Evaluating Asset Pricing Implications of DSGE Models

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Abstract

This paper conducts an econometric evaluation of structural macroeconomic asset pricing models. A one-sector dynamic stochastic general equilibrium model (DSGE) with habit formation and capital adjustment costs is considered. Based on the log-linearized DSGE model, a Gaussian probability model for the joint distribution of aggregate consumption, investment, and a vector of asset returns R_t is specified. We facilitate the stochastic discount factor M_t representation obtained from the DSGE model and impose the no-arbitrage condition $\mathbb{E}_{t-1}[M_t R_t] = 1$. In addition to the full general equilibrium model, we also consider consumption and production based partial equilibrium specifications, and a more general reference model. To evaluate the various asset pricing models we compute posterior model probabilities and loss function based measures of model adequacy.

JEL CLASSIFICATION: C11, C52, E32, G12

KEY WORDS Asset Pricing, Bayesian Econometrics, Business Cycle Fluctuations, Econometric Model Evaluation

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1 Introduction

In most theoretical frameworks in financial economics since the original work of Sharpe (1964), Lintner (1965), and Ross (1976), it has been argued that the expected return on any asset should be able to be expressed as a function of the particular asset's covariance with aggregate macroeconomic risks which underlie fluctuations the return to the market portfolio. Such models include various versions of the capital asset pricing model (CAPM) and the arbitrage pricing model (APT). A central approach to much of the recent research in macroeconomics, in particular business cycle theory, has been to explore how various versions of the dynamic stochastic general equilibrium (DSGE) paradigm can deliver multivariate stochastic process representations for macroeconomic aggregates which explain observed economic fluctuations in aggregate time series such as output, investment, and consumption. But originating with the work of Brock (1979, 1982) and Donaldson and Mehra (1984), it is well known that DSGE models are able to articulate the mapping between this sources of business cycle fluctuations and asset price movements. This work showed how many macroeconomic business cycle models can easily be turned into the various types of asset pricing models typically used in financial economics. In this sense, they expanded the set of joint restrictions on business cycle and asset pricing measurements which could be taken to economic data. A research agenda along these lines was suggested in Cochrane and Hansen (1992), and many recent papers have attempted to integrate asset pricing models and business cycle models within a unified theoretical framework. For example, Rouwenhorst (1995) and Jermann (1998) examine the implications of a simple one-sector DSGE model, while Boldrin *et al.* (1999) and Christiano and Fisher (1998) analyzed the joint behavior of asset returns and business cycle variables in a more complicated multi-sector model. In the latter three DSGE papers, great successes are reported in matching the stylized facts concerning the joint distribution of asset returns and business cycle measurements. In particular, the authors suggest that their models can account for the observed equity premium and average risk free rate without implying counterfactual high risk aversion of the economic agents.

An important additional advantage of DSGE models in the spirit of Brock (1982) is that they can easily be specialized to deliver partial equilibrium asset pricing models, commonly used in the financial economics literature. By imposing exogeneity on one or more of the sectors of a fully specified model economy, various asset pricing frameworks can be studied. In the so-called consumption based asset pricing models, e.g. Lucas (1978), Breeden (1979), and Hansen and Singleton (1982, 1983), the production sector is treated as exogenous. The households' intertemporal utility optimization problem is used to derive restrictions on the comovements of asset prices and consumption. Campbell (1996) and Campbell and Cochrane (1999a, 1999b) discuss recent work along this line, that finds the data to be consistent with model predictions. Alternatively, Cochrane (1991, 1996), Restoy and Rockinger (1994), Reffett (1998) and Kasa (1998) have formalized a complementary class of asset pricing models in which the household side is treated as exogenous. These partial equilibrium specifications are often referred to as production-based asset pricing (PBAP) models. Great successes have also been reported in this literature. Cochrane (1996) finds that a simple production based asset pricing model that relates asset returns to returns on aggregate investment performs as well as the CAPM and the Chen, Ross and Roll factor model and outperforms the simple consumption based model.

Among the many issues raised by recent work in asset pricing, two appear to be central in the literature on asset pricing implications of macroeconomic equilibrium models: (i) the models' ability to reproduce the observed premium on equity returns without generating counterfactual implications for other variables, such as consumption and the risk free rate. (ii) The ability of the stochastic discount factor M_t, that is obtained from the structural model, to price observed asset returns R_t, while generating economic fluctuations in consumption, investment, and output which are consistent with business cycle facts. In this paper we will focus on the second issue. We propose a unified theoretical and econometric framework for building and evaluating various forms of DSGE, CBAP, and PBAP models for macroeconomic and financial data. Instead of exploiting or testing the no-arbitrage restriction $\mathbb{E}_{t-1}[M_t R_t] = 1$ in a generalized methods of moments (GMM) framework, e.g. Hansen and Singleton (1982, 1983) and Cochrane (1996), we will consider fully specified probability models for vectors of macroeconomic variables and asset returns. We will facilitate the stochastic discount factor representations from DSGE, CBAP, or PBAP models and impose the no-arbitrage condition on the joint distribution. This approach will make the various specifications comparable, since all of them treat consumption, investment, and asset returns as endogenous, and do not rely on different sets of exogenous variables.

In general, it is assumed that asset returns and macroeconomic aggregates are jointly log-normally distributed, as for instance in Campbell and Cochrane (1999a, 1999b). The conditional log-normality assumption is more restrictive than the distributional assumption underlying the GMM approach but it will enable a very interesting likelihood based econometric analysis. To complete the specification of the probability models some auxiliary assumptions with respect to the time variation of conditional moments are necessary. Depending on the nature of these assumptions, the structural asset pricing models can be used to obtain conditional mean specifications for asset return data, conditional variance specifications, or a restriction on the time variation of conditional means and variances. In the first case, the analysis is related to the literature on predictability of stock returns, e.g. Kandal and Stambaugh (1996), Keim and Stambaugh (1986), and Kirby (1998). In the second case, the work is related to the extensive literature on models of conditional heteroskedasticity. Unlike in pure ARCH or GARCH models, in which time variation in second moments is modeled as autoregressive process, e.g. Engle (1982), Bollerslev (1986) and the subsequent literature, we are interested in examining whether the time variation could be linked to fluctuations of macroeconomic variables.

Our goal is to determine which type of fully specified or partially specified equilibrium models have the most realistic asset pricing implications. Through embedding the model restrictions into a joint probability distribution of macroeconomic and financial variables we will go a step further than it has been possible in methods of moments frameworks or in simple calibration excercises, e.g. Jermann (1998) and Boldrin *et al.* (1999). A crucial issue that has to be taken into account in the econometric evaluation is the potential misspecification of the structural asset pricing models. To cope with the potential misspecification we introduce a reduced form reference model and adopt the model evaluation approach proposed in Schorfheide (1999). By placing prior probabilities on the competing specifications and the reference model we create a mixture distribution for macroeconomic and financial variables. By conditioning on observed data, one obtains a posterior distribution for population characteristics and a posterior predictive distribution for future observations. This overall posterior distribution serves as a benchmark for the evaluation of the various structural models.

The paper is organized as follows. Section 2 provides the specification of our equilibrium model, that can be used to obtain DSGE, consumption based, and production based asset pricing models. Section 3 explains how the no-arbitrage condition is exploited to obtain various empirical models for a vector of macroeconomic and financial time series. Section 4 presents the empirical results. Section 5 concludes and discusses extensions of our work.

2 The Model Economy

Consider an economy which is formulated as a version of the model presented in Brock (1979, 1982) and Donaldson and Mehra (1984), amended to include capital adjustment costs as suggested in Cochrane (1991) and Jermann (1998). The model economy consists of a representative household and a firm. The households is endowed with ownership in the firm and a unit of time which it supplies inelasticly. The household makes consumption and savings decisions to maximize its expected lifetime discounted utility. The firm sells output goods, accumulates capital, and rents labor from households. It is endowed with a constant returns to scale production technology

$$f_t(K_t, N_t X_t) = T_t K_t^{\alpha} (N_t X_t)^{\alpha} \tag{1}$$

where K_t is the capital stock, which has been determined at time $t-1$, n_t is the input of labor, and X_t is a labor augmenting technological process. In this paper we assume that $\ln X_t = \gamma t$ is determinimistic. Total factor productivity T_t evolves according to a stationary $AR(1)$ process with innovation $\epsilon_{p,t}$. According to the transformation surface

$$\left[C_t^{\zeta} + \theta_t I_t^{\zeta}\right]^{1/\zeta} = f_t(K_t, N_t X_t) \tag{2}$$

the output can be transformed into consumption goods C_t or investment goods I_t. The specification could be interpreted as a reduced form of a two sector model (Huffman and Wynne, 1998). The process θ_t shifts the relative productivity of the investment good sector. We will assume that θ_t is an exogenous AR(1) process with steady state θ_*, driven by the innovation $\epsilon_{\theta,t}$. We assume that $\epsilon_t = [\epsilon_{p,t}, \epsilon_{\theta,t}] \sim iid\mathcal{N}(0, \Sigma_{\epsilon\epsilon})$. Unlike in Boldrin *et al.* (1999) the specification discussed in this section implies that capital can be moved frictionless from one sector to another.

The firm purchases current period investment goods to accumulate capital according to

$$K_{t+1} = (1-\delta)K_t + I_t \left(1 - \phi(I_t/K_t)\right) \tag{3}$$

where

$$\phi(I_t/K_t) = \frac{\eta}{2}\left[(I_t/K_t)^2 - \varphi\right]$$

is a capital adjustment cost, as in Cochrane (1991, 1996) and Jermann (1998).

2.1 Decision Problems

The representative household maximizes the sum of discounted expected future utility subject to a budget constraint. In period t, after realization of time t shocks, the household chooses consumption C_t, and allocates some wealth into an asset. A_t denotes the holdings of the asset and V_t its price. The household receives wage income

w_t per unit n_t of labor. Dividend payments are denoted by D_t. The household's problem can be formally stated as

$$\max_{\{C_t, A_{t+1}\}} \quad \mathbb{E}_0\left[\sum_{t=1}^{\infty} \beta^t \frac{(C_t - Z_t)^{1-\tau}}{1-\tau}\right] \tag{4}$$
$$s.t. \quad C_t + A'_{t+1}V_t \leq W_t N_t + A_t(V_t + D_t) \quad (\Lambda_t)$$
$$Z_t = \psi_1 Z_{t-1} + \psi_2(e^\gamma - \psi_1)C_{t-1} \quad (\Xi_t)$$

where Z_t denotes a habit stock. The higher consumption has been in the past, the lower is the utility obtained from a given level of consumption. Our simple model abstracts from the labor-leisure choice.

Let Λ and Ξ be the multipliers associated with the budget constraint and the habit evolution equations, respectively. The first order conditions for the strictly interior solution to this maximization problem are

$$\Lambda_t = (C_t - Z_t)^{-\tau} + \psi_2(e^\gamma - \psi_1)\beta \mathbb{E}_t[\Xi_{t+1}] \tag{5}$$
$$\Xi_t = -(C_t - Z_t)^{-\tau} + \psi_1 \beta \mathbb{E}_t[\Xi_{t+1}] \tag{6}$$
$$1 = \mathbb{E}_t[M_{t+1}R_{t+1}] \tag{7}$$

where M_t is the stochastic discount factor $\beta\Lambda_t/\Lambda_{t-1}$ and R_t is the asset return $(V_t + D_t)/V_{t-1}$.

The representative firm maximizes the expected value of the sum of its discounted net cash flow. Since the net cash flow is paid out as dividends to the shareholders, i.e., the representative household, the firm discounts date t revenues at the marginal utility of consumption $\beta_t\Lambda_t/\Lambda_0$. Formally, the firm solves the following problem

$$\max_{\{C_t^f, I_t^f, K_{t+1}, N_t^d\}} \quad \mathbb{E}_0\left[\sum_{t=0}^{\infty} \beta^t \frac{\Lambda_t}{\Lambda_0} D_t\right] \tag{8}$$
$$s.t. \quad D_t \leq C_t^f - W_t N_t^d$$
$$C_t + I_t = f(K_t, N_t X_t)$$
$$K_{t+1} \leq (1-\delta)K_t + I_t^f - \phi(I_t^f/K_t)K_t$$

The expression for dividends recognizes that investment goods are both sold and purchased by the firm. Tobin's q for this model is given by

$$Q_t = \theta_t \left(\frac{I_t}{K_t}\right)^{\zeta-1} \left(1 - \frac{3\eta}{2}(\frac{I_t^2}{K_t^2}) + \frac{\eta}{2}\varphi\right)^{-1} \tag{9}$$

The return on a one period investment is given by

$$R_{t+1}^I = \frac{Q_{t+1}}{Q_t}\left((1-\delta) + \eta\left(\frac{I_{t+1}}{K_{t+1}}\right)^3 + \frac{1}{Q_t}\left(1 + \theta_{t+1}\left(\frac{I_{t+1}}{K_{t+1}}\right)^\zeta\right)^{1-\frac{1}{\zeta}} \alpha T_{t+1} K_{t+1}^{\alpha-1} X_{t+1}\right. \tag{10}$$

Thus, we obtained a representation for the stochastic discount factor M_t from the household side, and derived a representation for the investment return R_t^I from the production side of the model economy. This stylized model economy with only one asset can be related to a more complex reality with many assets as follows. A necessary condition for the absence of arbitrage opportunities across a set of assets $j = 1, \ldots, J$, e.g. Hansen and Richard (1986), is the existence of a strictly positive random variable M_t with the property

$$\mathbb{E}_t[M_{t+1} R_{j,t+1}] = 1 \tag{11}$$

where $R_{j,t}$ is the return to the j'th asset. The model economy provides a stochastic representation for the stochastic discount factor. This paper focuses on the careful examination of the no arbitrage condition. We will use equilibrium relationships derived from the model economy together with Equation (11) to obtain a probabilistic representation of aggregate consumption, investment, and asset returns.

2.2 A Full General Equilibrium (DSGE) Model

If household and production side of the model economy are combined and the appropriate market clearing conditions are imposed, then a fully specified dynamic stochastic general equilibrium model is obtained. The model variables can be detrended by the deterministic trend $e^{\gamma t}$. Standard methods, e.g. Sims (1996), can be used to compute a log-linear approximation to the equilibrium of the model economy. The DSGE model provides a complete multivariate stochastic process representation for aggregate variables. Since it is driven by two stochastic shocks, the marginal distribution of consumption and investment will be non-singular.

2.3 A Consumption Based (CBAP) Model

To obtain a consumption based asset pricing model, the production side of the economy is regarded as exogenous. For instance, in Lucas (1978) model, consumers are receiving an endowment in every period. In Lucas' version there is no investment. However, to make all specifications comparable to each other we will introduce investment as exogenous process. The evolution of the capital stock is determined by the capital accumulation equation. The output $\tilde{C}_t$ that is available for consumption in any period t is determined via production function and transformation surface. As in Lucas' model, *ex post* the representative agent simply consumes the endowment in each period. The stochastic discount factor $M_t = \beta \Lambda_t / \Lambda_{t-1}$ can be obtained from the first order conditions of the household's problem

$$\max_{\{C_t, A_{t+1}\}} \quad I\!\!E_0 \left[\sum_{t=1}^{\infty} \beta^t \frac{(C_t - Z_t)^{1-\tau}}{1-\tau} \right] \tag{12}$$
$$s.t. \quad C_t + A_{t+1} V_t \leq \tilde{C}_t + A_t (V_t + D_t)$$
$$Z_t = \psi_1 Z_{t-1} + \psi_2 (e^{\gamma} - \psi_1) C_{t-1}$$

In our empirical illustration below we will follow a slightly different approach. Both investment and the consumption endowment $\tilde{C}_t$ are treated as jointly exogenous and are modeled as reduced form vector autoregression (VAR). The first-order conditions from the household's maximization problem, see Section 2.1, are used to map the consumption and investment process into the discount factor M_t.

2.4 A Production Based Model

To obtain a production based asset pricing model, the household side of the economy is regarded as exogenous. The firm solves the profit maximization problem (8) which leads to the investment return formula (10). *Ex post*, investment is determined by the difference between aggregate consumption and output. This is the production analog to Lucas' consumption based model. The investment return is a function of present and past values of consumption, investment, and the capital stock. Hence, the calculation of investment returns is similar to Cochrane's (1996) approach. For the empirical illustration we will treat consumption and investment as jointly exogenous and model them as VAR.

2.5 Reference Model

Consumption and investment are represented as a VAR that approximates an infinite dimensional moving average process. We simply assume that the stochastic discount factor can be expressed as a function of present and past values of the consumption and investment process. Unlike in the previous three specifications we do not assume particular functional forms that are potentially misspecified.

3 Econometric Specifications

Based upon the theoretical models introduced in the previous section, we will now specify joint probability models for aggregate consumption, investment, and a vector of asset returns. Let R_t denote an $J \times 1$ vector of one period asset returns with elements $R_{j,t}$. Moreover, define $r_t = \ln R_t$, $m_t = \ln M_t$, $w_t = [\ln C_t, \ln I_t]'$, and the $n \times 1$ vector $y_t = [w_t', r_t']$[1]. We will assume that asset returns and business cycle variables are jointly log-normally distributed. While this assumption is more restrictive than the moment assumptions that underlie a GMM analysis, it enables a likelihood based analysis. Moreover, for quarterly observations of portfolio returns the log-normality assumption is not unreasonable and quite common in the empirical finance literature. The empirical models will have the following state space form

$$y_t = Z\alpha_t + \delta_t + Fu_t \tag{13}$$

$$\alpha_t = T\alpha_{t-1} + \eta_t + G\epsilon_t \tag{14}$$

where $u_t = [\epsilon_t', \nu_t']$, $I\!E[\epsilon_t\epsilon_t'] = \Sigma_{\epsilon\epsilon}$, $I\!E[\epsilon_t\nu_t'] = \Sigma_{\epsilon\nu}$, and $I\!E[\nu_t\nu_t'] = \Sigma_{\nu\nu}$. Moreover, define $I\!E[u_tu_t'] = \Sigma_{uu}$, which is composed of $\Sigma_{\epsilon\epsilon}$, $\Sigma_{\epsilon\nu}$, and $\Sigma_{\nu\nu}$. We will use $I_{n\times n}$ to denote the $n \times n$ identity matrix, $0_{n\times m}$ to denote the $n \times m$ matrix of zeros, and $1_{n\times m}$ to denote the $n \times m$ matrix of ones. The likelihood function of the state space model can be evaluated with the Kalman Filter.

Under the assumption that asset returns and macroeconomic aggregates are conditionally log-normally distributed, the no-arbitrage condition (11) can be rewritten as follows

$$\ln I\!E_{t-1}[M_tR_t] = I\!E_{t-1}[\ln(M_tR_t)] + \frac{1}{2}var_{t-1}[\ln(M_tR_t)] = 0 \tag{15}$$

We will embody this restriction into the state space specifications below. It is assumed that the conditional expectation is taken with respect to the information set generated by all model variables, that is, both y_t and α_t.

3.1 DSGE Model

Define a $m \times 1$ vector $s_t = [c_t, i_t, m_t, k_{t+1}, t_t, r_t, \lambda_t, q_t, \theta_t, z_t, \xi_t]'$ of model variables. We will denote percentage deviations from the the deterministic steady state path s_t^* by $ds_t = s_t - s_t^*$. The DSGE model leads to the representation

$$ds_t = Dds_{t-1} + E\epsilon_t \tag{16}$$

where ϵ_t is a vector of structural disturbances. The elements of the matrices D and E are functions of the structural model parameters. Define the $1 \times m$ vector $\Gamma = [0, 0, 1, 0_{1\times(m-3)}]$ that selects the stochastic discount factor m_t from the vector s_t,

[1]Unless otherwise noted we will use lower case letters to denote logs of upper case variables.

that is, $m_t = \Gamma s_t$. The $(m+1)\times 1$ state vector α_t is of the form $\alpha_t = [I\!E_{t-1}[dm_t], ds_t']'$. The system matrices of the transition equation are given by

$$T = \begin{bmatrix} 0 & \Gamma D \\ 0_{m\times 1} & D \end{bmatrix}, \quad G = \begin{bmatrix} 0_{1\times 2} \\ E \end{bmatrix}$$

In the DSGE model $\eta_t = 0_{(m+1)\times 1}$.

Under the log-normality assumption the no-arbitrage condition can be rewritten as follows:

$$\begin{aligned} I\!E_{t-1}[r_{j,t}] &= -m^* - I\!E_{t-1}[dm_t] - \frac{1}{2} var_{t-1}[dr_{j,t}] \\ & \quad - \frac{1}{2} var_{t-1}[dm_t] - cov_{t-1}[dm_t, dr_{j,t}] \end{aligned} \tag{17}$$

We will incorporate this restriction into the specification of the measurement equation. The system matrices are

$$Z = \begin{bmatrix} 0 & 1 & 0 & 0_{1\times(m-2)} \\ 0 & 0 & 1 & 0_{1\times(m-2)} \\ -1_{J\times 1} & 0_{J\times 1} & 0_{J\times 1} & 0_{J\times(m-2)} \end{bmatrix}, \quad F = \begin{bmatrix} 0_{2\times 2} & 0_{2\times J} \\ 0_{J\times 2} & I_{J\times J} \end{bmatrix} \tag{18}$$

The vector δ_t is defined as

$$\delta_t = \begin{bmatrix} c^* + \gamma t \\ i^* + \gamma t \\ -m^* 1_{J\times 1} - \frac{1}{2} diag(\Sigma_{\nu\nu}) - \frac{1}{2}\Gamma E \Sigma_{\epsilon\epsilon}(\Gamma E)' 1_{J\times 1} - \Sigma_{\nu\epsilon}(\Gamma E)' \end{bmatrix} \tag{19}$$

where $diag(\Sigma_{\epsilon\epsilon})$ is the $J\times 1$ vector that contains the diagonal elements of $\Sigma_{\nu\nu}$. This completes the specification of the DSGE model.

3.2 Consumption Based Model

Consumption and investment follow a stationary VAR process with common deterministic trend $w_t^* = \Phi_0 + \gamma 1_{2\times 1} t$:

$$w_t - w_t^* = \Phi_1 dw_{t-1} + \ldots + \Phi_p dw_{t-p} + \epsilon_t \tag{20}$$

Note that the ϵ_t's do not have the interpretation of productivity and transformation curve shocks. However, they can be interpreted as linear combinations of the structural shocks in the DSGE model. The structural component of the CBAP model has the generic form

$$A dw_t + B ds_t = C dw_{t-1} + D ds_{t-1} \tag{21}$$

where s_t is an $m \times 1$ vector of unobserved variables $s_t = [m_t, \lambda_t, z_t, \xi_t]'$. Combining (21) with the VAR process (20) for consumption and investment yields

$$\begin{aligned} ds_t &= B^{-1}Dds_{t-1} + B^{-1}(C - A\Phi_1)dw_{t-1} \\ &\quad -B^{-1}A(\Phi_2 dw_{t-2} + \ldots + \Phi_p dw_{t-p}) - B^{-1}A\epsilon_t \end{aligned} \tag{22}$$

Define the $1 \times m$ vector $\Gamma = [1, 0_{1\times(m-1)}]$ that selects the stochastic discount factor m_t from the vector s_t. The state vector α_t is of the form $\alpha_t = [\mathbb{E}_{t-1}dm_t, ds_t']'$. The system matrices of the transition equation are given by

$$T = \begin{bmatrix} 0 & \Gamma B^{-1}D \\ 0_{m\times 1} & B^{-1}D \end{bmatrix}, \quad G = \begin{bmatrix} 0_{1\times 2} \\ -B^{-1}A \end{bmatrix} \tag{23}$$

Moreover, the vector η_t is

$$\eta_t = \begin{bmatrix} \Gamma \\ I_{m\times m} \end{bmatrix} [B^{-1}(C - A\Phi_1)dw_{t-1} - B^{-1}A(\Phi_2 dw_{t-2} + \ldots + \Phi_p dw_{t-p})] \tag{24}$$

The system matrices of the measurement equation are

$$Z = \begin{bmatrix} 0_{2\times 1} & 0_{2\times m} \\ -1_{J\times 1} & 0_{J\times m} \end{bmatrix}, \quad F = I_{n\times n} \tag{25}$$

The vector $\delta_t = [\delta_{w,t}', \delta_{r,t}']'$ is defined as

$$\delta_{w,t} = \Phi_0 + \gamma 1_{2\times 1} t \tag{26}$$

$$\begin{aligned} \delta_{r,t} &= -m^* 1_{J\times 1} - \frac{1}{2} diag(\Sigma_{\nu\nu}) \\ &\quad -\frac{1}{2}\Gamma B^{-1}A\Sigma_{\epsilon\epsilon}(\Gamma B^{-1}A)' 1_{J\times 1} + \Sigma_{\nu\epsilon}(\Gamma B^{-1}A)' \end{aligned} \tag{27}$$

This completes the specification of the CBAP model.

3.3 Production Based Model

PBAP and CBAP model share a similar structure. Consumption and investment follow a stationary VAR process with common deterministic trend $w_t^* = \Phi_0 + \gamma 1_{2\times 1} t$:

$$w_t - w_t^* = \Phi_1 dw_{t-1} + \ldots + \Phi_p dw_{t-p} + \epsilon_t \tag{28}$$

The structural component of the CBAP model has the generic form

$$Adw_t + Bds_t = Cdw_{t-1} + Dds_{t-1} \tag{29}$$

where s_t is the $m \times 1$ vector of variables $s_t = [r_t^I, q_t, k_{t+1}]'$. Define the $1 \times m$ vector $\Gamma = [1, 0_{1\times(m-1)}]$ that selects the investment return r_t^I from the vector s_t. The state vector α_t is of the form $\alpha_t = [\mathbb{E}_{t-1}[dr_t^I], ds_t']'$. The system matrices T, G, Z, F, and the vector η_t are given by Equations (23), (24), and (25) above.

The main difference between CBAP and PBAP lies in the implication of the no arbitrage restriction. The production based model does lead to a unique representation for the stochastic discount factor. Instead, one obtains a representation for the investment return r_t^I. However, it is possible to exploit the fact that under the absence of arbitrage there exists a random variable m_t that prices the investment return r_t^I, derived from the structural model, as well as the observed asset returns r_t. The no-arbitrage condition leads to

$$\mathbb{E}_{t-1}[m_t] = -\mathbb{E}_{t-1}[r_t^I] - \frac{1}{2}var_{t-1}[m_t] - \mathbb{E}_{t-1}[r_t^I] - \frac{1}{2}var_{t-1}[r_t^I] - cov_{t-1}[m_t, r_t^I] \quad (30)$$

Clearly, Equation (30) does not uniquely identify both the conditional mean and the conditional variance of the stochastic discount factor.[2] For our analysis it is actually not necessary to identify $\mathbb{E}_{t-1}[m_t]$ and $var_{t-1}[m_t]$. We can plug Equation (30) into Equation (17) and obtain the restriction

$$\begin{aligned} \mathbb{E}_{t-1}[r_t] &= r^{*I} - \mathbb{E}_{t-1}[dr_t^I] - \frac{1}{2}var_{t-1}[dr_t] \\ &\quad + \frac{1}{2}var_{t-1}[r_t^I] + cov_{t-1}[m_t, r_t^I] - cov_{t-1}[m_t, r_t] \end{aligned}$$

Under the additional assumption that the stochastic discount factor can be expressed as a function of the state in the previous period, ds_{t-1}, as well as past values of dw_t, and current ϵ_t we obtain

$$dm_t = f(ds_{t-1}, dw_{t-1}, \ldots, dw_{t-p}) + H\epsilon_t \quad (32)$$

Thus, $cov_{t-1}[dm_t, dr_t^I] = -G\Sigma_{\epsilon\epsilon}(\Gamma B^{-1}A)'$ and $cov_{t-1}[dm_t, dr_t] = \Sigma_{\nu\epsilon}G'$, which leads to $\delta_t = [\delta_{w,t}', \delta_{r,t}']'$ and

$$\begin{aligned} \delta_{w,t} &= \Phi_0 + \gamma 1_{2\times 1} t & (33) \\ \delta_{r,t} &= r^{*I} I_{J\times 1} - \frac{1}{2}diag(\Sigma_{\nu\nu}) + \frac{1}{2}\Gamma B^{-1} A \Sigma_{\epsilon\epsilon}(\Gamma B^{-1}A)' 1_{J\times 1} & (34) \\ &\quad - H\Sigma_{\epsilon\epsilon}(B^{-1}A\Gamma)1_{J\times 1} - \Sigma_{\nu\epsilon}H' \end{aligned}$$

[2] However, Equation (30) can be exploited to derive Hansen and Jaganathan (1991) type bounds. It is straightforward to show that the variance of the stochastic discount factor can be bounded by a function of its mean and the moments of the investment return:

$$\begin{aligned} &\min\left\{0, \left(\sqrt{2(\mathbb{E}_\tau[r_t^I] + \mathbb{E}_\tau[m_t])} - var_\tau^{1/2}[r_t^I]\right)^2\right\} & (31) \\ \leq\; & var_\tau[m_t] \leq \left(\sqrt{2(\mathbb{E}_\tau[r_t^I] + \mathbb{E}_\tau[m_t])} + var_\tau^{1/2}[r_t^I]\right)^2 \end{aligned}$$

A common assumption in the PBAP literature, e.g., Cochrane (1996) and Kasa (1998), to identify a stochastic discount factor is $m_t = -\ln r_t^I$. This assumption implies that $G = B^{-1}A$.

Alternatively, we are considering a specification where we do not make any assumptions with respect to $cov_{t-1}[m_t, r_t]$ and $cov_{t-1}[m_t, r_t^I]$ other than that they are constant over time. This leads to $\delta_{r,t} = \delta_r$ where δ_r is a $J \times 1$ vector of free parameters.

4 Empirical Analysis

Quarterly U.S. data from 1970:I to 1998:IV are used for the empirical analysis. The vector of observables y_t consists of log aggregate consumption of goods and services, log aggregate fixed investment, and log real returns of NYSE size portfolios 1, 5, 10.[3] The analysis is Bayesian. Our econometric models consist of a joint probability distribution for data and parameters. Prior probabilities are placed on the various model specifications. A pre-sample from 1962:II to 1969:IV is used to parameterize various prior distributions.

The DSGE model, which we will denote as $\mathcal{M}_1$ consists of the following parameters

$$\tilde{\theta}_{DSGE} = [\alpha, \beta^*, \delta, \ln T^*, \tau, \rho_p, \theta^*, \zeta, \rho_\theta, \eta, \psi_1, \psi_2]' \tag{35}$$

The coefficient matrices Z, F, T, G and the vectors δ_t and η_t of the state space representation are functions of $\tilde{\theta}_{DSGE}$. We will refer to these parameters as structural. The variance and covariance parameters are collected in the matrix

$$\Sigma_{uu} = \begin{bmatrix} \Sigma_{\epsilon\epsilon} & \Sigma_{\epsilon\nu} \\ \Sigma_{\nu\epsilon} & \Sigma_{\nu\nu} \end{bmatrix} \tag{36}$$

The non-redundant elements of Σ_{uu} are collected in the vector $\tilde{\theta}_\Sigma$. Let $\theta^{(1)} = [\tilde{\theta}_{DSGE}, \tilde{\theta}_\Sigma]$.

The marginal prior distributions for the structural parameters are summarized in columns 3 to 5 of Table 1. The shapes of the densities are chosen to match the domain of the structural parameters. The prior means correspond to values that are commonly used in the DSGE literature to calibrate structural models. As in previous studies, e.g. Canova (1994), Dejong *et al.* (1996, 1997), and Schorfheide

[3] The macroeconomic time series are extracted from the DRI database: consumption is the sum of GCNQ and GCSQ, investment is GIFQ. GPOP is used to convert the series into per capita terms. The financial series are obtained from the CRSP database. We follow Cochrane's (1996) approach to convert monthly nominal returns into quarterly returns.

(1999), it is assumed that all the structural parameters are *a priori* independent of each other. We use an Inverted-Wishart (IW) prior for Σ_{uu}

$$\Sigma_{uu} \sim IW(\nu, S) \tag{37}$$

with $\nu = 6$ and

$$S = \nu \begin{bmatrix} 0.01^2 I_{2\times 2} & 0_{2\times 3} \\ 0_{3\times 2} & \tilde{\Sigma}_{\nu\nu} \end{bmatrix}$$

where $\tilde{\Sigma}_{\nu\nu}$ is the sample covariance matrix of the log-returns r_t computed from the pre-sample.

The CBAP model, also denoted as $\mathcal{M}_2$, only uses a subset of the structural parameters of the DSGE model

$$\tilde{\theta}_{CBAP} = [\beta^*, \gamma, \tau, \psi_1, \psi_2]' \tag{38}$$

However, in addition to $\tilde{\theta}_{CBAP}$ it depends on the VAR parameters $C = [\Phi_0, \Phi_1, \dots, \Phi_p]$ and the covariance matrix Σ_{uu}. We use the same marginal prior densities for the structural parameters of the CBAP model as for the DSGE model. A prior for C and Σ_{uu} is constructed as follows. Let $\tilde{C}$ and $\tilde{\Sigma}_{\epsilon\epsilon}$ be the OLS estimates of C and $\Sigma_{\epsilon\epsilon}$ in Equation 20. based on the pre-sample. Then

$$\begin{aligned} \Sigma_{uu} &\sim IW(\nu, S) && (39) \\ C|\Sigma_{uu} &\sim \mathcal{N}\Big(\tilde{C}, \Sigma_{\epsilon\epsilon} \otimes (\nu \tilde{X}'\tilde{X}/\tilde{T})^{-1}\Big) && (40) \end{aligned}$$

where $\nu = 6$,

$$S = \nu \begin{bmatrix} \tilde{\Sigma}_{\epsilon\epsilon} & 0_{2\times 3} \\ 0_{3\times 2} & \tilde{\Sigma}_{\nu\nu} \end{bmatrix}$$

and $\tilde{X}$ is the matrix of regressors that corresponds to the coefficient matrix C. $\tilde{T}$ is the number of pre-sample observations. We are scaling the degrees of freedom for the Inverted Wishart distribution and the covariance matrix of the normal distribution to make the prior more diffuse. The parameter vector of the CBAP model is $\theta^{(2)} = [\tilde{\theta}_{CBAP}, \tilde{\theta}_C, tilde\theta_\Sigma]$. Parameter vector and prior for the PBAP model are defined in a similar fashion, with the exception that

$$\tilde{\theta}_{PBAP} = [\alpha, \beta^*, \gamma, \delta, \theta^*, \zeta, \eta]' \tag{41}$$

The reference model is fully linear. It is equipped with a conjugate Normal-IW prior. The parameters of the prior are estimated from the pre-sample. As above, the degrees of freedom for the IW prior are adjusted to be equal to 6 and the variance of the Gaussian prior is scaled by $\tilde{T}/6$.

Parameters		Prior			Posterior	
Name	Range	Density	Mean	(stdd)	Mode	(stdd)
		DSGE Model				
α	[0,1]	Beta	0.300	(0.050)	0.267	(0.001)
β^*		Fixed	0.990	N/A	0.990	N/A
γ	$I\!R$	Gaussian	0.005	(0.001)	0.004	(.0001)
δ		Fixed	0.025	N/A	0.025	N/A
$\ln T^*$	$I\!R$	Gaussian	6.000	(2.000)	6.221	(0.010)
τ	$I\!R^+$	Gamma	2.000	(1.000)	3.002	(0.404)
ρ_p	[0,1]	Beta	0.950	(0.025)	0.954	(0.013)
θ^*	$I\!R^+$	Gamma	1.000	(0.050)	1.003	(0.042)
ζ	$I\!R^+$	Gamma	1.500	(0.100)	1.428	(0.054)
ρ_θ	[0,1]	Beta	0.950	(0.025)	0.988	(0.004)
η	$I\!R^+$	Gamma	300.0	(50.00)	282.9	(23.06)
ψ_1	[0,1]	Uniform	0.500	(0.083)	0.936	(0.086)
ψ_2	$I\!R^+$	Gamma	0.200	(0.100)	0.229	(0.064)
σ_p	$I\!R^+$	Inv Gamma	0.010	(∞)	0.006	(.0003)
σ_θ	$I\!R^+$	Inv Gamma	0.010	(∞)	0.035	(0.002)
		CBAP Model				
β^*		Fixed	0.990	N/A	0.990	N/A
τ	$I\!R^+$	Gamma	2.000	(1.000)	1.464	(0.980)
ψ_1	[0,1]	Uniform	0.500	(0.083)	0.521	(0.080)
ψ_2	$I\!R^+$	Gamma	0.200	(0.100)	0.150	(0.798)
		PBAP Model				
α	[0,1]	Beta	0.300	(0.050)	0.292	(0.567)
β^*		Fixed	0.990	N/A	0.990	(N/A)
δ		Fixed	0.025	N/A	0.025	(N/A)
θ^*	$I\!R^+$	Gamma	1.000	(0.050)	0.978	(0.412)
ζ	$I\!R^+$	Gamma	1.500	(0.100)	1.346	(0.193)
η	$I\!R^+$	Gamma	300.0	(50.00)	150.1	(47.14)

Table 1: Prior and posterior distribution for the structural parameters of the DSGE, CBAP, and PBAP specifications. Posterior standard errors are calculated from Hessian matrix evaluated at the posterior mode. The parameters β and δ were fixed during the estimation.

We will subsequently adopt the following notation. Let Y_T denote the observations $y_1, \dots, y_T$. Prior and posterior model probabilities for models $\mathcal{M}_i$, $i = 1, 2, 3, *$ are denoted by $\pi_{i,0}$ and $\pi_{i,T}$, respectively. $\theta^{(i)}$ is the generic parameter vector for model $\mathcal{M}_i$, $p(\theta^{(i)}|\mathcal{M}_i)$ is its prior density, and $p(Y_T|\theta^{(i)}, \mathcal{M}_i)$ is the likelihood function.

4.1 Posterior Distributions and Model Probabilities

Since the posterior density $p(\theta^{(i)}|Y_T, \mathcal{M}_i)$ is proportional to the product of likelihood function and prior, we can obtain posterior mode estimates by

$$\hat{\theta}^{(i)}_{mode} = \text{argmax } p(Y_T|\theta^{(i)}, \mathcal{M}_i)p(\theta^{(i)}|\mathcal{M}_i) \tag{42}$$

The estimation results for the structural portion of the parameter vector are reported in columns 6 and 7 of Table 1. Standard errors are calculated from the inverse Hessian matrix, evaluated at the posterior mode.[4]

In several dimensions of the parameter space the likelihood function is not very informative and the prior is hardly updated. The Bayes estimation of the structural parameters can be interpreted as follows: find values of the structural parameters such that the model fits the data in a likelihood sense, without deviating too far from parameter values that are economically plausible.

Table 2 reports the marginal data densities $p(Y_T|\mathcal{M}_i)$ for the four models $\mathcal{M}_i$, $i = 1, \dots, 4$. The marginal data densities can be combined with prior probabilities $\pi_{i,0}$ to obtain posterior model probabilities $\pi_{i,T}$.

$$\pi_{i,T} = \frac{\pi_{i,0}p(Y_T|\mathcal{M}_i)}{\sum_{i=1}^4 \pi_{i,0}p(Y_T|\mathcal{M}_i)}, \quad p(Y_T|\mathcal{M}_i) = \int p(Y_T|\theta_i, \mathcal{M}_i)p(\theta_i|\mathcal{M}_i)d\theta_i \tag{43}$$

Here θ_i denotes the vector of parameters for model i, $p(Y_T|\theta_i, \mathcal{M}_i)$ is the likelihood function, and $p(\theta_i|\mathcal{M}_i)$ the prior density. The marginal data density for the reference model can be computed analytically, since we are using a conjugate prior for its coefficients. For the other three models we are using a Laplace approximation

$$\tilde{p}(Y_T|\mathcal{M}_i) = (2\pi)^{d/2}|\tilde{\Sigma}_i|^{1/2}p(Y_T|\tilde{\theta}_i, \mathcal{M}_i)p(\tilde{\theta}_i|\mathcal{M}_i) \tag{44}$$

based on a log-quadratic expansion around the posterior mode $\tilde{\theta}_i$. The covariance $\tilde{\Sigma}_i$ is calculated numerically from the Hessian matrix. The posterior probabilities can be used to rank the different models.

The marginal data densities indicate that the DSGE model performs slightly better than the reduced form model and the two partial equilibrium models. The ranking

[4]These preliminary calculations will be replaced by MCMC draws from the posterior.

Model	Laplace	Exact
DSGE	1293.40	
CBAP	1288.55	
PBAP	1276.68	
Reference		1290.25

Table 2: Marginal data densities for the DSGE, CBAP, PBAP specifications. The values for DSGE, CBAP, and PBAP are obtained by Laplace approximation, the value for the reference model is calculated exactly.

of the reference model is likely to change with a prior for the coefficients of the asset return equations that is more closely concentrated around zero. Most interestingly, the CBAP specification attains higher posterior probability than the PBAP model. This is contrary to the results in Cochrane (1996), who finds that the PBAP model outperforms the consumption based models. Part of the reason could be that it is more difficult to forecast investment returns based on past investment than changes in marginal utilities based on past consumption. Our approach of explicitly modeling joint distributions promises to yield new insights that go beyond a methods of moments or calibration analysis.

4.2 Stochastic Discount Factor

Conditional on observed data Y_T and parameter values $\theta^{(i)}$ it is possible to simulate the stochastic discount factor process. Let $\hat{\alpha}_{t|t} = I\!E[\alpha_t|Y_t]$ and $P_t = var[\alpha_t|Y_t]$. The Kalman Filter iterations generate the sequence $\{\hat{\alpha}_{t|t}, P_t\}_{t=1}^{T}$ which characterizes the distribution $p(\alpha_t|Y_t, \theta^{(i)}, \mathcal{M}_i)$. We will now derive a formulae to compute the moments of $\alpha_{t-1}|\alpha_t, Y_T, \theta^{(i)}, \mathcal{M}_i$ which can be used to simulate the sequence of state vectors $\alpha_1, \ldots, \alpha_T$ backwards, starting at $t = T$. This procedure is known as "smoothing", e.g., Hamilton (1994).

Consider the distribution of $[\alpha_{t-1}, y_t]$ conditional on α_t and Y_{t-1}. Using the formula for conditional means and variances of a multivariate normal distribution, it can be shown that

$$I\!E[\alpha_{t-1}|\alpha_t, Y_{t-1}] = \hat{\alpha}_{t-1|t-1} + P_{t-1}T'[TP_{t-1}T' + G\Sigma_{\epsilon\epsilon}G']^{-1}(\alpha_t - T\hat{\alpha}_{t-1|t-1} - \eta_t) \quad (45)$$

$$\begin{aligned} I\!E[y_t|\alpha_t, Y_{t-1}] &= Z(T\hat{\alpha}_{t-1|t-1} + \eta_t) + \delta_t \quad (46) \\ &\quad (ZTP_{t-1}T' + ZG\Sigma_{\epsilon\epsilon}G')[TP_{t-1}T' + G\Sigma_{\epsilon\epsilon}G']^{-1}(\alpha_t - T\hat{\alpha}_{t-1|t-1} - \eta_t) \end{aligned}$$

and

$$var\left[\begin{array}{c}\alpha_{t-1}\\ y_t\end{array}\middle|\alpha_t, Y_{t-1}\right] = \left[\begin{array}{cc}P_{t-1} - P_{t-1}T'[TP_{t-1}T' + G\Sigma_{\epsilon\epsilon}G']^{-1}TP_{t-1} & \\ -G\Sigma_{\epsilon u}[TP_{t-1}T' + G\Sigma_{\epsilon\epsilon}G']TP_{t-1} & \Sigma_{uu}\end{array}\right] \quad (47)$$

This leads to the following smoothing algorithm

$$\begin{aligned}\mathbb{E}[\alpha_{t-1}|\alpha_t, Y_t] &= \mathbb{E}[\alpha_{t-1}|\alpha_t, Y_{t-1}] \quad (48)\\ &\quad -(G\Sigma_{\epsilon u}[TP_{t-1}T' + G\Sigma_{\epsilon\epsilon}G']TP_{t-1})'\Sigma_{uu}^{-1}[y_t - \mathbb{E}[y_t|\alpha_t, Y_{t-1}]\end{aligned}$$

and

$$\begin{aligned}var[\alpha_{t-1}|\alpha_t, Y_t] &= P_{t-1} - P_{t-1}T'[TP_{t-1}T' + G\Sigma_{\epsilon\epsilon}G']^{-1}TP_{t-1}\\ &\quad -(G\Sigma_{\epsilon u}[TP_{t-1}T' + G\Sigma_{\epsilon\epsilon}G']TP_{t-1})'\Sigma_{uu}^{-1}\\ &\quad (G\Sigma_{\epsilon u}[TP_{t-1}T' + G\Sigma_{\epsilon\epsilon}G']TP_{t-1}) \quad (49)\end{aligned}$$

It can be verified that $p(\alpha_{t-1}|\alpha_t Y_t) = p(\alpha_{t-1}|\alpha_t, Y_T)$[5] We can simulate draws from the posterior distribution of the stochastic discount factor process (DSGE and CBAP model) or the investment return process (PBAP) model, by drawing from the posterior distribution of the parameters and then simulating the state vectors α_t conditional on the parameters.

[This subsection is incomplete. We are currently conducting the empirical analysis. We will also report various population moments for the discount factor, the asset returns and the aggregate variables.]

4.3 Loss Function Based Evaluation

In many cases structural models are designed to capture only limited aspects of reality and posterior probabilities are not the most interesting statistic to evaluate structural models. This argument is spelled out in detail in Schorfheide (1999). To judge asset pricing models we might not only be interested in one-step ahead forecasting performance, essentially captured by the marginal data densities, but rather in the economic loss of using one asset pricing model rather than another.

The evaluation approach can be formalized as follows: consider a decision δ and a loss function $L(\mathcal{S}_{T+h}, \varphi, \delta)$, that depends on the state of the world in period $T+h$ and some unobservable population characteristics φ. Let δ_i^* be the decision that is

[5]This is clearly true for the DSGE model, since δ_t and η_t are fixed. It is not quite true for PBAP and CBAP model. However, by expanding the state vector α_t the partial equilibrium models can be easily casted into a form for which the statement is true.

optimal under model $\mathcal{M}_i$. The various structural models $\mathcal{M}_i$ can be judged based on the posterior expected loss of the corresponding decisions δ_i^*

$$R(\delta_i^*|Y_T) = \int L(\mathcal{S}_{T+h}, \varphi, \delta_i^*) \left[\sum_{i=1,2,3,*} \pi_{i,T} p(\mathcal{S}_{T+h}, \varphi | Y_T, \mathcal{M}_i) \right] d(\mathcal{S}_{T+h}, \varphi) \quad (50)$$

This approach has been successfully applied to the evaluation of macroeconomic equilibrium models in Schorfheide (1999) and Chang *et al.* (1999).

In this subsection we will consider the following decision problem. A risk averse investor allocates his wealth to the assets $j = 1, \ldots, J$ to maximize the expected utility of next periods' wealth. Conditional on time T information we calculate the optimal portfolio allocation δ_i^* for the DSGE, CBAP, and PBAP model. To asses the different asset pricing models, we compute the posterior expected utility of next periods' wealth under the mixture of the structural models and the reference models. This evaluation procedure is closely related to the ideas in McCulloch and Rossi (1990), Kandel and Stambaugh (1996), and Avramov (1999). We assume that the investor has the same utility function as the representative agent in the model economy. The parametrization of the utility function is based on the DSGE model estimates. The level of wealth that is invested in period T is set equal to the steady state value of the assets held in our model economy.

[This subsection is incomplete. We are currently conducting the empirical analysis.]

5 Outlook

The prototypical model discussed in Section 2 is a simple one-sector model. Other authors, e.g. Boldrin *et al.* (1999), emphasize the importance of multi-sector specifications in which capital cannot move freely from one sector to the other. While these authors reported success in the model's ability to generate a realistic equity premium, it remains to be verified whether these models generate stochastic discount factors that are helpful for modeling the joint distribution of asset returns and macroeconomic variables in the sense that has been discussed in this proposal.

To obtain the CBAP and the PBAP specifications we modeled consumption and investment as exogenous vector autoregression. A theoretically more desirable but empirically potentially less successful approach is to exploit the restriction imposed on aggregate consumption and investment by the production function. Moreover, as there is a large literature on consumption based models that examines various forms of utility functions, e.g. with external or internal habit formation, it is worthwhile to expand the class of production based models and include alternative production technologies. A crucial aspect in the model development stage is to make sure that the business cycle implications of the model do not deteriorate as the asset pricing implications are improved. For this reason we emphasize the joint modeling of macroeconomic series and asset returns.

As a first step we assumed that the conditional second moments of the asset returns are constant over time. However, there is a large literature that emphasizes models of conditional heterskedasticity. The no-arbitrage condition implies that time variation of conditional covariances has to translate into time variation of the conditional mean. The next step is to incorporate conditional heterogeneity into our empirical model. While it is unlikely that our models will outperform sophisticated reduced form models of the multivariate GARCH or stochastic volatility class, it is interesting to determine whether the structural models can deliver a useful representation of conditional heteroskedasticity.

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A Approximating the Equilibrium

For the class of models we construct, under reasonable restrictions on the size of the adjustment cost, it can be shown that competitive equilibrium exist, and are at least once differentiable.[6] Therefore the use of a log-linear approximation can be justified on the grounds of the smoothness of the equilibrium. Sims (1996) method is used to obtain the equilibrium solution to the log-linearized model.

A.1 Detrending

To formulate the model is a stationary form, we stochastically detrend the endogenous variables by the productivity X. Detrended variables are denoted by a tilde. The Lagrange multipliers Λ, Ξ and Υ are discounted as follows: $\tilde{\Lambda} = \Lambda/X^{-\tau}$, $\tilde{\Xi} = \Xi/X^{-\tau}$, $\tilde{\Upsilon} = \Upsilon/X^{-\tau}$. The capital stock be normalized as $\tilde{K} = K/X$.

Now consider at stationary representation for the above restrictions embodied in the model. To do this, we first pick some parameter specifications for the primitives of the model. Given that the production technology $F(K, XN^d) = K^\alpha (XN^d)^{1-\alpha}$, then defining the stationary level of output to be $\tilde{Y} = \frac{Y}{X}$, the stationary production function can now be written as $\tilde{Y} = \tilde{K}^\alpha N^{1-\alpha}$. The stationary adjustment cost specification is $\phi(\tilde{I}/\tilde{K}) = \frac{\eta}{2}\left\{(\tilde{I}/\tilde{K})^2 - \varphi\right\}$, with $\phi'(\tilde{i}/\tilde{K}) = \eta(\tilde{I}/\tilde{K})$. Noting that $(X_t/X_{t=1})^\tau = \exp[-\tau(\gamma + \varepsilon_{x,t+1})]$, we can rescale the capital accumulation equation in (9) to obtain $\tilde{K}_{t+1} = (1-\delta)\tilde{K}_t + \tilde{I}_t - \phi(\tilde{I}_t/\tilde{K}_t) - \exp\{\gamma + \varepsilon_{x,t+1}\}$ and the stochastic discount factor $M_{t+1} = \tilde{M}_{t+1}\exp[-\tau(\gamma + \varepsilon_{x,t+1})]$. Dividends then are $\tilde{D}_t = \tilde{K}_t^\alpha N_t^{1-\alpha} - \tilde{W}_t N_t - \tilde{I}_t$, with $\tilde{W}_t N_t = (1-\alpha)\tilde{K}_t^\alpha N_t^{1-\alpha}$. Therefore $\tilde{D}_t = \alpha \tilde{K}_t^\alpha N_t^{1-\alpha} - \tilde{I}_t$. Tobin's Q_t and the investment return R_t^I have no trend, and are therefore given as $Q_t = \left(\tilde{I}_t/\tilde{K}_t\right)^{\zeta-1}\left(1 - \frac{3\eta}{2}(\tilde{I}_t/\tilde{K}_t)^2 + \frac{\eta\varphi}{2}\right)^{-1}$ and $R_{t+1}^I = \frac{Q_{t+1}}{Q_t}\left(1 - \delta + \eta\left[\tilde{i_{t+1}}/\tilde{K_{t+1}}\right]^3\right) + \frac{1}{Q_t}\left(1 + \theta_{t+1}\left[\tilde{I_{t+1}}/\tilde{K_{t+1}}\right]^\zeta\right)^{1-\frac{1}{\zeta}} \alpha P_{t+1} K_{t+1}^{\alpha-1}$. Then rewriting the conditional asset pricing model in (16) in stationary form, we obtain

$$\mathbb{E}_t\left[\tilde{M}_{t+1}\exp\{(1-\tau)(\gamma + \varepsilon_{x,t+1})\}R_{t+1}^I\right] = 1 \tag{51}$$

The detrended model can be log-linearized around its steady state.

[6] For versions of the model which are Pareto optimal (i.e., all versions except the one with external habit formation), existence and uniqueness of a continuous equilibrium manifold with respect to the vector of parameters can be shown to follow from now standard applications of the second welfare theorem to this equilibrium. As for the smoothness of equilibrium, we require that the adjustment cost not be too large (so as to make the value function no longer a strictly concave function of its first argument.) In such a case, the arguments of Araujo and Scheinkman (1977), Araujo (1991), and in particular Santos (1991) can be shown to apply.

A.2 The Full DSGE Model

A.2.1 Calculating the Steady State

We set the adjustment cost parameter φ so adjustment costs are zero in the steady state, i.e., $\varphi = \left(\frac{I^*}{K^*}\right)^2$ where any variable Z written in steady state is denoted as Z^*. From the capital accumulation equation, $\frac{I^*}{K^*} = \gamma + \delta - 1$ (where recall γ is the growth factor associate with the technological shock x). The steady state version of the firm's Euler equation is given as

$$\beta\gamma^{*\tau}\left\{1-\delta+\eta(\gamma+\delta-1)^3+\frac{1}{Q}\left(1+\theta^*(\frac{I^*}{C^*})^\zeta\right)^{1-\frac{1}{\zeta}}\alpha\rho^*(K^*)^{\alpha-1}\right\}=1 \tag{52}$$

Using the expression for the steady state capital stock which is $K^* = \gamma + 1 - \delta$, it can be shown that

$$\frac{1}{Q}\left(1+\theta^*(\frac{I^*}{C^*})^\zeta\right)^{1-\frac{1}{\zeta}} = \left(1-\eta(\gamma+1-\delta)^2\right)\left(\left(\frac{C^*}{I^*}\right)^\zeta+\theta^*\right)^{\frac{\zeta-1}{\zeta}} \tag{53}$$

Using the aggregator in the social feasibility condition, we obtain

$$\left(C^{*\zeta}+\theta^* I^{*\zeta}\right)^{\frac{1}{\zeta}} = \rho^* K^{*\alpha}$$

or

$$\left(\left(\frac{C^*}{I^*}\right)^\zeta+\theta^*\right)^{\frac{\zeta-1}{\zeta}} = \left(\frac{T^*}{\gamma+\delta-1}\right)^{\frac{\zeta-1}{\zeta}} K^{*(\alpha-1)(\zeta-1)} \tag{54}$$

Conbining these expressions, we obtain

$$\frac{1}{Q}\left(1+\theta^*(\frac{I^*}{C^*})^\zeta\right)^{1-\frac{1}{\zeta}} = \left(1-\eta(\gamma+\delta-1)^2\right)\left(\frac{T^*}{\gamma+\delta-1}\right)^{\frac{\zeta-1}{\zeta}} K^{*(\alpha-1)(\zeta-1)} \tag{55}$$

The firm's Euler equation then becomes

$$\begin{aligned} 1 \;=\;& \beta\gamma^{*\tau}\{1-\delta+\eta(\gamma+\delta-1)^3+ \\ & \alpha T^*\left(1-\eta(\gamma+\delta-1)^2\right)\left(\frac{T^*}{\gamma+\delta-1}\right)^{\zeta-1} K^{*(\alpha-1)\zeta}\} \end{aligned} \tag{56}$$

Therefore the steady state capital stock is

$$K^* = \left\{\frac{\frac{1}{\beta}\gamma^{*\tau}-(1-\delta)-\eta(\gamma+\delta-1)^3}{\alpha T^*(1-\eta(\gamma+\delta-1)^2)(\frac{T^*}{\gamma+\delta-1})^{\zeta-1}}\right\}^{\frac{1}{\zeta(1-\alpha)}} \tag{57}$$

with the steady state Tobin's Q given as

$$Q^* = \left(\frac{I^*}{C^*}\right)^{\zeta-1}\left(1-\eta(\frac{I^*}{K^*})^2\right)^{-1} \tag{58}$$

The steady state discount factor is $M^* = \beta$. The steady-state capital stock is obtained from the capital accumulation equation

$$I^* = K^*(\gamma^* + \delta - 1) \tag{59}$$

The steady state consumption can be determined from the aggregate resource constraint

$$C^* = \left((T^* K^{*\alpha})^\zeta - (\theta^* I^*)^\zeta\right)^{1/\zeta} \tag{60}$$

Under the three different specifications of habit we obtain

1. No habit:

$$\Lambda^* = C^{*-\tau} \tag{61}$$

2. No persistence in habit

$$\Lambda^* = C^{*-\tau}(1-\psi_2)^{-\tau}[1 - \psi_2 \beta \gamma^{*(1-\tau)}] \tag{62}$$

3. AR(1) habit

$$\begin{aligned} Z^* &= \psi_2 C^* & (63)\\ \Xi^* &= \frac{C^{*-\tau}(1-\psi_2)^{-\tau}}{\beta\psi_1\gamma^{*}-\tau - 1} & (64)\\ \Lambda^* &= C^{*-\tau}(1-\psi_2)^{-\tau} + \beta\psi_2 \Xi^* \gamma^{*-\tau}(\gamma^* - \psi_1) & (65) \end{aligned}$$

A.2.2 Log-linearizations

We can know discuss the loglinearization procedure we used. The linearized version of the capital accumulation equation in (XX) is given as

$$\gamma^* dk_{t+1} = \left[(1-\delta) + \eta\left(\frac{I^*}{K^*}\right)^3\right] dk_t + \left[\frac{I^*}{K^*}(1 - \eta\left(\frac{I^*}{K^*}\right)^2)\right] di_t \tag{66}$$

The linearized version of Tobin's q is given by

$$Q^* dq_t = [Q^*(1-\zeta)]\,(dc_t - di_t) + \kappa_1(di_t - dk_t) \tag{67}$$

where

$$\kappa_1 = \left(\frac{I^*}{C^*}\right)^{\zeta-1} \frac{[3\eta\left(\frac{I^*}{K^*}\right)^2]}{\left[1 - \eta\left(\frac{I^*}{K^*}\right)^2\right]^2}$$

We write the linearized investment return as

$$
\begin{aligned}
R^* dr_{t+1} &= \left[(1-\delta)+\eta\left(\frac{I^*}{K^*}\right)^3\right] dq_{t+1} - \left[(1-\delta)+\eta\left(\frac{I^*}{K^*}\right)^3 + \kappa_2\right] dq_t \\
&\quad +(3\eta+\zeta\kappa_3)di_t - \kappa_3\zeta dc_{t+1} + \kappa_3 d\theta_{t+1} + \kappa_2 dp_{t+1} \\
&\quad + \left[(\alpha-1)\kappa_2 - 3\eta\right] dk_{t+1} \qquad (68)
\end{aligned}
$$

where

$$
\begin{aligned}
\kappa_2 &= \frac{\alpha T^* K^{*(\alpha-1)}}{Q^*}\left[1-\theta^*(\frac{I^*}{C^*})^\zeta\right]^{1-\frac{1}{\zeta}} \\
\kappa_3 &= \frac{\alpha T^* K^{*(\alpha-1)}}{Q^*}\theta^*(\frac{I^*}{C^*})^\zeta(1-\frac{1}{\zeta})\left[1-\theta^*(\frac{I^*}{C^*})^\zeta\right]^{-\frac{1}{\zeta}}
\end{aligned}
$$

The log-linearization of the social transformation surface is

$$
\frac{1}{\zeta}(C^{*\zeta}+\theta^* I^{*\zeta})^{\frac{1}{\zeta}-1}\left[C^{*\zeta}dc_t + \zeta\theta I^{*\zeta} di_t + \theta I^* d\theta_t\right] = T^* K^{*\alpha}[dT_t + \alpha dk_t\} \qquad (69)
$$

Finally the loglinear version of the shock process for the innovation in the transformation surface is

$$
d\theta_t = \rho_\theta d\theta_{t-1} + \varepsilon_{\theta t} \qquad (70)
$$

The stochastic discount factor evolves according to

$$
dm_t = d\lambda_t - d\lambda_{t-1} \qquad (71)
$$

and $d\lambda_t$ is determined as follows:

1. No habit:

$$
d\lambda_t = -\tau dc_t \qquad (72)
$$

2. No persistence in habit

$$
\begin{aligned}
C^{*\tau}\lambda_* d\lambda_t &= -\tau(1-\psi_2)^{-\tau-1}[dc_t - \psi_2 dc_{t-1} + \psi_2\epsilon_{X,t}] \\
&\quad -\psi_2\tau\beta\gamma^{*(1-\tau)}(\mathbb{E}_t[dc_{t+1}] - \psi_2 dc_t) \qquad (73)
\end{aligned}
$$

3. AR(1) habit: Define the constants

$$
\begin{aligned}
\kappa_4 &= \frac{\psi_1+\psi_2(\gamma^*-\psi_1)}{1-\psi_2} \qquad (74) \\
\kappa_5 &= \frac{1}{\beta\psi_1\gamma^{-\tau}-1} \qquad (75) \\
\kappa_6 &= \frac{\beta\gamma^{*-\tau}[\psi_2(\gamma^*-\psi_1)+\psi_1]-1}{\beta\psi_1\gamma^{*-\tau}-1} \qquad (76)
\end{aligned}
$$

The log-linearized equations are of the form

$$\begin{aligned} dz_t &= \frac{\psi_1}{\gamma^*} dz_{t-1} + \frac{\gamma_* - \psi_1}{\gamma^*} dc_{t-1} - \epsilon_{X,t} && (77) \\ 0 &= \tau\kappa_4[dc_t - dz_t] + \psi_1\kappa_6 d\lambda_t - \psi_2(\gamma^* - \psi_1)\kappa_5 d\xi_t && (78) \\ \kappa_5 d\lambda_t &= \psi_2\beta(\gamma^* - \psi_1)\gamma^{-\tau}\kappa_4 I\!\!E_t[d\xi_{t+1}] - \frac{\tau}{1-\psi_2}[dc_t - \psi_2 dz_t] && (79) \end{aligned}$$

A.3 Consumption Based Model

A.3.1 Calculating the Steady-State

The steady states of consumption and investment C^*, I^* are exogenous. Thus, K^* can be obtained from the capital accumulation equation

$$K^* = I^*/(\gamma^* + \delta - 1) \quad (80)$$

Moreover, $M^* = \beta$. The determination of Λ^*, Ξ^*, Z^* is the same as in the general equilibrium case.

A.3.2 Log-linearizations

The CBAP model consists of the exogenous consumption and investment process

$$[dc_t, di_t]' = \Phi_1[dc_{t-1}, di_{t-1}]' + \ldots + \Phi_p[dc_{t-p}, di_{t-p}] + \epsilon_t \quad (81)$$

The stochastic discount factor evolves according to

$$dm_t = d\lambda_t - d\lambda_{t-1} \quad (82)$$

and $d\lambda_t$ is determined as follows:

1. No habit:

$$d\lambda_t = -\tau dc_t \quad (83)$$

2. No persistence in habit

$$\begin{aligned} c^{*\tau}\lambda_* d\lambda_t &= -\tau(1-\psi_2)^{-\tau-1}[dc_t - \psi_2 dc_{t-1} + \psi_2\epsilon_{X,t}] \\ &\quad -\psi_2\tau\beta\gamma^{*(1-\tau)}(I\!\!E_t[dc_{t+1}] - \psi_2 dc_t) && (84) \end{aligned}$$

3. AR(1) habit: Define the constants

$$\kappa_4 = \frac{\psi_1 + \psi_2(\gamma^* - \psi_1)}{1 - \psi_2} \quad (85)$$

$$\kappa_5 = \frac{1}{\beta\psi_1\gamma^{-\tau} - 1} \quad (86)$$

$$\kappa_6 = \frac{\beta\gamma^{*-\tau}[\psi_2(\gamma^* - \psi_1) + \psi_1] - 1}{\beta\psi_1\gamma^{*-\tau} - 1} \quad (87)$$

The log-linearized equations are of the form

$$dz_t = \frac{\psi_1}{\gamma^*} dz_{t-1} + \frac{\gamma_* - \psi_1}{\gamma^*} dc_{t-1} - \epsilon_{X,t} \quad (88)$$

$$0 = \tau\kappa_4[dc_t - dz_t] + \psi_1\kappa_6 d\lambda_t - \psi_2(\gamma^* - \psi_1)\kappa_5 d\xi_t \quad (89)$$

$$\kappa_5 d\lambda_t = \psi_2\beta(\gamma^* - \psi_1)\gamma^{-\tau}\kappa_4 \mathbb{E}_t[d\xi_{t+1}] - \frac{\tau}{1 - \psi_2}[dc_t - \psi_2 dz_t] \quad (90)$$

The cases 2 and 3 require the solution of a rational expectations system.

A.4 Production Based Model

A.4.1 Calculating the Steady-State

The steady states of consumption and investment C^*, I^* are exogenous. Thus, K^* can be obtained from the capital accumulation equation

$$K^* = I^*/(\gamma^* + \delta - 1) \quad (91)$$

The investment return is $R^* = 1/(\beta\gamma^{*\tau})$ and Tobin's q is

$$Q^* = \left(\frac{I^*}{C^*}\right)^{\zeta - 1} \left(1 - \eta(\frac{I^*}{K^*})^2\right)^{-1} \quad (92)$$

A.4.2 Log-linearizations

The CBAP model consists of the exogenous consumption and investment process

$$[dc_t, di_t]' = \Phi_1[dc_{t-1}, di_{t-1}]' + \ldots + \Phi_p[dc_{t-p}, di_{t-p}] + \epsilon_t \quad (93)$$

and a capital accumulation equation of the form

$$\gamma^* dk_{t+1} = \left[(1 - \delta) + \eta\left(\frac{I^*}{K^*}\right)^3\right] dk_t + \left[\frac{I^*}{K^*}(1 - \eta\left(\frac{I^*}{K^*}\right)^2)\right] di_t \quad (94)$$

The linearized version of Tobin's q is given by

$$Q^* dq_t = [Q^*(1 - \zeta)](dc_t - di_t) + \kappa_1(di_t - dk_t) \quad (95)$$

where

$$\kappa_1 = \left(\frac{I^*}{C^*}\right)^{\zeta-1} \frac{\left[3\eta\left(\frac{I^*}{K^*}\right)^2\right]}{\left[1-\eta\left(\frac{I^*}{K^*}\right)^2\right]^2}$$

The investment return evolves according to

$$\begin{aligned} R^* dr_{t+1} &= \left[(1-\delta)+\eta\left(\frac{I^*}{K^*}\right)^3\right] dq_{t+1} - \left[(1-\delta)+\eta\left(\frac{I^*}{K^*}\right)^3 + \kappa_2\right] dq_t \\ &\quad + (3\eta+\zeta\kappa_3) di_t - \kappa_3\zeta dc_{t+1} + \kappa_2 dp_{t+1} \\ &\quad + \left[(\alpha-1)\kappa_2 - 3\eta\right] dk_{t+1} \end{aligned} \tag{96}$$

where

$$\begin{aligned} \kappa_2 &= \frac{\alpha T^* K^{*(\alpha-1)}}{Q^*}\left[1-\theta^*(\frac{I^*}{C^*})^\zeta\right]^{1-\frac{1}{\zeta}} \\ \kappa_3 &= \frac{\alpha T^* K^{*(\alpha-1)}}{Q^*}\theta^*(\frac{I^*}{C^*})^\zeta(1-\frac{1}{\zeta})\left[1-\theta^*(\frac{I^*}{C^*})^\zeta\right]^{-\frac{1}{\zeta}} \end{aligned}$$