Building Gorman's Nest

Jeffrey T. LaFrance
Professor
University of California
Berkeley, California
USA
lafrance@are.berkeley.edu

Timothy K. M. Beatty
Assistant Professor
University of British Columbia
Vancouver, British Columbia
Canada
timothy.beatty@ubc.ca

Rulon D. Pope
Professor
Brigham Young University
Provo, Utah
USA
rulon_pope@byu.edu

Abstract

Gorman's class of Engel curve demand models is extended to incomplete systems. The Gorman class of aggregable incomplete demand systems admits any transformation of deflated income. A maximum rank of three for the demand equations is a corollary of Slutsky symmetry. Models of incomplete demand systems are developed that nest the rank of the demand system and functional form of the income variables, can be globally restricted to be weakly integrable, can be estimated consistently and efficiently with aggregate market data, and permit inferences on the impacts of policies on consumption and economic welfare for identifiable groups of consumers.

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Please address all correspondence to:

Professor Jeffrey T. LaFrance
Department of Agricultural and Resource Economics
207 Giannini Hall / MC 3310
University of California
Berkeley, CA 94720-3310
USA
lafrance@are.berkeley.edu

1. Introduction

There are good reasons to consider the effects of aggregation from micro units to the aggregate market level data in economic analysis. Policy effects vary across individuals. Preferences differ across individuals. The theory of choice applies to individuals. The impacts of government policies and market intervention schemes on prices, quantities, taxes, benefits and so forth unfold first at the market level and then for individuals. Consistent aggregation has been addressed for complete demand systems in Gorman (1953, 1961, 1981), Muellbauer (1975, 1976), Lewbel (1987, 1990), and van Daal and Merkies (1989). Of these, Gorman's (1981) remarkable and elegant contribution to the festschrift to Sir Richard Stone is the cornerstone to virtually all aggregation analyses in applied consumer behavior.

Following Muellbauer's (1975) extension of the Gorman polar form to a nonlinear function of income to obtain the price independent generalized linear (PIGL) and price independent generalized logarithmic (PIGLOG) functional forms, much progress has been made in the past 25 years on aggregation theory for complete demand systems. The Almost Ideal Demand System (AIDS) of Deaton and Muellbauer (1980) implements Muellbauer's results to produce demands with budget shares expressed as functions of linear and quadratic terms in the logarithm of prices and a linear term in the logarithm of income. The AIDS and its linear approximation (LA-AIDS) have been linchpins in applied demand analysis since their introduction. Most applications of the AIDS and LA-AIDS either assume separability and estimate a complete system of demands for a disaggregate group of commodities as functions of prices for the goods in the group and total expenditure on the group, or estimate a complete system of demands with highly aggregated commodities as functions of aggregate price indices and total consumption expenditures (hereafter, income, which we denote by m).

Shortly after the article by Deaton and Muellbauer, in a remarkable and elegant contribution to the festschrift to Sir Richard Stone, Gorman (1981) derived the set of functional forms for demand models that can be written in terms of any additive set of functions of income. Any complete system of demand equations in the class of *Gorman Engel curves* must satisfy two properties in addition to homogeneity, adding up and symmetry. First, if the number of independent functions of income is at least three, then the functions all must be either (a) polynomials in income, (b) polynomials in some non-integer power of income, (c) polynomials in the natural logarithm of income, or (d) a series of sine and cosine functions of the natural logarithm of income. Second, the number of linearly independent functions of income in this class of demand systems is at most three, where linear independence refers to the rank of the matrix of price functions that premultiply the income functions.

One of the important implications of these results is that all theoretically consistent complete demand systems that are compatible with aggregation across individual incomes and that have full column rank require at most three summary statistics from the distribution of income to estimate the demand parameters with aggregate data.

An open question of considerable interest is whether these results on aggregation in complete demand systems extend to incomplete demand systems, and if they do, what form this extension might take. In this paper, we extend Gorman's class of polynomial

Engel curve demand systems to incomplete demand systems. The extension admits any transformation of deflated income, expanding the PIGL/PIGLOG functional forms enormously. But a maximal rank of three for this class follows purely from Slutsky symmetry; neither adding up nor homogeneity play any role in the rank restriction. At least a quadratic polynomial is necessary to obtain rank three. A nondegenerate quadratic is sufficient. A significant benefit of the analysis is that the arguments leading to these conclusions clarify, verify, and extend Gorman's original conjecture that the quadratic is the most general nondegenerate class of full rank three aggregable demand models.

The models we develop for incomplete demand systems nest the rank of the demand system and functional form of the income variables, can be globally restricted to be weakly integrable (LaFrance and Hanemann, 1989), can be estimated with aggregate market data, and accommodate inferences on the impacts of policies on consumption and economic welfare of various identifiable groups of consumers.

2. Aggregation Theory for Complete Demand Systems

We begin with a fairly large amount of notation. Let $\boldsymbol{p} \in \mathbb{R}^{n_q}_{++}$ be the vector of market prices for the goods of interest, $\boldsymbol{q} \in \mathbb{R}^{n_q}_{+}$, let $\tilde{\boldsymbol{p}} \in \mathbb{R}^{n_{\tilde{q}}}_{++}$ be the vector of market prices for other goods, $\tilde{\boldsymbol{q}} \in \mathbb{R}^{n_{\tilde{q}}}_{+}$, let $m \in \mathbb{R}_{++}$ be income, let $s = \tilde{\boldsymbol{p}}'\tilde{\boldsymbol{q}} = m - \boldsymbol{p}'\boldsymbol{q} > 0$ be expenditure on other goods, let $\boldsymbol{z} \in \mathbb{R}^{J}$ be a vector of demand shifters, let $\pi(\tilde{\boldsymbol{p}})$ be a 1° homogeneous function of $\tilde{\boldsymbol{p}}$,[1] let $\boldsymbol{x} = [g_1(p_1/\pi(\tilde{\boldsymbol{p}}))\cdots g_{n_q}(p_{n_q}/\pi(\tilde{\boldsymbol{p}}))]' \equiv \boldsymbol{g}(\boldsymbol{p}/\pi(\tilde{\boldsymbol{p}}))$ be a vector of twice

[1] When the system is *complete*, $\tilde{\boldsymbol{p}}$ has no elements and in such a case we adopt the convention $\pi(\tilde{\boldsymbol{p}}) \equiv 1$.

continuously differentiable, strictly monotone functions, and let $y = f(m/\pi(\tilde{\boldsymbol{p}}))$ be a twice continuously differentiable, strictly monotone increasing transformation.

Suppose that we have a transformed demand system of the form

$$\frac{\partial y(\boldsymbol{x};\cdot)}{\partial \boldsymbol{x}} = \sum_{i=0}^{K} \alpha_i(\boldsymbol{x};\cdot) h_i\left(y(\boldsymbol{x};\cdot)\right), \tag{1}$$

where the "$\cdot$" after the semicolon indicates that the system may not be complete and depends on other variables in addition to $\boldsymbol{x}$. However, for notational clarity and compactness, we will omit this set of unspecified arguments in most of what follows. By a simple change of variables from $\boldsymbol{p}$ and m to $\boldsymbol{x} = \ln(\boldsymbol{p})$ and $y = \ln(m)$, Gorman (1981) showed three things about all complete demand systems in this class:

(i) Normalizing for a unique representation, accounting for adding up, and for some of the implications of symmetry, the nonlinear partial differential equations can be transformed into a set of homogeneous linear ordinary differential equations in functions of the natural logarithm of income. From the theory of differential equations, solutions to this system are of the form $h_i(m) = m^{\lambda_i}\left(\ln(m)\right)^i$, where each λ_i is a root of the characteristic polynomial for the linear ordinary differential equations. In general, such characteristic roots can be either real or complex, and complex roots come in conjugate pairs that may have both real and complex parts.

(ii) Given (i), if the rank of the $n_q \times K$ coefficient matrix $\boldsymbol{A}(\boldsymbol{x}) \equiv [\alpha_{ij}(\boldsymbol{x})]$ equals at least three, then symmetry implies that: (a) the characteristic roots are either purely real or purely complex (all roots of the form $\lambda_i = a_i + b_i\sqrt{-1}$ have $a_i = 0$ if

$b_i \neq 0$ and conversely, $b_i = 0$ if $a_i \neq 0$); (b) if any roots are real, there are no complex roots, and conversely; and (c) for real roots, there are no product terms of the form $m^{\alpha}\left(\ln(m)\right)^{\beta}$ with both $\alpha \neq 1$ and $\beta \neq 0$.

(iii) Given the functional form restrictions in (ii), symmetry implies that the rank of $\boldsymbol{A}(\boldsymbol{x})$ is at most three.

For rank three demand systems, this completely specifies the class of functional forms for the income terms. Only three mutually exclusive cases are possible: (a) $m\left(\ln(m)\right)^{r}$, r an integer; (b) $m^{1+\kappa}$, $\kappa \neq 0$; or (c) $m\sin\left(\tau\ln(m)\right)$ and $m\cos\left(\tau\ln(m)\right)$, $\tau > 0$, with both sine and cosine terms appearing as a conjugate complex pair. In other words, for rank three demand systems, the model must take one of the following three forms:

$$\boldsymbol{q} = \alpha_0(\boldsymbol{x})m + \sum_{j=1}^{K}\alpha_j(\boldsymbol{x})m[\ln(m)]^j\,; \tag{2}$$

$$\boldsymbol{q} = \alpha_0(\boldsymbol{x})m + \sum_{\kappa\in S}\beta_\tau(\boldsymbol{x})m^{1-\kappa} + \sum_{\kappa\in S}\gamma_\tau(\boldsymbol{x})m^{1+\kappa}\,, \tag{3}$$

where S is a set of nonzero constants; or

$$\boldsymbol{q} = \alpha_0(\boldsymbol{x})m + \sum_{\tau\in T}\beta_\tau(\boldsymbol{x})m\sin\left(\tau\ln(m)\right) + \sum_{\tau\in T}\gamma_\tau(\boldsymbol{x})m\cos\left(\tau\ln(m)\right), \tag{4}$$

where T is a set of positive constants. This includes PIGLOG models and extensions that are polynomials in $\ln(m)$, simple polynomials in income, and PIGL models and extensions that are polynomials in m^{κ}.

Demand models that have full rank (Lewbel, 1990) are characterized by the property that the rank of the matrix $\boldsymbol{A}(\boldsymbol{x})$ is equal to the number of its columns, that is, the number of different income functions, $h_j(y)$. Full rank one complete demand systems must be homothetic,

$$\boldsymbol{q} = \alpha_0(\boldsymbol{x})m, \tag{5}$$

due to adding up. In budget share form, full rank one systems are zero order polynomials in income. Muellbauer (1975, 1976) showed that all full rank two complete demand systems are either PIGL or PIGLOG; that is, either

$$\boldsymbol{q} = \alpha_0(\boldsymbol{x})m + \alpha_1(\boldsymbol{x})m^{1-\kappa}, \tag{6}$$

for some $\kappa \neq 0$, or

$$\boldsymbol{q} = \alpha_0(\boldsymbol{x})m + \alpha_1(\boldsymbol{x})m\ln(m). \tag{7}$$

If we note that Bernoulli's equation,

$$\frac{\partial e(\boldsymbol{x};\cdot)^{\kappa}}{\partial \boldsymbol{x}} = \kappa e(\boldsymbol{x};\cdot)^{\kappa-1}\left(\frac{\partial e(\boldsymbol{x};\cdot)}{\partial \boldsymbol{x}}\right) = \beta_0(\boldsymbol{x}) + \beta_1(\boldsymbol{x})\left[e(\boldsymbol{x},u)^{\kappa}/\kappa\right], \tag{8}$$

has the PIGL form,

$$\frac{\partial e(\boldsymbol{x};\cdot)}{\partial \boldsymbol{x}} = \alpha_0(\boldsymbol{x})e(\boldsymbol{x};\cdot) + \alpha_1(\boldsymbol{x})e(\boldsymbol{x},u)^{1-\kappa}, \tag{9}$$

while the logarithmic transformation,

$$\frac{\partial \ln\left[e(\boldsymbol{x};\cdot)\right]}{\partial \boldsymbol{x}} = \frac{\partial e(\boldsymbol{x};\cdot)/\partial \boldsymbol{x}}{e(\boldsymbol{x};\cdot)} = \alpha_0(\boldsymbol{x}) + \alpha_1(\boldsymbol{x})\ln\left[e(\boldsymbol{x};\cdot)\right], \tag{10}$$

has the PIGLOG form

$$\frac{\partial e(\boldsymbol{x};\cdot)}{\partial \boldsymbol{x}} = \alpha_0(\boldsymbol{x})e(\boldsymbol{x};\cdot) + \alpha_1(\boldsymbol{x})e(\boldsymbol{x};\cdot)\ln\left[e(\boldsymbol{x};\cdot)\right], \tag{11}$$

then we can see that full rank two complete systems are first-order polynomials in a single transformation of income.

Gorman (1981) conjectured that second-order polynomials are the most general nondegenerate full rank three complete demand systems. Pursuing this conjecture by exploiting the methods Van Daal and Merkies (1989) applied to the quadratic expenditure system, Lewbel (1990) showed that all full rank three members of the Gorman Engel curve class of complete demand systems are quadratic polynomials in one of three possible transformations of income: (a) the quadratic Bernouli equation,

$$\frac{\partial e(\boldsymbol{x};\cdot)^{\kappa}}{\partial \boldsymbol{x}} = \kappa e(\boldsymbol{x};\cdot)^{\kappa-1}\left(\frac{\partial e(\boldsymbol{x};\cdot)}{\partial \boldsymbol{x}}\right)$$

$$= \beta_0(\boldsymbol{x}) + \beta_1(\boldsymbol{x})\left[e(\boldsymbol{x},u)^{\kappa}/\kappa\right] + \beta_2(\boldsymbol{x})\left[e(\boldsymbol{x},u)^{\kappa}/\kappa\right]^2, \tag{12}$$

has the generalized PIGL form,

$$\frac{\partial e(\boldsymbol{x};\cdot)}{\partial \boldsymbol{x}} = \alpha_0(\boldsymbol{x})e(\boldsymbol{x};\cdot) + \alpha_1(\boldsymbol{x})e(\boldsymbol{x},u)^{1-\kappa} + \alpha_2(\boldsymbol{x})e(\boldsymbol{x},u)^{1+\kappa}; \tag{13}$$

(b) the quadratic logarithmic transformation,

$$\frac{\partial \ln\left[e(\boldsymbol{x};\cdot)\right]}{\partial \boldsymbol{x}} = \frac{\partial e(\boldsymbol{x};\cdot)/\partial \boldsymbol{x}}{e(\boldsymbol{x};\cdot)} = \alpha_0(\boldsymbol{x}) + \alpha_1(\boldsymbol{x})\ln\left[e(\boldsymbol{x};\cdot)\right] + \alpha_2(\boldsymbol{x})\left\{\ln\left[e(\boldsymbol{x};\cdot)\right]\right\}^2, \tag{14}$$

has the generalized PIGLOG form,

$$\frac{\partial e(\boldsymbol{x};\cdot)}{\partial \boldsymbol{x}} = \alpha_0(\boldsymbol{x})e(\boldsymbol{x};\cdot) + \alpha_1(\boldsymbol{x})e(\boldsymbol{x};\cdot)\ln\left[e(\boldsymbol{x};\cdot)\right] + \alpha_2(\boldsymbol{x})e(\boldsymbol{x};\cdot)\left\{\ln\left[e(\boldsymbol{x};\cdot)\right]\right\}^2; \tag{15}$$

and (c) the quadratic complex exponential transformation,[2]

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[2] Including $-(\iota\tau)^{-1}$ on the left is innocuous. The right can be multiplied by $-\iota\tau$ and absorbed into the complex conjugate price vectors without changing the final structure. Lewbel (1990) didn't derive or state (16)–(17), but they can be deduced easily enough from his arguments and a careful reading of Gorman (1981).

$$\frac{\partial}{\partial \boldsymbol{x}}\left[-(\iota\tau)^{-1} e(\boldsymbol{x};\cdot)^{-\iota\tau}\right] = e(\boldsymbol{x};\cdot)^{-\iota\tau-1}\frac{\partial e(\boldsymbol{x};\cdot)}{\partial \boldsymbol{x}}$$

$$= \alpha_0(\boldsymbol{x}) + \tfrac{1}{2}\left[\alpha_1(\boldsymbol{x}) - \iota\alpha_2(\boldsymbol{x})\right] e(\boldsymbol{x};\cdot)^{\iota\tau} + \tfrac{1}{2}\left[\alpha_1(\boldsymbol{x}) + \iota\alpha_2(\boldsymbol{x})\right]\left(e(\boldsymbol{x};\cdot)^{\iota\tau}\right)^2, \tag{16}$$

has the trigonometric form

$$\frac{\partial e(\boldsymbol{x};\cdot)}{\partial \boldsymbol{x}} = e(\boldsymbol{x};\cdot)\left\{\alpha_0(\boldsymbol{x}) + \alpha_1(\boldsymbol{x})\sin\left[\tau\ln\left(e(\boldsymbol{x};\cdot)\right)\right] + \alpha_2(\boldsymbol{x})\cos\left[\tau\ln\left(e(\boldsymbol{x};\cdot)\right)\right]\right\}, \tag{17}$$

where we use de Moivre's theorem to obtain the right-hand-side expressions,

$$e^{\pm\iota\tau x} = 1 + \tfrac{1}{1!}(\pm\iota\tau x) + \tfrac{1}{2!}(\pm\iota\tau x)^2 + \cdots$$

$$= \left[1 - \tfrac{1}{2!}(\tau x)^2 + \tfrac{1}{4!}(\tau x)^6 + \cdots\right] \pm \iota\left[\tfrac{1}{1!}(\tau x) - \tfrac{1}{3!}(\tau x)^3 + \tfrac{1}{5!}(\tau x)^5 + \cdots\right]$$

$$= \cos(\tau x) \pm \iota\sin(\tau x). \tag{18}$$

Thus, we have just shown that any full rank complete demand system in Gorman's class of Engel curves can be represented as an exact system of partial differential equations in a polynomial of a single function of income.

3. Aggregation Theory for Incomplete Demand Systems

All of the previous results on rank and functional form of the income terms in demand rely on 0° homogeneity and adding up for complete demand systems. But an incomplete information set is the rule in empirical demand analysis, not the exception. In every case, we are faced with a subset of quantity, price, and income or wealth data – even when one uses a demand model that has, say, total personal consumption expenditures as the source of the adding up condition. This measure of total expenditure does not include borrowing or saving, which means that the period-to-period budget constraint is missing at least one good (savings). Even if preferences are separable across periods, it is well-accepted and widely understood that expenditure is endogenous (e.g., Deaton). With borrowing and saving and the time dimension in consumption and demand, the consumer budget constraint is defined in terms of current wealth plus the discounted present value of expected future earnings relative to the discounted present value of current and expected future total consumption expenditures.

In addition, one is almost always interested in a much smaller subset of goods consumed than the complete list of possible items that can purchased and used by individuals in the economy. Usually we try to model a relatively small number of goods as a function of the prices of those goods, some measures of the costs of other goods, and either income or total expenditures on the goods of interest. Carrying out this kind of analysis in a coherent manner forces us to face up to the reality that the models we must develop and apply are incomplete.

We generically are only interested in, have data for, or are feasibly able to measure, estimate, and predict consumption behavior for a subset of the goods that make up the consumer's total budget. This has two essential, related, and unavoidable impacts on the demand models that we need to use in all of these cases. Only some of the goods are the objects of study, and the budget constraint becomes a strict inequality. And the demand equations will no longer be homogeneous of degree zero in the prices of the goods of interest and income, because the zero degree homogeneity condition arises from the adding up condition, which in turn is defined for all prices and income or wealth. Moreover, since we are only trying to model *some* of the goods purchased and consumed, there is no reason to require the demand equations for the goods that we don't (or can't) model to have the same functional structure as those that we do model. They might have the same form, but then again, they might not. We simply have no way of knowing or choosing which is the case, since we aren't in a position to measure, estimate, or predict these other demand functions with an incomplete information set. In other words, the residual claimant for the budget constraint is the generic variable total expenditure on all other goods, which by its very nature has a unit price and is measured only indirectly as the difference between the income variable and expenditures on the goods that we do measure. In almost all cases, therefore, if we're honest with ourselves about the true structure of the economic problems at hand, then we must admit that our demand systems are incomplete. The main point, however, is that, for these reasons and many others (some of which may become more clear as a result of the following arguments), incomplete systems are far more interesting than complete ones, in any case.

When the goods of interest form an incomplete demand system (Gorman, 1965; Epstein, 1982; LaFrance, 1985; and LaFrance and Hanemann, 1989), the results that are detailed in the previous section on the rank and functional form of the income terms of demand models that can be aggregated across individual income levels to market demands, cannot be applied. Moreover, as we will see from the results of this paper, there is no restriction on the class of income functions that meet the original definition for aggregation laid out by Gorman, so that the aggregation theory for complete demand systems cannot be extended to incomplete demand systems. One reason is that adding up plays two critical roles in Gorman‘s constructive argument. The constant function must be one of the income functions and the demands can be transformed to a square (complete) system of linear, homogeneous ordinary differential equations in functions of the logarithm of income. These two conditions lead to the general class of solutions in Gorman's full rank three case, with symmetry determining the final restrictions on functional form. Related, but somewhat distinct, is the fact that the functional form restrictions in the rank one and two cases are purely due to homogeneity in all prices and income. Since homogeneity does not apply in the same fashion to incomplete demand systems as it does to complete demand systems, we don't get any restrictions on the functional forms in the lower rank cases either.

At this point, it is probably worthwhile summarizing for comparison the main implications of aggregation in a complete demand system. All full rank one, two, or three (the highest possible rank) members of Gorman's class of Engel curves can be written as systems of exact partial differential equations in zero-, first- or second-order polynomials, respectively, in a single function of income. In addition, the possible class of transformations is limited. In rank one models, only the identity transformation works, $y \equiv m$; rank two has two possible cases, Bernoulli's transformation, $y \equiv m^{\kappa}$, and the logarithmic transformation, $y = \ln(m)$; while rank three has a third possibility for a pair of conjugate complex roots, $y = m^{\pm \iota\tau}$.

In the case of an incomplete demand system, we are able to show that the restriction on the functional form of the income transformation does not apply, even in the homothetic, rank one case. There are two related reasons for this increase in generality. First, homogeneity can be accommodated independently of the subset of prices of the goods of interest. A homothetic, weakly integrable, incomplete demand system can have a common income elasticity that differs from unity and need not be constant (LaFrance and Hanemann, 1989). Second, the budget constraint is a strict inequality – adding up does not apply to a proper subset of goods consumed. This influences demand models of all ranks, but its impact is perhaps greatest for rank three, and possibly even higher rank, incomplete demand systems.

Since Gorman's constructive argument shows us that all aggregable complete demand systems can be written as a system of partial differential equations in a polynomial of one among three possible functions of income, a natural extension is the class of incomplete demand systems that can be written as a polynomial of any order in any function of income. Proposition 1 shows that a maximal rank of three is a corollary to symmetry (the Appendix gives a constructive proof). A quadratic form is sufficient to obtain rank three, while the polynomial must be at least second-order to achieve rank three. In this sense, a quadratic form of a single function of income defines the most general non-degenerate class of full rank polynomial expenditure systems.

Proposition 1. If the possibly incomplete demand system has the polynomial form

$$\frac{\partial y(\boldsymbol{x};\cdot)}{\partial \boldsymbol{x}} = \sum_{i=0}^{K} \alpha_i(\boldsymbol{x};\cdot)\, y(\boldsymbol{x};\cdot)^i$$

and is weakly integrable, then there exist $\varphi_i : \mathbb{R}^{n_q} \to \mathbb{R},\ i = 2,\ldots,K$ *such that*

$$\alpha_i(\boldsymbol{x}) \equiv \varphi_i(\boldsymbol{x})\alpha_K(\boldsymbol{x})\ \forall\ i \geq 2\,.$$

In what follows, we apply this result to develop several full rank models of incomplete demand systems that can be used to nest both the rank and functional form in applied demand analysis. It warrants emphasizing, though, that these procedures are applicable only to incomplete demand systems. At least one of the goods in the consumer's bundle is left as the residual claimant for total expenditure, and therefore is not required to have the same functional form as the goods that are subjected to empirical analysis, and the specific model specifications derived in the sequel. When it is possible to relate the restricted functional forms for which a complete system can apply, we do so. However, it is in large part the adding up condition that drives the functional form restriction in the work by Gorman, Lewbel, Muellbauer, and others on aggregation in complete demand systems. Previous work of LaFrance and recent work by von Haefen, show that the functional form restrictions do not apply for any rank of incomplete demand system.

3.1 Nesting LA-AIDS, AIDS, and QAIDS within a QPIGL-IDS

In the two decades since its introduction by Deaton and Muellbauer, the AIDS has been widely used in demand analysis. The vast majority of empirical applications follows Deaton and Muellbauer's suggestion and replaces the translog price index that deflates income with Stone's index, which generates the LA-AIDS. Although Deaton and Muellbauer (1980: 317-320) cautioned against and avoided the practice, most empirical applications of the LA-AIDS include tests for and the imposition of an approximate version of Slutsky symmetry by restricting the log-price coefficient matrix to be symmetric. Examples include Anderson and Blundell (1983), Buse (1998), Moschini (1995), Moschini and Meilke (1989), and Pashardes (1993).[3]

In this subsection, we derive the conditions for integrability of LA-AIDS and a

[3] However, Browning and Meghir (1991) estimate a nonlinear AIDS by first estimating an LA-AIDS with a symmetric matrix of log-price coefficients to get starting values for the nonlinear AIDS.

simple method for nesting the homothetic integrable solution within a class of homothetic PIGL demand models. We then extend this nesting procedure to non-homothetic PIGL and QPIGL forms.

If it is integrable, the LA-AIDS can be written in matrix notation as

$$\boldsymbol{w} = \frac{\partial \ln e(\boldsymbol{p},u)}{\partial \ln \boldsymbol{p}} = \alpha + \boldsymbol{B} \ln \boldsymbol{p} + \gamma \left[\ln e(\boldsymbol{p},u) - (\ln \boldsymbol{p})' \frac{\partial \ln e(\boldsymbol{p},u)}{\partial \ln \boldsymbol{p}} \right] \tag{19}$$

where α and γ are n_q-vectors and $\boldsymbol{B}$ is an $n_q \times n_q$ matrix of parameters. At various points in the paper, it proves to be helpful to change variables from quantities, prices, expenditures, budget shares, and income to particular transformations of these variables. In the present situation, it is most useful to define $\boldsymbol{x} \equiv \ln(\boldsymbol{p})$ and $y(\boldsymbol{x},u) \equiv \ln[e(\boldsymbol{p}(\boldsymbol{x}),u)]$, where $\boldsymbol{p}(\boldsymbol{x}) \equiv [e^{x_1} \cdots e^{x_{n_q}}]'$. With these definitions, we can rewrite (19) in the form

$$(\boldsymbol{I} + \gamma \boldsymbol{x}') \frac{\partial y(\boldsymbol{x},u)}{\partial \boldsymbol{x}} = \alpha + \boldsymbol{B}\boldsymbol{x} + \gamma y(\boldsymbol{x},u) \, . \tag{20}$$

Our first result identifies conditions for local integrability of the LA-AIDS.[4]

Proposition 2. If the LA-AIDS is weakly integrable over an open set $\mathcal{N} \subset \mathbb{R}^{n_q}$ *with a nonempty interior and such that* $1 + \gamma' \boldsymbol{x} \neq 0 \;\forall\; \boldsymbol{x} \in \mathcal{N}$, *then either (a)* $\gamma \neq \boldsymbol{0}$ *and* $\boldsymbol{B} = \beta_0 \gamma \gamma'$ *for some* $\beta_0 \in \mathbb{R}$, *or (b)* $\gamma = \boldsymbol{0}$ *and* $\boldsymbol{B} = \boldsymbol{B}'$. *In case (a), the logarithmic expenditure function has the form*

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[4] This proposition only requires local conditions. However, $\mathcal{N}$ covers all of n_q–space except for an (n_q–1)–dimensional hyperplane, which has Lesbesgue measure zero in n_q–space.

$$y(\boldsymbol{x},u) = \alpha'\boldsymbol{x} + \beta_0\left[(1+\gamma'\boldsymbol{x})\ln(1+\gamma'\boldsymbol{x}) - \frac{\gamma'\boldsymbol{x}}{(1+\gamma'\boldsymbol{x})}\right] + (1+\gamma'\boldsymbol{x})\theta(\tilde{\boldsymbol{p}},u),$$

while in case (b) it has the form,

$$y(\boldsymbol{x},u) = \alpha'\boldsymbol{x} + \tfrac{1}{2}\boldsymbol{x}'\boldsymbol{B}\boldsymbol{x} + \theta(\tilde{\boldsymbol{p}},u),$$

where $\theta(\tilde{\boldsymbol{p}},u)$ *is* $0°$ *homogeneous in* $\tilde{\boldsymbol{p}}$ *and increasing in u, but otherwise cannot be identified.*

Case (b) produces a homothetic demand model and is the solution of interest.[5] In particular, this solution has the same structure as the homothetic Linear Incomplete Demand System (LIDS) in LaFrance (1985). By forgoing the same functional forms for all de-

[5] Case (a), where the log-income coefficients do not vanish and the matrix of log-price coefficients has rank one, is too restrictive to be of empirical interest. Nevertheless, this case reveals an interesting property. In particular, it is characterized by a system of linear identities among budget shares,

$$\boldsymbol{w} \equiv \alpha + \gamma(w_1 - \alpha_1)/\gamma_1,$$

where without loss in generality, $\gamma_1 \neq 0$. Recall that the linear expenditure system is characterized by a system of linear identities among expenditures,

$$\boldsymbol{e} \equiv \boldsymbol{P}\alpha + \gamma(e_1 - \alpha_1 p_1)/\gamma_1,$$

where $e_i \equiv p_i q_i$, $\boldsymbol{e} = [e_1 \cdots e_{n_q}]'$ is the n_q–vector of expenditures on the goods $\boldsymbol{q}$, and $\boldsymbol{P}$ = **diag**$[p_i]$. Similarly, LaFrance (1985) shows that weakly integrable, non-homothetic Linear Incomplete Demand Systems are characterized by linear identities in quantities,

$$\boldsymbol{q} \equiv \alpha + \gamma(q_1 - \alpha_1)/\gamma_1.$$

In this sense, (a) closes the set of models characterized by quantities demanded, expenditures, or budget shares lying on a ray in n_q–dimensional space.

mand equations, a minor consideration in almost all cases, we can nest a homothetic LA-AIDS and LIDS with Box-Cox transformations in an IDS framework. To see this, let the model apply to n_q out of $N \geq n_q+1$ goods. Define $m(\kappa) \equiv (m^{\kappa}-1)/\kappa$, $p_i(\lambda) \equiv (p_i^{\lambda}-1)/\lambda$, and $\boldsymbol{p}(\lambda) \equiv [p_1(\lambda) \cdots p_{n_q}(\lambda)]'$.[6] Assume that m and $\boldsymbol{p}$ are deflated, with a common deflator that is a known, positive valued and 1° homogeneous function of (at least some of) the prices of all other goods, $\pi(\tilde{\boldsymbol{p}})$. Under these conditions, we can write a class of weakly integrable, homothetic PIGL-IDS models in budget share form as

$$\boldsymbol{w} = m^{-\kappa}\boldsymbol{P}^{\lambda}[\alpha + \boldsymbol{B}\boldsymbol{p}(\lambda)], \tag{21}$$

where $\boldsymbol{P}^{\lambda} \equiv \mathbf{diag}[p_i^{\lambda}]$ is a diagonal matrix with typical diagonal element p_i^{λ}. Using the integration techniques detailed in LaFrance and Hanemann (1989), it can be shown that the expenditure function for this PIGL-IDS satisfies

$$\mathrm{e}(\boldsymbol{p}, \tilde{\boldsymbol{p}}, u) \equiv \pi(\tilde{\boldsymbol{p}})\left\{1 + \kappa\left[\alpha'\boldsymbol{p}(\lambda) + \tfrac{1}{2}\boldsymbol{p}(\lambda)'\boldsymbol{B}\boldsymbol{p}(\lambda) + \theta(\tilde{\boldsymbol{p}}, u)\right]\right\}^{1/\kappa}, \tag{22}$$

where $\theta(\tilde{\boldsymbol{p}}, u)$ is 0° homogeneous in $\tilde{\boldsymbol{p}}$ and increasing in u, but otherwise cannot be identified (LaFrance (1985); LaFrance and Hanemann (1989)). It also can be shown that the demands in (21) are homothetic, with income elasticities equal to $1-\kappa \;\forall\; \kappa \in \mathbb{R}$.

We next extend this way to nest the functional form and rank to full rank 2, non-homothetic, integrable AIDS-IDS models,

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[6] We could easily extend all of the results in this paper by allowing for each price to have its own Box-Cox parameter, say λ_i, $i = 1,\ldots,n_q$. None of the arguments would change, although the notational burden increases substantially.

$$\boldsymbol{w} = \alpha + \boldsymbol{B}\ln(\boldsymbol{p}) + \gamma\left[\ln(m) - \alpha_0 - \alpha'\ln(\boldsymbol{p}) - \tfrac{1}{2}\ln(\boldsymbol{p})'\boldsymbol{B}\ln(\boldsymbol{p})\right]. \tag{23}$$

To do this, we require a third result, which states that (23) is a special case of a complete class of incomplete demand models that can be characterized as follows. Let $y \equiv f(m)$ and $x_i \equiv g_i(p_i)$, $i = 1\ldots n_q$, where $f(\cdot)$ and $g_i(\cdot)$, $i = 1,\ldots,n_q$, are arbitrary strictly increasing and twice continuously differentiable functions on $\mathbb{R}_{++}$, and write the n_q–vector inverse of $\boldsymbol{g}(\cdot)$ as $\boldsymbol{p}(\boldsymbol{x})$. Suppose that, after an appropriate set of transformations, the demand functions for the goods $\boldsymbol{q}$ can be written as a linear function of $y(\boldsymbol{x}, \tilde{\boldsymbol{p}}, u) \equiv f\left(e(\boldsymbol{p}(\boldsymbol{x}), \tilde{\boldsymbol{p}}, u)\right)$ and linear and quadratic functions of $\boldsymbol{x}$, with no interaction terms between $\boldsymbol{x}$ and y,

$$\frac{\partial y(\boldsymbol{x}, \tilde{\boldsymbol{p}}, u)}{\partial x_i} = \alpha_i + \beta_i'\boldsymbol{x} + \tfrac{1}{2}\boldsymbol{x}'\Delta_i\boldsymbol{x} + \gamma_i y(\boldsymbol{x}, \tilde{\boldsymbol{p}}, u),\ i = 1,\ldots,n_q\,, \tag{24}$$

where, without loss in generality, $\gamma_1 \neq 0$ and each $n_q \times n_q$ matrix, Δ_i, is symmetric $\forall\ i$. We have the following.

Proposition 3. The system of partial differential equations in (24) *is weakly integrable if and only if it can be written in the form*

$$\frac{\partial y(\boldsymbol{x}, \tilde{\boldsymbol{p}}, u)}{\partial \boldsymbol{x}} = \tilde{\alpha} + \tilde{\boldsymbol{B}}\boldsymbol{p} + \gamma\left[y(\boldsymbol{x}, \tilde{\boldsymbol{p}}, u) - \alpha_0 - \tilde{\alpha}'\boldsymbol{x} - \tfrac{1}{2}\boldsymbol{x}'\tilde{\boldsymbol{B}}\boldsymbol{x}\right]$$

where α_0 *is a scalar (that may be a function of other prices),* $\tilde{\alpha} = \alpha - \alpha_0\gamma$ *is an* $n_q \times 1$ *vector,* $\tilde{\boldsymbol{B}}$ *is a symmetric* $n \times n$ *matrix that satisfies* $\tilde{\boldsymbol{B}} = \boldsymbol{B} + \gamma\alpha'$, *where* $\boldsymbol{B} = [\beta_1 \cdots \beta_{n_q}]$, *and* $\Delta_i = -\gamma_i\tilde{\boldsymbol{B}}\ \forall\ i$.

Note that if $f(\cdot)$ and all of the $g_i(\cdot)$ are natural logarithms, then we obtain the integrable AIDS-IDS. Moreover, the AIDS form is the only choice that has the structure of the proposition and can satisfy adding up at more than a single point. In other words, the AIDS model is the only functional form in this class that can be weakly integrable and a complete system. This elucidates one of the important differences between complete and incomplete demand systems. In this case, adding up ultimately determines the functional form of the complete demand system, while an incomplete demand system with the same basic structure admits any functional form.

For incomplete demand systems, since we don't have to live within the confines of a log-log form for the expenditure function, it is of interest to consider nesting it within a class of models. In particular, using the above Box-Cox definitions for $m(\kappa)$ and $\boldsymbol{p}(\lambda)$, we can write an integrable non-homothetic PIGL-IDS that is linear in the Box-Cox expenditure term and linear and quadratic in the Box-Cox price terms as

$$\boldsymbol{w} = m^{-\kappa}\boldsymbol{P}^{\lambda}\left\{\alpha + \boldsymbol{Bp}(\lambda) + \gamma\left[m(\kappa) - \alpha_0 - \alpha'\boldsymbol{p}(\lambda) - \tfrac{1}{2}\boldsymbol{p}(\lambda)'\boldsymbol{Bp}(\lambda)\right]\right\}, \tag{25}$$

where, for notational simplicity, we have dropped the tildes over the parameters.

Unlike the homothetic case, for all (κ, λ) pairs, this functional form allows one to estimate the income aggregation function through the Box-Cox parameter κ. If $\kappa = 0$ we obtain the integrable AIDS-IDS, if $\kappa = 1$ we obtain the linear-quadratic IDS (LQ-IDS) of LaFrance (1990), and for all (κ, λ) pairs we obtain an integrable PIGL-IDS.[7] Finally, it can be shown that the expenditure function for (25) is

$$\mathrm{e}(\boldsymbol{p}, \tilde{\boldsymbol{p}}, u) \equiv \pi(\tilde{\boldsymbol{p}})\left\{1 + \kappa\left[\alpha_0 + \alpha'\boldsymbol{p}(\lambda) + \tfrac{1}{2}\boldsymbol{p}(\lambda)'\boldsymbol{Bp}(\lambda) + \theta(\tilde{\boldsymbol{p}}, u)e^{\gamma'\boldsymbol{p}(\lambda)}\right]\right\}^{1/\kappa}, \tag{26}$$

[7] See Agnew (1998) for a comprehensive development and application of this full rank two PIGL-IDS.

where $\theta(\cdot)$ has the same properties as before. Note that this expenditure function simply generalizes the one for the homothetic case with the additional term α_0, which is often fixed at zero in applications, and the factor $e^{\gamma'\boldsymbol{p}(\lambda)}$, which produces the vector of non-homothetic income coefficients.

We conclude this subsection by extending these arguments to demand models that include linear and quadratic terms in the Box-Cox transformation of deflated income (QES-IDS). We proceed with this by extending the rank two PIGL-IDS expenditure function to one that is rank three and that generates a relatively simple form for the quadratic terms in the demand equations. A simple, and convenient, choice is a quasi-indirect utility function (Hausman (1981); LaFrance (1985); LaFrance and Hanemann (1989)) that can be written in a form that is consistent with the QES originally developed in Howe, Pollak, and Wales (1979),[8]

$$\varphi(\boldsymbol{p},m) = -\left\{\frac{1}{\left[m(\kappa)-\alpha_0-\alpha'\boldsymbol{p}(\lambda)-\frac{1}{2}\boldsymbol{p}(\lambda)'\boldsymbol{B}\boldsymbol{p}(\lambda)\right]}+\delta'\boldsymbol{p}(\lambda)\right\}e^{\gamma'\boldsymbol{p}(\lambda)}. \qquad (27).$$

Applying the methodology of LaFrance and Hanemann (1989), it can be shown that (27) is equivalent to an expenditure function of the form

$$e(\boldsymbol{p},\tilde{\boldsymbol{p}},u) \equiv \pi(\tilde{\boldsymbol{p}})\left\{1+\kappa\left[\alpha_0+\alpha'\boldsymbol{p}(\lambda)+\tfrac{1}{2}\boldsymbol{p}(\lambda)'\boldsymbol{B}\boldsymbol{p}(\lambda)-\frac{e^{\gamma'\boldsymbol{p}(\lambda)}}{\left(\delta'\boldsymbol{p}(\lambda)e^{\gamma'\boldsymbol{p}(\lambda)}+\theta(\tilde{\boldsymbol{p}},u)\right)}\right]\right\}^{1/\kappa}. \qquad (28)$$

---

[8] Solve (22) for θ, transform to $\tilde{\theta} = -1/\theta$ to get $\tilde{\varphi}(\boldsymbol{p},m) = -\,e^{\gamma'\boldsymbol{p}(\lambda)}\Big/[m(\kappa)-\alpha_0-\alpha'\boldsymbol{p}(\lambda)-\frac{1}{2}\boldsymbol{p}(\lambda)'\boldsymbol{B}\boldsymbol{p}(\lambda)]$, and add the term $-\delta'\boldsymbol{p}(\lambda)e^{\gamma'\boldsymbol{p}(\lambda)}$ to obtain (27).

That is, the QPIGL-IDS expenditure function generalizes the non-homothetic PIGL-IDS expenditure function by replacing $\theta(\tilde{\boldsymbol{p}},u)e^{\gamma'\boldsymbol{p}(\lambda)}$ with $-[\delta'\boldsymbol{p}(\lambda)+\theta(\tilde{\boldsymbol{p}},u)e^{-\gamma'\boldsymbol{p}(\lambda)}]^{-1}$, which produces the n_q-vector of parameters δ associated with the quadratic term in supernumerary income, in addition to the n_q-vector of parameters γ associated with the linear supernumerary income term.

An application of Roy's identity generates the QPIGL-IDS extension of the AIDS-IDS in budget share form as

$$\boldsymbol{w} = m^{-\kappa}\boldsymbol{P}^{\lambda}\Big\{\alpha + \boldsymbol{B}\boldsymbol{p}(\lambda) + \gamma\Big[m(\kappa) - \alpha_0 - \alpha'\boldsymbol{p}(\lambda) - \tfrac{1}{2}\boldsymbol{p}(\lambda)'\boldsymbol{B}\boldsymbol{p}(\lambda)\Big]$$

$$+\big[\boldsymbol{I} + \gamma'\boldsymbol{p}(\lambda)\big]\delta\Big[m(\kappa) - \alpha_0 - \alpha'\boldsymbol{p}(\lambda) - \tfrac{1}{2}\boldsymbol{p}(\lambda)'\boldsymbol{B}\boldsymbol{p}(\lambda)\Big]^2\Big\}. \tag{29}$$

Assuming that α and $\boldsymbol{B}$ do not vanish simultaneously, it follows that: (a) $\gamma \neq \mathbf{0}$, $\delta \neq \mathbf{0}$ is necessary and sufficient for a full rank three QPIGL-IDS; (b) $\gamma \neq \mathbf{0}$, $\delta = \mathbf{0}$ is necessary and sufficient for a full rank two, non-homothetic PIGL-IDS; (c) $\gamma = \mathbf{0}$, $\delta \neq \mathbf{0}$ is necessary and sufficient for a full rank two QPIGL-IDS that excludes the linear term; and (d) $\gamma = \delta = \mathbf{0}$ is necessary and sufficient for a homothetic PIGL-IDS. Thus, we obtain a rich class of models that permits nesting, testing and estimating the rank and functional form of the income terms in incomplete demand systems with a generalized AIDS structure.

3.3 Nested Rank 3 Extensions of Quadratic Utility Models

In this subsection, we apply the methods developed above to produce a full rank three generalization of the quadratic direct and translog indirect utility models (Christensen, Jorgenson, and Lau, 1975). We first define the functions

$$\varphi(\boldsymbol{x}, \tilde{\boldsymbol{p}}) = \boldsymbol{x}'\boldsymbol{B}\boldsymbol{x} + 2\gamma'\boldsymbol{x} + 1\,, \tag{30}$$

$$\theta(\boldsymbol{x},\tilde{\boldsymbol{p}},\boldsymbol{z}) = \alpha_0(\tilde{\boldsymbol{p}},\boldsymbol{z}) - \alpha(\tilde{\boldsymbol{p}},\boldsymbol{z})'\boldsymbol{x}, \tag{31}$$

where $\alpha(\tilde{\boldsymbol{p}},\boldsymbol{z})$ is a vector of functions of other prices and demographics, $\alpha_0(\tilde{\boldsymbol{p}},\boldsymbol{z})$ is a scalar function of other prices and demographics, with $\alpha(\tilde{\boldsymbol{p}},\boldsymbol{z})$ and $\alpha_0(\tilde{\boldsymbol{p}},\boldsymbol{z})$ 0° homogeneous in $\tilde{\boldsymbol{p}}$, $\boldsymbol{B}$ is an $n_q \times n_q$ matrix of parameters, and γ is a vector of parameters. Due to 0° homogeneity of $\alpha(\tilde{\boldsymbol{p}},\boldsymbol{z})$ and $\alpha_0(\tilde{\boldsymbol{p}},\boldsymbol{z})$ in $\tilde{\boldsymbol{p}}$, we can (and do) take $(\boldsymbol{p},\tilde{\boldsymbol{p}},m)$ to be deflated by $\pi(\tilde{\boldsymbol{p}})$ without any loss in generality, and absorb the common deflator in the price and income variables. The starting point for our application of Proposition 1 in this subsection is the class of indirect utility functions defined by

$$v(\boldsymbol{x},y,\tilde{\boldsymbol{p}},\boldsymbol{z}) = \psi\left\{-\frac{\sqrt{\varphi(\boldsymbol{x},\tilde{\boldsymbol{p}})}}{[y-\theta(\boldsymbol{x},\tilde{\boldsymbol{p}},\boldsymbol{z})]} - \frac{\delta'\boldsymbol{x}}{\sqrt{\varphi(\boldsymbol{x},\tilde{\boldsymbol{p}})}};\tilde{\boldsymbol{p}},\boldsymbol{z}\right\}. \tag{32}$$

Members of the class of incomplete demand systems generated by this indirect utility function include rank two translog and quadratic utility functions and rank three extensions that are quadratic in log-income and income, respectively.

Choices for $f(\cdot)$ and $\boldsymbol{g}(\cdot)$ that are both interesting and useful continue to be Box-Cox transformations, $y = (m^{\kappa}-1)/\kappa$ and $x_i = (p_i^{\lambda}-1)/\lambda$, $i = 1,\cdots,n_q$. When $\kappa = \lambda = 0$, we obtain a rank three extension of a translog model; when $\kappa = \lambda = 1$, we obtain a rank three extension of the quadratic model; and for all values of κ and λ, we obtain a full rank three QPIGL-IDS. Rank two versions are obtained when $\delta = \boldsymbol{0}$, while if $\theta(\boldsymbol{x},\tilde{\boldsymbol{p}},\boldsymbol{z}) \equiv 0$ and $\delta = \boldsymbol{0}$, we obtain rank one homothetic versions, again nesting the rank and functional form of the income terms within a single unifying framework.[9]

From the results of LaFrance and Hanemann (1989), to study the demands for the goods of interest, $\boldsymbol{q}$, we can focus exclusively on the quasi-indirect utility function,

$$\upsilon(\boldsymbol{x}, y, \tilde{\boldsymbol{p}}, \boldsymbol{z}) = -\frac{\sqrt{\varphi(\boldsymbol{x})}}{\left(y - \theta(\boldsymbol{x}, \tilde{\boldsymbol{p}}, \boldsymbol{z})\right)} - \frac{\delta' \boldsymbol{x}}{\sqrt{\varphi(\boldsymbol{x})}}. \tag{33}$$

Roy's identity applied to (33) therefore gives a rank three QPIGL-IDS,

$$\boldsymbol{q} = \left(\frac{dy}{dm}\right)^{-1} \frac{\partial \boldsymbol{x}'}{\partial \boldsymbol{p}} \left\{ \alpha + \left[1 - \delta' \boldsymbol{x}\left(\frac{y-\theta}{\varphi}\right)\right]\left(\frac{y-\theta}{\varphi}\right)(\boldsymbol{Bx} + \gamma) + \frac{(y-\theta)^2}{\varphi}\delta \right\}. \tag{34}$$

However, an alternative and useful view of this incomplete demand system arises from noting that the demands for $\boldsymbol{q}$ satisfy the partial differential equations,

$$\frac{\partial y}{\partial \boldsymbol{x}} = \alpha + \left[1 - \delta' \boldsymbol{x}\left(\frac{y-\theta}{\varphi}\right)\right]\left(\frac{y-\theta}{\varphi}\right)(\boldsymbol{Bx} + \gamma) + \frac{(y-\theta)^2}{\varphi}\delta. \tag{35}$$

By Lemma 2 in the Appendix, this functional form allows us to determine necessary and sufficient conditions for symmetry and sufficient conditions for concavity of y in $\boldsymbol{x}$, hence of e in $\boldsymbol{p}$, entirely from (35) for this class of incomplete demand models.

Calculating the second-order partial derivatives and careful (and, of course, quite tedious) grouping, canceling, and algebraic manipulation of various terms gives

$$\frac{\partial^2 y}{\partial \boldsymbol{x} \partial \boldsymbol{x}'} = \left[1 - \delta' \boldsymbol{x}\left(\frac{y-\theta}{\varphi}\right)\right]\left(\frac{y-\theta)}{\varphi}\right)\left[\boldsymbol{B} - \left(\frac{1}{\varphi}\right)(\boldsymbol{Bx} + \gamma)(\boldsymbol{Bx} + \gamma)'\right]$$

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[9] An advantage of both choices of preference functions in this paper is that the demands are conditionally linear in δ. This simplifies the interpretation, estimation and testing of the second-order income effects.

$$+\,2\frac{(y-\theta)^3}{\varphi^2}\left[\boldsymbol{I}-\left(\frac{1}{\varphi}\right)(\boldsymbol{Bx}+\gamma)\boldsymbol{x}'\right]\delta\delta'\left[\boldsymbol{I}-\left(\frac{1}{\varphi}\right)\boldsymbol{x}(\boldsymbol{Bx}+\gamma)'\right], \tag{36}$$

so that symmetry of $\boldsymbol{B}$ is necessary and sufficient for symmetry of $\frac{\partial^2 y}{\partial \boldsymbol{x}\partial \boldsymbol{x}'}$, and therefore also for symmetry of $\frac{\partial^2 e}{\partial \boldsymbol{p}\partial \boldsymbol{p}'}$ (see the Appendix for details).

Turning back to (33), note that we have used the transformation $\tilde{u}=-u^{-1}$ (again taken from Howe, Pollak, and Wales) of the Gorman polar form for the quasi-indirect utility function with $\delta=\mathbf{0}$ (i.e., we take the negative reciprocal of $(y-\theta)/\sqrt{\varphi}$, which is the generalized quadratic quasi-indirect utility function[10]). In this case $y=\theta$ is the bliss point and monotonicity requires $y-\theta<0$, while $\varphi>0$ is required for the radical to be well-defined in Gorman's choice for normalizing the utility index. When $\lVert\delta\rVert<\varepsilon$ for small enough $\varepsilon > 0$, we have $1-\delta'\boldsymbol{x}(y-\theta)/\varphi>0$. It turns out that this inequality must be satisfied, at least in a neighborhood of each data point, if preferences are well-behaved. This is equivalent to the condition that adding $-\delta'\boldsymbol{x}/\sqrt{\varphi}$ to $\tilde{u}=-u^{-1}$ does not change the sign of the (cardinal) utility index. Indeed, this condition is required for the Howe, Pollak, and Wales transformation from u to $-u^{-1}$ to remain well-defined and it can be shown that preferences become ill-behaved when it is violated.

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[10] We use the terminology "generalized quadratic" to refer to the fact that the quasi-indirect utility function is defined in terms of deflated and transformed prices and income, $\boldsymbol{x}$ and y, respectively, rather than directly in terms of $\boldsymbol{p}$ and m.

In any case, we would expect the second-order income effects to be small relative to the first-order income effects on the demands. In other words $\boldsymbol{\delta}'\boldsymbol{x}$ should be small relative to φ. The upshot is that, so long as $1-\boldsymbol{\delta}'\boldsymbol{x}(y-\theta)/\varphi > 0$, $y-\theta<0$ and $\varphi>0$, the second line of (36) is a symmetric, negative semidefinite, rank one matrix. Hence, Lemmas 1 and 2 in the Appendix show us that $\boldsymbol{B}=\boldsymbol{L}\boldsymbol{L}'$ is necessary, while $\boldsymbol{B}=\boldsymbol{L}\boldsymbol{L}'+\boldsymbol{\gamma}\boldsymbol{\gamma}'$ is sufficient, for $\dfrac{\partial^2 y}{\partial \boldsymbol{x}\partial \boldsymbol{x}'}$ to be symmetric and negative semidefinite. Under certain conditions for the functions y and φ, this is necessary and sufficient for weak integrability throughout on the open set

$$\Im \equiv \left\{(\boldsymbol{p},\tilde{\boldsymbol{p}},m,\boldsymbol{z}) \in \mathbb{R}^{n_q}_{++} \times \mathbb{R}^{n_{\tilde{q}}}_{++} \times \mathbb{R}_{++} \times \mathbb{R}^{J} : \varphi > 0, y-\theta < 0, 1-\boldsymbol{\delta}'\boldsymbol{x}(y-\theta)/\varphi > 0\right\}.$$

These curvature restrictions apply only to the parameters of the model and are straightforward to implement. We recently have experienced success applying them to U.S. food consumption (Beatty and LaFrance, 2001; LaFrance and Beatty, 2003).

4. Conclusions

In this paper, we have extended the existing literature on aggregation in demand analysis to incomplete demand systems. In stark contrast to complete demand systems, we find that there are no restrictions on the class of functional forms for the income variables that can satisfy weak integrability. On the other hand, in the large class of incomplete demand models that are specified as polynomials of any finite order of one function of income, the maximal rank of the incomplete demand system is three. This result follows purely from Slutsky symmetry, since adding up is obtained with the good or goods that are not the direct subject of study, while homogeneity can be sustained through some linearly homogeneous function of the prices of these other goods.

We also have used Box-Cox transformations of the prices of the goods of interest and a separate Box-Cox transformation on income to generate two large classes of nested functional forms. One makes it possible to test for the rank and functional form of generalized AIDS models. The other permits the same analysis to be applied to generalized translog/quadratic utility functional forms.

We have found both frameworks for nesting incomplete demand systems to be empirically tractable as well as substantial improvements over the traditional rank two alternatives (Beatty and LaFrance, 2001a, 2001b; LaFrance, Beatty, Pope and Agnew, 2000, 2002; and LaFrance and Beatty, 2004). For both classes of nested functional forms, rank three appears to be essential. In addition, the point estimates for the Box-Cox parameters on prices and income tend to fall much closer to unity (the extended QES case) than to zero (the extended QAIDS case). However, both restrictions ($\kappa = \lambda = 1$ or 0, respectively) are rejected at all reasonable levels of significance in every data set we have used to empirically investigate this question. We hope applied researchers find these models to be as informative and valuable as we have.

Our analysis has shown that all full rank one, two and three members of the Gorman class of complete demand systems can be written down as systems of exact partial differential equations in zero-, first- or second-order polynomials, respectively, in a single function of income. Moreover, the class of feasible functions is limited. Rank one models only permit the identity transformation, $y \equiv m$; rank two has two possible cases, $y \equiv m^{\kappa}$ and $y = \ln(m)$; and rank three admits a third possibility to accommodate a pair of conjugate complex roots, $y = m^{\pm\iota\tau}$.

For incomplete demand systems, even in the rank one case, these restrictions on functional form do not apply. There are two related reasons for this substantial increase in generality. First, homogeneity can be accommodated independently of the prices of the goods that are of primary interest. Relaxing this constraint affects even rank one models by generating a large class of homothetic incomplete demand systems. For example, the common income elasticity of demand for a homothetic subset of all goods consumed need not be unity or even constant (LaFrance and Hanemann, 1989). Both properties must hold in a homothetic complete demand system. Second, the budget constraint is a strict inequality – adding up does not apply. This influences demand models of all ranks. However, its impact is greatest for rank three and possibly even higher rank incomplete demand systems. Since Gorman's result hinges on adding up in addition to symmetry to obtain a complete system of linear ordinary differential equations, his functional form restrictions do not apply and cannot be extended to incomplete demand systems.

A natural extension of Gorman's framework is to the class of all generalized polynomial Engel curves. Proposition 1 then shows us that Gorman's rank result applies to incomplete demand systems as a corollary to symmetry. A quadratic form is sufficient for rank three. Obviously, it is necessary that the polynomial be at least second-order to achieve rank three. In this sense a quadratic form in a function of income defines the most general nondegenerate class of full rank generalized polynomial expenditure models, even for incomplete demand systems. Gorman's conjecture is indeed correct, even when we extend his framework as far as our abilities permit.

APPENDIX

Mathematical Preliminaries and Proofs

A.1. Semidefinite Matrices

Lemma 1. Let the $n \times n$ real-valued matrix $\boldsymbol{A}$ be symmetric and positive semidefinite. Then $\forall\ \boldsymbol{x} \in \mathbb{R}^n : \boldsymbol{x}'\boldsymbol{A}\boldsymbol{x} > 0$, the matrix $\boldsymbol{A} - (\boldsymbol{x}'\boldsymbol{A}\boldsymbol{x})^{-1}\boldsymbol{A}\boldsymbol{x}\boldsymbol{x}'\boldsymbol{A}$ is symmetric and positive semidefinite, with $\boldsymbol{x}$ contained in its null space.

Proof: Since $\boldsymbol{x}'\boldsymbol{A}\boldsymbol{x} > 0$ by hypothesis, $\forall\ \boldsymbol{z} \in \mathbb{R}^n$, $\boldsymbol{z}'\left[\boldsymbol{A} - (\boldsymbol{x}'\boldsymbol{A}\boldsymbol{x})^{-1}\boldsymbol{A}\boldsymbol{x}\boldsymbol{x}'\boldsymbol{A}\right]\boldsymbol{z} \geq 0$ if and only if $\boldsymbol{z}'\boldsymbol{A}\boldsymbol{z}(\boldsymbol{x}'\boldsymbol{A}\boldsymbol{x}) \geq (\boldsymbol{x}'\boldsymbol{A}\boldsymbol{z})^2$. Let the matrix $\boldsymbol{Q}$ satisfy $\boldsymbol{A} = \boldsymbol{Q}\boldsymbol{Q}'$ and define $\boldsymbol{v} = \boldsymbol{Q}'\boldsymbol{z}$ and $\boldsymbol{w} = \boldsymbol{Q}'\boldsymbol{x}$. Then $\boldsymbol{z}'\boldsymbol{A}\boldsymbol{z}(\boldsymbol{x}'\boldsymbol{A}\boldsymbol{x}) \geq (\boldsymbol{x}'\boldsymbol{A}\boldsymbol{z})^2$ if and only if $\boldsymbol{v}'\boldsymbol{v}(\boldsymbol{w}'\boldsymbol{w}) \geq (\boldsymbol{v}'\boldsymbol{w})^2$. The latter is an n–dimensional statement of the Cauchy-Schwarz inequality. Note that this inequality continues to apply even when some of the elements of $\boldsymbol{v}$ and or $\boldsymbol{w}$ vanish, which can occur if $\boldsymbol{A}$ has less than full rank. Finally, inspection verifies that

$$\left[\boldsymbol{A} - (\boldsymbol{x}'\boldsymbol{A}\boldsymbol{x})^{-1}\boldsymbol{A}\boldsymbol{x}\boldsymbol{x}'\boldsymbol{A}\right]\boldsymbol{x} = \boldsymbol{A}\boldsymbol{x} - \boldsymbol{A}\boldsymbol{x} = \boldsymbol{0},$$

so that $\boldsymbol{x}$ is contained in the null space of the matrix $\left[\boldsymbol{A} - (\boldsymbol{x}'\boldsymbol{A}\boldsymbol{x})^{-1}\boldsymbol{A}\boldsymbol{x}\boldsymbol{x}'\boldsymbol{A}\right]$. ■

We apply this result to the generalized quadratic/translog utility model by setting $\boldsymbol{A} = \begin{bmatrix} \boldsymbol{B} & \gamma \\ \gamma' & 1 \end{bmatrix}$ to obtain

$$\begin{bmatrix} \boldsymbol{B} & \gamma \\ \gamma' & 1 \end{bmatrix} - \left(\boldsymbol{x}'\boldsymbol{B}\boldsymbol{x} + 2\gamma'\boldsymbol{x} + 1\right)^{-1} \begin{bmatrix} (\boldsymbol{B}\boldsymbol{x}+\gamma)(\boldsymbol{B}\boldsymbol{x}+\gamma)' & (\gamma'\boldsymbol{x}+1)(\boldsymbol{B}\boldsymbol{x}+\gamma) \\ (\gamma'\boldsymbol{x}+1)(\boldsymbol{B}\boldsymbol{x}+\gamma)' & (\gamma'\boldsymbol{x}+1)^2 \end{bmatrix} \tag{37}$$

as a symmetric, positive semidefinite matrix with a zero eigen value in the direction of the vector $[\boldsymbol{x}' \quad 1]'$. It follows from this that the upper left $n_q \times n_q$ block,

$$\boldsymbol{B} - \left(\boldsymbol{x}'\boldsymbol{B}\boldsymbol{x} + 2\gamma'\boldsymbol{x} + 1\right)^{-1} (\boldsymbol{B}\boldsymbol{x} + \gamma)(\boldsymbol{B}\boldsymbol{x} + \gamma)',$$

is symmetric and positive semidefinite. This latter condition plus three additional properties of the demand model are necessary and sufficient for negative semidefiniteness of the Slutsky substitution matrix for the goods $\boldsymbol{q}$.

Lemma 2: A necessary condition for the symmetric matrix $\begin{bmatrix} \boldsymbol{B} & \gamma \\ \gamma' & 1 \end{bmatrix}$ *to be positive semidefinite is* $\boldsymbol{B} = \boldsymbol{L}\boldsymbol{L}'$, *where* $\boldsymbol{L}$ *is (upper) triangular, while a sufficient condition is* $\boldsymbol{B} = \boldsymbol{L}\boldsymbol{L}' + \gamma\gamma'$.

Proof: For necessity, note that if the complete matrix is positive semidefinite, then the upper left $n_q \times n_q$ submatix $\boldsymbol{B}$ must be as well. This implies the existence of a (possibly reduced rank) Choleski factorization $\boldsymbol{B}$ as $\boldsymbol{L}\boldsymbol{L}'$. For sufficiency, we simply note that

$$\begin{bmatrix} \boldsymbol{B} & \gamma \\ \gamma' & 1 \end{bmatrix} = \begin{bmatrix} \boldsymbol{L}\boldsymbol{L}' + \gamma\gamma' & \gamma \\ \gamma' & 1 \end{bmatrix} = \begin{bmatrix} \boldsymbol{L} & \gamma \\ \boldsymbol{0}' & 1 \end{bmatrix} \begin{bmatrix} \boldsymbol{L}' & \boldsymbol{0} \\ \gamma' & 1 \end{bmatrix}. \qquad \blacksquare$$

We apply this result to the generalized quadratic/translog utility model. When $\boldsymbol{L}$ is upper triangular, $\begin{bmatrix} \boldsymbol{B} & \gamma \\ \gamma' & 1 \end{bmatrix} = \begin{bmatrix} \boldsymbol{L} & \gamma \\ \boldsymbol{0}' & 1 \end{bmatrix} \begin{bmatrix} \boldsymbol{L}' & \boldsymbol{0} \\ \gamma' & 1 \end{bmatrix}$ is an upper triangular Choleski factorization of $\begin{bmatrix} \boldsymbol{B} & \gamma \\ \gamma' & 1 \end{bmatrix}$. Because the demand model is 0° homogeneous in the elements of δ and $\begin{bmatrix} \boldsymbol{B} & \gamma \\ \gamma' & 1 \end{bmatrix}$, normalizing the last element in the lower right corner of the latter to unity simply accommodates identification of the remaining model parameters.

A.2. Symmetry and Curvature

Lemma 3. Let $e(\boldsymbol{p},\tilde{\boldsymbol{p}},\boldsymbol{z},u)$ *be the deflated expenditure function, let* $y = f(e)$, *with* $f \in \mathbb{C}^2$, $f' > 0$, *and inverse* $m = \phi(y)$, *let* $x_i = g_i(p_i)$, $g_i \in \mathbb{C}^2$, $g_i' > 0$, $i = 1,\ldots,n_q$, *and write the deflated expenditure function as*

$$e(\boldsymbol{p},\tilde{\boldsymbol{p}},\boldsymbol{z},u) = \phi\left[y(\boldsymbol{g}(\boldsymbol{p}),\tilde{\boldsymbol{p}},\boldsymbol{z},u)\right].$$

Then (a) $\dfrac{\partial^2 e}{\partial \boldsymbol{p}\partial \boldsymbol{p}'}$ *is symmetric if and only if* $\dfrac{\partial^2 y}{\partial \boldsymbol{x}\partial \boldsymbol{x}'}$ *is symmetric; and (b) if* $\phi'' \le 0$, $g_i'' \le 0\ \forall\ i$, *and y is concave in* ***x***, *then e is concave in* ***p***.

Proof: We have

$$\frac{\partial e}{\partial \boldsymbol{p}} = \phi'(y)\,\mathbf{diag}\left[g_i'\right]\frac{\partial y}{\partial \boldsymbol{x}}, \tag{38}$$

so that

$$\frac{\partial^2 e}{\partial \boldsymbol{p}\partial \boldsymbol{p}'} = \phi''(y)\,\mathbf{diag}\left[g_i'\right]\frac{\partial y}{\partial \boldsymbol{x}}\frac{\partial y}{\partial \boldsymbol{x}'}\mathbf{diag}\left[g_i'\right] + \phi'(y)\,\mathbf{diag}\left[g_i''\frac{\partial y}{\partial x_i}\right]$$

$$+\ \phi'(y)\,\mathbf{diag}\left[g_i'\right]\frac{\partial^2 y}{\partial \boldsymbol{x}\partial \boldsymbol{x}'}\mathbf{diag}\left[g_i'\right]. \tag{39}$$

The first two terms on the right are automatically symmetric, so that symmetry of the left-hand-side is equivalent to symmetry of the Hessian matrix on the far right-hand-side. The first two matrices on the right are negative semidefinite when $\phi'' \le 0$ and $g_i'' \le 0\ \forall\ i$, so that if $\dfrac{\partial^2 y}{\partial \boldsymbol{x}\partial \boldsymbol{x}'}$ is negative semidefinite, then $\dfrac{\partial^2 e}{\partial \boldsymbol{p}\partial \boldsymbol{p}'}$ is as well. ■

In general terms, this result can be applied to both QPIGL-IDS models. When $\kappa = \lambda = 1$

in the above Box-Cox transformations of income and prices, $\phi'' = 0$ and $g_i'' = 0 \; \forall \; i$. Then the first two terms on the right of (39) vanish. In general, we have $\phi(y) = (1 + \kappa y)^{1/\kappa}$ and $g_i(p_i) = (p_i^\lambda - 1)/\lambda \; \forall \; i$, so that when $\kappa \geq 1$ and $\lambda \leq 1$, we have $\phi'' \leq 0$ and $g_i'' \leq 0 \; \forall \; i$ and the conditions of the Lemma are satisfied. More specifically, in the case of the generalized quadratic/translog utility model, an open neighborhood of $\kappa = \lambda = 1$ exists such that symmetry of $\boldsymbol{B}$ and the curvature conditions $\boldsymbol{B} = \boldsymbol{L}\boldsymbol{L}' + \gamma\gamma'$, $\boldsymbol{L}$ triangular, define *weak integrability* of the incomplete demand system throughout the set $\Im$.

A.3 Differential Equations and the PIGL/PIGLOG Functional Forms

Consider the quasi-linear ordinary differential equation

$$\frac{y'(x)}{y(x)} = \frac{d\ln\left(y(x)\right)}{dx} = \alpha(x) + \beta(x) f\left(y(x)\right). \tag{40}$$

This differential equation lies at the heart of the functional form question originally posed by Muellbauer (1975, 1976). In particular, the simplest form of this question is, "What is the class of functions $f\left(y\right)$ that can satisfy (40) and the 0° homogeneity condition,

$$\alpha'(x)x + \beta'(x)xf(y) + \beta(x)f'(y)y \equiv 0 \text{ ?"} \tag{41}$$

It turns out that there are only two possibilities: a special case of Bernoulli's equation,

$$\frac{y'}{y} = \alpha_0 + \beta_0 \left(\frac{y}{x}\right)^\kappa, \; \kappa \neq 0; \tag{42}$$

or a special case of the logarithmic transformation,

$$\frac{y'}{y} = \alpha_0 + \beta_0 \ln\left(\frac{y}{x}\right). \tag{43}$$

The reason for this can be obtained by analyzing the implications of (41) directly. First, consider the case where $\alpha'(x)x = 0$, so that $\alpha(x) = \alpha_0$, a constant. Then (41) reduces to

$$\beta'(x)xf(y) + \beta(x)f'(y)y \equiv 0, \tag{44}$$

or equivalently,

$$\frac{d\ln(f)}{d\ln(y)} = \frac{f'(y)}{f(y)}y = -\frac{\beta'(x)}{\beta(x)}x = -\frac{d\ln(\beta)}{d\ln(x)} = \kappa, \tag{45}$$

where κ is a constant because the left-hand-side is independent of x, while the right-hand-side is independent of y. Without any loss in generality, the solutions are $f(y) = y^{\kappa}$ and $\beta(x) = \beta_0 x^{-\kappa}$.

Now suppose that $\alpha'(x)x \neq 0$, so that

$$\beta'(x)xf(y) + \beta(x)f'(y)y = -\alpha'(x)x. \tag{46}$$

Since the right-hand-side is again independent of y, at least one of the terms on the left also must be independent of y. If $f'(y) = 0$, so that $f(y) = f_0$ is constant, we obtain the degenerate case where the functions of y on the right-hand-side of (40) are not linearly independent. Hence, it must be that $\beta'(x)x = 0$, i.e., $\beta(x) = \beta$, a constant, and

$$f'(y)y = \frac{df(y)}{d\ln(y)} = -\frac{\alpha'(x)x}{\beta} = \lambda, \tag{47}$$

where λ is a constant again because the left-hand-side is independent of x and the right-hand-side is independent of y. Solving the left side gives

$$f(y) = \lambda\ln(y) + \gamma, \tag{48}$$

while the right-hand-side can be rewritten as

$$\frac{d\alpha(x)}{d\ln(x)} = -\lambda\beta, \tag{49}$$

which has solution

$$\alpha(x) = \alpha - \lambda\beta \ln(x) . \tag{50}$$

Combining (48) and (50), we obtain (43), with $\alpha_0 = \alpha + \gamma$ and $\beta_0 = \beta\lambda$.

The implication is that, for ranks one and two demand models in this class, the admissible forms of $f(y)$ are completely determined by homogeneity.

When we consider *incomplete demand systems*, we do not necessarily have homogeneity (in terms of the subset of prices of interest) or adding up to restrict the functional forms. For Bernoulli's differential equation,

$$y' = \alpha(x)y + \beta(x)y^{1-\kappa},\ \kappa \neq 0 , \tag{51}$$

by noting that $\frac{d}{dx}\left(y^{\kappa}/\kappa\right) = y^{\kappa-1}y'$, we can rewrite this as the linear ordinary differential equation in $f(y) = y^{\kappa}/\kappa$,

$$\frac{d}{dx}\left(y^{\kappa}/\kappa\right) = y^{\kappa-1}y' = \left(\kappa\alpha(x)\right)\left(y^{\kappa}/\kappa\right) + \beta(x) , \tag{52}$$

with complete solution

$$y(x) = \left[\kappa e^{\int^{x}\kappa\alpha(s)ds}\left(\int^{x} e^{-\int^{s}\kappa\alpha(t)dt}\beta(s)ds + c\right)\right]^{1/\kappa} . \tag{53}$$

Similarly, the generic logarithmic first-order linear differential equation is

$$\frac{d\ln(y)}{dx} = \frac{y'}{y} = \alpha(x) + \beta(x)\ln\left(y\right) , \tag{54}$$

with complete solution given by

$$y(x) = \exp\left\{e^{\int^{x}\beta(s)ds}\left(\int^{x} e^{-\int^{s}\beta(t)dt}\alpha(s)ds + c\right)\right\} . \tag{55}$$

The generic nature of both of these differential equations is that they can be written as simple linear first-order ordinary differential equations,

$$\frac{df(y(x))}{dx} = \alpha(x) + \beta(x) f(y(x)) . \tag{56}$$

But when y is deflated income and the demands do not absorb all of the consumer's budget, homogeneity and adding up do not apply any restriction on the class of functions $f(y)$ that can solve this differential equation, and the complete class of solutions is

$$y(x) = f^{-1}\left[e^{\int^x \kappa\alpha(s)ds} \left(\int^x e^{-\int^s \kappa\alpha(t)dt} \beta(s) ds + c \right) \right] . \tag{57}$$

A.4 Proofs of the Propositions

Proposition 1. If the possibly incomplete demand system has the polynomial form

$$\frac{\partial y(\boldsymbol{x};\cdot)}{\partial \boldsymbol{x}} = \sum_{i=0}^{K} \boldsymbol{\alpha}_i(\boldsymbol{x};\cdot) y(\boldsymbol{x};\cdot)^i$$

and is weakly integrable, then there exist $\varphi_i : \mathbb{R}^{n_q} \to \mathbb{R},\ i = 2,\ldots,K$ *such that*

$$\boldsymbol{\alpha}_i(\boldsymbol{x}) \equiv \varphi_i(\boldsymbol{x}) \boldsymbol{\alpha}_K(\boldsymbol{x})\ \forall\ i \geq 2 .$$

Proof: Slutsky symmetry is equivalent to symmetry of the matrix

$$\sum_{i=0}^{K} \frac{\partial \boldsymbol{\alpha}_i}{\partial \boldsymbol{x}'} y^i + \sum_{i=1}^{K} \sum_{j=0}^{K} i \boldsymbol{\alpha}_i \boldsymbol{\alpha}'_j y^{i+j-1} , \tag{58}$$

where each $\partial \boldsymbol{\alpha}_i / \partial \boldsymbol{x}'$ is an $n_q \times n_q$ matrix. By continuity, symmetry requires that each like power of y has a symmetric coefficient matrix, and all of the matrices for powers K+1 through 2K-2 involve nontrivial symmetry conditions without involving any $\partial \boldsymbol{\alpha}_i / \partial \boldsymbol{x}'$ terms. The matrix for the term $y^{2K\text{-}1}$ only involves $\boldsymbol{\alpha}_K \boldsymbol{\alpha}'_K$, which is clearly symmetric.

Combine terms in like powers of y and apply a backward recursion beginning with the matrix for $y^{2K\text{-}2}$, so that

$$(K-1)\alpha_{K-1}\alpha'_K + K\alpha_K\alpha'_{K-1} \tag{59}$$

is symmetric if and only if $\alpha_{K-1} \equiv \varphi_{K-1}\alpha_K$, say, for some $\varphi_{K-1} : \mathbb{R}^{n_q} \to \mathbb{R}$. Similarly,

$$(K-2)\alpha_{K-2}\alpha'_K + (K-1)\alpha_{K-1}\alpha'_{K-1} + K\alpha_K\alpha'_{K-2} \tag{60}$$

is symmetric if and only if $\alpha_{K-2} \equiv \varphi_{K-2}\alpha_K$ for some $\varphi_{K-2} : \mathbb{R}^{n_q} \to \mathbb{R}$. Applying the recursive argument, consider the y^{2K-4} matrix for $K\geq 3$,

$$(K-3)\alpha_{K-3}\alpha'_K + (K-2)\alpha_{K-2}\alpha'_{K-1} + (K-1)\alpha_{K-1}\alpha'_{K-2} + K\alpha_K\alpha'_{K-3}\,. \tag{61}$$

Symmetry of the middle two terms follows from above, since $\alpha_{K-2}\alpha'_{K-1} = \varphi_{K-2}\varphi_{K-1}\alpha_K\alpha'_K$ and $\alpha_{K-1}\alpha'_{K-2} = \varphi_{K-1}\varphi_{K-2}\alpha_K\alpha'_K$. The matrix $(K-3)\left(\alpha_{K-3}\alpha'_K + \alpha_K\alpha'_{K-3}\right)$ is symmetric. Therefore, the matrix on y^{2K-4} is symmetric if and only if $3\alpha_K\alpha'_{K-3}$ is symmetric, which requires that $\alpha_{K-3} \equiv \varphi_{K-3}\alpha_K$ for some $\varphi_{K-3} : \mathbb{R}^{n_q} \to \mathbb{R}$; completing the argument for $K \leq 5$. If $K > 5$, for each j such that $4 \leq j \leq K\text{-}1$, ..., group like terms, substitute $\alpha_{K-i} \equiv \varphi_{K-i}\alpha_K$ for each $i < j$, and appeal to symmetry of the matrix $\left(\alpha_{K+1-j}\alpha'_K + \alpha_K\alpha'_{K+1-j}\right)$, which sequentially requires that each matrix of the following form is symmetric:

$$(j-1)\alpha_K\alpha'_{K+1-j} + \sum_{i=1}^{j-2}(K-i)\varphi_{K-i}\varphi_{K+1+i-j}\alpha_K\alpha'_K\,. \tag{62}$$

Each matrix is symmetric if and only if $\alpha_{K+1-j} \equiv \varphi_{K+1-j}\alpha_K$ for some $\varphi_{K+1-j} : \mathbb{R}^{n_q} \to \mathbb{R}$: $j = 4$ gives the condition for $\alpha_{K\text{-}3;}$ $j = K-1$ gives the condition for α_2; and $\forall\ K \geq 2$, $\alpha_i \equiv \varphi_i\alpha_K$, $i = 2,\ldots,K$, and the rank of $[\alpha_{ij}]$ is no more than three. □

Proposition 2. If the LA-AIDS is integrable over an open set $\mathcal{N} \subset \mathbb{R}^{n_q}$ with a non-empty interior and such that $1+\gamma'\boldsymbol{x} \neq 0 \ \forall \ \boldsymbol{x} \in \mathcal{N}$, then either (a) $\gamma \neq \mathbf{0}$ and $\boldsymbol{B} = \beta_0\gamma\gamma'$ for some $\beta_0 \in \mathbb{R}$, or (b) $\gamma = \mathbf{0}$ and $\boldsymbol{B} = \boldsymbol{B}'$. In case (a), the logarithmic expenditure function has the form

$$y(\boldsymbol{x},u) = \alpha'\boldsymbol{x} + \beta_0\left[(1+\gamma'\boldsymbol{x})\ln(1+\gamma'\boldsymbol{x}) - \frac{\gamma'\boldsymbol{x}}{(1+\gamma'\boldsymbol{x})}\right] + (1+\gamma'\boldsymbol{x})\theta(\tilde{\boldsymbol{p}},u),$$

while in case (b) it has the form,

$$y(\boldsymbol{x},u) = \alpha'\boldsymbol{x} + \tfrac{1}{2}\boldsymbol{x}'\boldsymbol{B}\boldsymbol{x} + \theta(\tilde{\boldsymbol{p}},u).$$

Proof: (This proof is taken entirely from LaFrance (2004)). It is straightforward to show that $|\boldsymbol{I}+\gamma\boldsymbol{x}'| = 1+\gamma'\boldsymbol{x}$, so that $\boldsymbol{I}+\gamma\boldsymbol{x}'$ is nonsingular and its inverse is $\boldsymbol{I}-\gamma\boldsymbol{x}'/(1+\gamma'\boldsymbol{x})$ if and only if $\gamma'\boldsymbol{x} \neq -1$. We therefore can write the LA-AIDS as a system of linear partial differential equations,

$$\frac{\partial y(\boldsymbol{x},u)}{\partial \boldsymbol{x}} - \gamma\frac{y(\boldsymbol{x},u)}{(1+\gamma'\boldsymbol{x})} = \left[\boldsymbol{I} - \frac{\gamma\boldsymbol{x}'}{(1+\gamma'\boldsymbol{x})}\right](\alpha + \boldsymbol{B}\boldsymbol{x}), \tag{63}$$

where use has been made of $[\boldsymbol{I}-\gamma\boldsymbol{x}'/(1+\gamma'\boldsymbol{x})]\gamma \equiv \gamma/(1+\gamma'\boldsymbol{x})$. Then, by simply noting that

$$\frac{\partial}{\partial \boldsymbol{x}}\left[\frac{y(\boldsymbol{x},u)}{1+\gamma'\boldsymbol{x}}\right] = \left[\frac{\partial y(\boldsymbol{x},u)}{\partial \boldsymbol{x}} - \gamma\frac{y(\boldsymbol{x},u)}{(1+\gamma'\boldsymbol{x})}\right]\frac{1}{(1+\gamma'\boldsymbol{x})}, \tag{64}$$

we can multiply by $1/(1+\gamma'\boldsymbol{x})$ to make the left-hand-side an exact differential. Consequently, Slutsky symmetry is equivalent to symmetry of the $n_q \times n_q$ matrix

$$\frac{\partial}{\partial \boldsymbol{x}'}\left\{\frac{1}{(1+\gamma'\boldsymbol{x})}\left[\boldsymbol{I} - \frac{\gamma\boldsymbol{x}'}{(1+\gamma'\boldsymbol{x})}\right](\alpha + \boldsymbol{B}\boldsymbol{x})\right\} =$$

$$\frac{\boldsymbol{B}}{(1+\gamma'\boldsymbol{x})} - \frac{\left[(\alpha + \boldsymbol{Bx})\gamma' + \gamma(\alpha' + \boldsymbol{x}'\boldsymbol{B}' + \boldsymbol{x}'\boldsymbol{B})\right]}{(1+\gamma'\boldsymbol{x})^2} - \frac{2(\alpha'\boldsymbol{x} + \boldsymbol{x}'\boldsymbol{Bx})\gamma\gamma'}{(1+\gamma'\boldsymbol{x})^3}. \tag{65}$$

Imposing symmetry on each of the terms associated with like powers of $(1+\gamma'\boldsymbol{x})$ and ignoring terms that are automatically symmetric, we obtain $\boldsymbol{B} = \boldsymbol{B}'$ and $\gamma\boldsymbol{x}'\boldsymbol{B} = \boldsymbol{B}'\boldsymbol{x}\gamma'$. There are two ways that these conditions are satisfied simultaneously $\forall\ \boldsymbol{x} \in \boldsymbol{\mathcal{N}}$: (i) $\gamma \neq \boldsymbol{0}$ and $\boldsymbol{B} = \beta_0\gamma\gamma'$ for some $\beta_0 \in \mathbb{R}$ (including $\beta_0 = 0$); and (ii) $\gamma = \boldsymbol{0}$ and $\boldsymbol{B} = \boldsymbol{B}'$. Case (i) gives the LA-AIDS model in the form

$$\frac{\partial y}{\partial \boldsymbol{x}} = \alpha + \beta_0\gamma\gamma'\boldsymbol{x} + \gamma\left(\frac{y - \alpha'\boldsymbol{x} - \beta_0(\gamma'\boldsymbol{x})^2}{1+\gamma'\boldsymbol{x}}\right) = \alpha + \gamma\left(\frac{y - \alpha'\boldsymbol{x} + \beta_0\gamma'\boldsymbol{x}}{1+\gamma'\boldsymbol{x}}\right). \tag{66}$$

This is a very simple system of linear first-order partial differential equations. Noting that

$$\frac{\partial}{\partial \boldsymbol{x}}\left(\frac{\gamma'\boldsymbol{x}}{1+\gamma'\boldsymbol{x}}\right) = \left[\gamma - \gamma\left(\frac{\gamma'\boldsymbol{x}}{1+\gamma'\boldsymbol{x}}\right)\right]\frac{1}{(1+\gamma'\boldsymbol{x})}, \tag{67}$$

and that

$$\frac{\partial}{\partial \boldsymbol{x}}\left\{\beta_0\left[\ln(1+\gamma'\boldsymbol{x}) - \left(\frac{\gamma'\boldsymbol{x}}{1+\gamma'\boldsymbol{x}}\right)\right]\right\} = \frac{\beta_0\gamma\gamma'\boldsymbol{x}}{(1+\gamma'\boldsymbol{x})^2}, \tag{68}$$

combining these two equations with (64), and integrating with respect to $\boldsymbol{x}$, we obtain the logarithmic expenditure function as

$$y(\boldsymbol{x}, u) = \alpha'\boldsymbol{x} + \beta_0\left[(1+\gamma'\boldsymbol{x})\ln(1+\gamma'\boldsymbol{x}) - \frac{\gamma'\boldsymbol{x}}{(1+\gamma'\boldsymbol{x})}\right] + (1+\gamma'\boldsymbol{x})\theta(\tilde{\boldsymbol{p}}, u). \tag{69}$$

Case (ii) generates the homothetic LA-AIDS demand model,

$$\frac{\partial y}{\partial \boldsymbol{x}} = \alpha + \boldsymbol{Bx}, \tag{70}$$

which gives the logarithmic expenditure function as

$$y(\boldsymbol{x},u) = \alpha'\boldsymbol{x} + \tfrac{1}{2}\boldsymbol{x}'\boldsymbol{B}\boldsymbol{x} + \theta(\tilde{\boldsymbol{p}},u)\,. \tag{71}$$

This establishes necessity of the parameter restrictions for integrability. On the other hand, $\gamma = \mathbf{0}$ trivially gives an LA-AIDS form. To show sufficiency when $\gamma \neq \mathbf{0}$, write

$$y - \boldsymbol{x}'\frac{\partial y}{\partial \boldsymbol{x}} = y - \boldsymbol{x}'\left[\alpha + \beta_0\gamma\gamma'\boldsymbol{x} + \gamma\left(\frac{y - \alpha'\boldsymbol{x} - \beta_0(\gamma'\boldsymbol{x})^2}{1+\gamma'\boldsymbol{x}}\right)\right] = \frac{y - \alpha'\boldsymbol{x} - \beta_0(\gamma'\boldsymbol{x})^2}{1+\gamma'\boldsymbol{x}}\,. \tag{72}$$

Direct substitution then gives the LA-AIDS form,

$$\frac{\partial y}{\partial \boldsymbol{x}} = \alpha + \beta_0\gamma\gamma'\boldsymbol{x} + \gamma\left(y - \boldsymbol{x}'\frac{\partial y}{\partial \boldsymbol{x}}\right). \qquad \square$$

Proposition 3. The system of partial differential equations in (24) *is weakly integrable if and only if it can be written in the form*

$$\frac{\partial y(\boldsymbol{x},\tilde{\boldsymbol{p}},u)}{\partial \boldsymbol{x}} = \tilde{\alpha} + \tilde{\boldsymbol{B}}\boldsymbol{p} + \gamma\left[\, y(\boldsymbol{x},\tilde{\boldsymbol{p}},u) - \alpha_0 - \tilde{\alpha}'\boldsymbol{x} - \tfrac{1}{2}\boldsymbol{x}'\tilde{\boldsymbol{B}}\boldsymbol{x}\right]$$

where α_0 *is a scalar (that may be a function of other prices),* $\tilde{\alpha} = \alpha - \alpha_0\gamma$ *is an* $n_q\times 1$ *vector,* $\tilde{\boldsymbol{B}}$ *is a symmetric* $n_q\times n_q$ *matrix that satisfies* $\tilde{\boldsymbol{B}} = \boldsymbol{B} + \gamma\alpha'$*, where* $\boldsymbol{B} = \left[\beta_1 \cdots \beta_{n_q}\right]$*, and* $\Delta_i = -\gamma_i\tilde{\boldsymbol{B}}\ \forall\ i$.

Proof: Symmetry of the Slutsky substitution terms is equivalent to symmetry of the $n_q\times n_q$ matrix with typical element

$$s_{ij} = \beta_{ij} + \sum_{k=1}^{n_q}\delta_{ijk}x_k + \gamma_i\left[\alpha_j + \sum_{k=1}^{n_q}\beta_{jk}x_k + \tfrac{1}{2}\sum_{k=1}^{n_q}\sum_{\ell=1}^{n_q}\delta_{jk\ell}x_kx_\ell + \gamma_j y\right]. \tag{73}$$

To show necessity, we will derive the implications of symmetry, $s_{ij} = s_{ji}\ \forall\ i,j$. These implications can be conveniently grouped into three sets:

(a) $$\beta_{ij} + \gamma_i \alpha_j = \beta_{ji} + \gamma_j \alpha_i;$$

(b) $$\sum_{k=1}^{n_q} (\delta_{ijk} + \gamma_i \beta_{jk}) x_k = \sum_{k=1}^{n_q} (\delta_{jik} + \gamma_j \beta_{ik}) x_k \text{ ; and}$$

(c) $$\gamma_i \sum_{k=1}^{n_q} \sum_{l=1}^{n_q} \delta_{jkl} x_k x_l = \gamma_j \sum_{k=1}^{n_q} \sum_{l=1}^{n_q} \delta_{ikl} x_k x_l .$$

From (a), it follows that $\alpha \equiv \hat{\alpha} - \gamma \hat{\alpha}_0$, where $\hat{\alpha}_i = (\beta_{i1} - \beta_{1i}) / \gamma_1$ and $\hat{\alpha}_0 = -\alpha_1 / \gamma_1$. Substituting the right-hand-side for each α_i back into (a) implies $B \equiv [\beta_{ij} + \gamma_i \hat{\alpha}_j]$ is symmetric, equivalently $\left[\beta_{ij}\right] = \boldsymbol{B} - \gamma\alpha'$ for some symmetric matrix $\boldsymbol{B}$. Now turning to (b), we will use a specialized result of LaFrance and Hanemann (1989, *Theorem 2*, p. 266) for these kinds of problems to obtain $\delta_{ijk} + \gamma_i \beta_{jk} = \delta_{jik} + \gamma_j \beta_{ik} \ \forall \ i, j, k$. We will return to this in a moment. First, however, we need to apply the same result of LaFrance and Hanemann to (c) to get $\gamma_i \delta_{jkl} = \gamma_j \delta_{ikl} \ \forall \ i, j, k, l$. This, in turn, implies that for each i, the $n_q \times n_q$ matrix $[\delta_{ikl}] = \gamma_i \boldsymbol{C}$ where $\boldsymbol{C}$ is a symmetric matrix with typical element $c_{kl} = \delta_{1kl} / \gamma_1$. Combining this with (b) gives $\gamma_i (c_{jk} + b_{jk}) = \gamma_j (c_{ik} + b_{ik}) \ \forall \ i, j, k$. Exploiting $\gamma_1 \neq 0$ and the symmetry of both $\boldsymbol{B}$ and $\boldsymbol{C}$ then gives $(b_{ij} + c_{ij}) = (b_{11} + c_{11}) \gamma_i \gamma_j / \gamma_1^2$, so that $\boldsymbol{B}$ and $\boldsymbol{C}$ are related by $\boldsymbol{C} = -(\boldsymbol{B} + \varepsilon\gamma\gamma')$, where $\varepsilon = -(b_{11} + c_{11}) / \gamma_1^2$. Combining all of these implications, the transformed demands can be written in matrix notation as

$$\begin{aligned} \frac{\partial y(\boldsymbol{x}, u)}{\partial \boldsymbol{x}} &= \hat{\alpha} - \hat{\alpha}_0 \gamma + (\boldsymbol{B} - \gamma \hat{\alpha}') \boldsymbol{x} - \tfrac{1}{2} \gamma \boldsymbol{x}' (\boldsymbol{B} + \varepsilon \gamma \gamma') \boldsymbol{x} + \gamma y(\boldsymbol{x}, u) \\ &= \hat{\alpha} + \boldsymbol{B}\boldsymbol{x} + \gamma \left[y(\boldsymbol{x}, u) - \hat{\alpha}_0 - \hat{\alpha}' \boldsymbol{x} - \tfrac{1}{2} \boldsymbol{x}' (\boldsymbol{B} + \varepsilon \gamma \gamma') \boldsymbol{x} \right]. \end{aligned} \tag{74}$$

Now, we note that adding and subtracting $\varepsilon\gamma$ and $\varepsilon\gamma\gamma'\boldsymbol{x}$ has no affect on the transformed demands. Therefore, let $\tilde{\alpha} = \hat{\alpha} + \varepsilon\gamma$, $\tilde{\boldsymbol{B}} = \boldsymbol{B} + \varepsilon\gamma\gamma'$, and $\alpha_0 = \hat{\alpha}_0 + \varepsilon$, and rewrite the partial differential equations in the equivalent form

$$\frac{\partial y(\boldsymbol{x},u)}{\partial \boldsymbol{x}} = \tilde{\alpha} + \tilde{\boldsymbol{B}}\boldsymbol{x} + \gamma\left[y(\boldsymbol{x},u) - \alpha_0 - \tilde{\alpha}'\boldsymbol{x} - \tfrac{1}{2}\boldsymbol{x}'\tilde{\boldsymbol{B}}\boldsymbol{x} \right]. \tag{75}$$

Finally, the integrating factor $\mathrm{e}^{-\gamma'\boldsymbol{x}}$ makes the partial differential equations exact,

$$\begin{aligned} \frac{\partial}{\partial \boldsymbol{x}}\left[y(\boldsymbol{x},u)\mathrm{e}^{-\gamma'\boldsymbol{x}} \right] &= \left[\tilde{\alpha} + \tilde{\boldsymbol{B}}\boldsymbol{x} - \gamma\left(\alpha_0 + \tilde{\alpha}'\boldsymbol{x} + \tfrac{1}{2}\boldsymbol{x}'\tilde{\boldsymbol{B}}\boldsymbol{x}\right)\right]\mathrm{e}^{-\gamma'\boldsymbol{x}} \\ &= \frac{\partial}{\partial \boldsymbol{x}}\left[\left(\alpha_0 + \tilde{\alpha}'\boldsymbol{x} + \tfrac{1}{2}\boldsymbol{x}'\tilde{\boldsymbol{B}}\boldsymbol{x}\right)\mathrm{e}^{-\gamma'\boldsymbol{x}} \right], \end{aligned} \tag{76}$$

with complete solution class given by

$$y(\boldsymbol{x},\tilde{\boldsymbol{p}},u) = \alpha_0 + \tilde{\alpha}'\boldsymbol{x} + \tfrac{1}{2}\boldsymbol{x}'\tilde{\boldsymbol{B}}\boldsymbol{x} + \theta(\tilde{\boldsymbol{p}},u)\mathrm{e}^{\gamma'\boldsymbol{x}}. \tag{77}$$

Sufficiency is demonstrated by applying Hotelling's/Shephard's lemma. □

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