BEHAVIOURAL COMPLEMENTARITY (NOT HETEROGENEITY) CAUSES THE LAW OF DEMAND.

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Abstract. Two results are presented. A negative one evidences that "generic heterogeneity" (such as characterised by the set of prices at which an agent is sensitive to price changes) is not sufficent to ensure that the Law of demand holds. A second result establishes that, given a population and its resulting aggregate behaviour, it is always possible to find a "complementary" set of agents whose behaviour would be sufficient to restore the Law of demand for the aggregate over the whole population. Finally, necessary and sufficient conditions for the law of demand to hold are restated in a space that makes easy an geometric interpretation of the results. While "behavioural heterogeneity" may well generate nice properties in the aggregate, it does *not* in a systematic manner.

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1. Introduction

Prices are commonly considered as the main determinants of individual decisions and as a fundamental coordinating device of the economy. In a purely competitive system, prices are even supposed to bring the coherence of individual decisions. However, the consistency of any economic model is likely to vanish if the properties of the aggregate demand (hence of equilibrium prices) are not well-determined. This explain why the subject is essential to economics.

It is well-kown that, despite the (already strong) assumptions of micro-economic theory (See Debreu 1974, Sonnenschein 1972, 1973, Mantel 1974, 1976), no other restrictions are placed on aggregate *excess* demand than homogeneity and Walras's Law. More recently, these results have been shown to extend to demand functions (See Chiappori & Eckeland 1999). This lead to the general conclusion that individual optimization *does not restrict sufficently* the aggregates in order to grant the weak axiom of revealed preferences, gross-substitutability or any other property that is required for the uniqueness and stability of (competitive) equilibrium.

This negative perspective is counter-balanced by other results of aggregation theory. Instead of imposing more stringent assumptions on *individual* behaviour, consider (*non-individualistic*) distributional assumptions on agents characteristics. Suitable distributions of consummers' characteristics can even "generate" properties for the aggregate that do not need to hold at the disaggregated level ! A clear example of such a phenomena is to be found in Hildenbrand (1983).

Among the distributional assumptions, the hypothesis of "preference dispersion" has been shown to be especially fruitfull (See Araujo & Mas-Colell, 1978, Dierker E., Dierker H. & Trockel 1984). This may explain the widespread belief that "behavioural heterogeneity" (*i.e.* dispersion of behavioural characteristics) may well generate the Law of demand in the aggregate. This is at least what is suggested by the (parametric) model of Grandmont (1992), extended in a "semi-parametric" setup by A. Kneip (1999) and extended further to its full generality by W. Hildenbrand and A. Kneip (1999).

The interest of the later results is straightforward: granting the law of demand under such a mild assumption as "behavioural heterogeneity" would fill the gap that still splits micro from macro-economics. It can almost be considered a re-foundations of modern economics. Unfortunatly, I have shown elsewhere that the representation of heterogeneity involved in those contributions is quite missleading in a way that strongly limits the validity of the results (See EBDV 1998, 1999 & 2000).

This paper pursues my line of enquiry and provides results along both perspective presented above. It first show that "generic heterogeneity" does not restrict further the set of "admissible" aggregate demand functions. As a result, it is not sufficient to ensure that the law of demand holds at the aggregate level. Second, for any population and its associated aggregated behaviour, it is possible to find a "complementary" group of individuals whose insertion into the initial population would drive the aggregate to be a Cobb-Douglas. The existence of "behavioural complementarities" implies, among others, that if a set of household is (globally) invariant under some "symmetry", the law of demand will hold. This confirms the view according to which "suitable" distribution of characteristics can "generate" new properties in the aggregate. Indeed, the later result does not require the initial demand function to be generated through utility maximisation. Finally necessary and sufficient conditions are provided for the law of demand to hold. This allows to extend the distributional results by introducing the notion of "almost complementarity behaviour" hence of "almost invariant sets" with a precise meaning of the latter concepts.

To sum up, while "behavioural heterogeneity" may sometimes generate nice properties at the aggregate, it does not in a systematic way: The law of demand holds for specific distribution of demand functions only. Consequently one would prefer the concept of "behavioural complementarities" that characterise the populations for which the Law of demand hold.
I argue that the geometric approach proposed here may be applied to numerous topics that goes well beyond demand aggregation.

2. Definitions, notations and results.

Consider an l-commodity economy and let $S^l = \left\{ p \in \Re^l_+ \mid \| p \| = \sum_{i=1}^l p_i = 1 \right\}$ be the set of positive price vectors with unit norm.

Definition 2.1. *A consumer is a pair $(\preceq, x)$ where $\preceq$ is a strictly-convex, monotone, continuous, complete preference preorder on $\Re^l_+$, and $x \in \Re^*_+$ its income.*

A function $f : S^l \to \Re^l_+$ is the individual demand funtion of the consumer $(\preceq, x)$ if for every p in S^l, $f(p)$ is the greatest element for $\preceq$ of $\left\{ q \in \Re^l_+ \mid p.q = x \right\}$. It is easy to show that an individual demand function f (1) is continuous, and (2) satisfies $p.f(p) = x$ for every p in S^l. The corresponding individual expenditure share function $w : S^l \to S^l$ is defined by $p \otimes f(p)/x$ where $\otimes$ denotes the tensorial product. It is easy to show that this function is continuous.

According to Chiappori and Ekeland (1999), any analytic mapping of $\Re^l_+$ satisfying Walras Law can be locally decomposed as the sum of l individual, utility maximizing demand functions. This result can be easily transposed to any analytic mapping of S^l. This leads to the following definition:

Definition 2.2. *A continuous function $w = S^l \to S^l$ is an expenditure share function.*

and to represent a consumer by a pair (w, x).

The aggregate expenditure share function W as defined by the means of the aggregate demand $F(p) = \sum_{h \in H} f^h\left(p, x^h\right)$ over a finite population H is a weigthed mean of the individual aggregate expenditure share:

$$W(p) = \frac{p}{X} \otimes \sum_{h \in H} f^h\left(p, x^h\right) = \sum_{h \in H} \frac{x^h}{X}\left[p \otimes f^h\left(p, x^h\right)/x^h\right] = \sum_{h \in H} \frac{x^h}{X} w^h(p)$$

where $X = \sum_{h \in H} x^h$. A natural question is: Does "behavioural heterogeneity" restrict further the set $\mathcal{C}\left(S^l\right)$ of admissible aggregate expenditure share function W ?

Before handling this problem, one should precise further what is intended by "behavioural heterogeneity". Following Hildenbrand and Kneip (1999), define $B^h_\alpha = \left\{p \in S^l \mid \sup_{i,j} \left\| p_j \partial_{p_j} w^h_i(p) \right\| \geq \alpha\right\}$, the set of prices over which the price elasticity of the individual expenditure share function is higher than α according to the supremum norm. Clearly, individuals that are sensitive to price changes over disjoint sets B^h_α display an heterogeneous behaviour. This consideration introduces the following result:

Proposition 2.1. *Given $\{U^j\}_{j \in J}$ a finite cover of S^l, for any expenditure share function $W(p)$ there exists a finite set of expenditure function $\{w^j\}_{j \in J}$ such that $B^j_\alpha \subset U^j$ for any $\alpha > 0$ and an associated set of positive real $\{x^j\}_{j \in J}$ that verifies $\sum_{j \in J} x^j = X$ and for which the equality*

$$W(p) = \sum_{j \in J} \frac{x^j}{X} w^j(p)$$

is verified over a compact set $K \subset S^l$ of strictly positive measure.

Heterogeneity such as measured by a distribution of the characteristic sets B_{α}^{h} does not significantly restrains the admissible aggregate behaviour. In particular there are no reasons for which the Law of demand should hold. Loosely, it is possible to get *any* expenditure share function as the aggregate of a population H, whatever its "degree of behavioural heterogeneity".

Does the later result mean that heterogeneity cannot bring regularities to the aggregate that do not exist at the micro-level? The next proposition shows that there exist *complementarities* across behaviour and that the latter (negative) result should not be overinterpreted. Heterogeneity as evaluated by the means of the characteristic sets B_{α}^{h} is to generic for obtaining interesting structure. However,

Proposition 2.2. *For any expenditure share function $w(p)$ there exists a (complementary) expenditure share function w^c and an associated income x^c such that the aggregate expenditure share function*

$$W(p) = \frac{x}{X} w(p) + \frac{x^c}{X} w^c(p)$$

is a Cobb-Douglas with aggregate revenu $X = x + x^c$.

In other words aggregation can "smoth" any aggregate behaviour to the most regular one, namely a Cobb-Douglas. In particular "irrational" individual behaviour such as represented by a demand function that violates the weak axiom of revealed preferences does not forbid such a property to hold in the aggregate. Note that, despite the semplicity of the proposition, the result is not fully obvious. Indeed the expenditure share $w^c(p) = [X.W(p) - x.w(p)] / x^c$ has to be in the simplex S^l for any $p \in S^l$.

By definition expenditure shares derived from Cobb-Douglas preferences remain unchanged. Hence the function can be represented by a fixed point a in S^l. Observe that both (original and complementary) functions are "symmetric" with respect to their aggregate. More generally,

Remark 2.1. *Any set $\{w^j(p)\}$ that is "globally invariant" by a bijective transformation T over S^l whose invariant is reduced to one point a, and in particular any orbit O_w of an expenditure share function by such a transformation, aggregates as the Cobb-Douglas a whenever income is identical over households.*

Obviously "strong" or "exact" complementarity such as introduced above is not necessary for the law of demand to hold. Indeed,

Proposition 2.3. *The law of demand holds if and only if*

$$\left(p^{-1} \otimes q^{-1} \otimes [p \otimes \Delta w - w \otimes \Delta p]\right) . (\Delta p) \leq 0$$

where Δ *is the difference operator,* $q = p - \Delta p$ *and the point . denotes the scalar product. For small variations of the price, this inequality can be restated:*

$$\left(w^{-1/2} \otimes dw - dp\right) . (dp) \leq 0.$$

In particular, the law of demand will hold whenever the image by w of S^l is "almost a point". More precisely, if any sphere $s_p(r)$ centered in p of half diameter r is included in an ellipsoïd $e_w(r\, w_i/p_i)$ of center $w(p)$ and hemi-axis $r\, w_i/p_i$. Denote that, in contrast with the assumptions generally made in the litterature, no minimum expenditure share is required.

Defining A the set of function w for which the law of demand holds, it is possible to extend the notion of behavioural complementarity. To any function w of $\mathcal{C}\left(S^l\right)$, one can now associate the (symmetric) set of "weakly complementary" functions $A^{wc}(w)$ such that the aggregate of w and any function of this set is included in A. By the same talking, one can introduce the concept of "weak invariance". Such a perspective sheds the light on very fact that it is not "generic heterogeneity" that might insure the law of demand but that the latter property holds for quite specific sets of admissible functions only.

The rest of the paper is dedicated to the proof and construction of these results.

3. Proof and construction of the results.

3.1. Proof of proposition 2.1: Let $V_w^j \subset S^l$ be the image of the set U^j by the application $w(p)$. Remark that if $w(p)$ is a Cobb-Douglas expenditure share function, the image set $w\left(S^l\right)$ hence all the sets V_w^j reduces to a point. In order to characterise these sets, define

$$m_k^j = \inf_{p \in U^j} w_k(p) = \inf_{v \in V_w^j} v_k.$$

Note that $\sum_{k=1}^l m_k^j \leq 1$ with strict equality when the minimum is reach at the same p, i.e. when V^j is a point. Denote by $s^j = 1 - \sum_k m_k^j$ the relative size of the subset V_w^j in S^l.

Assume that sum of the expenditure share changes to be explained by the J individuals is less than one:

Assumption 1.

$$\sum_{j \in J} s^j \leq 1.$$

This assumption derives from the fact that if one individual only is required to explain the changes in expenditure share when prices evolves *within* a set U^j, it requires for this individual a minimum fraction of the total income.
Assume furthermore the distribution of set V^j to be "not to widespread". More precisely, if J is the total number of set V^j, assume that:

Assumption 2.

$$m_k^j \leq \frac{1}{J-1} \sum_i m_k^i = \left(1 + \frac{1}{J-1}\right) \overline{m_k}$$

This assumption derives from the fact that if one individual only is required to explain the changes in expenditure share when prices shift *from one set U^j to another*, it requires a minimum fraction of the total income.

Define the set $A = \{(a^j, \alpha^j)\}_{j \in J}$ of Cobb-Douglas of income $\alpha^j = 1 - \beta\left(1 - s^j\right)$ and preferences defined by expenditure share a_k^j:

$$a_k^j = \frac{\beta}{\alpha^j} \left[\frac{1}{J-1} \sum_{i=1}^J m_k^i - m_k^j \right]$$

The coefficient $\beta = (J-1)/\sum_{j\in J}(1-s^j) \leq 1$ is set such a manner that $\sum_{j\in J}\alpha^j = 1$ and $\alpha^j \geq s^j$. One can verify that $a^j \in S^l$. Indeed $\sum_k a^j_k = 1$ and according to assumption 2 $a^j_k \geq 0$.

Denote $(a^{-j}, 1-\alpha^j)$ the Cobb-Douglas of income $1-\alpha^j$ and expenditure shares defined by aggregation over $J-1$ of the function a^i :

$$\left(1-\alpha^j\right) a_k^{-j} = \sum_{i\neq j}\alpha^i a_k^i = \beta\sum_{i\neq j}\left[\frac{1}{J-1}\sum_{u=1}^{J} m_k^u - m_k^i\right] = \beta\left(\sum_{u=1}^{J} m_k^u - \sum_{i\neq j} m_k^i\right) = \beta m_k^i$$

Let introduce now the function

$$w^j(p) = a^{-j} + \left(W(p) - a^{-j}\right)/\alpha^j$$

This function clearly verifies $w^j_k(p) \geq 0$ over U^j since $\alpha^j w^j_k(p) \geq \min_{p\in U^j} W(p) - \beta m^i_k = (1-\beta)\, m^i_k \geq 0$. Moreover $\sum_k w^j_k(p) = 1$ since it is the case for $W(p)$ and a^{-j}. Note that if the income of $w^j(p)$ is α^j the aggregate of $w^j(p)$ and the $J-1$ individuals represented by a^i, $i \neq j$ is exactly $W(p)$.

$$\alpha^j w^j(p) + \left(1-\alpha^j a_k^{-j}\right) = \alpha^j w^j(p) + \sum_{i\neq j}\alpha^{xi} a_k^i = \alpha^j w^j(p) + \left(1-\alpha^j\right) a_k^{-j} = W(p)$$

Define individual j as follows:

$$\begin{aligned} w^j(p) &= a^{-j} + \left(W(p) - a^{-j}\right)/\alpha^j \qquad \textit{over } U^j \\ w^j(p) &= a^j \qquad \textit{otherwise.} \end{aligned}$$

with income α^j as defined above. Heterogeneity is maximum because $\partial_p w^j(p) \equiv 0$ outside U^j. However, the set $\{w^j, \alpha^j\}_{j\in J}$ display the aggregate behaviour $W(p)$.

3.2. Proof of proposition 2.2. Let a be defined by $a_k = M_k/\left(\sum_{s=1}^{l} M_s\right)$ where M_k denotes $\max_p w_k(p)$. Clearly $a \in S^l$ since $\sum_{k=1}^{l} a_k = 1$ and $a_k \geq 0$. Define $x^c = x\left[\left(\sum_{s=1}^{l} M_s\right) - 1\right] \geq 0$ and the complementary behaviour $w^c(p)$ by

$$w^c(p) = \left(1 + \frac{x}{x^c}\right) a - \frac{x}{x^c} w(p)$$

Again, by definition $\sum_{k=1}^{l} w^c_k(p) = 1$ and

$$w^c_k(p) \geq \left(1 + \frac{x}{x^c}\right) a_k - \frac{x}{x^c} M_k \geq 0$$

with the equality that is reached at least for some p. Thus $w^c(p) \in S^l$ and both functions aggregate to the Cobb-Douglas a. Remark that x^c is the minimum income required to insure this result.

Notice again that, given a bijective transformation T whose invariant is reduced to one point a and a set $\{w^j\}$ globally invariant by the transformation $w(p) \mapsto T \circ w(p)$, the aggregate $a(p) = (1/J) \sum w^j(p)$ is such that

$$T \circ a(p) = T \circ \frac{1}{J} \sum w^j(p) = \frac{1}{J} \sum w^j(p) = a(p).$$

hence $a(p)$ is independent of p. In particular, this does hold for the orbits $\{w^j(p) = T^{\circ j} w((p))\}_{j=-\infty}^{j=\infty}$ any $w(p)$.

3.3. Proof of proposition 2.3. By definition the law of demand holds whenever

$$(p - q) . (F(p) - F(q)) \leq 0$$

where the point . denotes the scalar product. Remark that $\Delta w = w(p) - w(q) = (p - q) \otimes F(p) + q \otimes (F(p) - F(q))$. By commutativity of the tensorial product:

$$p \otimes q \otimes (F(p) - F(q)) = p \otimes (w(p) - w(q)) - (p - q) \otimes w(p)$$

Remark that if $p, q > 0$, the linear operator A defined by $Ax = p \otimes q \otimes x$ is definite positive. Its diagonality implies that $x.y = Ax.A^{-1}y$, hence

$$(p - q) . (F(p) - F(q)) \leq 0$$
$$\Longleftrightarrow$$
$$\left(p^{-1} \otimes q^{-1} \otimes [p \otimes \Delta w - w \otimes \Delta p]\right) . (\Delta p) \leq 0$$

Consider now the case where $\Delta p = p \otimes w^{-1/2} \otimes dp$ where dp is infinitesimal. In this case $q \approx p$ and the inequality boils down to $\left(w^{-1/2} \otimes dw - dp\right) . (dp) \leq 0$ straightforwardly. In order to give an intuition of previous formulae one can remark that $(\Delta w - p^{-1} \otimes w \otimes \Delta p) . (\Delta p) \leq 0$ is a sufficient condition for the law of demand to hold.

All these results find can easily geometric interpretation in the space S^l.

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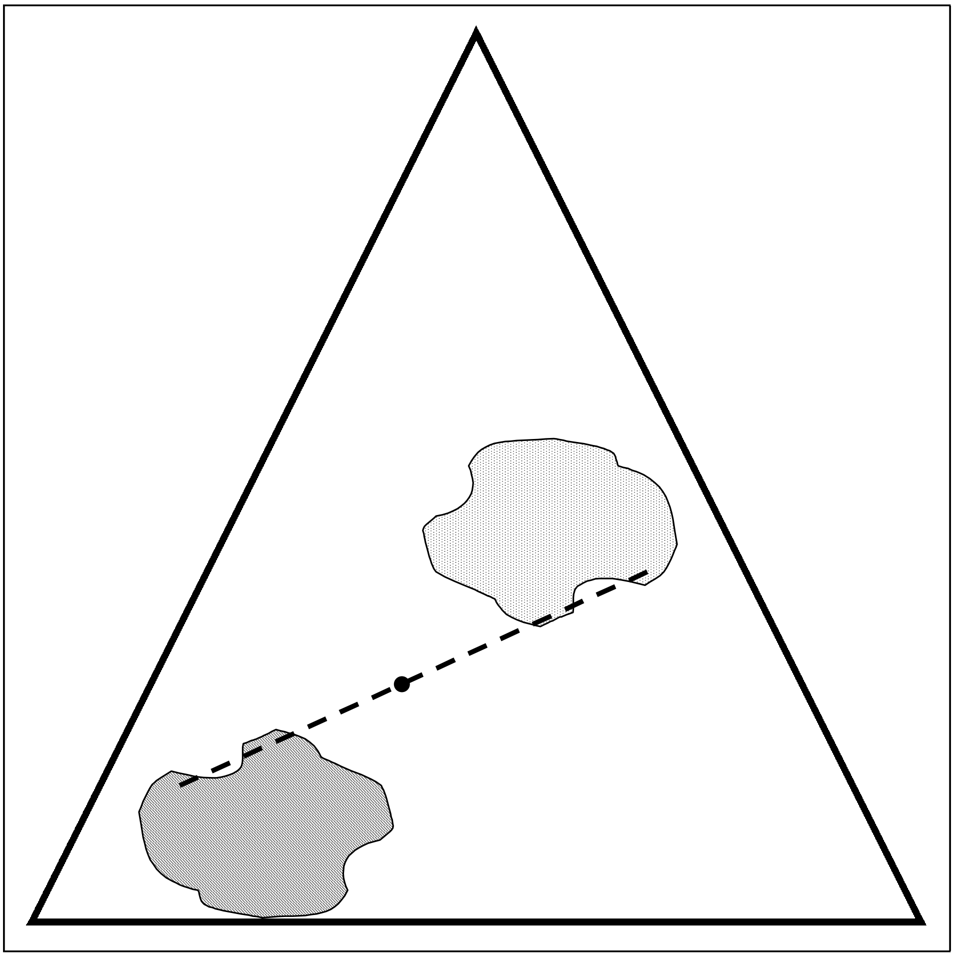

Figure 1. Complementary Sets

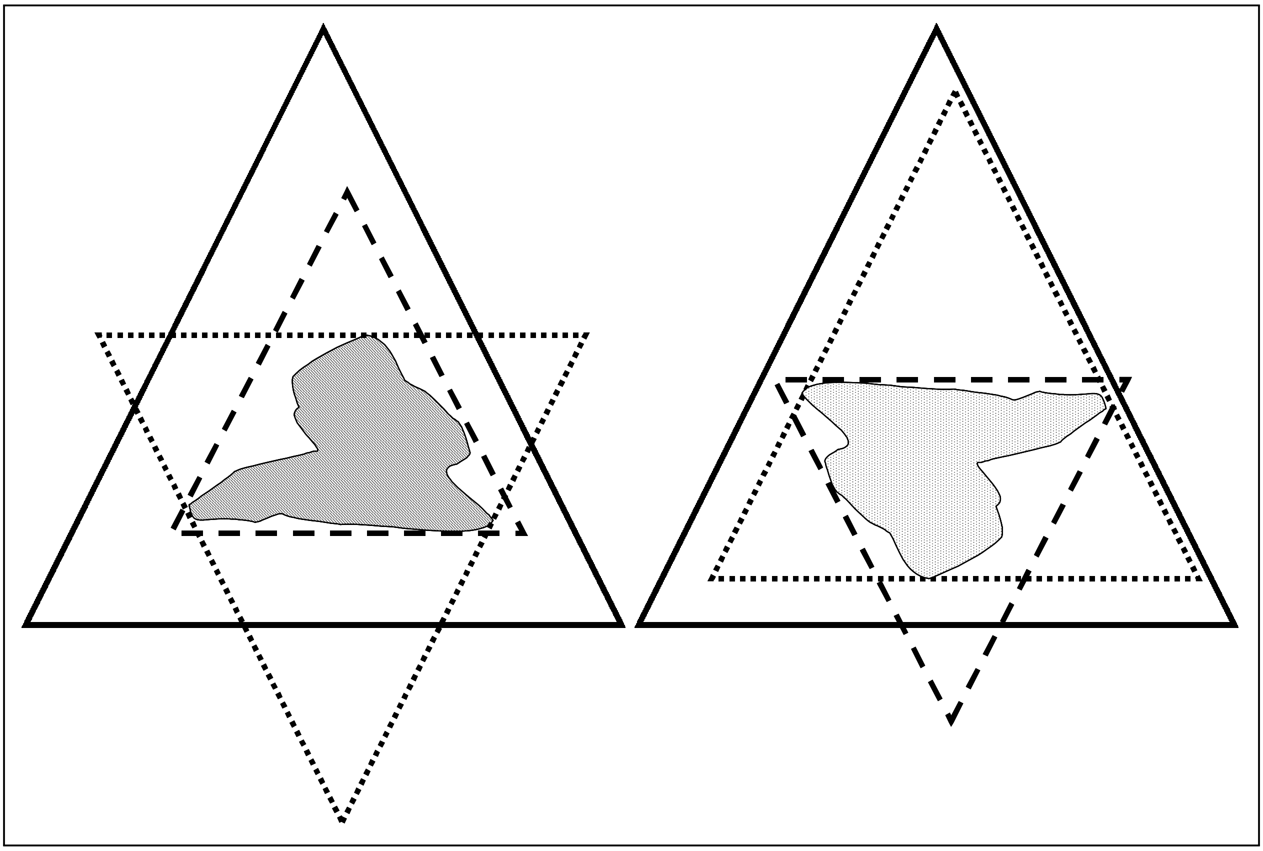

Figure 2. Minimum budget and complementary shares

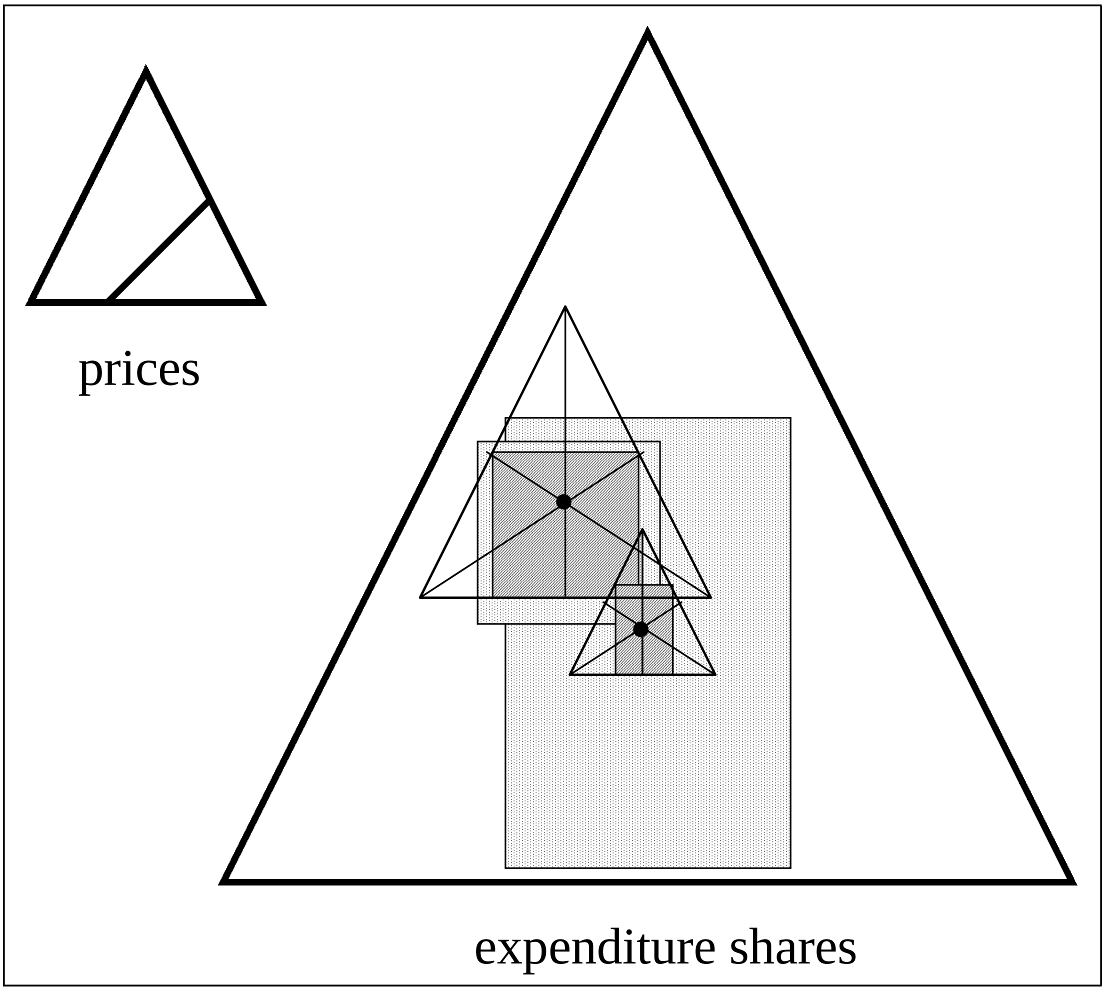

FIGURE 3. Heterogenous population and aggregate

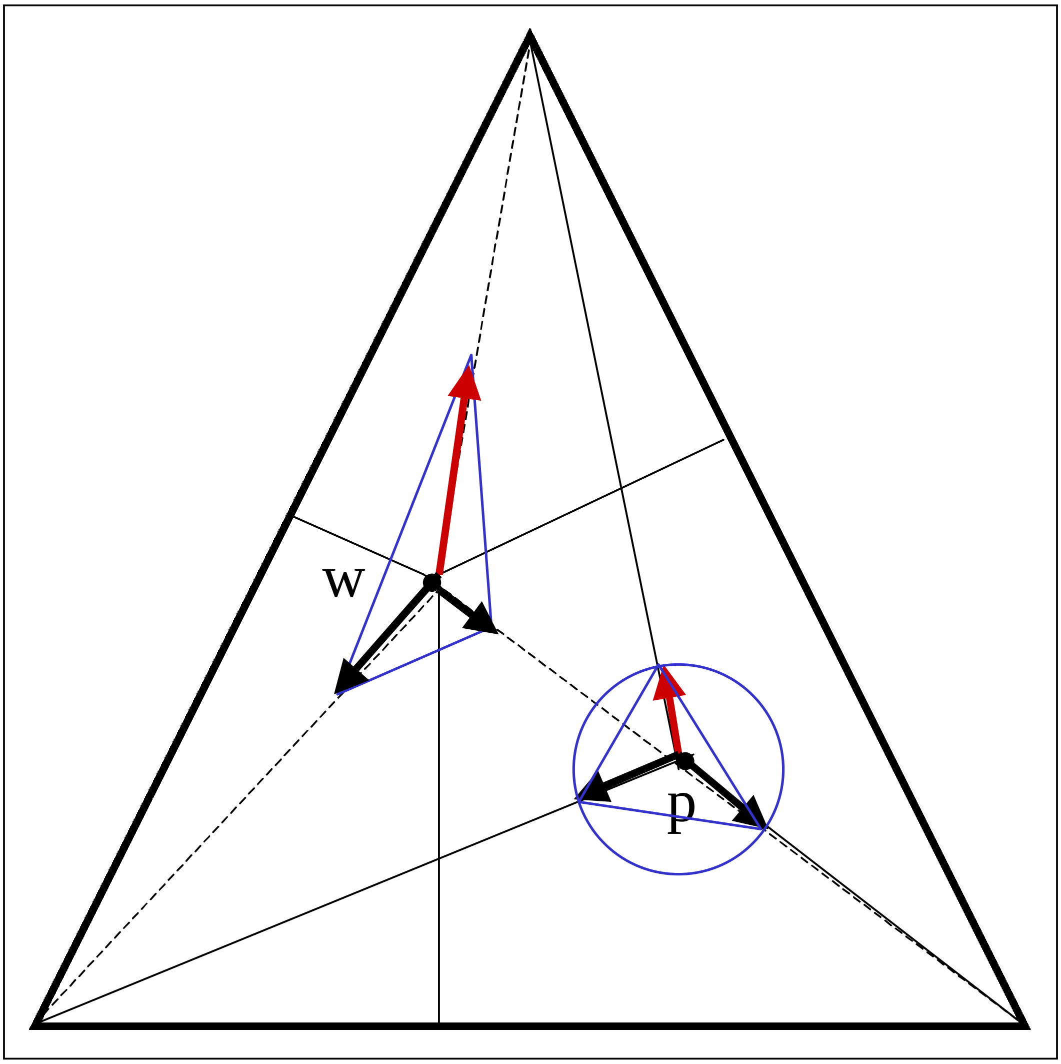

FIGURE 4. Tensorial product ...

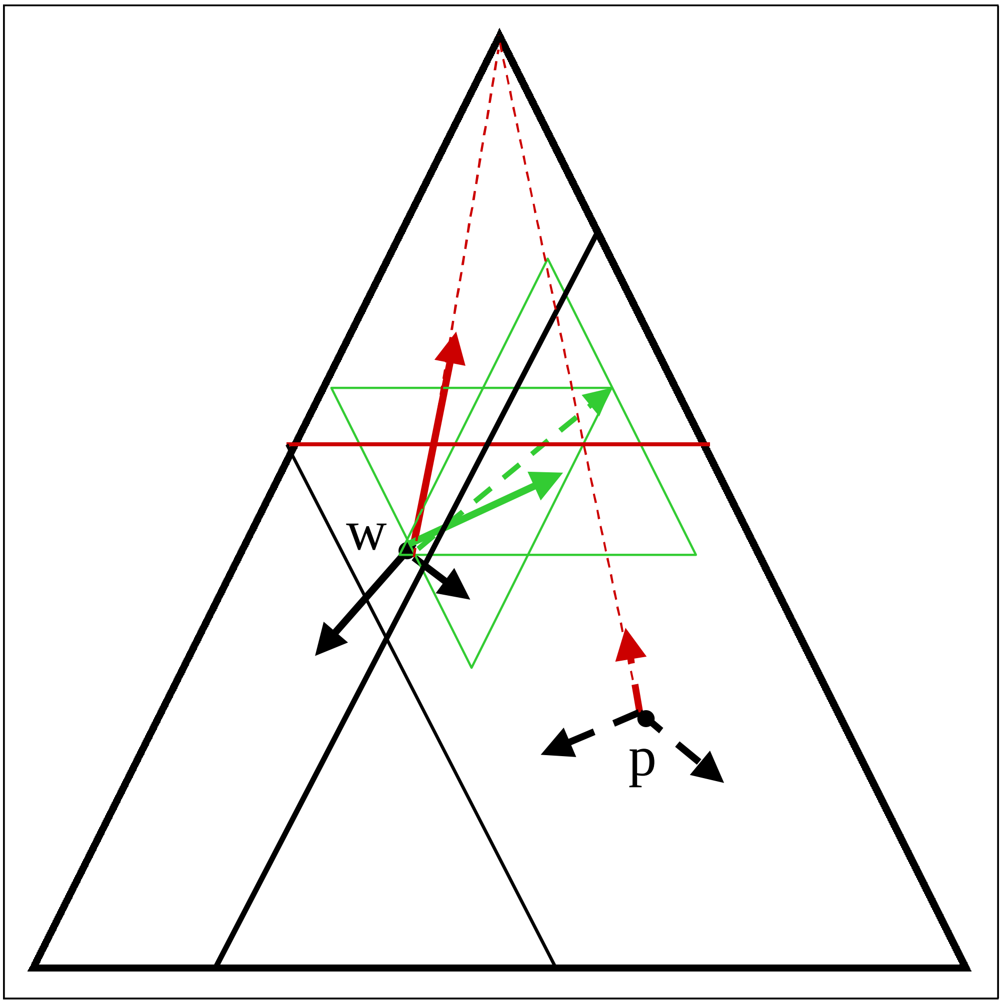

FIGURE 5. ... and the law of demand.

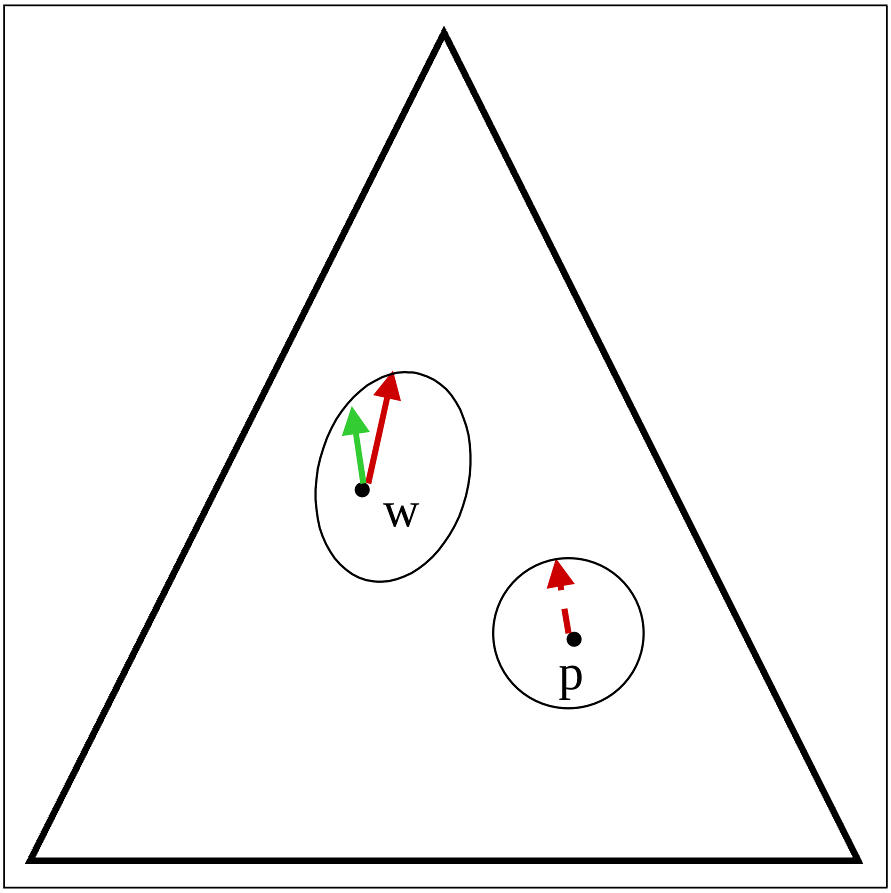

FIGURE 6. Visualizing the diagonal dominance of w price-elasticities.

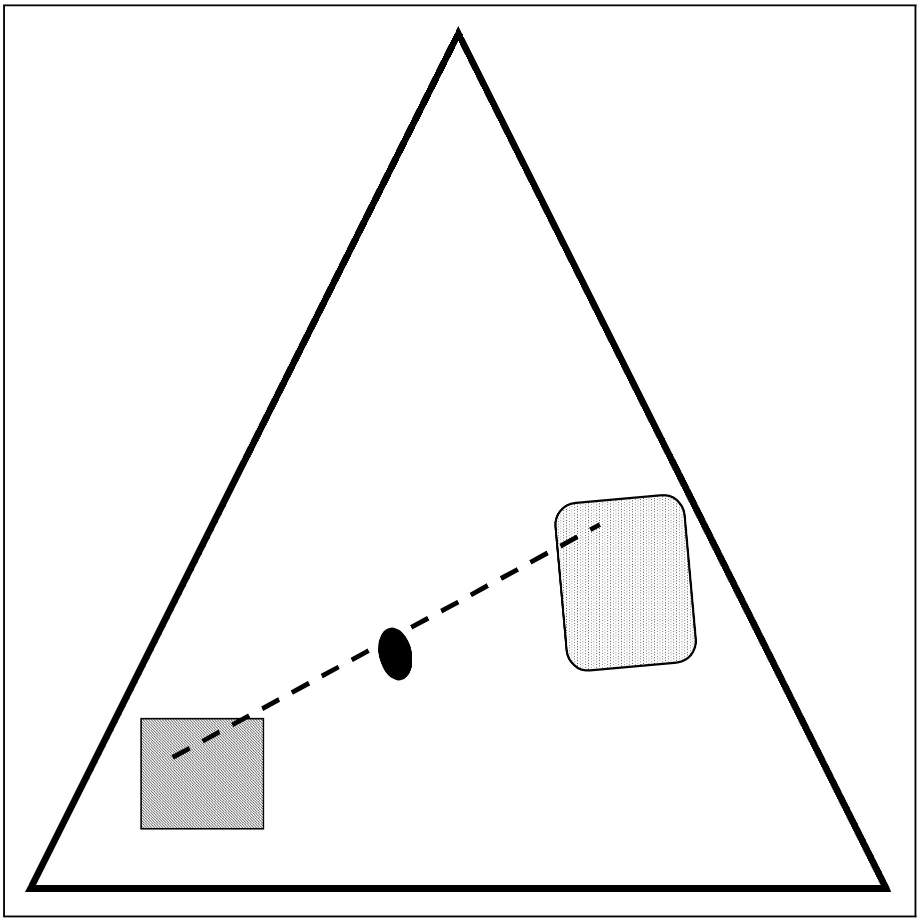

FIGURE 7. "weakly complementary" sets

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