Simultaneous-Offers Bargaining and the Deadline Effect[1]

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Abstract

A one-shot simultaneous-offers bargaining game is presented in which the unique pure strategy equilibrium offers are identical to those of the infinite-horizon Rubinstein alternating-offers game. For each player there is a small probability that his or her proposal will not arrive. A finitely-repeated version of the game with a small amount of (two-sided) incomplete information about disagreement payoffs is then used to explain the deadline effect. In any pure strategy equilibrium of this game agreement is reached only in the final period.

Keywords: Simultaneous-offers bargaining, Nash demand game, deadline effect.

1 Introduction

This paper has two purposes. Firstly, it analyzes a bargaining game with complete information in which the players make simultaneous offers. This game differs in the following two ways from the simultaneous-offer extensive forms found in the existing literature. (i) The offer process has a small amount of noise: for each player, there is a small probability that her demand (or, in an alternative interpretation, her acceptance of the other's demand) does not reach the other player. (ii) In the usual formulations of

[1] I am indebted for helpful comments to seminar audiences at LSE and Cambridge.

simultaneous-offers bargaining agreement is assumed to be reached automatically if the two players' demands are jointly feasible and, if not, each receives his disagreement payoff; in the game considered here, by contrast, each player, simultaneously, has an opportunity to accept or reject the other's proposal. The problem with existing simultaneous-offer models, in general, is their multiplicity of equilibrium outcomes. In the game presented here, on the other hand, there is a unique pure strategy perfect Bayesian equilibrium outcome. Although this is a one-shot game the equilibrium offers are identical to those of the infinite-horizon alternating offers bargaining game of Rubinstein (1982). This analysis provides a perturbed version of Nash's demand game (Nash (1953)) to complement those of Binmore (1987a) and Carlsson (1991) and provides added insight into the interpretation of the Nash bargaining solution.

The second purpose of the paper is to explain the deadline effect, the often-noticed phenomenon that when bargainers are subject to a deadline they frequently delay until the last possible moment before reaching agreement. The explanation given here is based on a two-sided, two-type incomplete information finitely-repeated version of the simultaneous-offers game described above. For each player there is an arbitrarily small probability that her disagreement payoff is high (more than half of, but less than the whole of, the available surplus). The result is that if the length of the game is not too great then pure strategy equilibria exist and, in any pure strategy equilibrium, agreement is reached only in the final period.

The most important non-cooperative bargaining game is the alternating-offers model analyzed by Rubinstein (1982). In reality there is usually no specific protocol constraining the identity of the first offeror or the order of moves, so it is natural to ask why, in the Rubinstein model, the players are treated in an ex ante asymmetric fashion[2] and what prevents player 2 from making a proposal in a period in which it is player 1's turn to propose. One answer is that, as the time between offers goes to zero, the first and second

[2]If there is an advantage to be had by making the first offer one would expect there to be a pre-negotiation over who should have the right to do so. Binmore (1987c) attributes this argument to Selten. See Evans (1997) for analysis of a coalitional bargaining game in which the players bid for the right to make the first offer.

equilibrium offers become close to one another and, in the limit, the ex ante asymmetry disappears (Binmore (1987b)). Yet, in some applications, it may be that the time between offers is relatively large because it takes a non-negligible time to formulate an offer and to consider an offer of the opponent. Furthermore, in some models the time at which a player makes an offer is chosen strategically by that player. For example, Admati and Perry (1987), in an incomplete information context, use such a model to explain delay and Ma and Manove (1993), as discussed more fully below, use one to explain the deadline effect. In such cases the assumption that a player can wait indefinitely before making her offer, thus imposing a cost on her opponent, without that opponent being allowed to make a proposal, is not an innocuous one. More generally, one would like to know if the standard bargaining results apply when the order of moves is relatively unconstrained.

In a discrete-time context, the most free-form representation of the bargaining process is one in which the players are allowed to make a proposal in every period. In the Nash demand game (Nash (1952)) each player makes a demand; if the demands are compatible, each obtains what he or she has demanded and, otherwise, each gets his or her disagreement payoff. It is easy to see that there is a Nash equilibrium of this game corresponding to each feasible, Pareto-efficient pair of utilities which Pareto-dominates the disagreement point. Nash also suggested a "smoothed" version of this game in which, after a feasible pair of proposals x and y have been made, the probability that they are implemented is $p(x, y)$ where p varies smoothly with x and y. Binmore (1987a) and Carlsson (1991) have analyzed versions of this game. In one version of it, the randomness arises because there is some uncertainty in the minds of the players about the actual extent of the feasible set of utilities. In the other, a small error term is added to each bargainer's intended demand so that it may be, for example, that the pair of intended demands (x, y) is feasible but, because of the added noise, the pair of actual demands is not and the players therefore get their disagreement payoffs. The result is that, in the Pareto-dominant Nash equilibrium, the payoffs approach the Nash bargaining solution as the errors go to zero. A drawback of this result is that it depends on fairly special assumptions about the distribution of the noise and the nature of the convergence of the distributions (see Carlsson (1991)). Furthermore, some multiplicity of equilibrium remains: there are trivial equilibria in which no agreement is reached and also, perhaps, non-trivial ones which do not converge to the Nash bargaining solution (they converge to trivial equilibria).

The game analyzed here is a different modification of the Nash demand game, the single-stage version of which is as follows. The players each make a proposal, simultaneously. On receiving the other's proposal, each has to decide, again simultaneously, whether to accept it. This step embodies the idea that a contract which player i has proposed to j is not binding unless player j actually signs it. If only one of the proposals is accepted then that proposal is implemented. If both are accepted then one of them is chosen at random to be implemented (for most of the paper it is assumed that each is chosen with probability one half). If each player rejects the other's proposal then they get their disagreement payoffs. For each player, there is a small strictly positive probability, ε, that her demand does not reach the other player. In an alternative, equivalent, game the demands arrive with probability one but there is probability ε that player i's acceptance of j's demand gets lost. There are various possible interpretations of this formulation. For example, it may be that the two parties are negotiating at a distance, perhaps by telephone or fax. Alternatively, the bargainers meet face-to-face to present their proposals but a player cannot accept the other's proposal until he has consulted a principal who is not present at the negotiations and this communication introduces a small probability of error.

This game has a unique pure strategy perfect Bayesian equilibrium (PBE) outcome. In equilibrium each accepts the other's proposal and, as ε converges to zero, both equilibrium proposals converge to the Nash bargaining solution. The equilibrium proposals $(x_1(\varepsilon), x_2(\varepsilon))$ (both expressed in terms of the share of the pie going to player 1) have the property that i ($i = 1, 2$), having proposed $x_i(\varepsilon)$, is indifferent between accepting $x_j(\varepsilon)$ ($j \neq i$) and rejecting it. For example, player 1 is indifferent between accepting $x_2(\varepsilon)$, which is slightly less than her proposal $x_1(\varepsilon)$, and, by holding out for $x_1(\varepsilon)$ (which she knows will be accepted if it arrives), running the small risk of getting the disagreement payoff because $x_1(\varepsilon)$ has been lost in the post. Even though this is a single-period game, the two proposals are the same as the equilibrium proposals of Rubinstein's infinite-horizon alternating-offers game with discount factor, or breakdown probability, equal to $(1-\varepsilon)/(1+\varepsilon)$. I also study the finite horizon repeated version of the game. Again there is a unique pure strategy PBE outcome and in equilibrium both players accept the first proposal.

The above results are derived in Sections 2. In Section 3 a version of the model with two-sided incomplete information is used to explain the deadline effect. It often happens that parties engaged in bargaining have to reach agreement, if at all, before some exogenously imposed deadline. For example, two litigants may negotiate in advance of a trial, the date of which is fixed. Or two parties may have a joint project which they must embark on by a certain date if it is not to be exploited by some third party. For a third example, a buyer and a seller may be bargaining over the production and sale of a good which, if they do not agree on terms by a certain date, will become obsolete. It has often been noted that in such situations there is a pronounced deadline effect: a deal is often struck only at the last possible moment. Roth, Murnighan and Schoumaker (1988) report the results of several bargaining experiments in which they found a strong and robust effect of this kind. There are many explanations of delay in the bargaining literature[3] but few explanations of the deadline effect. Fershtman and Seidmann (1993) derive this effect in a game in which a bargainer is not allowed to accept an offer less than or equal to an offer which he or she has previously rejected. Spier (1992) analyzes bargaining between a plaintiff and a defendant in advance of a trial. As she points out, this differs from other bargaining contexts in that discounting does not imply that delay

[3]Complete information bargaining games which exhibit delay include Dekel (1990), which employs forward induction arguments, Fernandez and Glazer (1991), in which the bargainers play a game after each rejection of an offer, Jehiel and Moldovanu (1995), in which there are several buyers and externalities between them, and Compte and Jehiel (1998), in which outside options are history-dependent. Incomplete information models with delay involve screening, signalling or wars of attrition (e.g., Cramton (1984, 1992) Admati and Perry (1987), Chatterjee and Samuelson (1987)). Kennan and Wilson (1993) provides a good survey.

destroys surplus. She shows that there is a significant deadline effect when there is a per-period cost of bargaining, one-sided incomplete information and all offers are made by the uninformed player. Essentially the player who makes the offers has an incentive to make a take-it-or-leave-it offer in the last period. Ma and Manove (1993) study an alternating-offer model in which the player whose turn it is to make an offer is allowed to delay it for strategic effect and in which a random time elapses between the moment when an offer is made and the time when it arrives at the respondent. In this model it is optimal to wait until the deadline is close in order to present the other player with a comparatively unfavourable offer which he prefers to accept because there is a risk that his counter-offer would not arrive before the deadline. One problem with this explanation is that it may depend on an artificial aspect of the bargaining protocol, namely the alternation of offers. When one player delays making his offer, the other player has an interest in making one himself.

This paper offers a different explanation, based on a reputation argument. By comparison with the papers mentioned above it has the advantages that players are treated symmetrically and are allowed to make a proposal at any time and to accept any proposal made by the other player. The two players play the simultaneous-offers game described above, repeated T times. With probability close to one the disagreement payoffs are zero but, for each player, there is an arbitrarily small probability η that that player gets payoff γ ($\frac{1}{2} < \gamma < 1$) if there is no agreement by the deadline (γ is the value at the deadline date, so that, if δ is the discount factor, the disagreement payoff evaluated at the start of the game is $\delta^{T-1}\gamma$). For example, there may be a small probability that the seller has an alternative use for his capital equipment. The result is that, if η is low and T is not too high then pure PBE exist in which no agreement is reached before the very last period. In such equilibria both parties maintain intransigent positions until period T, when they agree on an equal split of the surplus. Although this equilibrium can only exist if T is not too large, delay can be very long and can, through discounting, destroy up to half of the available surplus. This phenomenon can occur even if it is common knowledge that gains from immediate trade exist (that is, if $\delta^{T-1}\gamma < \frac{1}{2}$). If it is not common knowledge that gains from trade exist then a stronger result obtains: *every* pure strategy PBE involves trade only in the final period. One advantage of this result is that it does not depend on restrictions on beliefs (only on the restriction to pure strategy equilibria).

These equilibria represent a kind of war of attrition: if a player suggests a compromise agreement then it becomes common knowledge that she is not the high type and this puts her at a disadvantage. Therefore neither will do so, even though, in equilibrium, a low-type player has no chance of obtaining the same payoff as the high type. In other papers on bargaining which have a war of attrition character (Perry (1986), Ordover and Rubinstein (1986), Chatterjee and Samuelson (1987), Compte and Jehiel (1997)) each party randomizes in each period between playing "weak" and playing "tough". Eventually, at some random time, one party concedes and the other "wins", the result being an extreme agreement. In the game presented here, by contrast, neither concedes and the final agreement represents a compromise between the two positions, a result which accords more closely with empirical and experimental evidence. It is the existence of the deadline which allows a pure strategy war of attrition to take place.

2 The Complete Information Bargaining Game

Two players have a joint project which has a surplus of unit size and they bargain over the division of this surplus. Each simultaneously makes a proposal as to how the surplus should be distributed. For each player i ($i = 1, 2$) there is a probability $\varepsilon > 0$ that i's proposal does not reach the other player. The event that 1's proposal fails to arrive is independent of the event that 2's proposal fails to arrive. If a player receives the other player's proposal then he or she decides either to accept or to reject it. If only one proposal is accepted then that proposal is implemented while if each accepts the other's proposal then one of the two proposals is chosen at random, with equal probabilities, to be implemented. If neither proposal is accepted then the disagreement outcome results, giving payoff v_i to player i ($i = 1, 2$).

Player i ($i = 1, 2$) has a von Neumann-Morgenstern utility function $u_i(.)$ which is twice differentiable, strictly increasing and concave. These functions are normalized so that $u_i(0) = 0$ and $u_i(1) = 1$. It is assumed that $v_1 + v_2 < 1$, which ensures that there is some net value in their relationship. Let β_i be the proposal which gives i her disagreement payoff: that is, $u_1(\beta_1) = v_1$ and $u_2(1 - \beta_2) = v_2$. Since u_i is concave and satisfies $u_i(0) = 0$ and $u_i(1) = 1$, we have $u_i(x) \geq x$ for all x. This implies that $v_1 \geq \beta_1$ and $v_2 \geq 1 - \beta_2$ which in turn means, since $v_1 + v_2 < 1$, that $\beta_1 < \beta_2$.

Letting the two proposals be x_1 and x_2 (both expressed in terms of player 1's share), 1's payoff in the event that both proposals get through and both are accepted is thus $(1/2)u_1(x_1) + (1/2)u_1(x_2)$ while 2's payoff is $(1/2)u_2(1 - x_1) + (1/2)u_2(1 - x_2)$. In this game a pure strategy s_i for player i consists of a proposal $x(s_i)$ together with, for each possible pair of proposals (x_1, x_2), an acceptance rule $a_i(x_j \mid x_i, s_i)$ $(j \neq i)$. The latter is a function taking values 0 and 1: if $a_1(x_2 \mid x_1, s_1) = 1$ then, conditional on having proposed x_1 (not necessarily the equilibrium proposal), player 1 will accept x_2 if it arrives and if $a_1(x_2 \mid x_1, s_1) = 0$ she will reject. Let this game be denoted by $G(v_1, v_2)$. The term *equilibrium* henceforth means perfect Bayesian equilibrium (this is an appropriate solution concept because at the moment when a player has to decide whether to accept the other's proposal he does not know whether his own proposal has arrived). In a pure strategy perfect Bayesian equilibrium of $G(v_1, v_2)$ each player's strategy is a best response to the other's, both at the proposal stage and also, after every pair of proposals, at the acceptance stage.

When player 1 receives the proposal x_2 from player 2 she is unsure whether her own proposal x_1 has arrived or not. Therefore, even if 2's strategy, having proposed x_2, is to accept x_1, 1 must, in making her acceptance decision, take into account the risk that x_1 has not arrived. Given that player 2 plays the pure strategy s_2, let player 1's expected payoff, after she has proposed x_1 and she has received the proposal x_2 from player 2, be $\Pi_1(a \mid x_1, x_2, s_2)$ if she accepts x_2 and $\Pi_1(r \mid x_1, x_2, s_2)$ if she rejects. Then

$$\Pi_1(a \mid x_1, x_2, s_2) = (1 - \varepsilon)a_2(x_1 \mid x_2, s_2)(1/2)[u_1(x_1) + u_1(x_2)]$$

$$+[1 - (1 - \varepsilon)a_2(x_1 \mid x_2, s_2)]u_1(x_2)$$

and

$$\Pi_1(r \mid x_1, x_2, s_2) = (1-\varepsilon)a_2(x_1 \mid x_2, s_2)u_1(x_1) + [1-(1-\varepsilon)a_2(x_1 \mid x_2, s_2)]v_1.$$

Therefore

$$\Pi_1(a \mid x_1, x_2, s_2) - \Pi_1(r \mid x_1, x_2, s_2) = (1-\varepsilon)a_2(x_1 \mid x_2, s_2)(1/2)[u_1(x_2) - u_1(x_1)]$$

$$+[1-(1-\varepsilon)a_2(x_1 \mid x_2, s_2)][u_1(x_2) - v_1].$$

Clearly, if $a_2(x_1 \mid x_2, s_2) = 0$ then it is weakly better for 1 to accept x_2 if $u_1(x_2) \geq v_1$. If $a_2(x_1 \mid x_2, s_2) = 1$, player 1 weakly prefers to accept x_2 if $u_1(x_2) \geq \alpha u_1(x_1) + (1-\alpha)v_1$, where $\alpha = (1-\varepsilon)(1+\varepsilon)^{-1}$. This, together with similar reasoning for player 2, establishes the following lemma.

Lemma 1 *Suppose that, in $G(v_1, v_2)$, the players play the pair of pure strategies (s_1, s_2). Given any (x_1, x_2),*

(i) If $a_2(x_1 \mid x_2, s_2) = 1$ then 1 weakly (strictly) prefers to accept x_2 if and only if $u_1(x_2) \geq (>) \alpha u_1(x_1) + (1-\alpha)v_1$.

(ii)If $a_2(x_1 \mid x_2, s_2) = 0$ then 1 weakly (strictly) prefers to accept x_2 if and only if $u_1(x_2) \geq (>) v_1$.

(iii) If $a_1(x_2 \mid x_1, s_1) = 1$ then 2 weakly (strictly) prefers to accept x_1 if and only if $u_2(1-x_1) \geq (>) \alpha u_2(1-x_2) + (1-\alpha)v_2$.

(iv)If $a_1(x_2 \mid x_1, s_1) = 0$ then 2 weakly (strictly) prefers to accept x_1 if and only if $u_2(1-x_1) \geq (>) v_2$.

Hence, in particular,

(v) If $u_1(x_2) > \alpha u_1(x_1) + (1-\alpha)v_1 \geq v_1$ then player 1 strictly prefers to accept x_2.

(vi) If $u_2(1-x_1) > \alpha u_2(1-x_2) + (1-\alpha)v_2 \geq v_2$ then player 2 strictly prefers to accept x_1.

In addition, we have

Lemma 2 *In any pure strategy equilibrium of $G(v_1, v_2)$ each player's proposal is accepted by the other.*

The proof of Lemma 2 is in the Appendix. The essential point is that if only one of the equilibrium proposals were to be accepted, say $x_1 > \beta_1$, then player 2 could deviate and offer very slightly less than x_1, which 1 must then accept because of the risk of x_1 not arriving. This would improve 2's expected payoff.

Suppose that (s_1^*, s_2^*) is a pure strategy equilibrium of $G(v_1, v_2)$ and that the equilibrium proposals are x_1^* and x_2^*. Then $u_1(x_1^*) \geq v_1$, otherwise 1 can do better by proposing v_1 (since, by Lemma 2, x_1^* will be accepted in equilibrium). Therefore, by Lemma 1(v), having made proposal x_1^* and received any proposal x_2 such that $u_1(x_2) > \alpha u_1(x_1^*) + (1-\alpha)v_1$, 1 must accept x_2. This means that x_2^* cannot satisfy this inequality: if it did, 2 should make a lower proposal. Hence $u_1(x_2^*) - v_1 \leq \alpha[u_1(x_1^*) - v_1]$. If $u_1(x_2^*) - v_1 < \alpha[u_1(x_1^*) - v_1]$, i.e., $u_1(x_2^*) < \alpha u_1(x_1^*) + (1-\alpha)v_1$, then, since x_1^* is accepted, 1 should reject x_2^* by Lemma 1(i). This contradicts Lemma 2. Therefore

$$u_1(x_2^*) - v_1 = \alpha[u_1(x_1^*) - v_1]. \tag{1}$$

By a symmetrical argument,

$$u_2(1 - x_1^*) - v_2 = \alpha[u_2(1 - x_2^*) - v_2], \tag{2}$$

so that the Nash product $[u_1(x) - v_1][u_2(1-x) - v_2]$ takes the same value at x_1^* as at x_2^*.

Furthermore, if (x_1^*, x_2^*) satisfies (1) and (2) there is indeed a pure strategy equilibrium in which 1 proposes x_1^*, 2 proposes x_2^* and both proposals are accepted; one such equilibrium is described in detail in the proof of Theorem 1 below. The details of the off-equilibrium-path portions of the equilibrium are complicated but it should be clear from (1) that if 1 has proposed x_1^* and expects it to be accepted then the lowest proposal which she will accept is x_2^* and so it is optimal for 2 to propose x_2^*. Similarly, if 2 has proposed x_2^* then, from (2) , the highest proposal which he will accept is x_1^*.

Although this is a single-period game, it is closely related to the Rubinstein infinite-horizon alternating-offers game. In fact, if the disagreement payoffs v_1 and v_2 are both zero then it is immediate from (1) and (2) that x_i^* is the Rubinstein proposal of player i when the discount factor is α. Similarly, in the alternating-offers model with exogenous probability of breakdown p, it is well known that the equilibrium offers x_1 and x_2 have the property that 1 is indifferent between x_2 and a lottery giving x_1 with probability p and the disagreement outcome with probability $(1-p)$ and 2 is indifferent between $1-x_1$ and a lottery giving $1-x_2$ with probability p and the disagreement outcome with probability $(1-p)$. Therefore x_i^* is the Rubinstein proposal of player i when the breakdown probability is α. It follows that Rubinstein's analysis does not depend on repetition or on the alternating-offers protocol. It is a standard result that (1) and (2) have a unique solution and that this converges, as $\alpha \to 1$, to the Nash bargaining solution. This clearly follows from the fact that the Nash product $F(x) = [u_1(x) - v_1][u_2(1-x) - v_2]$ is strictly concave and satisfies $F(\beta_1) = F(\beta_2) = 0$ which implies that, given $\alpha < 1$ (close enough to 1), there is a unique pair (x_1^*, x_2^*) such that $F(x_1^*) = F(x_2^*)$ and $u_1(x_2^*) - v_1 = \alpha[u_1(x_1^*) - v_1]$. As $\alpha \to 1$ x_1^* converges to the maximum of F.

Theorem 1 *If ε is sufficiently small then $G(v_1, v_2)$ has a unique pure strategy equilibrium proposal pair. This is the unique solution, (x_1^*, x_2^*), to equations (1) and (2). In equilibrium both proposals are accepted. As $\varepsilon \to 0$ ($\alpha \to 1$), x_1^* and x_2^* both converge to the Nash bargaining solution.*

Proof. In Appendix.

Applying this to the case in which both players are risk-neutral, i.e., $u_1(x) = x$ and $u_2(x) = 1 - x$: in a pure strategy equilibrium 1 proposes

$$x_1^* = (\frac{1}{2})[(1+\varepsilon) - (1+\varepsilon)v_2 + (1-\varepsilon)v_1]$$

and 2 proposes

$$x_2^* = (\frac{1}{2})[(1-\varepsilon) - (1-\varepsilon)v_2 + (1+\varepsilon)v_1].$$

1's equilibrium expected payoff is $(1-\varepsilon^2)[(1/2)(x_1^* + x_2^*)] + \varepsilon^2 v_1$ since her expected payoff conditional on at least one proposal reaching its destination is $(1/2)(x_1^* + x_2^*)$. This simplifies to $(1-\varepsilon^2)[(1/2)(1 + v_1 - v_2)] + \varepsilon^2 v_1$. Similarly, 2's expected payoff is $(1 -$

$\varepsilon^2)[(1/2)(1+v_2-v_1)]+\varepsilon^2 v_2$. It follows that, in the limit as $\varepsilon \to 0$, the two players share the surplus equally: i gets $v_i + (1/2)(1-v_i-v_j)$ $(i \neq j)$.

Remark 1 Suppose that the probability of an offer failing to arrive differs between the two players. Let the probability that i's offer fails to arrive be ε_i. Then it is easy to see that the two equilibrium proposals x_1^* and x_2^* satisfy the equations

$$u_1(x_2^*) - v_1 = \alpha_1(u_1(x_1^*) - v_1)$$

and

$$u_2(1-x_1^*) - v_2 = \alpha_2(u_2(1-x_2^*) - v_2)$$

where $\alpha_i = (1-\varepsilon_i)/(1+\varepsilon_i)$. For example, if utilities are linear and $v_i = 0$ $(i = 1, 2)$ then player 1 proposes $(1-\alpha_2)/(1-\alpha_1\alpha_2)$, which is the Rubinstein proposal when the discount factors are α_1 and α_2. A player with a greater value of ε is at a disadvantage because he or she is under more pressure to accept the other player's proposal.

Remark 2 So far it has been assumed that if both proposals are accepted then each has probability 1/2 of being implemented. Suppose now that in that event the probability of player 1's proposal being implemented is θ and for player 2 the probability is $1-\theta$, where $\theta \in [0,1]$. Then the two equilibrium proposals x_1^* and x_2^* satisfy

$$u_1(x_2^*) - v_1 = \alpha_1(\theta)(u_1(x_1^*) - v_1)$$

and

$$u_2(1-x_1^*) - v_2 = \alpha_2(\theta)(u_2(1-x_2^*) - v_2)$$

where

$$\alpha_1(\theta) = \frac{(1-\theta)(1-\varepsilon)}{1-\theta(1-\varepsilon)}$$

and

$$\alpha_2(\theta) = \frac{\theta(1-\varepsilon)}{\theta(1-\varepsilon)+\varepsilon}.$$

In the linear utility case with $v_1 = v_2 = 0$, x_1^* therefore equals $(1-\alpha_2(\theta))/(1-\alpha_1(\theta)\alpha_2(\theta))$, which reduces to $1-\theta+\varepsilon\theta$. x_2^* equals $\alpha_1(\theta)x_1^*$, i.e. $(1-\theta)(1-\varepsilon)$. So, for small ε, player 1's payoff is approximately $1-\theta$. If the probability that her own proposal is implemented is very high then she gets a very low payoff. The reason is that in that case player 2 has less incentive to accept player 1's proposal and so, correspondingly, player 1 has more incentive to accept 2's. A similar phenomenon was noted by Carlsson (1991) in connection with unequal distribution of the surplus.

The finitely repeated game

It is straightforward to generalize the model to a situation in which the game is repeated finitely many times. Suppose that there are finitely many (T) discrete periods indexed $1, ..., T$. In each period each player makes a proposal as in the single-period game analyzed above. As before, there is probability $1 - \varepsilon$ that a given proposal will arrive in the current period; if it fails to arrive in the current period then it will never arrive. For simplicity, it is assumed here that at the start of each period each player knows the whole history to date (including the size of any offer which failed to arrive). This means that all information is public at the start of each stage. If either player accepts an offer the game ends immediately. The players discount future payoffs using a common discount factor $\delta \in (0, 1]$. I confine attention to the case in which the players are risk-neutral. If no offers have been accepted before time t and at time t player 2 accepts player 1's offer of $x_1 \in [0, 1]$ while player 1 does not accept player 2's offer, then the payoffs are $\delta^{t-1}x_1$ for player 1 and $\delta^{t-1}(1 - x_1)$ for player 2. If player 1, at time t, accepts player 2's offer of x_2 and player 2 does not accept 1's offer, then 1's payoff is $\delta^{t-1}x_2$ and player 2's is $\delta^{t-1}(1 - x_2)$. If both offers, x_1 and x_2, are accepted at time t then player 1 gets $\delta^{t-1}\left(\frac{x_1+x_2}{2}\right)$ and player 2 gets $\delta^{t-1}\left(\frac{(1-x_1)+(1-x_2)}{2}\right)$. If no offers are accepted by the end of time T player i's payoff is $\delta^{T-1}v_i$, where, as before, $v_1 + v_2 < 1$. Let this repeated game be called $G^T(v_1, v_2)$.

A pure strategy for player i in the bargaining game $G^T(v_1, v_2)$ consists of (i) a function prescribing, for each $t = 1, 2, ..., T$, and for each partial history up to and including $t - 1$ (if $t = 1$ the only partial history is the null history) of proposals which were rejected or did not arrive, a proposal to be made in the current period t and (ii) a function prescribing, for each partial history as described above plus i's t-period offer and j's t-period offer, whether or not to accept j's offer. Let S_i^T be the set of pure strategies for i. As above, *equilibrium* refers to perfect Bayesian equilibrium.

Theorem 2 *If ε is sufficiently small, $G^T(v_1, v_2)$ has a unique pure strategy equilibrium sequence of offers. In equilibrium each player accepts the first offer to arrive. As $\varepsilon \to 0$, player i's equilibrium payoff converges to $\delta^{T-1}v_i + (1/2)(1 - \delta^{T-1}v_i - \delta^{T-1}v_j)$ $(i \neq j)$.*

Proof. The proof is by induction on the number of periods remaining. The induction hypothesis is that, when there are $t < T$ periods remaining, there is some $\varepsilon_t > 0$ such that, for any $\varepsilon < \varepsilon_t$, there is a unique pure strategy equilibrium pair of proposals, both of which are accepted if they arrive, and that the equilibrium payoffs $V_t^i(v_1, v_2)$ $(i = 1, 2)$ are given by

$$V_t^1(v_1, v_2) = (1 - \varepsilon^2)\frac{1 - (\delta\varepsilon^2)^t}{2(1 - \delta\varepsilon^2)} + \frac{\delta^{t-1}(1 + \varepsilon^{2t})v_1}{2} - \frac{\delta^{t-1}(1 - \varepsilon^{2t})v_2}{2}$$

and

$$V_t^2(v_1, v_2) = (1 - \varepsilon^2)\frac{1 - (\delta\varepsilon^2)^t}{2(1 - \delta\varepsilon^2)} + \frac{\delta^{t-1}(1 + \varepsilon^{2t})v_2}{2} - \frac{\delta^{t-1}(1 - \varepsilon^{2t})v_1}{2}.$$

By Theorem 1, this is true for $t = 1$. Suppose that it is true for $t < T$. Then, if $\varepsilon < \varepsilon_t$, in any pure strategy equilibrium of $G^T(v_1, v_2)$ the continuation payoffs when there are t periods to go must be $V_t^1(v_1, v_2)$ and $V_t^2(v_1, v_2)$. Clearly, $V_t^1(v_1, v_2) + V_t^2(v_1, v_2) < 1$ because there is strictly positive probability that no proposals will arrive. Therefore, by Theorem 1, there is $\varepsilon_{t+1} > 0$ such that, if $\varepsilon < \varepsilon_{t+1}$, the unique pair of pure strategy equilibrium proposals when there are $t + 1$ periods to go, both of which are accepted, gives expected payoffs $V_{t+1}^i(v_1, v_2) = (1/2)(1 - \varepsilon^2)[1 + \delta V_t^i(v_1, v_2) - \delta V_t^j(v_1, v_2)] + \varepsilon^2 \delta V_t^i(v_1, v_2)$ $(i \neq j)$. Substituting for $V_t^1(v_1, v_2)$ and $V_t^2(v_1, v_2)$ establishes that the induction hypothesis is true for $t + 1$. ■

As discussed in the Introduction, Ma and Manove (1993) explain the deadline effect using an alternating-offer model in which the player whose turn it is to make an offer can choose the moment at which he makes it and there is then a random delay before it arrives. The deadline effect arises because of the incentive to delay making an offer until near the deadline in order to put pressure on the respondent. Theorem 2 suggests that if players are instead able to offer at any time[4], rather than having to wait for the other player to offer first, randomness in the arrival of offers does not cause delay in agreement, at least in a complete-information model. The next section demonstrates that a different conclusion arises in an incomplete-information version of the current model.

3 A Bargaining Game with Incomplete Information about Disagreement Payoffs

Suppose now that the players play the finitely-repeated game $G^T(v_1, v_2)$ with the difference that it is not common knowledge that the disagreement payoffs are v_1 and v_2 respectively for players 1 and 2. Instead, there is probability $1 - \eta_1$ that player 1 has disagreement payoff zero and probability η_1 that she has disagreement payoff γ, where $\frac{1}{2} < \gamma < 1$; independently, player 2 has probability $1 - \eta_2$ of being the normal type (disagreement payoff zero) and probability η_2 of being the high type (disagreement payoff γ). Here η_1 and η_2 are small but strictly positive. As before, the history of the game is commonly known at the outset of each stage and payoffs are discounted according to a common discount factor δ. The low (i.e., normal) type of player i is denoted by L_i and the high type by H_i. Let this game of incomplete information be denoted by $\Gamma^T(\eta_1, \eta_2, \varepsilon)$.

The set of pure strategies for player i is the same as in the complete information game. A typical history up to and including stage $T - t$ (i.e., when there are t stages to go) is denoted by h^{T-t}. A *system of beliefs* p specifies, for each t, h^{T-t}, x_1 and x_2, values $p_i(h^{T-t})$ and $p_i(h^{T-t}, x_i)$ $(i = 1, 2)$. $p_i(h^{T-t})$ is the probability which $j \neq i$ assigns after history h^{T-t} to the event that i is the high type and $p_i(h^{T-t}, x_i)$ is the corresponding probability if j then receives the proposal x_i. A perfect Bayesian equilibrium in pure strategies (σ, p) is a profile of pure strategies (one for each type of each player) and a system of beliefs such that the beliefs are consistent with the strategies and each strategy is optimal after every history given the other strategies and the beliefs.

Given a strategy profile $\sigma^T = (\sigma^T_{H_1}, \sigma^T_{L_1}, \sigma^T_{H_2}, \sigma^T_{L_2}) \in (S^T_1)^2 \times (S^T_2)^2$ and a system

[4]In a discrete-time framework; Ma and Manove's model is set in continuous time.

of beliefs p, let $W^t_{H_i}(h^{T-t}, p_j(h^{T-t}), \sigma^T, \varepsilon)$ be H_i's expected continuation payoff at the start of period $T-t+1$ if these strategies are played, the history so far is h^{T-t} and H_i has belief $p_j(h^{T-t})$. $W^t_{L_i}(h^{T-t}, p_j(h^{T-t}), \sigma^T, \varepsilon)$ is defined analogously. $W^T_{H_i}(\eta_j, \sigma^T, \varepsilon)$ and $W^T_{L_i}(\eta_j, \sigma^T, \varepsilon)$ are the corresponding *ex ante* expected payoffs. Also, let $W^t_{H_i}(p_j, \sigma^t, \varepsilon)$ be H_i's expected continuation payoff when there are t periods to go if his belief is p_j and the strategy profile $\sigma^t \in (S^t_1)^2 \times (S^t_2)^2$ is played during the remainder of the game (as if the game were starting at this stage). Given strategy profile σ^T, $(\sigma^T_{-H_i}, \sigma)$ is the strategy profile in which H_i plays strategy σ and H_j, L_j and L_i play according to σ^T (similarly for a deviation by L_i).

This section is devoted firstly to constructing a pure strategy equilibrium in which there is no agreement until the final period and then to showing that all pure strategy equilibria have this character if it is not common knowledge that there are gains from trade. The description of the equilibrium is lengthy largely because of the need to show that the equilibrium strategies are pure off the equilibrium path, in particular after two simultaneous non-equilibrium proposals have been made. Note also that the non-normal (i.e., high) types are modelled here as rational players who vary their actions with the history. In other studies of perturbed bargaining games (e.g., Compte and Jehiel (1997)), the non-normal type is an obstinate bargainer who always makes the same high demand and rejects any proposal which gives him less than this.

In the exhibited equilibrium each player i believes that the other player is the high type with probability η_j, 0 or 1, depending on the history, and once a player's belief reaches either 0 or 1 it remains at that value regardless of future events. In Propositions 1-5 I construct the equilibrium continuations for the various subgames which can arise. Along the equilibrium path neither player makes a serious offer (and high and low types pool) in periods $1, ...T-1$ so that at the start of the final stage i believes that the other is the high type with probability η_j $(i \neq j)$. In the final stage they play an equilibrium (σ^1_S, p^1_S) of the resulting one-period game of incomplete information which is separating on both sides. In this equilibrium, which is set out in Proposition 1, the low types simply play the pure strategy equilibrium of the complete information game in which it is common knowledge that both players are the low type. If ε is small then the high types can do no better than make an unacceptable proposal and get their disagreement payoff.

Proposition 1 *Let $\varepsilon < 2\gamma - 1$. Suppose that η_1 and η_2 are strictly positive and sufficiently small. Then $\Gamma^1(\eta_1, \eta_2, \varepsilon)$ has a pure strategy equilibrium $(\sigma_S^1(\varepsilon), p_S^1)$ in which H_1 proposes 1, H_2 proposes 0, L_1 proposes $(1/2)(1+\varepsilon)$ and L_2 proposes $(1/2)(1-\varepsilon)$. L_1 and L_2 accept each other's proposals. H_1 and H_2's proposals are rejected by both types. The equilibrium payoffs are*

$$W^1_{L_1}(\eta_2, \sigma_S^1(\varepsilon), \varepsilon) = (1-\eta_2)\frac{(1-\varepsilon^2)}{2},$$

$$W^1_{L_2}(\eta_1, \sigma_S^1(\varepsilon), \varepsilon) = (1-\eta_1)\frac{(1-\varepsilon^2)}{2},$$

$$W^1_{H_1}(\eta_2, \sigma_S^1(\varepsilon), \varepsilon) = \gamma,$$

and

$$W^1_{H_2}(\eta_1, \sigma_S^1(\varepsilon), \varepsilon) = \gamma.$$

Showing that the stipulated offers are optimal is straightforward (and essentially the same as the demonstration above for the complete information case); the details are in the Appendix.

Suppose now that one player, say player 2, has revealed himself as the low type while the other player has not and that the final stage has been reached without agreement. Proposition 2 sets out the equilibrium $(\sigma_P^1(\varepsilon), p_P^1)$ which is played in that event. In this equilibrium the two types of player 1 make a pooling offer $\widehat{x}_1(\varepsilon)$ which L_2 accepts. L_2's proposal is $\alpha\widehat{x}_1(\varepsilon)$, which is accepted by L_1 but rejected by H_1 (who would only accept $x_2 \geq \alpha\widehat{x}_1(\varepsilon) + (1-\alpha)\gamma$). $\widehat{x}_1(\varepsilon)$ is the largest proposal which is acceptable to L_2 given that player 1 will accept $\alpha\widehat{x}_1(\varepsilon)$ with probability $1-\eta_1$ conditional on this offer arriving. As $\varepsilon \to 0$, $\widehat{x}_1(\varepsilon) \to 1$ and so player 1 is able to extract approximately all the surplus as a result of the small asymmetry of information.

Proposition 2 *Suppose that η_1 satisfies $0 < \eta_1 < 2\gamma - 1$. Given small enough ε, $\Gamma^1(\eta_1, 0, \varepsilon)$ has a pure strategy equilibrium $(\sigma_P^1(\varepsilon), p_P^1)$ such that, as $\varepsilon \to 0$, $W_{L_1}^1(0, \sigma_P^1(\varepsilon), \varepsilon) \to 1$, $W_{H_1}^1(0, \sigma_P^1(\varepsilon), \varepsilon) \to 1$ and $W_{L_2}^1(\eta_1, \sigma_P^1(\varepsilon), \varepsilon) \to 0$. $\Gamma^1(0, \eta_2, \varepsilon)$ has an equilibrium symmetric to this.*

Proof. In Appendix.

The next proposition sets out the continuation equilibrium from the start of any stage at which one player is believed to be the low type and the other is believed to be the high type. The beliefs always remain the same subsequently and the strategies are the same as in the complete information game between a player with disagreement payoff zero and one with disagreement payoff γ.

Proposition 3 *For small ε, $\Gamma^t(1, 0, \varepsilon)$ $(t \leq T)$ has a pure strategy equilibrium $(\sigma_{10}^t(\varepsilon), p_{10}^t)$ in which the equilibrium payoffs are*

$$W_{L_2}^t(1, \sigma_{10}^t(\varepsilon), \varepsilon) = (1 - \varepsilon^2)\frac{1 - (\delta\varepsilon^2)^t}{2(1 - \delta\varepsilon^2)} - \frac{\delta^{t-1}(1 - \varepsilon^{2t})\gamma}{2},$$

$$W_{H_1}^t(0, \sigma_{10}^t(\varepsilon), \varepsilon) = (1 - \varepsilon^2)\frac{1 - (\delta\varepsilon^2)^t}{2(1 - \delta\varepsilon^2)} + \frac{\delta^{t-1}(1 + \varepsilon^{2t})\gamma}{2},$$

$$W_{L_1}^t(0, \sigma_{10}^t(\varepsilon), \varepsilon) = (1 - \varepsilon^2)\frac{1 - (\delta\varepsilon^2)^t}{2(1 - \delta\varepsilon^2)} + \frac{\delta^{t-1}(1 - \varepsilon^{2t})\gamma}{2}$$

and

$$W_{H_2}^t(1, \sigma_{10}^t(\varepsilon), \varepsilon) = \delta^{t-1}\gamma.$$

In this equilibrium, H_1 and L_2 each propose approximately $(\frac{1}{2})(1 + \delta^{t-1}\gamma)$, giving approximate payoffs of $(\frac{1}{2})(1 + \delta^{t-1}\gamma)$ and $(\frac{1}{2})(1 - \delta^{t-1}\gamma)$. $\Gamma^t(0, 1, \varepsilon)$ has an equilibrium symmetric to this.

Proof. Let $(\sigma_{10}^t(\varepsilon), p_{10}^t)$ be the following profile of strategies and beliefs. After any history, 1 believes that 2 is the low type and 2 believes that 1 is the high type. H_1 and L_2 play equilibrium pure strategies of $G^t(\gamma, 0)$, the complete-information game of t periods in which 1 has a disagreement payoff of γ and 2 has a disagreement payoff of zero. In $G^t(\gamma, 0)$ these are best responses to each other after every history and are therefore sequentially optimal for the beliefs specified. L_1 plays an arbitrary pure sequential best response to L_2's strategy and H_2 plays an arbitrary pure sequential best response to H_1's strategy. The expressions for $W^t_{L_2}(1, \sigma^t_{10}(\varepsilon), \varepsilon)$ and $W^t_{H_1}(0, \sigma^t_{10}(\varepsilon), \varepsilon)$ follow from Theorem 2. Since H_1's proposals are all greater than $\frac{1}{2}$ and she rejects anything less, H_2 can do no better than wait for his disagreement payoff, hence $W^t_{H_2}(1, \sigma^t_{10}(\varepsilon), \varepsilon)$. If L_1 mimics H_1 her expected payoff is $W^t_{H_1}(0, \sigma^t_{10}(\varepsilon), \varepsilon) - \varepsilon^{2t}\gamma$ because she will then get the same payoff as H_1 would unless there is no agreement, which, since the first proposal to arrive will be accepted, happens with probability ε^{2t}. Suppose that there is some strategy $\widetilde{\sigma}_1$ which gives L_1 a higher payoff than this, i.e.,

$$W^t_{L_1}(0, (\sigma^t_{10,-L_1}(\varepsilon), \widetilde{\sigma}_1), \varepsilon) > W^t_{H_1}(0, \sigma^t_{10}(\varepsilon), \varepsilon) - \varepsilon^{2t}\gamma.$$

Let ϕ be the probability that, if 1 plays $\widetilde{\sigma}_1$ against L_2's strategy $\sigma^t_{10,L_2}(\varepsilon)$, there is no agreement, so that $\phi \geq \varepsilon^{2t}$. Then H_1's expected payoff from playing $\widetilde{\sigma}_1$,

$$W^t_{H_1}(0, (\sigma^t_{10,-H_1}(\varepsilon), \widetilde{\sigma}_1), \varepsilon),$$

is equal to

$$W^t_{L_1}(0, (\sigma^t_{10,-L_1}(\varepsilon), \widetilde{\sigma}_1), \varepsilon) + \phi\gamma.$$

Therefore

$$W^t_{H_1}(0, (\sigma^t_{10,-H_1}(\varepsilon), \widetilde{\sigma}_1), \varepsilon) \geq W^t_{L_1}(0, (\sigma^t_{10,-L_1}(\varepsilon), \widetilde{\sigma}_1), \varepsilon) + \varepsilon^{2t}\gamma > W^t_{H_1}(0, \sigma^t_{10}(\varepsilon), \varepsilon),$$

which contradicts the fact that $W^t_{H_1}(0, \sigma^t_{10}(\varepsilon), \varepsilon)$ is H_1's maximal payoff. This shows that the best that L_1 can do is mimic H_1, hence that L_1 has the stated payoff. ■

The next proposition describes the equilibrium continuation in any subgame in which each believes the other to be the low type.

Proposition 4 *For small ε, $\Gamma^t(0, 0, \varepsilon)$ ($t \leq T$) has a pure strategy equilibrium $(\sigma^t_{00}(\varepsilon), p^t_{00})$ in which the equilibrium payoffs are*

$$W^t_{L_1}(0, \sigma^t_{00}(\varepsilon), \varepsilon) \quad = \quad (1 - \varepsilon^2)\frac{1 - (\delta\varepsilon^2)^t}{2(1 - \delta\varepsilon^2)},$$

$$
\begin{aligned}
W^t_{L_2}(0,\sigma^t_{00}(\varepsilon),\varepsilon) &= (1-\varepsilon^2)\frac{1-(\delta\varepsilon^2)^t}{2(1-\delta\varepsilon^2)}, \\
W^t_{H_1}(0,\sigma^t_{00}(\varepsilon),\varepsilon) &= \delta^{t-1}\gamma, \\
W^t_{H_2}(0,\sigma^t_{00}(\varepsilon),\varepsilon) &= \delta^{t-1}\gamma.
\end{aligned}
$$

In this equilibrium, L_1 and L_2 each offer approximately 1/2 and each offer is accepted.

Proof. Let $(\sigma^t_{00}(\varepsilon), p^t_{00})$ be the following profile of strategies and beliefs. After any history, 1 believes that 2 is the low type and 2 believes that 1 is the low type. L_1 and L_2 play pure equilibrium strategies of $G^t(0,0)$, the complete-information game of t periods in which 1 and 2 each has a disagreement payoff of zero (see Theorem 2). In $G^t(0,0)$ these are best responses to each other after every history and are therefore sequentially optimal for the beliefs specified. H_1 plays an arbitrary sequential pure best response to L_2's strategy and H_2 plays an arbitrary sequential pure best response to L_1's strategy. The expressions for $W^t_{L_1}(0,\sigma^t_{00}(\varepsilon),\varepsilon)$ and $W^t_{L_2}(0,\sigma^t_{00}(\varepsilon),\varepsilon)$ follow from Theorem 2. H_1 and H_2 can do no better than wait for their disagreement payoffs, hence $W^t_{H_1}(0,\sigma^t_{00}(\varepsilon),\varepsilon)$ and $W^t_{H_2}(0,\sigma^t_{00}(\varepsilon),\varepsilon)$. ■

Next, in Proposition 5, I describe the continuation strategies when there are t periods to go and one player has been revealed as the low type while the other has not. The strategies are t-period analogues of the pooling equilibrium of Proposition 2. Both types of player 1 (in the case in which it is 2 who is believed to be the low type with probability 1) make the same high demand, which L_2 accepts; L_2 proposes slightly less than this and both types of player 1 accept. One main difference between this (when $t > 1$) and the equilibrium of Proposition 2 is that both types of player 1 accept player 2's proposal: this is because 2 will believe that 1 is the high type with probability 1 if she rejects it and the continuation equilibrium profile will then be $\sigma^{t-1}_{10}(\varepsilon)$ (as described in Proposition 3), which would be worse for player 1 than accepting the proposal.

Proposition 5 *Let $\delta^{T-1}(\frac{1}{2}+\gamma) > \frac{1}{2}$ and let $0 < \eta_1 < 2\gamma - 1$. For small ε, $\Gamma^t(\eta_1, 0, \varepsilon)$ $(t \geq 1)$ has a pure strategy equilibrium $(\sigma^t_P(\varepsilon), p^t_P)$ such that*

$$
\lim_{\varepsilon\to 0} W^t_{H_1}(0,\sigma^t_P(\varepsilon),\varepsilon) = (\frac{1}{2})(1+\delta^{t-1}),
$$

$$
\begin{aligned}
\lim_{\varepsilon\to 0} W^t_{L_1}(0, \sigma^t_P(\varepsilon), \varepsilon) &= (\frac{1}{2})(1+\delta^{t-1}), \\
\lim_{\varepsilon\to 0} W^t_{H_2}(\eta_1, \sigma^t_P(\varepsilon), \varepsilon) &= \delta^{t-1}\gamma, \\
\lim_{\varepsilon\to 0} W^t_{L_2}(\eta_1, \sigma^t_P(\varepsilon), \varepsilon) &= (\frac{1}{2})(1-\delta^{t-1}).
\end{aligned}
$$

$\Gamma^t(0, \eta_2, \varepsilon)$ has an equilibrium symmetric to this.

Proof. in Appendix.

Combining the results above, we have:

Theorem 3 *Let $\delta^{T-1} + \frac{\delta}{2} > 1$. If η_1, η_2 and ε are sufficiently small, $\Gamma^T(\eta_1, \eta_2, \varepsilon)$ has a pure strategy equilibrium $(\sigma^T_S(\varepsilon), p^T_S)$ in which no proposal is accepted before the final period and, in the final period, each low type accepts the other's proposal.*

Proof. Let $(\sigma^T_S(\varepsilon), p^T_S)$ be the following strategy profile and beliefs.

In period 1 H_1 and L_1 both propose 1 and H_2 and L_2 both propose 0. These offers are rejected by both types. In subsequent periods, except the final one, the proposals are the same as in the first period, and these proposals are rejected, as long as neither player has previously deviated. In the final period, if there has previously been no deviation by either player, the equilibrium of $\Gamma^1(\eta_1, \eta_2, \varepsilon)$ described in Proposition 1 is played: i.e., each low type proposes approximately $\frac{1}{2}$ and accepts the other's proposal, while the high types demand the whole pie and reject the equilibrium proposals of both types of the other player.

If neither player has deviated from the strategy above then player j ($j \neq i$) believes that i is the high type with probability η_i. If, at any stage before T, when there has previously been no deviation by either player, a player makes a proposal other than the one specified above (i.e., demands less than the whole pie), then the other player believes that the deviating player is the low type with probability 1, and continues to believe so for the rest of the game, for all subsequent histories. Therefore, immediately after the first deviation (at stage $T-t$, $1 \le t \le T-1$), the possible belief states are $(0,0)$, $(\eta_1, 0)$ or $(0, \eta_2)$. If the beliefs are $(0,0)$, the continuation strategies are $\sigma^{t+1}_{00}(\varepsilon)$, the equilibrium of $\Gamma^{t+1}(0,0,\varepsilon)$ described in Proposition 4. If they are $(\eta_1, 0)$, (i.e., if 1 has proposed 1 and 2 has proposed $x_2 \neq 0$), then H_2 and L_2 both reject 1's proposal; if $x_2 < \delta W^t_{L_1}(0, \sigma^t_P(\varepsilon), \varepsilon)$, H_1 and L_1 both reject x_2 and otherwise they both accept. If $x_2 < \delta W^t_{L_1}(0, \sigma^t_P(\varepsilon), \varepsilon)$ and 1 rejects then 2 believes that 1 is the high type with probability η_1 and the continuation strategies from the next period are $\sigma^t_P(\varepsilon)$, the equilibrium of $\Gamma^t(\eta_1, 0, \varepsilon)$ described in Proposition 5 (note that $\delta^{T-1} + \frac{\delta}{2} > 1$ implies that $\delta^{T-1}(\frac{1}{2} + \gamma) > \frac{1}{2}$ since $\gamma > \frac{1}{2}$). If $x_2 \geq \delta W^t_{L_1}(0, \sigma^t_P(\varepsilon), \varepsilon)$ and 1 rejects x_2 then 2 believes that 1 is the high type with probability 1 and the continuation strategies from the next period are $\sigma^t_{10}(\varepsilon)$, the equilibrium of $\Gamma^t(1, 0, \varepsilon)$ described in Proposition 3. If the beliefs are $(0, \eta_2)$ the strategies are symmetrical to the above.

From the first period after a deviation the strategies are in equilibrium by Propositions 3,4 and 5. In the final period, after no deviations, they are in equilibrium by Proposition 1. Clearly it can never be optimal for either type of player 1 to accept a proposal of 0 or for either type of player 2 to accept a proposal of 1 since the expected continuation payoff of each player is strictly positive. Suppose that the first deviation is at period $T-t$, when there are $t+1$ periods to go. Suppose that player 2 deviates by proposing $x_2 > 0$. If $x_2 < \delta W^t_{L_1}(0, \sigma^t_P(\varepsilon), \varepsilon)$, it is optimal for H_1 and L_1 to reject since the continuation payoffs from next period are then $W^t_{L_1}(0, \sigma^t_P(\varepsilon), \varepsilon)$ and $W^t_{H_1}(0, \sigma^t_P(\varepsilon), \varepsilon) > W^t_{L_1}(0, \sigma^t_P(\varepsilon), \varepsilon)$. If $x_2 \geq \delta W^t_{L_1}(0, \sigma^t_P(\varepsilon), \varepsilon)$ and 1 rejects then the continuation payoff for each type of 1 is, for small ε, approximately $\frac{\delta}{2}(1 + \delta^{t-1}\gamma)$ by Proposition 3. Since, for small ε, this is less than $\delta W^t_{L_1}(0, \sigma^t_P(\varepsilon), \varepsilon)$ by Proposition 5, it is optimal for both to accept. If player 2 deviates by proposing $x_1 < 1$, the argument is symmetrical. If both simultaneously make deviant proposals, the continuation strategies are in equilibrium by Proposition 4.

It remains to show that neither player can profit by making a deviant proposal before period T. By symmetry, it suffices to consider L_2 and H_2. Take L_2 first and consider a deviation at $T-t$ where $1 \leq t \leq T-1$. His expected payoff, discounted to this period, if he conforms to the above strategy is, for small ε, approximately $(1-\eta_1)\frac{\delta^t}{2}$. If he makes a proposal $x_2 \neq 0$ which 1 will reject, his expected payoff is approximately $\frac{\delta}{2}(1 - \delta^{t-1})$, and if he makes the lowest acceptable proposal ($\delta W^t_{L_1}(0, \sigma^t_P(\varepsilon), \varepsilon)$) his expected payoff is approximately

$$1 - \lim_{\varepsilon \to 0} \delta W^t_{L_1}(0, \sigma^t_P(\varepsilon), \varepsilon) = 1 - \frac{\delta}{2}(1 + \delta^{t-1}) = 1 - \frac{\delta}{2} - \frac{\delta^t}{2}.$$

By assumption, $\delta^t + \frac{\delta}{2} > 1$, which implies that, for small enough η_1,

$$(1 - \eta_1)\frac{\delta^t}{2} > 1 - \frac{\delta}{2} - \frac{\delta^t}{2}$$

and

$$(1 - \eta_1)\frac{\delta^t}{2} > \frac{\delta}{2}(1 - \delta^{t-1}).$$

Therefore it is not optimal for L_2 to make a deviant proposal. If H_2 makes such a proposal his expected payoff is either $\delta^t \gamma$ by Proposition 5 or $1 - \frac{\delta}{2} - \frac{\delta^t}{2}$, as above. If he does not deviate his expected payoff is $\delta^t \gamma > \frac{\delta^t}{2} > 1 - \frac{\delta}{2} - \frac{\delta^t}{2}$. This establishes that the stated strategies form an equilibrium. ■

For high values of the discount factor this equilibrium can involve very long delay. The loss of surplus which results (if both players are the normal type) is $1 - \delta^{T-1}$, which is limited by the condition $\delta^{T-1} + \frac{\delta}{2} > 1$. For δ close to 1, T can therefore be long enough for almost half of the surplus to be lost. Notice also that this equilibrium can exist even if it is common knowledge that there are gains from trade, i.e., if $\delta^{T-1}\gamma < \frac{1}{2}$.

Finally, the next theorem shows that if it is not common knowledge that there are gains from trade ($\delta^{T-1}\gamma > \frac{1}{2}$) then no pure strategy equilibrium exists in which agreement is reached before the final period. For simplicity, I only consider histories in which every proposal arrives (which happens, for small ε, with probability close to 1).

Theorem 4 *Suppose that $\delta^{T-1}\gamma > \frac{1}{2}$. In any pure strategy equilibrium of $\Gamma^T(\eta_1, \eta_2, \varepsilon)$, with η_1, η_2 and ε all small, there is no agreement before period T in any history in which all proposals arrive.*

Proof. Take a pure strategy equilibrium with strategy profile $\sigma = (\sigma_{H_1}, \sigma_{L_1}, \sigma_{H_2}, \sigma_{L_2})$.

Suppose that, if these strategies are played and if all proposals arrive, L_1 and L_2 will reach agreement in period $t < T$. Suppose that L_1 were to mimic H_1's strategy from the outset. Conditional on 2 being the low type and all offers arriving (which has probability close to 1 if ε and η_2 are close to zero), either (a) agreement will be reached before T and L_1's payoff will be the same as H_1's would be or (b) period T will be reached with 2 believing that 1 is the high type (since L_1 would have reached agreement at t) and 1 believing that 2 is the low type with probability at least $1 - \eta_2$. If (b) obtains, then 1 will (with probability $(1 - \varepsilon^2)$) reach agreement with L_2 (i.e, H_1 will accept L_2's offer if it arrives, or L_2 will accept H_1's offer, or both). This is because otherwise L_2's expected payoff would be zero, but L_2 can get a strictly positive payoff by proposing x_2 such that $1 > x_2 > \alpha + (1 - \alpha)\gamma$, which H_1 must accept by Lemma 1. Therefore we conclude that L_1's payoff is approximately the same as H_1's would be. Since this deviation cannot be profitable, it must be that, for small ε and η_2, L_1's payoff is bounded below, approximately, by H_1's payoff. Hence L_1 in equilibrium gets more than $\frac{1}{2}$ because H_1's expected payoff is at least equal to her disagreement payoff $\delta^{T-1}\gamma > \frac{1}{2}$. By symmetry, L_2's expected payoff is also greater than $\frac{1}{2}$. Since this is impossible because the total surplus is 1, the assumption that L_1 and L_2 reach agreement before T is contradicted. It is easy to see that H_1 and H_2 will not reach agreement either. ■

4 Conclusion

The first part of this paper can be thought of as a contribution to the Nash program: the search for non-cooperative foundations for cooperative solution concepts such as the Nash bargaining solution. It also gives some additional insight into the interpretation of the Nash bargaining solution. Aumann and Kurz (1977) show that the Nash outcome is the point at which each player is equally bold, where a player's boldness is the maximum probability which makes him willing to risk losing the whole gain against an additional gain. The game analyzed here gives a non-cooperative foundation for this idea. In equilibrium each player is indifferent between accepting and rejecting the other's proposal. Therefore, in equilibrium, each is just willing to accept the lottery consisting of his own proposal with probability $1 - \varepsilon$ and the disagreement payoff with probability ε. Similarly, Rubinstein, Safra and Thomson (1992) define the (ordinal)-Nash solution in terms of the players' willingness to risk a breakdown of negotiations.

The paper also provides a theoretical explanation of the deadline effect in a context in which players are free to make offers, and to accept any offer, at any time. The effect stems from the fact that a small amount of incomplete information has a large impact on the split negotiated in the final-stage subgame. This has the implication that, even though there is only a very small prior probability that a given bargainer is strong, he has to protect that reputation by refusing to compromise until the last possible moment.

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Appendix

Proof of Lemma 2 Suppose, to the contrary, that, in some equilibrium (s_1^*, s_2^*), 1's equilibrium proposal x_1^* is rejected by 2. If 2's equilibrium proposal $x_2^* > \beta_2$ then player 1 must accept x_2^* (since $x_2^* > \beta_2 > \beta_1$) and so player 2's expected payoff, $(1-\varepsilon)u_2(1-x_2^*)+\varepsilon v_2$, is less than 2's reservation payoff v_2. This contradiction establishes that $x_2^* \leq \beta_2$. Consider in turn the four cases: (a) $\beta_1 < x_2^* < \beta_2$, (b) $x_2^* = \beta_2$, (c) $x_2^* < \beta_1$ and (d) $x_2^* = \beta_1$. In case (a), 1 accepts x_2^* after making the proposal x_1^* by Lemma 1(ii) and has expected payoff $(1-\varepsilon)u_1(x_2^*)+\varepsilon v_1$. Since $x_2^* < \beta_2$, we have $u_2(1-x_2^*) > v_2$ and there exists $x_1 > x_2^*$ such that $1-x_2^* > 1-x_1 > 1-\beta_2$ and $u_2(1-x_1) > \alpha u_2(1-x_2^*)+(1-\alpha)v_2$. If player 1 proposes x_1 2 must, having proposed x_2^*, accept it by Lemma 1(vi). This gives 1 an expected payoff (if she rejects x_2^*) of $(1-\varepsilon)u_1(x_1)+\varepsilon v_1 > (1-\varepsilon)u_1(x_2^*)+\varepsilon v_1$ and so the deviation is profitable. In case (b), 1's expected payoff is $(1-\varepsilon)u_1(x_2^*)+\varepsilon v_1 = (1-\varepsilon)u_1(\beta_2)+\varepsilon v_1$. Having proposed β_2, 2 must accept any proposal x_1 such that $x_1 < \beta_2$. If 1 makes such a proposal and accepts 2's proposal of β_2, 1's expected payoff is

$$(1-\varepsilon)^2(1/2)[u_1(x_1)+u_1(\beta_2)]+\varepsilon(1-\varepsilon)u_1(x_1)+\varepsilon(1-\varepsilon)u_1(\beta_2)+\varepsilon^2 v_1.$$

For x_1 close to β_2 this quantity is approximately $(1-\varepsilon^2)u_1(\beta_2)+\varepsilon^2 v_1 > (1-\varepsilon)u_1(\beta_2)+\varepsilon v_1$ and so the deviation is profitable. (Note that $u_1(\beta_2) > v_1$ because $\beta_2 > \beta_1$). In case (c), 1's expected payoff is v_1 since she will reject x_2^* if $x_2^* < \beta_1$. If 1 proposes x_1 such that $x_1 > \beta_1$ then, by Lemma 1(i), she will strictly prefer to reject x_2^* if $a_2(x_1 \mid x_2^*, s_2^*) = 1$. She will also reject x_2^* if $a_2(x_1 \mid x_2^*, s_2^*) = 0$ by Lemma 1(ii). Therefore, if $\beta_1 < x_1 < \beta_2$, 2 must accept x_1 after proposing x_2^*. This implies that 1 has a profitable deviation: propose x_1 such that $\beta_1 < x_1 < \beta_2$ and reject x_2^*. In case (d), there exists $x_1 > \beta_1$ such that

$$u_2(1-x_1) > \alpha u_2(1-\beta_1) + (1-\alpha)u_2(1-\beta_2).$$

If 1 proposes x_1 then, by Lemma 1(vi), 2 must, having proposed $x_2^* = \beta_1$, accept. This deviation increases 1's expected payoff.

This establishes that 1's proposal must be accepted in equilibrium. Symmetrical arguments show that 2's proposal must also be accepted. ■

Proof of Theorem 1 The argument in the text establishes that (x_1^*, x_2^*) must satisfy equations (1) and (2) and that, given $\alpha \in (0,1)$, there exists a unique solution to (1) and (2) and that this converges to the Nash bargaining solution as $\alpha \to 1$.

It remains to show that a pure strategy equilibrium exists. Consider the following strategy pair.

Proposals: 1 proposes x_1^* and 2 proposes x_2^*.

Acceptance strategies after non-deviant proposals: After proposing x_1^*, 1 accepts x_2 if and only if $x_2 \geq x_2^*$; after proposing x_2^*, 2 accepts x_1 if and only if $x_1 \leq x_1^*$.

Acceptance strategies after deviant proposals: After any proposal $x_1 \neq x_1^*$, 1 accepts x_2^*. After any proposal $x_2 \neq x_2^*$, 2 accepts x_1^*. Suppose that x_1 and x_2 have been proposed, where $x_1 \neq x_1^*$ and $x_2 \neq x_2^*$.

(a) if $u_1(x_2) \geq v_1$ and $u_2(1-x_1) < \alpha u_2(1-x_2) + (1-\alpha)v_2$, 1 accepts and 2 rejects; (b) if $u_1(x_2) \geq v_1, u_2(1-x_1) \geq \alpha u_2(1-x_2) + (1-\alpha)v_2$ and $u_1(x_2) \geq \alpha u_1(x_1) + (1-\alpha)v_1$, 1 and 2 both accept; (c) if $u_1(x_2) \geq v_1$, $u_2(1-x_1) \geq \alpha u_2(1-x_2) + (1-\alpha)v_2$ and $u_1(x_2) < \alpha u_1(x_1) + (1-\alpha)v_1$, 1 rejects and 2 accepts; (d) if $u_1(x_2) < v_1$ and $u_2(1-x_1) < v_2$, both 1 and 2 reject; (e) if $u_1(x_2) < v_1$, $u_2(1-x_1) \geq v_2$, and $u_1(x_2) < \alpha u_1(x_1) + (1-\alpha)v_1$, 1 rejects and 2 accepts; (f) if $u_1(x_2) < v_1$, $u_2(1-x_1) \geq v_2$, and $u_1(x_2) \geq \alpha u_1(x_1) + (1-\alpha)v_1$, both 1 and 2 accept.

Given 2's strategy, it is optimal for 1 to propose x_1^* because this is the highest proposal which 2 will accept, and $x_1^* > x_2^* > \beta_1$. Similarly, x_2^* is optimal for 2.

After making the proposal x_1^* and receiving any proposal x_2, 1 knows that her proposal will be accepted, if it arrives. Therefore, by Lemma 1(i), it is optimal for her to accept x_2 if $u_1(x_2) \geq \alpha u_1(x_1^*) + (1-\alpha)v_1$. Hence, by (1), it is optimal to accept if $x_2 \geq x_2^*$. A symmetric argument applies to 2. This shows that the acceptance strategies after non-deviant proposals are optimal.

I show next that the acceptance strategies after deviant proposals are optimal. If 1 proposes $x_1 > x_1^*$ and 2 proposes x_2^*, 2 rejects and so it is optimal for 1 to accept x_2^*. If 1 proposes $x_1 < x_1^*$ and 2 proposes x_2^*, 2 accepts x_1. Since $u_1(x_2^*) = \alpha u_1(x_1^*) + (1-\alpha)v_1$ and $u_1(x_1) < u_1(x_1^*)$, we have $u_1(x_2^*) > \alpha u_1(x_1) + (1-\alpha)v_1$. Therefore, by Lemma 1(i), it is optimal for 1 to accept x_2^*. Symmetric arguments show that 2 should accept x_1^*. Now suppose that $x_1 \neq x_1^*$ and $x_2 \neq x_2^*$. In case (a), 1's strategy is obviously optimal and it is optimal for 2 to reject by Lemma 1(iii). In (b), both should accept by Lemma 1(i) and Lemma 1(iii). In (c), 1 should reject by Lemma 1(i). Since $\alpha u_1(x_1) + (1-\alpha)v_1 > u_1(x_2) \geq v_1$, we have $u_1(x_2) < u_1(x_1)$ and so $u_2(1-x_2) > u_2(1-x_1)$. Since $u_2(1-x_1) \geq \alpha u_2(1-x_2) + (1-\alpha)v_2$, $u_2(1-x_1) > v_2$. This implies that 2 should accept. Case (d): obvious. Case (e): 1 should reject by Lemma 1(i) and 2's acceptance is clearly optimal. In case (f), 1 should accept by Lemma 1(i). Since $u_1(x_2) < v_1$ and $u_1(x_2) \geq \alpha u_1(x_1) + (1-\alpha)v_1$, $u_1(x_2) > u_1(x_1)$ and so $u_2(1-x_2) < u_2(1-x_1)$. Also $u_2(1-x_1) \geq v_2$. Therefore $u_2(1-x_1) > \alpha u_2(1-x_2) + (1-\alpha)v_2$. Now apply Lemma 1(iii).

This completes the proof that the strategies described form an equilibrium. ■

Proof of Proposition 1 Let the following define the profile $(\sigma_S^1(\varepsilon), p_S^1)$ of strategies and beliefs. The proposals are as specified in the statement of the theorem.

Beliefs: After any proposal $x_2 \neq 0$, player 1 (of either type) believes that 2 is the low type with probability 1; after $x_2 = 0$, she believes that 2 is the high type with probability 1. After any proposal $x_1 \neq 1$, player 2 believes that 1 is the low type with probability 1 and after $x_1 = 1$, he believes that 1 is the high type with probability 1.

Acceptance strategies: for pairs of proposals which include at least one which has positive probability according to $\sigma_S^1(\varepsilon)$, the following table sets out the acceptance decisions. A means accept, R reject.

	H_1	H_2	L_1	L_2
$x_1 = 1$ and $x_2 \geq \gamma$	A	R	A	R
$x_1 = 1$ and $0 < x_2 < \gamma$	R	R	A	R
$x_1 = 1$ and $x_2 = 0$	R	R	R	R
$x_1 \leq 1 - \gamma$; and $x_2 = 0$	R	A	R	A
$1 > x_1 > 1 - \gamma$ and $x_2 = 0$	R	R	R	A
$x_1 = (\frac{1}{2})(1 + \varepsilon)$ and $x_2 \geq (\frac{1}{2})(1 - \varepsilon)$			A	A
$x_1 = (\frac{1}{2})(1 + \varepsilon)$ and $0 < x_2 < (\frac{1}{2})(1 - \varepsilon)$			R	A
$x_1 < (\frac{1}{2})(1 + \varepsilon)$ and $x_2 = (\frac{1}{2})(1 - \varepsilon)$			A	A
$1 > x_1 > (\frac{1}{2})(1 + \varepsilon)$ and $x_2 = (\frac{1}{2})(1 - \varepsilon)$			A	R

In particular, (a) (i) After proposing 1, H_1 accepts x_2 if and only if $x_2 \geq \gamma$. (ii) After proposing 0, H_2 accepts x_1 if and only if $1 - x_1 \geq \gamma$. (iii) After proposing $(1/2)(1+\varepsilon)$, L_1 accepts x_2 if and only if $x_2 \geq (1/2)(1 - \varepsilon)$. (iv) After proposing $(1/2)(1 - \varepsilon)$, L_2 accepts x_1 if and only if $x_1 \leq (1/2)(1 + \varepsilon)$.

After other pairs of proposals:

(b) Suppose that (x_1, x_2) has been proposed, where $x_1 \neq 1$, $x_1 \neq (1/2)(1 + \varepsilon)$, $x_2 \neq 0$, and $x_2 \neq (1/2)(1 - \varepsilon)$. (i) If $1 - x_1 < \alpha(1 - x_2)$, L_1 accepts and L_2 rejects. (ii) If $1 - x_1 \geq \alpha(1 - x_2)$ and $x_2 \geq \alpha x_1$, L_1 accepts and L_2 accepts. (iii) If $1 - x_1 \geq \alpha(1 - x_2)$ and $x_2 < \alpha x_1$, L_1 rejects and L_2 accepts.

(c) Suppose that (x_1, x_2) has been proposed, where $x_1 \neq 1$ and $x_2 \neq 0$. (i) H_1 accepts x_2 if and only if accepting x_2 is weakly preferred to rejecting it, on the assumption that player 2 is the low type and follows the acceptance rules defined above. (ii) H_2 accepts x_1 if and only if accepting x_1 is weakly preferred to rejecting it, on the assumption that player 1 is the low type and follows the acceptance rules defined above.

On the positive probability paths induced by $\sigma_S^1(\varepsilon)$, H_2 rejects 1's offer and H_1 rejects 2's offer, since $(1/2)(1 - \varepsilon) < \gamma$ and $(1/2)(1 + \varepsilon) > 1 - \gamma$. H_2's and H_1's offers are rejected with probability 1. Therefore the strategy profile gives expected payoffs as specified.

The beliefs are clearly consistent with the strategies. By (a)(ii) and (a)(iv) any proposal x_1 which will be accepted with positive probability is strictly less than γ, since $(1/2)(1+\varepsilon) < \gamma$ and $1-\gamma < \gamma$. Therefore H_1 cannot do better than propose 1. Similarly, H_2 cannot improve on the proposal of 0. L_1's optimal proposal is either $(1/2)(1+\varepsilon)$ (the highest which L_2 will accept, by (a)(iv)), $1-\gamma$ (the highest which H_2 will accept, by (a)(ii)), or some x_1 which will be rejected by both types. $(1/2)(1+\varepsilon)$ gives an expected payoff of $V^* = (1-\eta_2)(1/2)(1-\varepsilon^2)$; $(1-\gamma)$ gives an expected payoff of

$$\widehat{V} = (1-\eta_2)\{(1/2)[(1-\gamma)+(1/2)(1-\varepsilon)]\}(1-\varepsilon^2) + \eta_2(1-\gamma)(1-\varepsilon)$$

(L_1 must accept $(1/2)(1-\varepsilon)$ after proposing $(1-\gamma)$ since $(1/2)(1-\varepsilon) > (1-\gamma)$); an offer which will be rejected gives

$$\widetilde{V} = (1-\eta_2)(1-\varepsilon)(1/2)(1-\varepsilon).$$

Since $1-\varepsilon^2 > (1-\varepsilon)^2$, $V^* > \widetilde{V}$. Since $\lim_{\eta_2\to 0} V^* = (1/2)(1-\varepsilon^2)$ and

$$\lim_{\eta_2\to 0} \widehat{V} = (1/2)[(1-\gamma)+(1/2)(1-\varepsilon)](1-\varepsilon^2)$$

and since $\gamma > (1/2)(1-\varepsilon)$, we have $V^* > \widehat{V}$ for small enough η_2. Therefore, for such η_2, L_1's specified proposal is optimal. Similarly, L_2's specified proposal is optimal for small enough η_1.

Using Lemma 1, it is straightforward to verify that the acceptance strategies are optimal given the beliefs specified. ■

Proof of Proposition 2 Let $(\sigma_P^1(\varepsilon), p_P^1)$ be the following profile of strategies and beliefs.

Proposals: H_1 and L_1 both propose $\widehat{x}_1(\varepsilon) = [\eta_1(1-\varepsilon^2)+\varepsilon(1+\varepsilon)][\eta_1(1-\varepsilon)+2\varepsilon]^{-1}$. H_2 proposes 0. L_2 proposes $\alpha\widehat{x}_1(\varepsilon)$.

Beliefs: After any proposal player 1 (of either type) believes that 2 is the low type with probability 1. After any proposal $x_1 < \widehat{x}_1$, player 2 believes that 1 is the low type with probability 1 and after $x_1 \geq \widehat{x}_1$, he believes that 1 is the high type with probability η_1.

Acceptance strategies: for pairs of proposals which include at least one which has positive probability according to $\sigma_P^1(\varepsilon)$, the following table sets out the acceptance decisions of H_1, L_1 and L_2.

	H_1	L_1	L_2
$x_1 = \widehat{x}_1, x_2 \geq \alpha\widehat{x}_1 + (1-\alpha)\gamma$ and $1 - \widehat{x}_1 \geq \alpha(1 - x_2)$	A	A	A
$x_1 = \widehat{x}_1, x_2 \geq \alpha\widehat{x}_1 + (1-\alpha)\gamma$ and $1 - \widehat{x}_1 < \alpha(1 - x_2)$	A	A	R
$x_1 = \widehat{x}_1$ and $\alpha\widehat{x}_1 \leq x_2 < \alpha\widehat{x}_1 + (1-\alpha)\gamma$	R	A	A
$x_1 = \widehat{x}_1$ and $0 < x_2 < \alpha\widehat{x}_1$	R	R	A
$x_1 < 1$ and $x_2 = 0$	R	R	A
$x_1 = 1$ and $x_2 = 0$	R	R	R
$x_1 \leq 1 - \alpha(1 - \alpha\widehat{x}_1),\ x_2 = \alpha\widehat{x}_1$and $x_2 \geq \alpha x_1 + (1-\alpha)\gamma$	A	A	A
$x_1 \leq 1 - \alpha(1 - \alpha\widehat{x}_1), x_2 = \alpha\widehat{x}_1$and $x_2 < \alpha x_1 + (1-\alpha)\gamma$	R	A	A
$x_1 > 1 - \alpha(1 - \alpha\widehat{x}_1), x_1 \neq \widehat{x}_1$ and $x_2 = \alpha\widehat{x}_1$	A	A	R

In particular: (a) (i) After proposing $\widehat{x}_1$, H_1 accepts x_2 if and only if $x_2 \geq \alpha\widehat{x}_1 + (1-\alpha)\gamma$. (ii) After proposing $\widehat{x}_1$, L_1 accepts x_2 if and only if $x_2 \geq \alpha\widehat{x}_1$. (iii) After proposing $\alpha\widehat{x}_1$, L_2 accepts $\widehat{x}_1$. (iv) After proposing $\alpha\widehat{x}_1$, L_2 accepts $x_1 \neq \widehat{x}_1$ if and only if $1 - x_1 \geq \alpha(1 - \alpha\widehat{x}_1)$.

Acceptance strategies after other pairs of proposals:

(b) Suppose that (x_1, x_2) has been proposed, where $x_1 < \widehat{x}_1$, $x_2 \neq 0$ and $x_2 \neq \alpha\widehat{x}_1$.

	H_1	L_1	L_2
$1 - x_1 < \alpha(1 - x_2)$ and $x_2 \geq \gamma$	A	A	R
$1 - x_1 < \alpha(1 - x_2)$ and $x_2 < \gamma$	R	A	R
$1 - x_1 \geq \alpha(1 - x_2)$ and $x_2 \geq \alpha x_1 + (1-\alpha)\gamma$	A	A	A
$1 - x_1 \geq \alpha(1 - x_2)$ and $\alpha x_1 + (1-\alpha)\gamma > x_2 \geq \alpha x_1$	R	A	A
$1 - x_1 \geq \alpha(1 - x_2)$ and $x_2 < \alpha x_1$	R	R	A

(c) Suppose that (x_1, x_2) has been proposed, where $x_1 > \widehat{x}_1, x_2 \neq 0$ and $x_2 \neq \alpha\widehat{x}_1$.

	H_1	L_1	L_2
$x_2 < \alpha x_1$	R	R	A
$x_2 \geq \alpha x_1$ and $1 - x_1 \leq \alpha(1 - x_2)$	A	A	R
$x_2 > \alpha x_1 + (1-\alpha)\gamma$ and $1 - x_1 > \alpha(1 - x_2)$	A	A	A
$\alpha x_1 \leq x_2 \leq \alpha x_1 + (1-\alpha)\gamma$ and $1 - x_1 > \alpha(1 - x_2)$	R	A	A

H_2's acceptance strategy:

(d) After any pair of proposals (x_1, x_2), H_2 accepts x_1 if and only if accepting x_1 is weakly preferred to rejecting it, given the beliefs and acceptance strategies specified above.

Clearly, the beliefs are consistent with the strategies and it is straightforward to verify, using Lemma 1, that each acceptance rule set out above is optimal given the beliefs and the other acceptance rules. The only cases which require further argument are lines 3 and 8 of the first table. Consider line 3, i.e., suppose that 2 has proposed x_2 such that $\alpha\widehat{x}_1 \leq x_2 < \alpha\widehat{x}_1 + (1-\alpha)\gamma$ and 1 has made the proposal $\widehat{x}_1$. L_2 believes that x_2 will be accepted (if it arrives) with probability $1-\eta_1$ because H_1 will reject it and L_1 will accept. $\widehat{x}_1$ has the property that L_2 is indifferent between accepting and rejecting it, given that he has proposed $\alpha\widehat{x}_1$ and that this will be accepted with probability $1-\eta_1$: if L_2 rejects $\widehat{x}_1$ his expected payoff is $(1-\varepsilon)(1-\eta_1)(1-\alpha\widehat{x}_1)$ and, if he accepts,

$$\varepsilon(1-\widehat{x}_1) + (1-\varepsilon)\eta_1(1-\widehat{x}_1) + (1-\varepsilon)(1-\eta_1)(1/2)(1-\widehat{x}_1+1-\alpha\widehat{x}_1).$$

Routine calculation shows that these are equal. Since he is indifferent between accepting and rejecting $\widehat{x}_1$ after proposing $\alpha\widehat{x}_1$, he strictly prefers to accept $\widehat{x}_1$ after proposing $x_2 > \alpha\widehat{x}_1$, if it will be accepted with probability $1-\eta_1$.

Now consider line 8 of the first table, i.e., suppose that 2 has proposed $\alpha\widehat{x}_1$ and 1 has proposed $x_1 \neq \widehat{x}_1$ such that $x_1 \leq 1-\alpha(1-\alpha\widehat{x}_1)$ and $\alpha\widehat{x}_1 < \alpha x_1 + (1-\alpha)\gamma$. H_1's and L_2's strategies are optimal by Lemma 1. Note that if $x_1 > \widetilde{x}_1 = (\frac{1}{2})(1+\varepsilon)$, then

$$1 - x_1 < \alpha(1-\alpha x_1).$$

This is because $\widetilde{x}_1$ is defined by the relation $f(\widetilde{x}_1) = f(\alpha\widetilde{x}_1)$ where f is the function F defined in section 2, specialized to the case of risk-neutral utility and $v_1 = v_2 = 0$, i.e., $f(x) = x(1-x)$. If $x_1 > \widetilde{x}_1$ then, by strict concavity of f, $f(x_1) < f(\alpha x_1)$. That is, $x_1(1-x_1) < \alpha x_1(1-\alpha x_1)$. Therefore, since $\widehat{x}_1 > \widetilde{x}_1$ for sufficiently small ε,

$$\widehat{x}_1 > 1 - \alpha(1-\alpha\widehat{x}_1). \qquad (A1)$$

$(A1)$ implies that $x_1 < \widehat{x}_1$ and so it is optimal for L_1, having proposed x_1, to accept $\alpha\widehat{x}_1$ by Lemma 1.

For small ε, $\widehat{x}_1$ is approximately equal to 1 and so $\widehat{x}_1 > \gamma$. Also, from $(A1)$, $\widehat{x}_1 > 1-\alpha(1-\alpha\widehat{x}_1)$. Therefore it is optimal for H_1 and L_1 to propose $\widehat{x}_1$ because this is the highest proposal which 2 will accept and higher than both γ and 2's expected offer $(\alpha\widehat{x}_1)$. H_2's proposal is optimal because there is no chance of 1 proposing or accepting anything below $1-\gamma$ (for small ε both $\widehat{x}_1$ and $\alpha\widehat{x}_1$ are close to 1). In completing the assessment of L_2's optimal strategy we need to consider four possible courses of action: (i) propose something which will be rejected with probability 1 and accept $\widehat{x}_1$; (ii) propose $\alpha\widehat{x}_1$ and accept $\widehat{x}_1$, giving *ex ante* expected payoff

$$\begin{aligned}\overline{V}_{L_2} &= \varepsilon(1-\varepsilon)(1-\eta_1)[1-\alpha\widehat{x}_1]+\varepsilon(1-\varepsilon)[1-\widehat{x}_1] \\ &\quad +(1-\varepsilon)^2(1-\eta_1)(1/2)(1-\widehat{x}_1+1-\alpha\widehat{x}_1) \\ &\quad +(1-\varepsilon)^2\eta_1[1-\widehat{x}_1].\end{aligned}$$

(proposing $\alpha\widehat{x}_1$ and rejecting $\widehat{x}_1$ would give the same expected payoff, as shown above). (iii) propose $\alpha\widehat{x}_1+(1-\alpha)\gamma$, the lowest proposal which will be accepted by both types, and accept $\widehat{x}_1$, giving expected payoff

$$\begin{aligned}V'_{L_2} &= \varepsilon(1-\varepsilon)[1-\alpha\widehat{x}_1-(1-\alpha)\gamma] \\ &\quad +\varepsilon(1-\varepsilon)[1-\widehat{x}_1]+(1-\varepsilon)^2(1/2)(1-\widehat{x}_1+1-\alpha\widehat{x}_1-(1-\alpha)\gamma);\end{aligned}$$

(iv) propose $\alpha\widehat{x}_1+(1-\alpha)\gamma$ and reject $\widehat{x}_1$, giving expected payoff

$$V^*_{L_2} = (1-\varepsilon)[1-\alpha\widehat{x}_1-(1-\alpha)\gamma].$$

(i) is clearly inferior to (ii). Ignoring terms in ε^2,

$$(\overline{V}_{L_2}-V'_{L_2}) \simeq \eta_1(1-\widehat{x}_1)-(\frac{\eta_1}{2})(2-\widehat{x}_1(1+\alpha))+(\frac{1}{2})(1-\alpha)\gamma-\varepsilon\eta_1(1-\widehat{x}_1),$$

and so, since $1+\alpha = 2(1+\varepsilon)^{-1}$ and $1-\alpha = 2\varepsilon(1+\varepsilon)^{-1}$,

$$(\overline{V}_{L_2}-V'_{L_2}) \simeq \varepsilon(1+\varepsilon)^{-1}(\gamma-\eta_1\widehat{x}_1)-\varepsilon\eta_1(1-\widehat{x}_1).$$

As $\varepsilon \to 0$, $\widehat{x}_1 \to 1$ and hence $\varepsilon^{-1}(\overline{V}_{L_2} - V'_{L_2}) \to (\gamma - \eta_1)$, which is strictly positive. Therefore, for small enough ε, (iii) is inferior to (ii). Similarly, ignoring terms in ε^2,

$$(\overline{V}_{L_2} - V^*_{L_2}) \simeq (\frac{1+\eta_1}{2})\widehat{x}_1(\alpha - 1) - \varepsilon\eta_1(1-\widehat{x}_1) + \varepsilon(1-\alpha\widehat{x}_1) + (1-\varepsilon)(1-\alpha)\gamma.$$

Therefore, as $\varepsilon \to 0$, $\varepsilon^{-1}(\overline{V}_{L_2} - V^*_{L_2}) \to (2\gamma - 1 - \eta_1)$ so (iv) is inferior to (ii) because $\eta_1 < 2\gamma - 1$. This establishes that L_2's proposal is optimal for small ε.

The equilibrium payoffs of L_1 and H_1 converge to 1 as $\varepsilon \to 0$ because $\widehat{x}_1 \to 1$ and $\alpha\widehat{x}_1 \to 1$. ■

Proof of Proposition 5 The proof is by induction on t. Suppose that, for small ε, $\Gamma^{t-1}(\eta_1, 0, \varepsilon)$ $(t \geq 2)$ has a pure strategy equilibrium $(\sigma_P^{t-1}(\varepsilon), p_P^{t-1})$ in which the payoffs of H_1, L_1, H_2 and L_2 are respectively $V_{H_1}(\varepsilon), V_{L_1}(\varepsilon), \delta^{t-2}\gamma$ and $V_{L_2}(\varepsilon)$, where $V_{H_1}(\varepsilon) \geq V_{L_1}(\varepsilon)$ and, as $\varepsilon \to 0$, $V_{H_1}(\varepsilon) \to (\frac{1}{2})(1+\delta^{t-2}), V_{L_1}(\varepsilon) \to (\frac{1}{2})(1+\delta^{t-2})$ and $V_{L_2}(\varepsilon) \to (\frac{1}{2})(1-\delta^{t-2})$. By Proposition 2, this is true for $t = 2$. We will construct, for each small enough ε, an equilibrium for $\Gamma^t(\eta_1, 0, \varepsilon)$ such that, as $\varepsilon \to 0$, the payoffs converge to the expressions given. Let $(\sigma_P^t(\varepsilon), p_P^t)$ be defined as follows.

Proposals in the first period (i.e., when there are t periods to go)

H_1 and L_1 both propose $\widetilde{x}_1(\varepsilon) = (\frac{1}{2})[(1+\varepsilon) - (1+\varepsilon)\delta V_{L_2}(\varepsilon) + (1-\varepsilon)\delta V_{L_1}(\varepsilon)]$.

L_2 proposes $\widetilde{x}_2(\varepsilon) = (\frac{1}{2})[(1-\varepsilon) - (1-\varepsilon)\delta V_{L_2}(\varepsilon) + (1+\varepsilon)\delta V_{L_1}(\varepsilon)]$.

(Note that these are the proposals made in the single-period complete information game between 1 and 2 with $v_1 = \delta V_{L_1}(\varepsilon)$ and $v_2 = \delta V_{L_2}(\varepsilon)$).

H_2 proposes 0.

Beliefs after first-period events

(1) 1 believes that 2 is the low type at the start of the second period (i.e., when there are $t-1$ periods to go), no matter what happens in the first period, assuming that the game continues to the second period.

(2) If 1 proposes $x_1 > \widetilde{x}_1(\varepsilon)$, 2 believes that 1 is the high type, and continues to

believe this at the start of the second period; if 1 proposes $x_1 < \tilde{x}_1(\varepsilon)$, 2 believes that 1 is the low type, and continues to believe this at the start of the second period (regardless, for example, of whether 1 has rejected a high proposal x_2); if 1 proposes $\tilde{x}_1(\varepsilon)$, 2 believes that 1 is the high type with probability η_1.

(3) If 1 has proposed $\tilde{x}_1(\varepsilon)$, or if 1's offer does not arrive, and 1 rejects x_2, then (i) if $x_2 \geq \tilde{x}_2(\varepsilon)$, 2 believes that 1 is the high type with probability 1; (ii) if $x_2 < \tilde{x}_2(\varepsilon)$, 2 believes that 1 is the high type with probability η_1.

Continuation strategies from the start of the second period

If the game does not end in the first period there are three possible belief states at the start of the second period: (a) 1 is believed to be the high type, in which case the strategy profile $\sigma_{10}^{t-1}(\varepsilon)$ described in Proposition 3 is played; (b) 1 is believed to be the low type, in which case the strategy profile $\sigma_{00}^{t-1}(\varepsilon)$ described in Proposition 4 is played; (c) 1 is believed to be the high type with probability η_1, in which case $\sigma_P^{t-1}(\varepsilon)$ is played, giving continuation payoffs $V_{H_1}(\varepsilon), V_{L_1}(\varepsilon), \delta^{t-2}\gamma$ and $V_{L_2}(\varepsilon)$ respectively to H_1, L_1, H_2 and L_2.

Acceptance strategies in the first period

(i) If 1 proposes $\tilde{x}_1(\varepsilon)$ and 2 proposes $x_2 \geq \tilde{x}_2(\varepsilon)$, H_1, L_1 and L_2 all accept;

(ii) If 1 proposes $\tilde{x}_1(\varepsilon)$ and 2 proposes $x_2 < \tilde{x}_2(\varepsilon)$, H_1 and L_1 reject and L_2 accepts;

(iii) If 1 proposes $x_1 < \tilde{x}_1(\varepsilon)$ and 2 proposes $\tilde{x}_2(\varepsilon)$, H_1 and L_1accept and L_2 accepts if and only if accepting x_1 is weakly preferred to rejecting, on the assumption that 1 will accept and the continuation, if any, will be $\sigma_{00}^{t-1}(\varepsilon)$;

(iv) If 1 proposes $x_1 > \tilde{x}_1(\varepsilon)$ and 2 proposes $\tilde{x}_2(\varepsilon)$, H_1 and L_1 accept and L_2 rejects;

(v) If 1 proposes $x_1 \neq \tilde{x}_1(\varepsilon)$ and 2 proposes $x_2 \neq \tilde{x}_2(\varepsilon)$, then (a) if $x_1 > \tilde{x}_1(\varepsilon)$ the acceptance strategies are as in the first period of the strategy profile $\sigma_{10}^{t}(\varepsilon)$ described in Proposition 3; and (b) if $x_1 < \tilde{x}_1(\varepsilon)$ the acceptance strategies are as in the first period of the strategy profile $\sigma_{00}^{t}(\varepsilon)$ described in Proposition 4;

(vi) H_2's acceptance strategy is a pure best response to the acceptance strategies above, given the specified beliefs and continuation strategies.

The beliefs are clearly consistent with the strategies and the continuation strategies are clearly in equilibrium after each possible first-period history. Given the acceptance strategies, H_1's and L_1's proposal of $\widetilde{x}_1(\varepsilon)$ is optimal because L_2 will accept this but reject anything higher while H_1 and L_1 will accept $\widetilde{x}_2(\varepsilon)$, which is less than $\widetilde{x}_1(\varepsilon)$. Equivalent reasoning shows that it is optimal for L_2 to propose $\widetilde{x}_2(\varepsilon)$. H_2's reservation payoff is $\delta^{t-1}\gamma$, by the induction hypothesis. Therefore he should not propose or accept anything higher than $1-\delta^{t-1}\gamma$. As $\varepsilon \to 0$, $\widetilde{x}_1(\varepsilon) \to \frac{1}{2}(1+\delta^{t-1})$ and $\widetilde{x}_2(\varepsilon) \to \frac{1}{2}(1+\delta^{t-1})$. By assumption, $\delta^{t-1}(\frac{1}{2}+\gamma) > \frac{1}{2}$, so $\frac{1}{2}(1+\delta^{t-1}) > 1-\delta^{t-1}\gamma$, which implies that, if ε is small enough, H_2 should not make an acceptable proposal, or accept 1's offer. Therefore it is optimal to propose 0.

Next, we have to check that the acceptance strategies are optimal, given the continuations specified. Note that, if 1's rejection payoff (i.e., her payoff if she rejects) is either $\widetilde{x}_1(\varepsilon)$ (if her offer arrives) or $\delta V_{L_1}(\varepsilon)$ (if it does not) then, by the complete information analysis, 1 is indifferent between accepting and rejecting $\widetilde{x}_2(\varepsilon)$. Similarly, if 2's rejection payoff is either $1-\widetilde{x}_2(\varepsilon)$ or $\delta V_{L_2}(\varepsilon)$ then 2 is indifferent between accepting and rejecting $\widetilde{x}_1(\varepsilon)$. Case (i): L_2's rejection payoff is either $1-x_2 \leq 1-\widetilde{x}_2(\varepsilon)$ (if his proposal arrives) or $\delta V_{L_2}(\varepsilon)$ (if it does not); therefore it is optimal to accept $\widetilde{x}_1(\varepsilon)$. H_1's rejection payoff is either $\widetilde{x}_1(\varepsilon)$ or approximately (for small ε) $\frac{\delta}{2}(1+\delta^{t-2}\gamma)$ while L_1's is slightly less than this, since rejection leads to $\Gamma^{t-1}(1,0,\varepsilon)$ if $\widetilde{x}_1(\varepsilon)$ does not arrive; therefore, by Lemma 1, it is optimal to accept x_2 if $\delta V_{L_1}(\varepsilon) > \frac{\delta}{2}(1+\delta^{t-2}\gamma)$. For small enough ε, this is guaranteed if $\frac{1}{2}(1+\delta^{t-2}) > \frac{1}{2}(1+\delta^{t-2}\gamma)$. The latter is true since $\gamma < 1$. Case (ii): H_1's rejection payoff is either $\widetilde{x}_1(\varepsilon)$ or $\delta V_{H_1}(\varepsilon) > \delta V_{L_1}(\varepsilon)$; L_1's is either $\widetilde{x}_1(\varepsilon)$ or $\delta V_{L_1}(\varepsilon)$. Therefore it is optimal for each to reject x_2. L_2's rejection payoff is $\delta V_{L_2}(\varepsilon) < 1-\widetilde{x}_1(\varepsilon)$, so it is optimal to accept $\widetilde{x}_1(\varepsilon)$. Case (iii): L_2's strategy is obviously optimal. If L_2's strategy is to accept x_1 it is optimal for H_1 and L_1 to accept $\widetilde{x}_2(\varepsilon)$ if the rejection payoff (if x_1 does not arrive) is less than $\delta V_{L_1}(\varepsilon)$. Similarly, if L_2's strategy is to reject x_1 it is optimal for $H_1$and L_1 to accept $\widetilde{x}_2(\varepsilon)$ if the rejection payoff is less than $\delta V_{L_1}(\varepsilon)$ since $\delta V_{L_1}(\varepsilon) < \widetilde{x}_2(\varepsilon)$. Rejection (if x_1 does not arrive) leads to $\Gamma^{t-1}(0,0,\varepsilon)$ and so, for small ε, H_1's rejection payoff is $\delta^{t-1}\gamma$ and L_1's is approximately $\frac{\delta}{2}$.

$$\lim_{\varepsilon\to 0}\delta V_{L_1}(\varepsilon) = \frac{\delta}{2}(1+\delta^{t-2}) > \frac{\delta}{2}$$

and, since $\gamma < 1$,

$$\frac{\delta}{2}(1+\delta^{t-2}) > \delta^{t-1}\gamma.$$

Therefore, for small enough ε, it is optimal to accept $\widetilde{x}_2(\varepsilon)$. Case (iv): L_2's proposal will be accepted. It is optimal for him to reject $x_1 > \widetilde{x}_1(\varepsilon)$ if L_2's rejection payoff (assuming 2's offer does not arrive) is at least $\delta V_{L_2}(\varepsilon)$. The rejection continuation is $\Gamma^{t-1}(1,0,\varepsilon)$ and so, by Proposition 3, L_2's rejection payoff is approximately $\frac{\delta}{2}(1-\delta^{t-2}\gamma)$.

$$\lim_{\varepsilon\to 0}\delta V_{L_2}(\varepsilon) = \frac{\delta}{2}(1-\delta^{t-2}) < \frac{\delta}{2}(1-\delta^{t-2}\gamma)$$

and so, for small ε, L_2 should reject any $x_1 > \widetilde{x}_1(\varepsilon)$. Given that x_1 will be rejected, H_1 and L_1 should, for small ε, accept $\widetilde{x}_2(\varepsilon)$ if $\widetilde{x}_2(\varepsilon) > \frac{\delta}{2}(1+\delta^{t-2}\gamma)$ (this is approximately 1's rejection payoff by Proposition 3).

$$\lim_{\varepsilon\to 0}\widetilde{x}_2(\varepsilon) = \frac{1}{2}[1-\frac{\delta}{2}(1-\delta^{t-2})+\frac{\delta}{2}(1+\delta^{t-2})] = \frac{1}{2}[1+\delta^{t-1}] > \frac{\delta}{2}(1+\delta^{t-2}\gamma)$$

since $\gamma < 1$. This shows that acceptance is optimal for H_1 and L_1. Cases (v) and (vi) are obvious.

As $\varepsilon \to 0$, H_1's and L_1's payoff from this strategy profile converges to

$$\frac{1}{2}[\lim_{\varepsilon\to 0}(\widetilde{x}_1(\varepsilon)+\widetilde{x}_2(\varepsilon))] = \frac{1}{2}[1+\delta^{t-1}].$$

Similarly, L_2's payoff converges to $\frac{1}{2}[1-\delta^{t-1}]$. This completes the proof. ■