u}_{t}\right)\right|=\left|\int_{\widetilde{u}_{t}}^{u_{t}}\frac{dr_{t+1}^{f}\left(u\right)}{du}du\right|$$

Once again, whether $s_{t+1}^{A}\left(u^{-}\right)>s_{t+1}^{A}\left(u^{+}\right)$ depends on whether the larger integrand of $s^{A}\left(u^{-}\right)$ outweighs the larger range of $s_{t+1}^{A}\left(u^{+}\right)$.

CCH for $(\overline{u}_t, \widetilde{u}_t)$ First consider a pair of recessionary threshold and mean-inducing innovation $(\overline{u}_t, \widetilde{u}_t)$. Furthermore, assume that both consumption growth is positively serially correlated, and the mean-inducing innovation $\widetilde{u}_t$ for the ARCH bond return is negative $\widetilde{u}_t < 0$.[6] In what follows we first examine under which conditions negative innovations induce a larger deviation in the bond return from its mean than do equal magnitude positive innovations. That is, we wish to find pairs (u^+, u^-) so that:

$$s_{t+1}^{A}\left(u^{-}\right)>s_{t+1}\left(u^{+}\right) \tag{23}$$

Then, in a second step, we will examine when those negative innovations for which (23) holds will also be recessionary ones, creating a link to CCH.

Table 4.2 below summarizes the relationship between the absolute deviations induced by equal magnitude pairs of innovations (u^+, u^-). Begin by noting that for all very small magnitude innovations $|u_t| \leq |\widetilde{u}_t|$, the absolute deviation is larger for u^+, so that $s_{t+1}^{A}(u^-) < s_{t+1}^{A}(u^+)$. This is illustrated in Figure (6): the area under the ARCH line between $\widetilde{u}_t$ and u^- is fully contained within the corresponding area between $\widetilde{u}_t$ and u^+. That is, for all $|u_t| \leq |\widetilde{u}_t|$, the "range effect" outweighs the "integrand effect", and absolute deviations due to u^- are smaller than those due to u^+.

[6] Recall from the discussion above that this is the more difficult case, since it is the one in which range effects arise which work against CCH. Thus, all results carry over a fortiori to $\widetilde{u}_t > 0$.

Moreover, it is easy to see that for innovation pairs with magnitude only slightly greater than $|\widetilde{u}_t|$, the range effect continues to dominate, so that $s_{t+1}^A(u^-) < s_{t+1}^A(u^+)$. (This appeals to a continuity argument.) Further, as long as the ARCH line has negative slope, there exists some innovation $\widetilde{u}_t - \varepsilon = u^-$ such that the range effect and integrand effect cancel one another out, so that $s_{t+1}^A(u^-) = s_{t+1}^A(u^+)$. [The precise value of ε is derived in Appendix A.2.] Taken together, this implies that for sufficiently large magnitude innovations $|u_t| > |\widetilde{u}_t - \varepsilon|$ the integrand effect dominates, so that $s_{t+1}^A(u^-) > s_{t+1}^A(u^+)$. In summary, small magnitude innovations will have procyclical effects on volatility, while large magnitude innovations will have countercyclical effects on volatility.

Table 4.2: Innovations and Absolute Deviations

$\widetilde{u}_t < 0$

Innovation Magnitude		Absolute Deviations in r_{t+1}^f	
Small	$\|u^-\| < \|\widetilde{u}_t - \varepsilon\|$	Smaller for u^-	$s_{t+1}^A(u^-) < s_{t+1}^A(u^+)$
Borderline	$\|u^-\| = \|\widetilde{u}_t - \varepsilon\|$	Equally Sized	$s_{t+1}^A(u^-) = s_{t+1}^A(u^+)$
Large	$\|u^-\| > \|\widetilde{u}_t - \varepsilon\|$	Larger for u^-	$s_{t+1}^A(u^-) > s_{t+1}^A(u^+)$

Now consider the CCH property for given $\widetilde{u}_t$ and given recessionary threshold $\overline{u}_t$: CCH holds for $(\widetilde{u}_t, \overline{u}_t)$ whenever

$$E\left\{s_{t+1}^A(u_t) \mid u_t < \overline{u}_t; \widetilde{u}_t\right\} > E\left\{s_{t+1}^A(u_t) \mid u_t > \overline{u}_t; \widetilde{u}_t\right\} \tag{24}$$

. These expectations may be written as

$$\begin{aligned} E\left\{s_{t+1}^A(u_t) \mid u_t < \overline{u}_t; \widetilde{u}_t\right\} &= \int_{-\infty}^{\overline{u}} s_{t+1}^A(u_t)\, dP(u_t) \\ &= \int_{-\infty}^{\widetilde{u}_t-\varepsilon} s_{t+1}^A(u_t)\, dP(u_t) + \int_{\widetilde{u}_t-\varepsilon}^{\overline{u}} s_{t+1}^A(u_t)\, dP(u_t) \end{aligned}$$

and correspondingly

$$E\left\{s_{t+1}^A(u_t) \mid u_t > \overline{u}_t; \widetilde{u}_t\right\} = \int_{-\widetilde{u}_t+\varepsilon}^{\infty} s_{t+1}^A(u_t)\, dP(u) + \int_{\overline{u}}^{-\widetilde{u}_t+\varepsilon} s_{t+1}^A(u_t)\, dP(u)$$

Using these integrals, one can see that CCH will hold for $(\widetilde{u}_t, \overline{u}_t)$ whenever

$$\begin{aligned} &\left[\int_{-\infty}^{\widetilde{u}_t-\varepsilon} s_{t+1}^A(u_t)\, dP(u_t) - \int_{-\widetilde{u}_t+\varepsilon}^{\infty} s_{t+1}^A(u_t)\, dP(u)\right] \\ > &\left[\int_{\overline{u}}^{-\widetilde{u}_t+\varepsilon} s_{t+1}^A(u_t)\, dP(u) - \int_{\widetilde{u}_t-\varepsilon}^{\overline{u}} s_{t+1}^A(u_t)\, dP(u_t)\right] > 0 \end{aligned} \tag{25}$$

That is, whenever the countercyclical effects on volatility of large magnitude shocks outweigh the procyclical effects of small magnitude shocks, then heteroscedasticity is countercyclical overall, and (25) holds. That is, if the countercyclical effects of "tail" shocks outweigh the procyclical effects in "body" shocks, CCH holds for $(\widetilde{u}_t, \overline{u}_t)$.

Thus, the smaller is the magnitude of the innovation $\widetilde{u}_t - \varepsilon$ at which CCH takes over, the more likely is (25) to hold. More precisely, the smaller is the region $[\widetilde{u}_t - \varepsilon, -\widetilde{u}_t + \varepsilon]$ in which the procyclical range effect dominates, the larger is the region where CCH dominates, and the more likely is CCH to dominate on the whole. The value for ε is derived in Appendix A.2: it is decreasing in γ and α, so that a greater slope -$\gamma^2\alpha$ of the ARCH line makes CCH more likely to hold.[7] Further, ε is decreasing in $\widetilde{u}_t$, the mean-inducing innovation. That is, the closer the mean-inducing innovation is to zero, the smaller is its magnitude, and thus the smaller is the region in which PCH dominates. Put another way, the smaller is $|\widetilde{u}_t|$, the smaller are the range effects, and thus the more easily can the countercyclical integrand effect dominate. As a result, small magnitude $\widetilde{u}_t$ make CCH more likely to hold.

In addition, the placement of the recessionary threshold $\overline{u}_t$ is crucial for CCH. In particular, CCH will certainly hold if the recessionary threshold is equal to $\widetilde{u}_t - \varepsilon$. This is because now the absolute deviations due to recessionary shocks will be greater than those due to *any* equally large expansionary shocks. This follows from the discussion above. To see this, divide the set of negative innovations into recessionary u^-_{rec} and expansionary ones $u^-_{\exp}$. Also, define corresponding large and small magnitude positive innovations as $u^{++} = -u^-_{rec}$ and $u^+ = -u^-_{\exp}$ respectively. Thus, recessionary innovations are u^-_{rec}, while all others are expansionary, as illustrated in Figure 4(b). We proceed in four steps:

1. $\overline{u}_t < \widetilde{u}_t - \varepsilon < 0$ guarantees that for all recessionary u^-_{rec}, $s^A(u^-_{rec}) > s^A(u^{++})$, where $u^{++} = -u^-_{rec}$. Thus, $Es^A(u^-_{rec}) > Es^A(u^{++})$.

2. Now we need to guarantee that adding the small magnitude expansionary innovations u^-_{rec} and u^+ to the expansionary expectation will not counterbalance the effect in 1. Since $s^A(u^+)$ is monotonically increasing in the magnitude of u^+, $u^{++} > u^+$ guarantees that $s^A(u^{++}) > s^A(u^+)$

3. Moreover, for all expansionary $u^-_{\exp}$, $s^A(u^+) > s^A(u^-_{\exp})$. Thus, by 2. and transitivity, $s^A(u^-_{rec}) > s^A(u^-_{\exp})$.

Putting all of these steps together, one obtains that

$$Es^A\left(\underbrace{u^-_{rec}}_{u_t \text{ recessionary}}\right) > Es^A\left(\underbrace{u^{++} \cup u^+ \cup u^-_{\exp}}_{u_t \text{ expansionary}}\right)$$

which means that the expected absolute deviation induced by recessionary innovations is greater than that induced by expansionary innovations. Thus, CCH holds for $(\overline{u}_t, \widetilde{u}_t)$ when $\overline{u}_t < \widetilde{u}_t - \varepsilon$.

Moreover, a fortiori, CCH will also hold if the recessionary threshold is lies below $\widetilde{u}_t - \varepsilon$. Thus, we obtain the following sufficient condition for CCH to hold

[7] Indeed, the sensitivity analysis of the numerical example in the next section reflects these relationships.

for $(\widetilde{u}_t, \overline{u}_t)$

$$\overline{u}_t \leq \widetilde{u}_t \quad \Longrightarrow \quad \text{CCH holds for } (\widetilde{u}_t, \overline{u}_t) \tag{26}$$

CCH overall Finally, recall that both the mean-inducing innovation $\widetilde{u}_t$ and the recessionary threshold $\overline{u}_t$ are functions of $\log x_{t-1}$, and are thus changing over time. That is, equation (26) only ensures that CCH holds for a given pair $(\widetilde{u}_t, \overline{u}_t)$, not for all pairs $(\widetilde{u}_t, \overline{u}_t)$. To ensure that CCH holds overall, one must take expectations over both $\widetilde{u}_t$ and $\overline{u}_t$. Doing this would allow one to find parameter constellations $(c, \rho, \xi, \alpha, \gamma, \beta)$ for which CCH holds. Rather than finding such six-dimensional parameter regions, we discuss briefly which parameter choices are propitious for CCH, and then defer to numerical simulations.

First note that for any given parameter constellation $(\rho, \xi, \alpha, \gamma, \beta)$ the sufficient condition for CCH to hold may be expressed as a lower bound on the economy´s growth unconditional mean growth rate μ_x. This follows directly from (26): since $E\overline{u}_t = -\mu_x$. Loosely, sufficiently large growth rates imply that (26) will hold often enough for CCH to hold overall.

Thus, the factors favoring CCH overall[8] are high mean consumption growth rates μ_x, as well as high coefficients of risk aversion γ, and large positive serial correlation in volatility α. Risk aversion and serial correlation in volatility work together to induce the asymmetric reactions to positive and negative innovations in the first place. If consumption growth is positive, then it takes relatively large magnitude negative shocks to throw the economy into recession. As discussed above, large magnitude negative shocks are precisely the kind which will be amplified by ARCH-forecasting of variance. Thus, to the extent that large magnitude negative shocks coincide with recessionary ones, CCH will hold.

5 Simulation Results

In order to examine whether the degree of cyclical variation in volatilities is quantitatively significant and on the order of magnitude of the empirically observed values we perform a numerical exercise. Bond returns are calculated directly from equation (r^f_{t+1} - ARCH), while equity returns must be obtained by numerical approximation, as detailed in Appendix B.3.

In order to obtain numerical values for asset returns one needs to choose values for six parameters. The four parameters of the AR(1)-ARCH(1) process are pinned down by the data. They are chosen to match unconditional moments in monthly U.S. data on growth in the consumption of non-durables and services.[9] The only free parameters are the discount factor δ and the risk aversion

[8]Recall that we are concentrating the discussion on the case where $\rho > 0$.

[9]Monthly data are used to choose parameters for the consumption growth process, since Schwert (1989)'s estimates are based upon monthly data as well. The monthly consumption data, however, has dubious time series properties (cf. Wilcox (1992)). The parameters which are difficult to pin down are the two serial correlations. Rather than basing the numerical exercise on meaningless estimates, the basic parameter set takes conservative values for α and ρ which work *against* CCH. Moreover, in the sensitivity analysis, we will argue that our results

coefficient γ. We examine three possible values for the degree of relative risk aversion, namely $\gamma \in \{1.5, 2, 4\}$. These choices are quite conservative, and are well within the range generally considered in the macrofinance literature. δ is set to 0.99: since the discount factor is only a scaling parameter with no effect on the variances, there is no point in varying it. For further details on the calibration see Appendix B.

Table 5.1: Countercyclical Heteroscedasticity in Bond Returns ARCH(1) versus no ARCH

γ	$\%\Delta s$ no ARCH	$\%\Delta s$ ARCH(1)
1.5	1 %	32 %
2	1 %	32 %
4	1 %	33 %
data	1859-1987	29 %
	1953-1987	134 %

Results are summarized in Tables 5.1 and 5.2. The tables compare the percentage increase in volatility during recession $\%\Delta s$ in the canonical constant variance and the ARCH(1) models. Clearly, Table 5.1 shows that the constant variance model is *not* able to account for any significant degree of CCH in bond returns. In contrast, the ARCH(1) model matches the estimates from the data quite well. The results are similarly positive for equity returns. From Table 5.2 one can see that it is also the ARCH(1) model which accounts for significantly more CCH in equity returns than the no-ARCH model. The following subsections present these simulation results in considerably greater detail, along with a sensitivity analysis.

Table 5.2: Countercyclical Heteroscedasticity in Equity Returns ARCH(1) versus no ARCH

γ	$\%\Delta s$ no ARCH	$\%\Delta s$ ARCH(1)
1.5	68 %	115 %
2	59 %	94 %
4	45 %	37 %
data	1859-1987	61 %
	1920-1952	234 %
	1953-1987	68 %

are quite robust to variations in the autocorrelations. Finally, all simulations have also been performed for the calibration [presented in Appendix B.1] to quarterly data. These results do *not* vary in any significant way from those of the monthly simulations presented here, and are available upon request.

5.1 Countercyclical Heteroscedasticity in Bond Returns

We find that bond returns are indeed significantly more volatile during recessions than during expansions in the artificial economy. To compare volatility in recessions and expansions, we follow Schwert (1989) in regressing the absolute deviation of the bond return r_{t+1}^f from its unconditional mean on a constant and a contraction dummy variable. $Contr_t$ takes on the value 1 whenever the innovation at t is recessionary $u_t \in U_t^{rec}$, and the value 0 otherwise:

$$\left| r_{t+1}^f (u_t) - E r_{t+1}^f \right| = \beta_1 + \beta_2 \cdot contr_t + \varepsilon_t \tag{27}$$

Clearly, an estimate $\widehat{\beta}_2$ which is positive and significant implies that volatility of the bond return is greater during recessions. Further, the ratio $\widehat{\beta}_2/\widehat{\beta}_1$ provides an estimate of the percentage increase in volatility during recessions over expansions.

The ARCH(1) model *can* account for significant degrees of greater bond return volatility during recession. Simulation results for the ARCH(1) model, calibrated to U.S. monthly data, are presented in Table 5.3.[10] γ gives the coefficient of relative risk aversion, while the first two columns represent OLS estimates of the coefficients of regression equation (27), with corresponding White corrected-for-heteroscedasticity t-values in brackets below. Note that $\widehat{\beta}_1$ is an estimate of the standard deviation of bond returns during expansions s^{exp}, while $\widehat{\beta}_2$ is an estimate of the *increase* in standard deviation during recession. $\widehat{\beta}_2$ is positive and highly significant, reflecting a significant increase in volatility during recession. This can also be seen in the fourth column, which presents estimates of the standard deviation during recession, $s^{rec} = \widehat{\beta}_1 + \widehat{\beta}_2$. The estimate of s^{rec} is clearly greater than its recessionary counterpart.

Table 5.3: Bond Return Volatility in Recession and Expansion ARCH(1) model

	$s^{\exp}$	$s^{rec}-s^{\exp}$	s^{rec}	$\%\Delta s$
γ	$\widehat{\beta}_1$	$\widehat{\beta}_2$	$\widehat{\beta}_1+\widehat{\beta}_2$	$\widehat{\beta}_2/\widehat{\beta}_1$
1.5	9.26×10^{-4}	3.01×10^{-4}	1.22×10^{-3}	32 %
	[42.08]	[6.93]		
2	1.23×10^{-3}	3.90×10^{-4}	1.62×10^{-3}	32 %
	[42.03]	[6.75]		
4	2.47×10^{-3}	8.01×10^{-4}	3.27×10^{-3}	33 %
	[41.57]	[6.82]		

Finally, the last column quantifies the amount by which volatility is greater during recession, by presenting the percentage increase in standard deviations during recession. We take this value $\%\Delta s$ as a measure of the degree of countercyclical heteroscedasticity: the estimate for the ARCH(1) model is about 33%

[10] All simulation results are averages over 40 runs of 5000 periods each.

at all levels of risk aversion considered. Thus, the degree of CCH generated by the when it rains it pours-ARCH(1) model approximates well the 29% estimated by Schwert(1989) for the period 1859-1987.

In contrast, the canonical constant variance model can clearly *not* account for any significant degree of CCH in bond returns. $\widehat{\beta}_2$ is slightly positive but not significant, indicating that there is no significant increase in volatility during recession. This is also reflected in the fact that the estimate of standard deviation during recession, $s^{rec} = \widehat{\beta}_1 + \widehat{\beta}_2$, is not significantly different from its expansionary counterpart $s^{\exp}$. Finally, the degree of CCH ($\%\Delta s$) generated by the canonical constant variance model is only about 1 %. Clearly, without ARCH it is not possible to account for any significant degree of CCH.

Table 5.4: Volatility of Bond Returns in Recession and Expansion
Canonical Constant Variance Model

	$s^{\exp}$	$s^{rec}-s^{\exp}$	s^{rec}	$\%\Delta s$
γ	$\widehat{\beta}_1$	$\widehat{\beta}_2$	$\widehat{\beta}_1 + \widehat{\beta}_2$	$\widehat{\beta}_2/\widehat{\beta}_1$
1.5	1.09×10^{-3}	1.60×10^{-5}	1.11×10^{-3}	1 %
	[43.69]	[0.64]		
2	1.45×10^{-3}	1.50×10^{-5}	1.47×10^{-3}	1 %
	[43.42]	[0.45]		
4	2.90×10^{-3}	4.18×10^{-5}	2.94×10^{-3}	1 %
	[44.88]	[0.52]		

In Table 5.5 we present Schwert's (1989) estimates of CCH in monthly bond returns, to check whether the ARCH(1) model can generate excess volatility in recessions that is on the same order of magnitude as that in the data. Indeed, we find that the ARCH(1) model can generate degrees of CCH which approximate the 29% observed in the US economy over the entire 1859-1987 period. The amount of CCH generated does, however, fall short of matching the 134% by which volatility in short-term interest rates during recessions exceeds that during expansions over the post-war period.[11]

Table 5.5:Increase in Bond Return Volatility during Recession
U.S. monthly data [Source: Schwert (1989)]

period	**1859-1987**	**1859-1919**	**1920-1952**	**1953-1987**
$\%\ \Delta s$	29 %	15 %	16 %	134 %

5.2 Sensitivity Analysis

The sensitivity analysis is motivated by two concerns. First and foremost, the theoretical discussion of Section 4 showed that whether CCH holds or not depends upon parameter values. Secondly, due to well-known problems with serial

[11] Note that Schwert (1989)'s estimates are, however, considerably less significant than those of the simulation data. t-values for the two periods used for comparison here are only about 1.4.

correlation of measurement errors in monthly consumption growth estimates[12], it is safer to treat the serial correlations in consumption growth and volatility as unknown or highly uncertain. Thus, it is important to establish that the CCH results are not reversed or destroyed by some values for ρ and α. Indeed, the only way to damage the CCH result, without doing away with ARCH(1) entirely, is to set ρ equal to zero. Even then, however, the degree of CCH remains significant and on the order of magnitude of the data estimates.

5.2.1 Mean Consumption Growth μ_x

Sensitivity of the CCH results to variations in mean consumption growth μ_x is particularly important, as stressed in the theoretical analysis in Section 4. The greater is the average growth rate of the economy, the more strongly negative a shock must be (on average) in order to throw the economy into recession. Thus, the greater is μ_x, the more likely are recessions to be associated with the precisely the kind of large negative innovations which favor CCH. Table 5.6 illustrates this point: when the economy is not growing, there is no CCH. Furthermore, the degree of CCH is clearly increasing in the unconditional mean growth rate of the economy μ_x, confirming the analysis in Section 4.

Table 5.6: Bond Return Volatility in Recession and Expansion ARCH(1) Calibration with $\gamma = 1.5$ and Varying μ_x

	$s^{\exp}$	$s^{rec} - s^{\exp}$	s^{rec}	$\%\Delta s$
μ_x	$\widehat{\beta}_1$	$\widehat{\beta}_2$	$\widehat{\beta}_1 + \widehat{\beta}_2$	$\widehat{\beta}_2/\widehat{\beta}_1$
0.00	9.97×10^{-4}	7.29×10^{-6}	1.00×10^{-3}	0.8 %
	[41.90]	[0.22]		
0.001575	9.26×10^{-4}	3.01×10^{-4}	1.22×10^{-3}	32 %
	[42.08]	[6.93]		
0.00315	9.48×10^{-4}	3.67×10^{-4}	1.31×10^{-3}	39 %
	[40.60]	[6.00]		
0.0063	9.71×10^{-4}	5.30×10^{-4}	1.44×10^{-3}	55 %
	[37.95]	[4.85]		
1859-1987				29 %

5.2.2 Serial Correlation in Consumption Growth ρ

The serial correlation of consumption growth ρ cannot be reliably estimated from monthly series. However, it turns out that CCH is not particularly sensitive to ρ. This robustness is reflected in Table 5.7, which presents simulation results for serial correlations values ranging from -0.48 to 0.48. The only way to endanger the CCH results is to set serial correlation equal to zero: even then, the increase in volatility during recession $\widehat{\beta}_2$ remains significant at the 5 % level. Thus, as long as the true monthly consumption growth rate series does show

[12] See Wilcox (1992) for a thorough discussion.

some amount of serial correlation, we can be quite confident in the CCH results for bond returns.

Table 5.7: Bond Return Volatility in Recession and Expansion ARCH(1) Calibration with $\gamma = 1.5$ and Varying ρ

	$s^{\exp}$	$s^{rec} - s^{\exp}$	s^{rec}	$\%\Delta s$
ρ	$\widehat{\beta}_1$	$\widehat{\beta}_2$	$\widehat{\beta}_1 + \widehat{\beta}_2$	$\widehat{\beta}_2/\widehat{\beta}_1$
-0.48	1.87×10^{-3}	8.26×10^{-4}	2.51×10^{-3}	44 %
	[44.44]	[10.45]		
-0.24	9.26×10^{-4}	2.91×10^{-4}	1.21×10^{-3}	31 %
	[42.11]	[6.72]		
0.0	5.33×10^{-6}	9.29×10^{-7}	6.26×10^{-6}	18 %
	[24.00]	[1.90]		
0.24	9.26×10^{-4}	3.01×10^{-4}	1.22×10^{-3}	32 %
	[42.08]	[6.93]		
0.48	1.87×10^{-3}	8.28×10^{-4}	2.70×10^{-3}	44 %
	[44.38]	[10.44]		

5.2.3 Serial Correlation in Variances α

To examine sensitivity of our results to the degree of serial correlation in volatilities, α is varied between 0.10 and 0.90. Simulation results from this exercise are presented in Table 5.9 below. It is easy to see that the degree of CCH is increasing in serial correlation. This is not surprising, since greater α lead to a more steeply sloped $\frac{dr^f(u)}{du}$ line, favoring CCH. Increasing the serial correlation allows us to approach the 134 % by which bond returns were more volatile in recession over the post-war period 1953-1987.

Table 5.9: Bond Return Volatility in Recession and Expansion ARCH(1) Calibration with $\gamma = 1.5$ and Varying α

	$s^{\exp}$	$s^{rec} - s^{\exp}$	s^{rec}	$\%\Delta s$
α	$\widehat{\beta}_1$	$\widehat{\beta}_2$	$\widehat{\beta}_1 + \widehat{\beta}_2$	$\widehat{\beta}_2/\widehat{\beta}_1$
0.10	9.50×10^{-4}	2.72×10^{-4}	1.22×10^{-3}	29 %
	[43.68]	[6.41]		
0.20	9.26×10^{-4}	3.01×10^{-4}	1.22×10^{-3}	32 %
	[42.08]	[6.93]		
0.50	8.17×10^{-4}	3.69×10^{-4}	1.18×10^{-3}	45 %
	[34.24]	[7.46]		
0.90	4.37×10^{-4}	5.07×10^{-4}	9.44×10^{-4}	115 %
	[19.38]	[7.22]		

5.3 Countercyclical Heteroscedasticity in Equity Returns

Time-varying second moments also lead to significantly greater degrees of CCH in equity returns. Again, the "when it rains it pours" mechanism greatly increases the amount of CCH which the model can account for. To see this, compare the degree of CCH in the constant variance model (the last column of Table 5.11) to the degree of CCH in the ARCH(1) model (Table 5.10). [For details on the simulation method, see Appendix B.3]

Table 5.10: Equity Return Volatility in Recession and Expansion ARCH(1) model

	$s^{\exp}$	$s^{rec}-s^{\exp}$	s^{rec}	$\%\Delta s$
γ	$\widehat{\beta}_1$	$\widehat{\beta}_2$	$\widehat{\beta}_1+\widehat{\beta}_2$	$\widehat{\beta}_2/\widehat{\beta}_1$
1.5	0.00200	0.00229	0.00429	115 %
	[39.90]	[23.29]		
2	0.00198	0.00187	0.00386	94 %
	[41.86]	[20.19]		
4	0.00245	0.00088	0.00333	37 %
	[41.85]	[7.62]		

At low levels of risk aversion, the estimated increase in volatility during recession $\widehat{\beta}_2$ is positive and highly significant for both models.[13] However, the degree of CCH generated by the ARCH(1) model is more than twice as great as that generated by the constant variance model: 140 % in the ARCH(1) case, as opposed to 68 % in the constant variance case for $\gamma = 1.5$. These values compare favorably with those estimated by Schwert (1989) for U.S. data and reported in Table 5.12 below.

Table 5.11: Equity Return Volatility in Recession and Expansion Constant Variance Model

	$s^{\exp}$	$s^{rec}-s^{\exp}$	s^{rec}	$\%\Delta s$
γ	$\widehat{\beta}_1$	$\widehat{\beta}_2$	$\widehat{\beta}_1+\widehat{\beta}_2$	$\widehat{\beta}_2/\widehat{\beta}_1$
1.5	0.00282	0.00192	0.00474	68 %
	[38.25]	[26.08]		
2	0.00347	0.00206	0.00553	59 %
	[38.53]	[25.98]		
4	0.00489	0.00219	0.00708	45 %
	[38.83]	[21.42]		

That the constant variance model also exhibits some degree of cyclical variation in equity return volatility is not surprising, when one considers that non-linearities were seen to be driving the CCH in bond returns. Since constant-variance equity returns are already non-linear, they too can be expected to

[13] That some degree of CCH is also found in non-ARCH equity returns is not surprising. Recall that it was the non-linearity of the ARCH bond return which was driving the CCH result there. Since the non-ARCH equity return is also non-linear, it is plausible that it should also display some amount of CCH.

display some degree of cyclical variation in their volatility. That ARCH(1) equity returns display even greater degrees of CCH reflects the added degree of non-linearity contributed by the time-varying second moments.

Table 5.12: Increase in Equity Return Volatility during Recession U.S. monthly data [Source: Schwert (1989)]

period	**1859-1987**	**1859-1919**	**1920-1952**	**1953-1987**
$\%\Delta s$	61%	-6%	234%	68%

5.4 Habit Persistence

The equity return CCH results should, however, be treated with some of caution. To see why, first note that the degree of CCH which both the ARCH(1) and the constant variance model can generate is *decreasing* in the degree of risk aversion. This, in turn, may be related to the relationship between risk aversion and risk premia.

To be more precise, equity returns may be written as the sum of the bond return and the equity risk premium as

$$r^e_{t+1} = r^f_{t+1} + \underbrace{\left(r^e_{t+1} - r^f_{t+1}\right)}_{\text{risk premium}} \tag{28}$$

Bond returns r^f_{t+1} are decreasing in the variance of the underlying asset (due to precautionary effects). Risk premia, on the other hand, are *increasing* in the volatility of the underlying asset. The more variable the stream of payoffs, the more a risk averse agent will have to be compensated for holding it. The total effect is ambiguous.

For low levels of risk aversion, equity returns turn out to be highly correlated with bond returns ($corr\left(R^e_t, R^f_{t+1}\right) \approx 0.98$), implying that the total effect has equity returns *decreasing* in the volatility of the underlying asset. Recalling the theoretical discussion of Section 4, it is precisely this negative relationship between volatility and returns which allows to generate the large degrees of CCH documented in Table 5.10. The tight correlation between bond and equity returns is also reflected, however, in the extremely small equity premia generated by the power utility model studied here at low levels of risk aversion. Furthermore, when risk aversion in increased to moderate levels ($\gamma = 4.0$), the CCH generated by the power utility model begins to evaporate. Thus, one might be tempted to suspect that the greater volatility of equity returns in recession is intimately linked to the extremely low - and counterfactual - equity premia associated with the basic Lucas(1978)-Mehra/Prescott (1985) model.

It turns out that this suspicion is unfounded. In Ebell (2000), I check whether it is possible to generate endogenously *both* large equity premia and more volatile equity returns during recessions. In particular, I extend Cochrane and Campbell (1999)'s habit persistence model to include AR(1)-ARCH(1) consumption growth. When $\gamma = 0.9$[14], this model can account for both realistic equity premia of 6.05 % per annum *and* equity returns that are about 118% more volatile during recessions.

6 Conclusions

The main contribution of this paper has been to show that introducing time-varying second moments into a consumption CAPM framework can induce cyclical variation in asset returns. Moreover, the degree of countercyclical heteroscedasticity generated by the model is quantitatively significant and empirically relevant. CCH in bond and equity returns in the simulated model is similar to that found in the data for an ARCH(1) parameterization with low to moderate levels of risk aversion.

Furthermore, the *when it rains it pours* mechanism has the potential to explain cyclical variation in the volatilities of more general economic variables. Although we concentrate on explaining countercyclical heteroscedasticity in financial markets, our framework is in no way specific to financial markets. All sorts of forcing processes may have time-varying second moments, and ARCH driving processes may be integrated into any number of models. Moreover, there are no limits *per se* on which type of variables are generated endogenously. Thus, the approach presented here could also be used to integrate ARCH-forecasting into models with production or models with endogenous labor choice, to name just two.

In particular, we make the following conjecture: endogenous variable which may be expressed as quadratic functions of the innovation to the driving process will tend to display asymmetric volatility. Whether this asymmetric volatility is counter- or pro-cyclical will depend upon whether the endogenous variable reacts positively or negatively to variance.

The reasoning behind this conjecture is simple. Recall that the asymmetric volatility results are driven by the fact that endogenous variables are *quadratic* rather than linear functions of the innovations. Endogenous variables that are linear functions of the innovations react symmetrically to positive and negative innovations. In contrast, endogenous variables that are quadratic in the innovations react asymmetrically to positive and negative innovations. Thus, to the extent that a given endogenous variable may be expressed (or approximated) as a quadratic function of the innovation, it should also exhibit asymmetric volatility. More precisely, say that some endogenous variable y_{t+1} may be expressed (or approximated) by a quadratic equation in the innovation to its driving pro-

[14] Those familiar with the habit persistence literature will note that this actually improves somewhat on Campbell and Cochrane (1999)'s constant variance results. Setting all other parameters equal, the constant variance model generates equity premia of about 6% per annum when $\gamma = 2.0$. This translates into steady state risk aversion of $\frac{\gamma}{\bar{S}} = \frac{2.0}{0.057} = 35.09$, whose large value has been criticized. In the AR(1)-ARCH(1) model, $\gamma = 0.9$ is required to generate 6% per annum equity premia, which is equivalent to steady state risk aversion of 15.79, a considerably more reasonable value.

cess w_t such as

$$\begin{aligned} y_{t+1} &= a + bw_t + cw_t^2 \qquad c \neq 0 \\ \frac{dy_{t+1}}{dw_t} &= b + 2cw_t \end{aligned}$$

Figure (8) shows the reaction in y_t to an innovation w_t when the endogenous variable reacts positively to the innovation w_t ($b > 0$), but *negatively* to variance $c < 0$. In this case, the shaded region under the $\frac{dy_{t+1}}{dw_t}$ line between w^- and zero is clearly greater than that between w^+ and zero, reflecting the stronger reaction of y_{t+1} to negative innovations. Thus, y_{t+1} behaves like a bond return, and volatility will tend to be greater during recessions.

If, on the other hand, the endogenous variable reacts positively to variance ($c > 0$), the volatility asymmetry is likely to be reversed. To see this, note that in Figure (9), the slope of the $\frac{dy_{t+1}}{dw_t}$ line is positive. Thus, the area under this line will be greater for w^+ than for w^-, and volatility will tend to be greater during expansions. For variables that are decreasing in the level w_t ($b < 0$), the above results will be reversed. The combination $b, c < 0$ will favor procyclical heteroscedasticity, while the combination $b < 0$ and $c > 0$ will favor countercyclical heteroscedasticity.

We take this conjecture as a guide to future research. If the conjecture proves correct, then introducing time-varying second moments into other stochastic models will allow the *when it rains it pours* mechanism to account for counter- or procyclical heteroscedasticity in a wide range of variables. The ability to explain the presence of countercyclical heteroscedasticity in non-financial variables may be useful in more than just an explaining-the-data sense. Recent work by Storesletten, Telmer and Yaron (1999) take countercyclical heteroscedasticity in idiosyncratic income shocks (to use the term they coined) as given in an OLG framework, and show that this setup is capable of matching observed equity premia for low levels of risk aversion. Thus, via cyclical variation in income shocks, there may be an interesting link between equity premia and cyclical variation in equity returns, which deserves further study.

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Appendix A: Mean-Inducing Innovations $\widetilde{u}_t$

Define the innovation $\widetilde{u}_t$ as that which induces the bond return to be exactly equal to its mean. That is

$$r^f_{t+1}(\widetilde{u}_t) = Er^f_{t+1}$$

A.1.1.No-ARCH case

In the no-ARCH case, the innovation $\widetilde{u}_t$ which induces the mean bond return Er^f satisfies:

$$\begin{aligned} r^f_{t+1}(\widetilde{u}_t) &= -\log\beta + \gamma\left[c(1+\rho) + \rho^2\log x_{t-1} + \widetilde{u}_t\right] - \frac{\gamma^2}{2}\sigma_x^2 \\ &= -\log\beta + \gamma\underbrace{\frac{c}{1-\rho}}_{E\log x_t} - \frac{\gamma^2}{2}\sigma_x^2 = Er^f \end{aligned}$$

so that the mean-inducing innovation may be written as

$$\widetilde{u}_t = \frac{c}{1-\rho} - c(1+\rho) - \rho^2\log x_{t-1}$$

The idea is simple: the greater the growth rate at date $t-1$, the greater the current riskfree rate r^f_t. The greater the current riskfree rate, the smaller (or more negative) the innovation required to bring the future riskfree rate back down to its unconditional mean value.

Moreover, since $\log x_t$ is distributed as $N\left(\frac{c}{1-\rho}, \sigma_x^2\right)$, it is easy that the mean inducing innovation will be distributed normally as $\widetilde{u}_t \sim N\left(0, \rho^4\sigma_x^2\right)$.

A.1.2.ARCH case

In the ARCH case, the innovation $\widetilde{u}_t$ which induces the mean bond return Er^f satisfies:

$$\begin{aligned} r^f_{t+1}(\widetilde{u}_t) &= -\log\beta + \gamma\left[c(1+\rho) + \rho^2\log x_{t-1} + \widetilde{u}_t\right] - \frac{\gamma^2}{2}\left[\xi + \alpha\widetilde{u}_t^2\right] \\ &= -\log\beta + \gamma\underbrace{\frac{c}{1-\rho}}_{E\log x_t} - \frac{\gamma^2}{2}\underbrace{\frac{\xi}{(1-\rho^2)(1-\alpha)}}_{var\log x_t} = Er^f \end{aligned}$$

Rearranging terms, this implies that the mean-inducing innovation satisfies the following quadratic equation:

$$\underbrace{-\gamma\rho^2\left[\frac{c}{1-\rho} - \log x_{t-1}\right] + \frac{\gamma^2}{2}\xi\left[\frac{1-(1-\rho^2)(1-\alpha)}{(1-\rho^2)(1-\alpha)}\right]}_{c(\log x_{t-1})} + \gamma\rho\widetilde{u}_t - \frac{\gamma^2}{2}\alpha\widetilde{u}_t^2 = 0$$

Making use of the quadratic formula, one may obtain an expression for the conditional mean-inducing innovation $\widetilde{u}_t$ as

$$\begin{aligned}\widetilde{u}_{t,1} &= \frac{\rho - \sqrt{\rho^2 + 2\alpha \cdot c\left(\log x_{t-1}\right)}}{\gamma\alpha} \\ \widetilde{u}_{t,2} &= \frac{\rho + \sqrt{\rho^2 + 2\alpha \cdot c\left(\log x_{t-1}\right)}}{\gamma\alpha}\end{aligned}$$

Note that $Ec\left(\log x_{t-1}\right) = \frac{\gamma^2}{2}\xi\left[\frac{1-\left(1-\rho^2\right)(1-\alpha)}{(1-\rho^2)(1-\alpha)}\right] > 0$. From Figure (10) it is easy to see that it is the smaller root $\widetilde{u}_{t,1} = \widetilde{u}_t$ which is relevant. This smaller root has an negative expected value, since $E\sqrt{\rho^2 + 2\alpha c\left(\log x_{t-1}\right)} > \rho$

$$E\widetilde{u}_t = \frac{\rho}{\gamma\alpha} - \frac{E\sqrt{\rho^2 + 2\alpha \cdot c\left(\log x_{t-1}\right)}}{\gamma\alpha} < 0$$

Note, however, that it is not always possible to guarantee that $\widetilde{u}_t$ is real. In particular, $\widetilde{u}_t$ will be complex whenever $2\alpha c\left(\log x_{t-1}\right) + \rho^2 < 0$, which is the case whenever:

$$-2\alpha\gamma\rho^2\left[\frac{c}{1-\rho} - \log x_{t-1}\right] + \alpha\gamma^2\xi\left[\frac{1-\left(1-\rho^2\right)(1-\alpha)}{\left(1-\rho^2\right)(1-\alpha)}\right] < -\rho^2$$

The problem is that when $\log x_{t-1}$ is extremely large and negative, then an extremely large and positive innovation is needed to bring the next period's interest rate back up to its mean via the smoothing effect. However, any innovation which is that large, may also be so large as to increase the variance so greatly via the precautionary effect, that the overall effect will be negative. That is, there is no way to bring the interest rate up to its mean in one fell swoop. This is, however, an exceedingly rare occurrence: it is necessary that $\log x_{t-1}$ be so small that:

$$\log x_{t-1} < \frac{c}{1-\rho} - \left[\frac{1}{2\alpha\gamma} + \frac{\gamma\xi}{2\rho^2}\left[\frac{1-\left(1-\rho^2\right)(1-\alpha)}{\left(1-\rho^2\right)(1-\alpha)}\right]\right]$$

Using the parameters from the calibration to the U.S. quarterly data given in Appendix B, one can see that complex values for $\widetilde{u}_t$ only occur for $\gamma = 1.5$ when the consumption growth rate is more than 77 (yes, seventy-seven) standard deviations below its mean.

Appendix A.2: Calculation of ε

We are looking for ε such that

$$s^A_{t+1}\left(\widetilde{u}_t - \varepsilon\right) = s^A_{t+1}\left(-\widetilde{u}_t + \varepsilon\right)$$

Recalling that

$$s^A_{t+1}(u_t) = \left| r^f_{t+1}(u_t) - r^f_{t+1}(\widetilde{u}_t) \right| = \left| \int_{\widetilde{u}_t}^{u_t} \left[\gamma\rho - \gamma^2 \alpha u \right] du \right|$$

leads to

$$\begin{aligned} s^A_{t+1}(\widetilde{u}_t - \varepsilon) &= \left| \int_{\widetilde{u}_t}^{\widetilde{u}_t - \varepsilon} \left[\gamma\rho - \gamma^2 \alpha u \right] du \right| \\ s^A_{t+1}(-\widetilde{u}_t + \varepsilon) &= \left| \int_{\widetilde{u}_t}^{-\widetilde{u}_t + \varepsilon} \left[\gamma\rho - \gamma^2 \alpha u \right] du \right| \end{aligned}$$

Solving the integrals, we can write

$$s^A_{t+1}(-\widetilde{u}_t + \varepsilon) = \begin{array}{ll} (\varepsilon - 2\widetilde{u}_t)\left(\gamma\rho - \frac{\gamma^2}{2}\alpha\varepsilon\right) & \text{if } 0 < \rho < \frac{\gamma}{2}\alpha\varepsilon \\ -(\varepsilon - 2\widetilde{u}_t)\left(\gamma\rho - \frac{\gamma^2}{2}\alpha\varepsilon\right) & \text{if } \rho > \frac{\gamma}{2}\alpha\varepsilon > 0 \end{array}$$

$$s^A_{t+1}(\widetilde{u}_t - \varepsilon) = \varepsilon\gamma\rho + \frac{\gamma^2}{2}\alpha\varepsilon(\varepsilon - 2\widetilde{u}_t)$$

1. $\rho < \frac{\gamma}{2}\alpha\varepsilon$

   In this case, ε must satisfy

$$\gamma^2 \alpha \varepsilon^2 - 2\gamma^2 \alpha \varepsilon \widetilde{u}_t + 2\gamma\rho\widetilde{u}_t = 0$$

Using the quadratic formula, and the fact that we are interested in the positive root, one may obtain:

$$\varepsilon = \widetilde{u}_t + \sqrt{\widetilde{u}_t^2 - \underbrace{\frac{2\rho}{\gamma\alpha}\widetilde{u}_t}_{<0}} > 0$$

In this case, ε is clearly decreasing in risk aversion γ and serial correlation in variances α, that is, ε is decreasing in the slope of the $\frac{dr^f_{t+1}(u)}{du}$ -ARCH line. Moreover, ε is increasing in ρ and in the magnitude of the mean-inducing innovation $\widetilde{u}_t$.

2. $\rho > \frac{\gamma}{2}\alpha\varepsilon$

$$\varepsilon = -\widetilde{u}_t$$

In this case, ε only depends upon $\widetilde{u}_t$. Once again, ε is increasing in the magnitude of $\widetilde{u}_t$.

Appendix B.1: Quarterly ARCH(1) Parameterization

It is necessary to choose values for a total of six parameters in order to compute solutions to this model. These may be broken down into two groups. In the first group are the two parameters of the AR(1) process governing consumption growth (c, ρ), in addition to the two parameters of ARCH(1) process governing the variance (ξ, α). These AR(1)-ARCH(1) parameters are pinned down by the data. The second group of parameters are the two (free) preference parameters risk aversion γ and the discount factor δ.

AR(1)-ARCH(1) Parameters

AR(1)-ARCH(1) parameters are chosen to match unconditional moments and first-order serial correlations of U.S. quarterly growth in consumption of non-durables and services. The unconditional moments to match may be seen in Table B.1: they are the estimates reported in Kandel and Stambaugh (1990), based on Breeden et. al.'s (1989) data set, covering the period 1929-1982.

Table B.1: Unconditional Moments of Consumption Growth Breeden, et. al. 1929-1982

μ_x	0.00452
σ_x	0.0129

The first-order autocorrelations are chosen to match estimates from an AR(1)-ARCH(1) regression using the NIPA data set. Parameters estimates are given in Table B.2, with standard errors reported in square brackets.

Table B.2: AR(1)-ARCH(1) Estimates for Consumption Process

AR(1) equation: $\log x_t = c + \rho \log x_{t-1} + \varepsilon_{1,t}$

$\widehat{c}$	$\widehat{\rho}$
6.6×10^{-3}	0.24
[9.8×10^{-4}]	[0.091]

ARCH(1) equation: $\widehat{\varepsilon}_{1,t}^2 = \xi + \alpha u_{t-1}^2 + \varepsilon_{2,t}$

$\widehat{\xi}$	$\widehat{\alpha}$
5.1×10^{-3}	0.33
[5.0×10^{-6}]	[0.11]

The calibration strategy is to first set first-order autocorrelations (ρ, α) equal to their estimated values from Table B.2, and to then use properties of the autoregressive processes to find values for the constants which are consistent with the unconditional moments in Table B.1 and the serial correlation estimates in Table B.2. These consistent values for (c, ξ) are given in Table B.3 below:

Table B.3: Constants consistent with Unconditional Moments

c	$\mu_x (1-\widehat{\rho})$	3.4×10^{-3}
ξ	$\sigma_x^2 \left(1-\widehat{\rho}^2\right)(1-\widehat{\alpha})$	1.1×10^{-4}

Preference Parameters

The only free parameters are then the two preference parameters risk aversion and discount factor. Here we choose a discount factor of $\delta = 0.99$ and vary the risk aversion coefficient to satisfy $\gamma \in \{1.5, 2, 4\}$. Note that these are conservative values for γ: Mehra and Prescott (1985) considers risk aversion coefficients of up to 10, while Kandel and Stambaugh (1989) argue that risk aversion coefficients on the order of 29 might be reasonable.

Appendix B.2: Monthly ARCH(1) Parameterization

It is necessary to choose values for a total of six parameters in order to compute solutions to this model. These may be broken down into two groups. In the first group are the two parameters of the AR(1) process governing consumption growth (c, ρ), in addition to the two parameters of ARCH(1) process governing the variance (ξ, α). These AR(1)-ARCH(1) parameters are pinned down by the data. The second group of parameters are the two (free) preference parameters risk aversion γ and the discount factor δ.

AR(1)-ARCH(1) Parameters

AR(1)-ARCH(1) parameters are chosen to match unconditional moments. The unconditional moments to match may be seen in Table B.4: they are the estimates based upon NIPA data, covering the period 1959:01-1999:06 [source: Bureau of Economic Analysis, U.S. Department of Commerce].

Table B.4: Unconditional Moments of Consumption Growth NIPA data, 1959:01-1999:06

μ_x	0.0017
σ_x	0.0038

Since the dubious time series properties of monthly consumption growth estimates are well-known (see Wilcox (1992)), no serious attempt is made to estimate ρ and α. Rather than making what are most likely meaningless estimates, we choose conservative values of ρ and α, and then allow them to vary widely in the sensitivity analysis of Section 5. We assume that serial correlation in consumption growth ρ is about equal in monthly and quarterly data. Then, we make a very conservative estimate of the serial correlation in volatility. We choose $\alpha = 0.20$, which is lower than the more reliable estimates from quarterly data. Note that this conservative choice works against CCH, and thus against our results.

Table B.5: Basic Parameter Values

ρ	α
0.24	0.20

Again, the calibration strategy is now to use properties of the autoregressive processes to find values for the constants which are consistent with the unconditional moments in Table B.4 and the serial correlations in Table B.5. These consistent values for (c, ξ) are given in Table B.6 below:

Table B.6: Constants consistent with Unconditional Moments

c	$\mu_x(1-\rho)$	1.3×10^{-3}
ξ	$\sigma_x^2(1-\rho^2)(1-\alpha)$	1.1×10^{-5}

Preference Parameters

The only free parameters are then the two preference parameters risk aversion and discount factor. Here we choose a discount factor of $\delta = 0.99$ and vary the risk aversion coefficient to satisfy $\gamma \in \{1.5, 2, 4\}$. Note that these are conservative values for γ: Mehra and Prescott (1985) considers risk aversion coefficients of up to 10, while Kandel and Stambaugh (1989) argue that risk aversion coefficients on the order of 29 might be reasonable.

Appendix B.3: Numerical Approximation of Equilibrium Equity Returns

In order to obtain the equilibrium sequence of equity returns, we must first find the sequence of price-dividend ratios $\left\{\frac{p_t}{d_t}\right\}_t$ which satisfies the Euler equation (5). This may be achieved by iterating on equation (5) using the parameterized expectations approach (PEA) developed by Marcet and Marshall (1994). This algorithm finds a parameterization of expectations $\Psi\left(x_t, u_t^2; \psi\right) = E_t\left\{x_{t+1}^{1-\gamma}\left(\frac{p_{t+1}}{d_{t+1}}+1\right)\right\} = \frac{1}{\beta}\frac{p_t}{d_t}$ which is consistent both with the exogenous growth rates and endogenous price-dividend ratios. That is, the algorithm first assumes some functional form (in our case an exponential one) by which values of the state variables $\left(x_t, u_t^2\right)$ are transformed into expectations:

$$\Psi\left(x_t, u_t^2; \psi\right) = \psi_1 \exp\left\{\psi_2 x_t + \psi_3 u_t^2\right\}$$

Now the series of price-dividend ratios generated by these expectations $\Psi\left(x_t, \widehat{\sigma}_{t+1}^2; \psi\right)$ can be calculated as

$$\frac{p_t}{d_t}(\psi) = \beta \cdot \psi_1 \exp\left\{\psi_2 x_t + \psi_3 u_t^2\right\}$$

Next, the consistency of $\left\{\frac{p_t}{d_t}(\psi)\right\}_t$ needs to be checked. This may be done by imposing rational expectations, and then finding the RE price-dividend ratios as

$$\frac{p_t}{d_t}_{RE} = \beta E_t \left\{ x_{t+1}^{1-\gamma} \left(\frac{p_{t+1}}{d_{t+1}}(\psi) + 1 \right) \right\} \tag{29}$$

Loosely speaking, a fixed point in this algorithm is then the series $\left\{\frac{p_t}{d_t}(\psi)\right\}_t$ which implies itself. In particular, $\left\{\frac{p_t}{d_t}\right\}$ is a PEA solution to the Euler equation if non-linear least squared regressions of the equation

$$\frac{1}{\beta}\frac{p_t}{d_t}_{RE} = \zeta_1 \exp\left\{\zeta_2 x_t + \zeta_3 u_t^2\right\}$$

produce estimates $\left(\widehat{\zeta}_1, \widehat{\zeta}_2, \widehat{\zeta}_3\right)$ which are close enough to those values which generated the price-dividend ratios in the first place, namely (ψ_1, ψ_2, ψ_3).

From the sequence of equilibrium price-dividend ratios, it is easy to recover the sequence of equilibrium equity returns as

$$r_{t+1}^e = x_{t+1} \cdot \frac{\frac{p_{t+1}}{d_{t+1}} + 1}{\frac{p_t}{d_t}} \tag{30}$$