Identification in Empirical Games*

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Abstract

We study problems of identification of an econometric model for empirical games in the most general way. Global identification results have already been found for example in some specific auctions. We work with local identification principle, which enable us to write general identification results. In our framework we have symmetric players and iid private information. The benchmark model is a one with the distribution of the private information and an additionnal dissociated parameter (e.g. parameter of risk aversion) as parameters of interest. Eventually we extend the study to the case of partial observability.

1 Introduction

The problem of identification in empirical games is an already existing issue but which did not find any satisfactory answer. What has not been done and seems interesting is to find a general framework, applicable to any structural model of games with I players, an observable x_i, $i = \overline{1, I}$, as the result of a transformation φ_i of an unobservable ξ_i, which is an iid private information with a distribution F_θ and of an additional parameter λ_i.

$$x_i = \varphi_i \left(\xi_i, F_\theta, \lambda \right)$$

In our framework we have symmetric players and iid private information. This is a strong limitation with respect to what we can see from the real life, but we recall that no general identification conditions have been established until now, and it seems obvious to start with this benchmark model. Moreover we face here a problem of local

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identification. One could argue that some works have already be done which establish global identification results. This is true and well done for example in Laffont, Hossard and Vuong, Campo, Perrigne and Vuong, Donald and Paarsch, but this is at the cost of generality. With local identification we can establish results for a broad class of games.

In the work of Florens, Protopopescu and Richard (FPR), two conditions was found to establish the possibility to identify F_θ when we assume symmetry among players, in the parametric case or in the nonparametric one.

Our work goes a little bit further in two ways. First, we give an identification condition for any parameters of the transformation φ (for example a parameter of risk aversion) and not only parameters of the distribution function. Second, this condition is valid in the parametric as well as in the nonparametric case.

We begin with a general theorem which states the first identification condition. Some illustrations follow. The first is IPV (Independent Private Value) Dutch and first-price sealed-bid auction. We consider the case of potential bidders who are identical ex ante, who are potentially risk averse w.r.t. winning the object, and who have the same HARA (Hyperbolic Absolute Risk Aversion) utility function. Here, our parameters of interest are then $\theta = F$ and λ the coefficient of risk aversion. We establish that there is no identification. In the parametric case there is two applications. The parameters of interest are λ and θ the vector of the parameters associated to F_θ. First, we show that in a procurement model with the cost as private information we have identification. Then we study a model of duopoly for which we find that there is no identification.

In the last section we have an extension of the general theorem to the case of partial observability. Two examples, which are inspired by the models studied for example by Wolak or Lavergne and Thomas, illustrate the case of partial observability with exclusion. We conclude with an application which is a simpler version of the model by Florens, Hugo and Richard for procurements in the space industry.

2 General framework

Following FPR work , we consider that the I players draw **iid** private signals ξ_i, $i = \overline{1, I}$, following a distribution F_θ which is assumed to be common knowledge to all and completely determined by θ.

Let $\xi_i \in \Xi = \left[\underline{\xi}, \overline{\xi}\right]$, $F_\theta \in \mathcal{F}_\varphi$,

$\mathcal{F}_\varphi = \mathcal{F} \cap D_\varphi$ with $\mathcal{F}$ the subset of distribution function in $C^q(\Xi)$ and D_φ an appropriate vector subspace of $C^q(\Xi)$ (in order to guarantee the monotonicity of φ in ξ given F_θ).

The ξ_i are then transformed into observable actions (bids) x_i by means of a transformation φ_i, say $x_i = \varphi_i(\xi_i, F_\theta, \lambda)$.

In our study the function $\varphi : \Xi \times C^q(\Xi) \times R^r$ is common to all players (the **game is symmetric**)

$$x_i = \varphi(\xi_i, F_\theta, \lambda)$$

Let $\mathbf{\Phi}$ denote the operator

$$\begin{aligned} \mathbf{\Phi} &: \Theta \times R^r \longrightarrow C^q(\Xi) ; \theta \times \lambda \longrightarrow \mathbf{\Phi}(\theta, \lambda) \\ \mathbf{\Phi}(\theta, \lambda)(\xi) &= \varphi(\xi, \theta, \lambda) \overset{say}{=} \varphi_{\theta,\lambda}(\xi) \end{aligned}$$

$d_F\ \mathbf{\Phi}_{F,\lambda}(H)$ denotes the Fréchet differential of $\mathbf{\Phi}(F, .)$ in the direction of H,
$d\lambda\mathbf{\Phi}\ (\beta)$ denotes the Fréchet differential of $\mathbf{\Phi}(., \lambda)$ in the direction of β.
In practice Fréchet differentials can be computed as Gâteaux differentials, i.e.

$$d_F\mathbf{\Phi}_{F,\lambda}(H) = \lim_{\lambda \longrightarrow 0^+} \frac{d}{d\lambda} \mathbf{\Phi}(F + \lambda H, \lambda)$$

(under some conditions verified by assumption)

3 General identification principle

The parameters of interest are $(\theta, \lambda) \in \Theta \times R$, while the observations (x_i), $i = \overline{1,n}$ are iid.

Let defined Θ as a open subset of a Banach space (in order to have the differentiability).

Remark 1 *$\theta = F$ corresponds to the nonparametric case, let say semiparametric because of λ, and $\theta \in R^p$ to the parametric one.*

Definition 1 *The parameters (θ_1, λ_1) and (θ_2, λ_2) are observationaly equivalent ($(\theta_1, \lambda_1) \sim (\theta_2, \lambda_2)$) iff $G_1 = G_2$ where $G_i = F_{\theta i} \circ \varphi^{-1}_{F_{\theta i}, \lambda_i}$* $\quad (i = 1, 2)$

Definition 2 *the parameters $(\theta, \lambda) \in \Theta \times R^r$ are globally identified iff*

$$\forall\ (\theta_*, \lambda_*) \in \Theta \times R^r,\ (\theta_*, \lambda_*) \sim (\theta, \lambda) \Longrightarrow (\theta_*, \lambda_*) = (\theta, \lambda)$$

Definition 3 *The parameters $(\theta, \lambda) \in \Theta \times R^r$ are locally identified iff there exists a neighborhood $\mathcal{V}(\theta, \lambda)$ of (θ, λ) in $\Theta \times R^r$ relative to the norm $\|.\|_{(2,qo)??}$ such that*

$$\forall\ (\theta_*, \lambda_*) \in \mathcal{V}(\theta, \lambda),\ (\theta_*, \lambda_*) \sim (\theta, \lambda) \Longrightarrow (\theta_*, \lambda_*) = (\theta, \lambda)$$

Lemma 1 *(θ_2, λ_2) is observationaly equivalent to (θ_1, λ_1) iff*

$$F_{\theta_2} - G_1 \circ \varphi_{F_{\theta 2}, \lambda_2} = 0 \tag{1}$$

Proof. (i) if $(\theta_2, \lambda_2) \sim (\theta_1, \lambda_1)$ then $G_1 = G_2$ and $G_1 \circ \varphi_{F_{\theta 2}, \lambda_2} = G_2 \circ \varphi_{F_{\theta 2}, \lambda_2} = F_{\theta 2}$

(ii) from equation (1) it follows that $G_2 = F_{\theta 2} \circ \varphi^{-1}_{F_{\theta 2}, \lambda_2} = G_1$ ■

Theorem 2 *The model* $(\varphi, \theta, \lambda)$ *is locally identified if the operator :* $T_{\theta,\lambda} = d_\theta F_\theta + \frac{f}{\partial \varphi_{\theta,\lambda}} \left(d_\theta \mathbf{\Phi}_{\theta,\lambda} - d_\lambda \mathbf{\Phi}_{\theta,\lambda} \right)$ *is one to one* $\forall \left(\theta, \lambda\right) . \in \Theta \times R$.

Proof. consider the application

$$\begin{array}{rcl} A & : & (\Theta \times R^r) \times (\Theta \times R^r) \longrightarrow \mathcal{D}_\varphi \\ \left[(\theta, \lambda) ; (\theta_*, \lambda_*) \right] & \longmapsto & A \left((\theta, \lambda) ; (\theta_*, \lambda_*) \right) = F_{\theta *} - G_{F,\lambda} \circ \varphi_{F_{\theta *}, \lambda *} \end{array}$$

Note that $A \left((\theta, \lambda) ; (\theta, \lambda) \right) = 0$. It follows from the implicit function theorem that, if the Fréchet differential of A w.r.t. (θ_*, λ_*) is invertible at $\left((\theta, \lambda); (\theta, \lambda) \right)$, then there exists a neighborhood $\mathcal{V}(\theta, \lambda)$ of (θ, λ) such that $(\theta_*, \lambda_*) = (\theta, \lambda)$ is the unique solution to the equation $A \left((\theta, \lambda) ; (\theta_*, \lambda_*) \right) = 0$ for $(\theta_*, \lambda_*) \in \mathcal{V}(\theta, \lambda)$, in which case it follows from Lemma 1 that (θ, λ) is locally identified. The Fréchet differential of A w.r.t. (θ_*, λ_*) at $\left((\theta, \lambda) ; (\theta, \lambda) \right)$ is characterized by the operator

$$d_\theta F_\theta - \left(\partial G_{F_{\theta,\lambda}} \circ \varphi_{F_\theta, \lambda} \right) \left(d_\theta \mathbf{\Phi}_{\theta,\lambda} - d_\lambda \mathbf{\Phi}_{\theta,\lambda} \right)$$

■

4 The Nonparametric Case

Our parameters of interest are then $\theta = F$ and λ.

IPV Dutch and first-price sealed-bid auctions represent a quiet general type of auctions and it is interesting to see what type of result we get.

4.1 IPV Dutch and First-Price Sealed-Bid Auctions

Following Donald and Paarsch , we consider the case of potential bidders who are identical ex ante, who are potentially risk averse w.r.t. winning the object, and who have the same HARA (Hyperbolic Absolute Risk Aversion) utility function

$$U(Y) = \lambda Y^{1/\lambda} \text{ where } \lambda \geq 1$$

In our case, we take as a parameter the coefficient of risk aversion λ.

$\lambda \geq 1$ and take values in R.

Dutch and first-price sealed-bid auctions are also strategically equivalent. Following Donald and Paarsch we have in both case

$$\varphi_{F,\lambda}(\xi) = \xi - \frac{\int_{\xi_0}^{\xi} F^{m\lambda}(v)\, dv}{F^{m\lambda}(\xi)}$$

$(F, \lambda) \in \mathcal{D}_\varphi \times R$ and we actually face a semiparametric identification problem.

4.1.1 Computation of $d_F \mathbf{\Phi}_{F,\lambda}(H)$

We recall that

$$d_F \mathbf{\Phi}_{F,\lambda}(H) = \lim_{\lambda \to 0^+} \tfrac{d}{d\lambda} \mathbf{\Phi}(F + \lambda H, \lambda)$$

$$\tfrac{d}{d\lambda} \mathbf{\Phi}(F + \lambda H, \lambda)(\xi) = \tfrac{d}{d\lambda} \varphi(\xi, F + \lambda H, \lambda)$$

$$= \tfrac{d}{d\lambda} \left[\xi - \frac{\int_{\xi_0}^{\xi} [F(v) + \lambda H(v)]^{m\lambda}\, dv}{[F(\xi) + \lambda H(\xi)]^{m\lambda}} \right]$$

$$= - \frac{\int_{\xi_0}^{\xi} [F(v) + \lambda H(v)]^{m\lambda}\, dv\, (-m\lambda H(\xi))}{[F(\xi) + \lambda H(\xi)]^{m\lambda+1}} - \frac{\int_{\xi_0}^{\xi} m\lambda H(v) [F(v) + \lambda H(v)]^{m\lambda-1}\, dv}{[F(\xi) + \lambda H(\xi)]^{m\lambda}}$$

Then we find

$$d_F \mathbf{\Phi}_{F,\lambda}(H)(\xi) = \frac{m\lambda}{[F(\xi)]^{m\lambda+1}} \times \left\{ H(\xi) \int_{\xi_0}^{\xi} F^{m\lambda}(v)\, dv - F(\xi) \int_{\xi_0}^{\xi} H(v) F^{m\lambda-1}(v)\, dv \right\}$$

4.1.2 Computation of $d_\lambda \mathbf{\Phi}_{F,\lambda}$

We have $d_\lambda \mathbf{\Phi}_{F,\lambda}(\beta) = \lim_{\lambda \to 0^+} \tfrac{d}{d\lambda} \mathbf{\Phi}(F, \lambda + \lambda\beta)$

$$\tfrac{d}{d\lambda} \mathbf{\Phi}(F, \lambda + \lambda\beta)(\xi) = \tfrac{d}{d\lambda} \varphi(\xi, F, \lambda + \lambda\beta)$$

$$= \tfrac{d}{d\lambda} \left[\xi - \frac{\int_{\xi_0}^{\xi} [F(v)]^{m(\lambda+\lambda\beta)}\, dv}{[F(\xi)]^{m(\lambda+\lambda\beta)}} \right]$$

$$= \frac{m\beta \ln F(\xi) \int_{\xi_0}^{\xi} [F(v)]^{m(\lambda+\lambda\beta)}\, dv}{[F(\xi)]^{m(\lambda+\lambda\beta)}}$$

$$- \frac{\int_{\xi_0}^{\xi} m\beta \ln F(v) [F(v)]^{m(\lambda+\lambda\beta)}\, dv}{[F(\xi)]^{m(\lambda+\lambda\beta)}}$$

Then we find
$d_\lambda \mathbf{\Phi}_{F,\lambda}(\beta)(\xi) = \frac{m\beta}{[F(\xi)]^{m\lambda}} \left\{ \ln F(\xi) \int_{\xi_0}^{\xi} [F(v)]^{m\lambda} dv - \int_{\xi_0}^{\xi} \ln F(v) [F(v)]^{m\lambda} dv \right\}$

4.1.3 Computation of $T_{F,\lambda}(H,\beta)(\xi)$

We can compute

$$\begin{aligned} \partial\varphi_{F,\lambda}(\xi) &= \frac{\partial}{\partial\xi}\varphi_{F,\lambda}(\xi) = 1 + \frac{m\lambda f(\xi)}{F^{m\lambda+1}(\xi)} \int_{\xi_0}^{\xi} F^{m\lambda}(u)\, du - \frac{F^{m\lambda}(\xi)}{F^{m\lambda}(\xi)} \\ &= \frac{m\lambda f(\xi)}{F^{m\lambda+1}(\xi)} \int_{\xi_0}^{\xi} F^{m\lambda}(u)\, du \end{aligned}$$

and then deduce

1. $\frac{f}{\partial\varphi_{F,\lambda}} d_F \mathbf{\Phi}_{F,\lambda}(H)(\xi) = H(\xi) - \frac{F(\xi)}{\int_{\xi_0}^{\xi} [F(v)]^{m\lambda} dv} \int_{\xi_0}^{\xi} H(v) F^{m\lambda-1}(v)\, dv$

2. $\frac{f}{\partial\varphi_{F,\lambda}} d_\lambda \mathbf{\Phi}_{F,\lambda}(\beta)(\xi) = \frac{F(\xi)\beta}{\lambda} \left\{ \ln F(\xi) - \frac{\int_{\xi_0}^{\xi} \ln F(v) [F(v)]^{m\lambda} dv}{\int_{\xi_0}^{\xi} [F(v)]^{m\lambda} dv} \right\}$

Finally we find :
$T_{F,\lambda}(H,\beta)(\xi)$
$= H(\xi) - H(\xi)$
$+ \frac{F(\xi)}{\int_{\xi_0}^{\xi} [F(v)]^{m\lambda} dv} \int_{\xi_0}^{\xi} H(v) F^{m\lambda-1}(v)\, dv - \frac{F(\xi)\beta}{\lambda} \left\{ \ln F(\xi) - \frac{\int_{\xi_0}^{\xi} \ln F(v) [F(v)]^{m\lambda} dv}{\int_{\xi_0}^{\xi} [F(v)]^{m\lambda} dv} \right\}$
$= \frac{F(\xi)}{\int_{\xi_0}^{\xi} [F(v)]^{m\lambda} dv}$
$\times \left\{ \int_{\xi_0}^{\xi} H(v) F^{m\lambda-1}(v)\, dv - \frac{\beta}{\lambda} \left[\ln F(\xi) \int_{\xi_0}^{\xi} [F(v)]^{m\lambda} dv - \int_{\xi_0}^{\xi} \ln F(v) [F(v)]^{m\lambda} dv \right] \right\}$

Remark 2 *We need $\lambda \neq 0$. Here, because we have a parameter of risk aversion, $\lambda \geq 1$*

In order to determine if the model is identified in the semiparametric case, we need to study the injectivity of $T_{F,\lambda}(H,\beta)$.

Actually we can show that it is not injective. To see this let take the Ker of this operator. This correspond to the set $Ker\left[T_{F,\lambda}(H,\beta)\right] = \left\{(H,\beta)/T_{F,\lambda}(H,\beta)(\xi) = 0\right\}$.

$$\begin{aligned} T_{F,\lambda}(H,\beta)(\xi) &= 0 \Leftrightarrow \beta C(F,\xi) = \lambda \int_{\xi_0}^{\xi} H(v)\, F^{m\lambda-1}(v)\, dv \qquad (2) \\ \text{with } C(F,\xi) &= \ln F(\xi) \int_{\xi_0}^{\xi} [F(v)]^{m\lambda} dv - \int_{\xi_0}^{\xi} \ln F(v)\, [F(v)]^{m\lambda} dv \end{aligned}$$

Any H and β verifying (2) must also verify $\frac{\partial}{\partial \xi} T_{F,\lambda}(H,\beta)(\xi) = 0$. This imply $\beta \frac{\partial}{\partial \xi} C(F,\xi) = \lambda H(\xi)\, F^{m\lambda-1}(\xi)$. Rewriting it we have

$$H(\xi) = \frac{\beta}{\lambda F^{m\lambda-1}(\xi)} \frac{\partial}{\partial \xi} C(F,\xi) \qquad (3)$$

$$\begin{aligned} \text{with } \frac{\partial}{\partial \xi} C(F,\xi) &= \int_{\xi_0}^{\xi} [F(v)]^{m\lambda} dv \frac{f(\xi)}{F(\xi)} + \ln F(\xi) \left\{ [F(\xi)]^{m\lambda} - [F(\xi)]^{m\lambda} \right\} \\ &= \int_{\xi_0}^{\xi} [F(v)]^{m\lambda} dv \frac{f(\xi)}{F(\xi)} \end{aligned}$$

It follows from (3) that $\forall \beta$, $\exists H(\xi)$ s.t. $(H,\beta) \in Ker\left[T_{F,\lambda}(H,\beta)\right]$. This implies that $\dim\left\{Ker\left[T_{F,\lambda}(H,\beta)\right]\right\} \geq \dim\{\beta\} = 1$ and then that $T_{F,\lambda}(H,\beta)$ is not injective[1].

Then we have shown the following result :

Proposition 3 *In an IPV Dutch or First-Price Sealed Bid model of auction, with F the distribution of private information and λ the parameter of risk aversion as parameters of interest, the semiparametric identification is not possible.*

5 The Parametric Case

Here, the parameter of interest is $(\theta, \lambda) \in R^{p+r}$.

If we take the same type of auction than in the semiparametric case, this only means that we have to specify a parametric distribution $F_{(\theta)}$.

5.1 IPV Procurement

We are interested in the distribution of the cost c. Being in a parametric case we specify a distribution F_θ fully characterized by the parameter θ.

[1]"$Ker\left[T_{F,\eta}(H,\beta)\right]$ has a huge number of elements"

In the case of a procurement the participants will play

$$\varphi_{F_\theta,\lambda}(\xi) = \xi - \frac{\int_\xi^{+\infty} [1 - F_\theta(v)]^{m\lambda} \, dv}{F^{m\lambda}(\xi)}$$

Let take a general form for the distribution as an exponential : $c \sim \mathcal{E}(\theta)$, $\theta > 0$, $c > 0$.

Then $F_\theta(\xi) = 1 - \exp(-\theta\xi)$ and $f(\xi) = \theta \exp(-\theta\xi)$ and we find

$$\varphi_{F_\theta,\lambda}(\xi) = \xi - \frac{1}{\theta m\lambda}$$

5.1.1 Computation of $d_\theta F\left(\hat{\theta}\right)$

Here this term is not reduced to the identity.

$$d_\theta F\left(\widehat{\theta}\right)(\xi) = \frac{\partial \exp(-\theta m\lambda\xi)}{\partial\theta} \cdot \widehat{\theta} = -m\lambda\xi \exp(-\theta m\lambda\xi)\,\widehat{\theta}$$

5.1.2 Computation of $d_\theta \mathbf{\Phi}_{\theta,\lambda}\left(\hat{\theta}\right)$

$$d_\theta \mathbf{\Phi}_{\theta,\lambda}\left(\hat{\theta}\right)(\xi) = \frac{\partial\varphi_{F_\theta,\lambda}(\xi)}{\partial\theta} \cdot \widehat{\theta} = \frac{\widehat{\theta}}{\theta^2 m\lambda}$$

5.1.3 Computation of $d_\lambda \mathbf{\Phi}_{\theta,\lambda}(\lambda)$

$$d_\lambda \mathbf{\Phi}_{\theta,\lambda}(\lambda)(\xi) = \frac{\lambda}{\theta m\lambda^2}$$

5.1.4 Computation of $T_{F,\lambda}(H,\beta)(\xi)$

Here we simply have $\dfrac{f(\xi)}{\partial\varphi_{F_\theta,\lambda}(\xi)} = f(\xi) = \theta \exp(-\theta\xi)$

Then

$$T_{\theta,\lambda}\left(\widehat{\theta},\lambda\right)(\xi) = -m\lambda\xi \exp(-\theta m\lambda\xi)\,\widehat{\theta} + \left[\frac{\widehat{\theta}}{\theta^2 m\lambda} + \frac{\lambda}{\theta m\lambda^2}\right]\theta \exp(-\theta\xi)$$

As in the nonparametric case, we study the set $Ker\left[T_{\theta,\lambda}\left(\hat{\theta},\lambda\right)\right]$.

$$T_{\theta,\lambda}\left(\hat{\theta},\lambda\right)(\xi) = 0 \Leftrightarrow \xi\hat{\theta} = \frac{\exp(\theta m\lambda\xi)}{\theta m^2\lambda^2}\left[\frac{\hat{\theta}}{\theta} + \frac{\lambda}{\lambda}\right]$$

If we differentiate twice w.r.t. ξ we obtain

$$\frac{\hat{\theta}}{\theta}+\frac{\lambda}{\lambda}=0$$

then we have $\hat{\theta}=\lambda=0$ which leads to the injectivity of the operator $T_{\theta,\lambda}$.

In the IPV procurement model with a cost $c\sim\mathcal{E}(\theta)$, θ and the parameter of risk aversion λ are identified.

5.2 The Duopoly Case

Not only the auction models are of interest. A good illustration could be then this one : two firms with private costs c_i $(i=1,2)$ compete in a Cournot game. The private costs are iid and follow a normal distribution, $c_i\sim\mathcal{N}\left(\mu,\sigma^2\right)$. The cost functions are linear :

$$C\left(Q_i\right)=c_iQ_i$$

the quantity to choose are supposed to be an affine function of the cost :

$$Q_i=g-hc_i$$

and the inverse demand function is written :

$$P=a-b\left(Q_1+Q_2\right)$$

For firm i the program is to maximize its expected revenue :

$$\underset{Q_i}{Max}\ \ E\left[PQ_i-C\left(Q_i\right)\right]$$

After the first order condition we find :$Q_i=\dfrac{a-bg}{2b}+\dfrac{h}{2}\mu-\dfrac{1}{2b}c_i$

In this case, our parameters of interest are the ones of the distribution, μ and σ, and only one of the coefficient, b.

By analogy with the previous presentation we can say that $\theta=(\mu,\sigma)$ and $\lambda=b$.

To facilitate the notation we will note c_i as c and we will denote φ as the response function of the firm,

$$\varphi\left(c\right)=\frac{a-bg}{2b}+\frac{h}{2}\mu-\frac{1}{2b}c$$

We recall that the distribution function is

$$F\left(c\right)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\frac{c-\mu}{\sigma}}\exp\left(-\frac{u^2}{2}\right)dt$$

5.2.1 Computation of $d_\theta F\left(\hat{\theta}\right)$

$$d_\theta F\left(\widehat{\theta}\right)(c) = \frac{\partial F(c)}{\partial \mu}\cdot\widehat{\mu} + \frac{\partial F(c)}{\partial \sigma}\cdot\widehat{\sigma}$$
$$= \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{1}{2}\left[\frac{c-\mu}{\sigma}\right]^2\right)\left\{-\frac{\widehat{\mu}}{\sigma} - \frac{(c-\mu)}{\sigma^2}\widehat{\sigma}\right\}$$

5.2.2 Computation of $d_\theta \mathbf{\Phi}_{\theta,\lambda}\left(\hat{\theta}\right)$

$$d_\theta \mathbf{\Phi}_\theta\left(\hat{\theta}\right)(c) = \frac{\partial \varphi_\theta(\xi)}{\partial \mu}\cdot\widehat{\mu} = \frac{h}{2}\widehat{\mu}$$

5.2.3 Computation of $d_\lambda \mathbf{\Phi}_{\theta,\lambda}(\lambda)$

$$d_\lambda \mathbf{\Phi}_{\theta,\lambda}(\lambda)(\xi) = \frac{\partial \varphi_\theta(\xi)}{\partial b}\cdot\widehat{b} = -\frac{(a-c)}{2b^2}\widehat{b}$$

5.2.4 Computation of $T_{F,\lambda}(H,\beta)(\xi)$

Here we simply have $\dfrac{f(c)}{\partial \varphi_{F_\theta,\lambda}(c)} = -f(c)\,2b$

Then

$$T_\theta\left(\hat{\theta}\right)(c)$$
$$= \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{1}{2}\left[\frac{c-\mu}{\sigma}\right]^2\right)\left\{-\frac{\widehat{\mu}}{\sigma} + \frac{(\mu-c)}{\sigma^2}\widehat{\sigma} - \frac{bh}{\sigma}\widehat{\mu} + \frac{(a-c)}{\sigma b}\widehat{b}\right\}$$

we study the set $Ker\left[T_\theta\left(\hat{\theta}\right)\right]$.

$$T_\theta\left(\hat{\theta}\right)(c) = 0 \Longrightarrow \widehat{\sigma} = -\frac{\widehat{b}}{b} \tag{4}$$

Then the Ker is not reduced to $(0,0,0)$, T_θ is not injective.

In this duopoly model with private cost $c \sim \mathcal{N}\left(\mu,\sigma^2\right)$ and the coefficient term b of the inverse demand function, as parameters of interest, there is no identification.

Remark 3 *If all the parameters (a,b,g,h) of the quantity response are to be identified, there is no chance to have identification either.*

Remark 4 *If only the parameters (μ,σ) of the distribution function are to be identified, there is identification.*

6 Partial Observability

Generally speaking, it seems that in the semiparametric case we don't have easily identification.

One way to identify and to generalize our theorem is to take into account the case of partial observability.

Actually we consider the case where ξ_i, x_i and φ can be partitioned in the following way :

$\xi_i = \begin{pmatrix} \eta_i \\ z_i \end{pmatrix}$, $x_i = \varphi(\xi_i, \lambda) = \begin{pmatrix} y_i \\ z_i \end{pmatrix} = \begin{pmatrix} \varphi_y(\eta_i, z_i, \lambda) \\ z_i \end{pmatrix}$

where $z_i \in R^q$ is observable as the realizations of a variable z.

to facilitate the notation the indexes "i" will not be written

F can be decomposed in F_z the distribution of the z's and F^z the distribution of η conditional on z

$$F = F_z F^z$$

Let M and C fully characterize respectively F_z and F^z.

Remark 5 *In the parametric case, M and C would correspond to the parameters of the distributions F_z and F^z. In the semiparametric case, M and C would correspond to F_z and F^z.*

Our parameter of interest is then $\theta = (M, C, \lambda)$.

From this point we can proceed in the same way than in the general identification part, with some changes in the arguments.

G, the joint distribution of the observables (y_i, z_i), can be written

$$G = F_z G^z$$

Note that $G(\hat{y}, z) = (F_z G^z) \circ (\hat{y}, z) = F_z(z) G^z(\hat{y}, z) = F_z(z) F^z(\varphi^{-1}(\hat{y}, z)) = (F_z F^z) \circ (\varphi^{-1}(\hat{y}, z))$, the last equality because w.r.t. z, φ^{-1} corresponds to the identity.

Then a short version of the Lemma is now

$$\theta_* \sim \theta \Longleftrightarrow F_*^z F_{*z} - G \circ \varphi_* = 0 \tag{5}$$

Theorem 4 *The model (φ, θ) is locally identified if the operator $T_\theta = F^z d_M F_z + F_z d_C F^z + \frac{f}{\partial \varphi} d_\theta \mathbf{\Phi}_\theta$ is one to one $\forall \theta . \in \Theta$.*

Proof. consider the application

$$\begin{array}{ccc} A & : & \Theta \times \Theta \longrightarrow \mathcal{D}_{\varphi} \\ [\theta;\theta_*] & \longmapsto & A(\theta;\theta_*,) = F_*^z F_{*z} - G \circ \varphi_* \end{array}$$

Note that $A(\theta;\theta) = 0$. It follows from the implicit function theorem that, if the Fréchet differential of A w.r.t. θ_* is one to one at $((\theta,\eta);(\theta,\eta))$, then there exists a neighborhood $\mathcal{V}(\theta)$ of θ such that $\theta_* = \theta$ is the unique solution to the equation $A(\theta;\theta_*) = 0$ for $\theta_* \in \mathcal{V}(\theta)$, in which case it follows from (4) that θ is locally identified. The Fréchet differential of A w.r.t. θ_* at $(\theta;\theta)$ is characterized by the operator

$$d_\theta (F_z F^z) - (\partial G \circ \varphi)(d_\theta \mathbf{\Phi}_\theta)$$

Moreover, we know that $g \circ \varphi = \dfrac{f^{z_2}}{\partial\varphi/\partial\eta}$. Then ■

7 Partial Observability with Exclusion

Here we are in the case where z is decomposed in two observables z_1 and z_2.

$\xi = \begin{pmatrix} \eta \\ z_1 \\ z_2 \end{pmatrix} \in \Xi \times R^{k_1} \times R^{k_2}$, $x = \begin{pmatrix} y \\ z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} \varphi_y(\eta, z, \lambda, F^{z_2}) \\ z_1 \\ z_2 \end{pmatrix} \in R \times R^{k_1} \times R^{k_2}$.

We recall or establish the following assumptions :

φ_y is monotone increasing in η

$\lambda \in R$

F^{z_2} is the conditional law of η knowing z_2 with a density $f^{z_2}(\eta) = f(\eta/z_2)$

The most important points are that η is independent of z_1 conditionally on z_2, and z_1 and z_2 are measurably separated[2]. This actually state the exclusion.

Our parameter of interest is $\theta = (\lambda, F^{z_2})$

Remark 6 *z_1 is not necessarily continuous. If it is a discrete variable, then we should take differences instead of differentials for the calculus. Concerning λ, we could generalize the result to the case where this parameter is not a real number. This just a way to simplify the presentation.*

Lemma 5 *In the case of partial observability with exclusion, θ_2 is observationally equivalent to θ_1 iff*

$$F_{\theta_2} - G_1\left(\varphi_y\left(\eta, z, \lambda_2, F_2^{z_2}\right)/\tilde{z}\right) = 0 \tag{6}$$

[2]$\forall\alpha(z_2)$ differentiable function of z_2, $\dfrac{\partial}{\partial z_1}\alpha(z_2) = 0$ a.s. in the sens of the law of Z

Proof. (i) if $\theta_2 \sim \theta_1$ then $G_1 = G_2$ and $G_1 \circ \varphi_{F_{\theta 2}} = G_2 \circ \varphi_{F_{\theta 2}} = F_{\theta 2}$

(ii) from equation (4) it follows that $G_2 = F_{\theta 2} \circ \varphi^{-1}_{F_{\theta 2}, \eta_2} = G_1$ ■

Theorem 6 *The linear operator of interest is* $T_\theta\left(\tilde{\theta}\right) = \dfrac{\partial}{\partial z_1}\left[d_\theta \mathbf{\Phi}_\theta\left(\tilde{\theta}\right)\right]\dfrac{\partial \varphi_y}{\partial a} - \dfrac{\partial^2 \varphi_y}{\partial z_1 \partial \lambda} d_\theta \mathbf{\Phi}_\theta\left(\tilde{\theta}\right).$ *If* $T_\theta\left(\tilde{\theta}\right) = 0$ *implies* $\tilde{\lambda} = 0$ *and if the model is locally identified with* λ *known, then the model is locally identified.*

Proof. We have

$G\left(y/\tilde{z}_1, \tilde{z}_2\right) = P\left(y \leq \tilde{y}/z_1 = \tilde{z}_{1,} z_2 = \tilde{z}_2\right)$

$= P\left(\varphi_y\left(\eta, z, \lambda, F^{z_2}\right) \leq \tilde{y}/z_1 = \tilde{z}_{1,} z_2 = \tilde{z}_2\right) = P\left(\eta \leq \tilde{\eta}/z_1 = \tilde{z}_{1,} z_2 = \tilde{z}_2\right)$

$= P\left(\eta \leq \tilde{\eta}/z_2 = \tilde{z}_2\right)$

$= F^{z_2}\left(\tilde{\eta}\right)$

Consider the application

$$\begin{array}{rcl} A & : & \Theta \times \Theta \longrightarrow \mathcal{D}_\varphi \\ \left[\theta; \theta_*\right] & \longmapsto & A\left(\theta; \theta_*\right) = F_{\theta *} - G\left(\varphi_y\left(\eta, z, \lambda_*, F_*^{z_2}\right)/\tilde{z}_1, \tilde{z}_2\right) \end{array}$$

Note that $A\left(\theta; \theta\right) = 0$. It follows from the implicit function theorem that, if the Fréchet differential of A w.r.t. $\left(\theta_*, \eta_*\right)$ is one to one at $\left(\left(\theta, \eta\right); \left(\theta, \eta\right)\right)$, then there exists a neighborhood $\mathcal{V}(\theta)$ of (θ) such that $\theta_* = \theta$ is the unique solution to the equation $A\left(\theta; \theta_*\right) = 0$ for $\theta_* \in \mathcal{V}(\theta)$, in which case it follows that (θ) is locally identified. The Fréchet differential of A w.r.t. θ_*at $\left(\theta; \theta\right)$ is characterized by the operator

$$d_\lambda A_\theta\left(\tilde{\lambda}\right) + d_{F^{z_2}} A_\theta\left(\tilde{F}^{z_2}\right) = \frac{\partial \varphi_y}{\partial \lambda} g\left(\varphi_y\right)\tilde{\lambda} + d_{F^{z_2}} G_\theta\left(\tilde{F}^{z_2}\right) g\left(\varphi_y\right) - \tilde{F}^{z_2} \tag{7}$$

Moreover, we know that $g \circ \varphi = \dfrac{f^{z_2}}{\partial \varphi / \partial \eta}$

Then if we have

$$\left\{\frac{\partial \varphi_y}{\partial \lambda}\tilde{\lambda}\frac{f^{z_2}}{\partial \varphi/\partial \eta} + d_{F^{z_2}} \mathbf{\Phi}_\theta\left(\tilde{F}^{z_2}\right)\frac{f^{z_2}}{\partial \varphi_y/\partial \eta} - \tilde{F}^{z_2} = 0\right\} \Longrightarrow \tilde{\lambda} = 0,\ \tilde{F}^{z_2} = 0 \tag{8}$$

the Fréchet differential of A w.r.t. θ_* at $\left(\theta; \theta\right)$ would be one to one, in which case it follows from Lemma 2 that θ is locally identified.

The idea now is to differentiate (6) w.r.t. z_1. Then the condition now is

$$\left\{\frac{\partial}{\partial z_1}\left[\left(\frac{\partial \varphi_y}{\partial \lambda}\tilde{\lambda} + d_{F^{z_2}} \mathbf{\Phi}_\theta\left(\tilde{F}^{z_2}\right)\right)/\frac{\partial \varphi_y}{\partial \eta}\right] = 0\right\} \Longrightarrow \tilde{\lambda} = 0,\ \tilde{F}^{z_2} = 0$$

with the notation $\theta = (\lambda, F^{z_2})$ we can rewrite it as

$$\left\{ \frac{\partial}{\partial z_1} \left[d_\theta \mathbf{\Phi}_\theta \left(\tilde{\theta} \right) / \frac{\partial \varphi_y}{\partial \eta} \right] = 0 \right\} \Longrightarrow \tilde{\theta} = 0$$

which leads finally to

$$\frac{\partial}{\partial z_1} \left[d_\theta \mathbf{\Phi}_\theta \left(\tilde{\theta} \right) \right] \frac{\partial \varphi_y}{\partial \eta} - \frac{\partial^2 \varphi_y}{\partial z_1 \partial \eta} d_\theta \mathbf{\Phi}_\theta \left(\tilde{\theta} \right) = 0$$

and if this implies that $\tilde{\lambda} = 0$, we face an already known problem. ■

Example 1 *Let take this type of model :*

$$y = m(z_1, \lambda) + \Psi(\eta, F^{z_2}) \equiv \varphi_y(\eta, z, \lambda, F^{z_2})$$

then the identification condition is written

$$\left\{ \frac{\partial}{\partial z_1} d_\theta \mathbf{\Phi}_\theta \left(\tilde{\theta} \right) = \frac{\partial^2 m}{\partial z_1 \partial \lambda} \tilde{\lambda} = 0 \right\} \Longrightarrow \tilde{\lambda} = \tilde{F}^{z_2} = 0$$

which leads to

$$\frac{\partial^2 m}{\partial z_1 \partial \lambda} \neq 0$$

and we end with a reasonable condition to establish that $\tilde{\lambda} = 0$.and then to face a common problem.

Example 2 *What happens now when we face a complete nonparametric problem of this type :*

$$y = m(z_1) + \Psi(\eta, F^{z_2}) \equiv \varphi_y(\eta, z, \lambda, F^{z_2})$$

with $\theta = (m, F^{z_2})$. In this case the identification condition is

$$\left\{ \frac{\partial}{\partial z_1} \left[d_m \mathbf{\Phi}_\theta (\tilde{m}) + d_{F^{z_2}} \mathbf{\Phi}_\theta \left(\tilde{F}^{z_2} \right) \right] = 0 \right\} \Longrightarrow \tilde{\theta} = 0$$

the left hand side equals $\frac{\partial}{\partial z_1} \tilde{m}(z_1) = 0$, then we end with $\tilde{m}(z_1) = cst \Longrightarrow \tilde{\theta} = 0$. Actually with $\tilde{m}(z_1) = cst$, we face now a much more simplified problem.

7.1 Application to a Semiparametric Model of Procurement

A good application for this section is a model of procurement as in the work of Florens, Hugo, Richard. The interest here is that it takes into account the number of participants (as usual) and the quality of the projects. Theses variables are supposed to be known at least ex-ante. In our presentation the number of participants N corresponds to z_1 and the quality Q_0 to z_2.

We suppose that all the firms are qualified and that their strategical choice is the price P of their offer, depending on their private cost c, its distribution conditionally on Q_0 $F(c/Q_0)$,the number of participants N, and the parameter of risk aversion λ. We can deduce the symmetric equilibrium $P = \varphi_y(c)$:

$$P = c + \frac{\int_c^{\bar{c}} S^{N\lambda}(v/Q_0)\,dv}{S^{N\lambda}(c/Q_0)}$$

with $S(c/Q_0) = 1 - F(c/Q_0)$. Here our parameter of interest is $\theta = (\lambda, S)$.

For the ease of notation we will denote $S(c/Q_0)$ by $S(c)$.

We are in the case where z_1 is a discret variable. Let denote $B(N)$ the left hand side of (8)as a function of N. the identification condition must be true for every N. Then an equivalent left hand side term in the identification condition is $B(N) = B(1)$,which leads after computation to the following identification condition :

$$\begin{aligned} & \frac{\tilde{\lambda}}{\lambda}\left\{\int_c^{\bar{c}} S^{N\lambda}(v)\ln S(v/Q_0)\,dv \int_c^{\bar{c}} S^{\lambda}(v)\,dv - \int_c^{\bar{c}} S^{\lambda}(v/Q_0)\ln S(v)\,dv \int_c^{\bar{c}} S^{\lambda N}(v)\,dv\right\} \\ = & \int_c^{\bar{c}} S^{\lambda N}(v)\,dv \int_c^{\bar{c}} \tilde{S}(v)\,S^{\lambda-1}(v)\,dv - \int_c^{\bar{c}} S^{\lambda}(v)\,dv \int_c^{\bar{c}} \tilde{S}(v)\,S^{\lambda N-1}(v)\,dv \\ \Longrightarrow & \left\{\tilde{\lambda} = 0,\ \tilde{S} = 0\right\} \end{aligned}$$

I will show that there is no identification. For this I will just show a counter-example. Let take any $\left(\tilde{\lambda}, \tilde{S}\right)$ which verify

$$\tilde{S}(v) = S(v) - \frac{\tilde{\lambda}}{\lambda} S(v)\ln S(v)$$

we can check that these $\left(\tilde{\lambda}, \tilde{S}\right)$ verify $B(N) = B(1)$, but $\left(\tilde{\lambda}, \tilde{S}\right)$ is not necessarily equal to $(0, 0)$.

Proposition 7 *In the semiparametric model of procurement defined above with the parameter of risk aversion λ and $S(c/Q_0)$ the survival function of c conditionnally on the quality as parameters of interests, there is no identification.*

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