Business Language for Agents with Asymmetric Perceptions

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Abstract

This paper addresses the relationship between individual perceptions and the uses of a business language. Perceptions are modeled explicitly, and are not common knowledge. A business language enables individuals with different perceptions to trade.

I present a formal criterion for faithfulness of the business language among heterogeneous agents. Roughly, the language is heterogeneously faithful if different agents who observe the same real-world object can perceive it in a way that leads them to make the same report. Different business languages lead to different possible equilibria, and thus can be Pareto-ranked. In particular, heterogeneously faithful languages are compared with one where agents can fully disclose what they perceive.

Key Terms: Reporting, Faithfulness, Full Disclosure, Perceptions.
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Contents

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1 Introduction

For trade to occur, both sides of a trade need to have a shared understanding of what is being traded. In an Arrow-Debreu world, this is captured by assuming the commodity space is common knowledge. There may be restrictions on what is observable or contractable, but the usual assumption is that if different agents see the same thing, they will agree on what they observe. Implicitly, this means that either there is no need for a business language when goods are commonly observable, or that reports can be given costlessly and in arbitrarily fine detail.

This assumption is made not because of its plausibility but because of technical convenience. It has been known at least since Weber [56] that perceptions do not perfectly correspond to the real world, and that variations in physical characteristics of real-world objects may be too subtle for one individual to detect but apparent to another. Experimental evidence of perceptual differences in numerical judgements, and of the difficulty involved in modeling these differences according to a psychological law, is discussed in Dickhaut and Eggleton [19]. What happens if we dispense with a common view of the world, and acknowledge that agents only understand the world as they subjectively perceive it—can any of neoclassical economic theory be salvaged?

This paper models agents who perceive the world privately and subjectively. Preferences are defined on the agent's subjective conceptions of the world, and not on alternatives as they are presented in an objective sense. Trade occurs through a shared language, whose vocabulary and syntax are agreed upon, but where the meanings of the terms are subjectively understood.

The main questions addressed are:

1. To what extent can a business language salvage any of neoclassical economic theory when agents have private, subjective perceptions?

2. Can different business languages be compared—for example, by Pareto ranking the equilibria possible under each language—when the environment is not common knowledge?

Each agent has a set of *conceptions*, i.e., of concepts the the agent can in principle perceive. Conceptions are subjective, which means that no one can observe anyone else's conceptions. An agent's conceptions do not need to correspond to reality: they may be incomplete, vague, or wrong.

When an agent observes a real-world object, the agent sees a member of the set of conceptions; this is what is meant by *perception.* There may be more than one way that the same object can appear to the same person. Conversely, many objects may differ too subtly for the agent to detect. Perception is introduced as a binary relation between the real-world objects and the agent's conceptions, capturing this many-to-many nature. Similarly, reporting in the shared language is introduced as a binary relation between the agent's conceptions and the terms in the shared language.

For agents to trade using a business language, there needs to be some sense in which different terms can be said to mean the same thing (at least approximately) to different users of the language. This idea leads to a criterion of *heterogeneous faithfulness.* That is, the language is faithful among agents with heterogeneous perceptions if the agents, when faced with the same real-world object, can report it in the same ways. This notion is the driving force behind the results in this paper.

This criterion of heterogeneous faithfulness is contrasted with an alternative criterion of completeness or *fullness.* A language can fully represent an agent's conceptions if every function from the agent's conceptions to any other object is identical to a function from the reports the agent makes in the shared language. Both criteria appear in various forms in the literature, but heterogeneous faithfulness turns out to have more desirable properties for enabling agents to trade.

The welfare properties of different business languages are evaluated in stages. First, I construct a consumer's choice problem in a pure exchange setting, where the consumer's preferences and choices are defined over the set of conceptions. In other words, the consumer is interested in choices as they privately appear, and not as they are objectively presented in the real-world. The consumer cannot determine exactly which conceptions are available, however, as trade occurs in the shared language.

Once the consumer's problem is posed, I consider the competitive equilibria that can arise. With an appropriate definition of Pareto dominance, the equilibria can be partially ordered. One business language can then be said to Pareto dominate another based on the equilibria that the two languages make possible.

The structure of this paper is as follows. The next section discusses related literature. This is followed by a section defining the model of perceptions. Reporting in the shared language is then introduced, after which the results on perceptions and reporting are presented. In particular, the criterion of complete or full reporting is compared with heterogeneous faithfulness. Preferences and the consumer's budget problem are then defined, followed by the definitions of Pareto dominance and the discussion of competitive equilibria. A final section gives concluding remarks. In an appendix, I show how the model here is fully constructive.

2 Background and Literature Review

2.1 Disclosure

The model presented here is related to the disclosure model presented in Demski and Sappington [17, 18]. The issue addressed here differs from that in Demski and Sappington, in that their interest is in when a reporting language enables an agent to *fully reveal* a private signal, the meaning of which is known to all parties. That is, the Demski-Sappington model applies to the special case of the model here where agents have homogenous perceptions.

It will be useful for comparisons later to view the Demski-Sappington model formally. At date t, a firm knows its history of cash flows and of informative signals from each date in $\{0, \dots, t\}$. Demski and Sappington write the firm's information set as $H_t \equiv \{(x_\tau, y_\tau)_{s=0}^t\}$, where x_τ is the net cash flow and y_τ is the firm's signal at date τ. A state of the world determines the history of the signals the firm will see, which belong to some commonly known set Y, and of cash flows the firm will realize. Thus, for each t, the state of the world determines $h_t \in H_t$. A function η_t maps the state of the world to the period t signal.

Let T be the set of possible reports, and let Ω be the set of states of the world. Demski and Sappington define the earnings disclosure as fully revealing if, upon observing the cash flow and the earnings report, the market can infer the current period signal. That is, the earnings disclosure fully reveals the firm's signal if the market has an interpretation function $T \xrightarrow{\iota} Y$ that makes the following diagram commute at each date t:

$$\begin{array}{ccc} \Omega & \xrightarrow{\eta_t} & Y \\ {\scriptstyle h_t}\big\downarrow & & \big\uparrow{\scriptstyle \iota} \\ H_t & \xrightarrow{\langle x_t, I_t \rangle} & T \end{array} \tag{1}$$

In other words, the disclosure is fully revealing if the market gets the same information from interpreting the report the firm makes, based on the history the firm observes, as if the market could observe the signal directly. Mathematically, this says that $\eta_t = \iota \circ \langle x_t, I_t \rangle \circ h_t$.

Vickry [54] was quick to observe that the nature of Y (that is, of what the market can perceive) largely determines the class of earnings calculations that are fully revealing. In the special cases where $Y = \varnothing$ or where Y is a singleton, every possible earnings calculation I_t is fully-revealing: either the market automatically knows that the firm cannot see any signal (because Y is empty), or the firm automatically knows the unique signal that the firm can observe. Vickrey calls this situation *weak transparency.* Similarly, if the cash flow is a sufficient statistic for the signal, then any earnings calculation (or none at all) is fully-revealing; Vickrey calls this *strong transparency.*

If the system is not weakly transparent in Vickrey's sense, then fully revealing disclosure means that if the market were to see the signal and the cash flow, it would make the same earnings report that the firm would make. So if the cash flow is publicly known and the system is not weakly transparent, the earnings calculation is fully revealing if the following modification of Diagram 1 commutes, where ι^{-1} is the inverse map of ι:

$$\begin{array}{ccc} \Omega & \xrightarrow{\eta_t} & Y \\ {\scriptstyle h_t}\downarrow & & \downarrow{\scriptstyle \langle x_t, \iota^{-1} \rangle} \\ H_t & \xrightarrow{\langle x_t, I_t \rangle} & T \end{array} \tag{2}$$

Thus the reporting language is fully revealing if each report means the same thing to each agent. Unless the set of signals is degenerate in Vickrey's sense, a fully revealing earnings disclosure evidently must also be faithful in the sense described in the introduction. It will become clear that this is because the firm and the market have identical perceptions.

2.2 Perceptual Differences and Hardness

In the Demski-Sappington model, the firm and the market both understand the set of signals in the same way. The ability of the firm to tell the market its signal depends entirely on the reporting language and the signals themselves.

The model of *information hardness* in Kirschenheiter [31], based on the notion of Ijiri [29], links reporting to perceptions. There is a collection X of "true signals," which are not directly observed. Agents have a partition of X, and see each signal as the unique set to which it belongs.[1] Individuals communicate through a shared language. As in the present setting, each individual has a private notion of what terms in the shared language mean.

[1] Thus, Kirschenheiter's model is connected to Blackwell's informativeness. For a similar approach, cf. Marschak and Miyasawa [33] and Gjesdal [22, 23].

Information is *hard* in Kirschenheiter's setting if two people observing the same signal issue the same message. This contrasts with the the notion of faithfulness here, which requires only that two people observing the same object *can* issue the same report. The difference reflects the fact that signals are uniquely observable in Kirschenheiter but may be perceived many ways in the present model.

Kirschenheiter's mapping from the individual's information partitions to the shared language is also single-valued. That is, the reporting system weakly coarsens the individual's information partition, but there are no signals in Kirschenheiter's world where the report is ambiguous.

The formal details of Kirschenheiter's model can be presented in a diagram similar to Diagram 2. Let X be the collection of "true signals;" let $\mathfrak{P}(X)$ be the class of partitions of X. Agent i has a partition $S_i \in \mathfrak{P}(X)$, and perceives $x \in X$ as the set $a \in S_i$ that contains it. Individual i also has a map r_i that assigns each set in S_i to a unique term in the shared language T. Information is hard iff the following diagram commutes for every pair of individuals i, j:

$$\begin{array}{ccc} X & \xrightarrow{\subset} & S_i \\ {\scriptstyle\subset}\downarrow & & \downarrow{\scriptstyle \mathrm{r_i}} \\ S_j & \xrightarrow{\mathrm{r_j}} & T \end{array} \tag{3}$$

Except for labeling, Diagram 3 is identical to Diagram 2. The interpretations in Kirschenheiter differ from those in Demski and Sappington, but the essense of both models, and of the model here, is the above diagram. Thus, the commutative square seems to capture an essential feature of faithful reporting.

3 Perceptions

3.1 Perceptions as a Relation

Perception is modeled here as a binary relation between the objective, external world and an agent's private understanding of the world. There are real-world objects in nature (signals, commodity baskets, etc.), which affect an agent's decision problem. However, the agent cannot perfectly distinguish all possible real-world objects.

Each agent has a set of *conceptions*, that is, of subjective notions about the different possible choices. To each real-world object, there is an associated subset of the agent's conceptions; this subset gives the agent's possible perceptions of the real-world object. Conversely, to each conception, there is an associated subset of real-world objects; these are objects the agent can perceive in the same way.

An example may clarify. Suppose an individual is aware of wines with various bodies. A given body of wine may be indistinguishible to the agent from a wine that the agent has previously classified as light-bodied. That is, upon tasting from both bottles, the agent may be unable to say which wine is fuller in body. The first bottle of wine is then perceptible as light-bodied.

The same wine might also be indistinguishible from a third wine, which the agent has previously classfied as full-bodied. For example, the body of the first wine may be in between that of the second and third. This means that the first wine can also be perceived as full-bodied. The wines perceptible as light-bodied are then the first two, and those perceptible as full-bodied are the second and third. All three are perceptible as wines.

The example illustrates that perceptions are not uniquely defined, and that being perceptible the same way is not an equivalence relation. Instead, it is a tolerance relation (see [45]). Two objects that can be perceived in the same way will be thought of here as belonging to the same neighborhood.

3.2 A Formal Model of Perceptions

Throughout this work, I assume that there is a set I of agents, with typical agents $i, j \in I$. Associated with each $i \in I$ is a set S_i, called i's *internal conceptions*, or sometimes just i's conceptions.

There is a collection[2], X, interpreted as the real-world objects. No one sees X; instead, each agent i observes S_i, but for $j \neq i$, agent i does not know S_j.

For each i, a relation $\Vdash_i$ ("can be perceived as") connects X with S_i. For $x \in X$ and $a \in S_i$, if

$$x \Vdash_i a,$$

then I say that agent i *can perceive* real-world object x as internal conception a.

Formally, the definition of perception is as follows:

[2]This is not necessarily a set, and generally not a set in the constructive sense; see the appendix for detais.

Definition 3.1. *An agent's* perceptions *are a triple* $\langle X, \Vdash_i, S_i \rangle$*, where* $\Vdash_i$ *is a binary relation between the real-world objects* X *and the agent's conceptions* S_i*.*

Example 3.1. *In the Demski-Sappington model,* X *corresponds to the collection of possible states of the world, while* S_i *is the set of net cash flow and signal histories* H_t *the firm can potentially observe. Their function* h_t *gives a unique history at each date* t*, so for fixed* t *the associated relation is the graph of* h_t*.*

The agent's perceptions are assumed to satisfy a completeness condition:

Assumption 3.1. *For each* $i \in I$ *and each* $x \in X$*, there is an* $a \in S_i$ *such that* $x \Vdash_i a$*.*

Assumption 3.1 requires the agent to have a coarse enough conception to cover every real-world object. This assumption is necessary for any non-trivial discussion of reporting. Intuitively, if the agent cannot perceive some $x \in X$ in any way—even as the most generic possible conception—then there is nothing for the agent to report.

Additionally, I require that perceptions satisfy the following consistency condition:

Assumption 3.2. *For each* $i \in I$*, each* $x \in X$*, and each* $a, b \in S_i$*, if both* a *and* b *are possible perceptions of* x*, then there is a conception* $c \in S_i$ *such that anything perceptible as* c *is perceptible both as* a *and as* b*. That is,*

$$(\forall x \in X)(\forall a, b \in S_i)((x \Vdash_i a \wedge x \Vdash_i b)$$

$$\rightarrow ((\exists c \in S_i)(\forall y \in X)(y \Vdash_i c \rightarrow y \Vdash_i a \wedge y \Vdash_i b)))$$

Assumption 3.2 requires that, if the agent has two conceptions that overlap, then the agent has a conception of their overlap.

3.3 Perceptions of Many Objects

The model so far discusses the perception of a single real-world object. However, the power of the model comes from its ability to describe agents who can perceive many things in the same way, and who can have many perceptions of the same object. To make this precise, I introduce some definitions.[3]

[3]For motivation of these definitions, see Sambin and Gebellato [43] and Sambin [42].

Definition 3.2. *Let $\langle Y, R, T \rangle$ be a triple, where Y and T are arbitrary collections and R is a binary relation between them. The correspondence r associated with R is given by, for every $y \in Y$,*

$$r(y) \equiv \{t \in T | yRt\}.$$

In particular, for agent i's perceptions, the symbol $\Vdash_i$ is used both for the relation, as above, and for the associated correspondence; which use is meant will be clear from context. So for $x \in X$, agent i has associated with x a collection of ways that i can perceive x:

$$\Vdash_i (x) \equiv \{a \in S_i | x \Vdash_i a\}.$$

Instead of fixing the left side of a relation, one can also fix the right side. In this way, one can define a correspondence from $\Vdash_i$ goes the opposite direction, from $S_i \longrightarrow X$. This correspondence is the inverse image of the agent's perceptions:

Definition 3.3. *Let $\langle Y, R, T \rangle$ be a triple, where R is a binary relation between Y and T. The correspondence r^- associated with the inverse image along R is given by, for every $t \in T$,*

$$r^-(t) \equiv \{y \in Y | yRt\}.$$

For agent i's perceptions, the symbol $\Vdash_i^-$ is used for this correspondence. Thus, given a conception $a \in S_i$, the objects that are perceptible as a are

$$\Vdash_i^- (a) \equiv \{x \in X | x \Vdash_i a\}.$$

The correspondences $\Vdash_i (\cdot)$ and $\Vdash_i^- (\cdot)$ describe how an individual object can be perceived, and what an individual conception can be the perception of. The following definitions extend these correspondences, in order to facilitate discussion of perceptions of arbitrary collections.

Definition 3.4. *Let $\langle Y, R, T \rangle$ be a triple, where R is a binary relation between Y and T and $r(\cdot)$ is the associated correspondence. Let $\widehat{Y} \subseteq Y$. The* strong *image of $\widehat{Y}$ along $r(\cdot)$ is*

$$\{t \in T | (\forall y \in Y)(t \in r(y) \rightarrow y \in \widehat{Y})\}.$$

The weak *image of $\widehat{Y}$ along $r(\cdot)$ is*

$$\{t \in T | (\exists y \in Y)(t \in r(y) \wedge y \in \widehat{Y})\}.$$

In the case of perceptions, the strong image of $D \subseteq X$ is denoted by $\Box D$, while the weak image is denoted by $\Diamond D$.[4]

The mappings from S_i to X are defined symmetrically. The strong image of $U \subseteq S_i$ along the inverse of $\Vdash_i$ is called the *restriction* of U, written rest U, while the weak image is called the *extent* of U, written ext U.

Spelling these out, we have, for $U \subseteq S_i, D \subseteq X$,

$$\Diamond D \equiv \{a \in S_i | (\exists x \in X)(x \Vdash_i a \wedge x \in D)\}$$

$$\text{ext } U \equiv \{x \in X | (\exists a \in S_i)(x \Vdash_i a \wedge a \in U)\}$$

for the weak images, and

$$\Box D \equiv \{a \in S_i | (\forall x \in X)(x \Vdash_i a \rightarrow x \in D)\}$$

$$\text{rest } U \equiv \{x \in X | (\forall a \in S_i)(x \Vdash_i a \rightarrow a \in U)\}$$

for the strong images. For singletons $a \in S_i$ and $x \in X$, we have

$$\text{ext } \{\text{a}\} = \Vdash_i^- (a)$$

and

$$\Diamond\{x\} = \Vdash_i (x).$$

Thus, there are four correspondences between X and S_i that characterize agent i's perceptions. To reiterate: for an arbitrary collection of real-world objects D, $\Diamond D$ gives the conceptions that can be perceptions of something in D, while $\Box D$ gives the conceptions that can *only* be perceptions of something in D. For a singleton $\{x\} \subset X$, $\Box\{x\}$ is empty unless x can be distinguished from every other real-world object. On the other hand, $\Diamond\{x\}$ is inhabited, due to Assumption 3.1.

Symmetrically, for an arbitrary collection of conceptions U, ext U gives the collection of real-world objects that can be perceived as something in U, while rest U gives the real-world objects that can *only* be perceived as something in U. For a singleton $\{a\} \subseteq S_i$, rest {a} is empty unless a is a perception of an object that can be distinguished from every other real-world object. It is possible that ext {a} is empty as well, since the agent may have a conception of something that does not actually exist. This can happen because i does not see X.

[4]The notation is by analogy with alethic modal logic, where $\Box$ is read as "necessarily" and $\Diamond$ is read as "possibly."

3.4 The Perceptual Topology

The correspondences described in the previous section arise from the agent's perceptions. In this section, I discuss the composition of these mappings, which give rise to operators from $X \longrightarrow X$.

These operators induce a topology on X, called the *perceptual topology*. Intuitively, two members of X are "close" to each other if they can be perceived the same way. Thus, the agent's set of conceptions S_i is viewed as a collection of neighborhoods over X, which forms the base of the topology induced by $\Vdash_i$.

To describe the perceptual topology, I first recall the usual definition of an open set:

Definition 3.5 (Open Sets). *Let $\langle Y, T \rangle$ be a topological space, where T is a base of the topology on Y. A subset $C \subseteq Y$ is* open *iff, for every $y \in Y$, if y is in C then y has a neighborhood $t \in T$ such that $t \subseteq C$.*

The members of S_i can be viewed as the names of neighborhoods in X.[5] In particular, each $a \in S_i$ is associaetd with

$$\Vdash_i^- (a) = \{x \in X | x \Vdash_i a\},$$

so all of S_i is associated with the neighborhood structure

$$\{\{x \in X | x \Vdash_i a\} | a \in S_i\} = \{\Vdash_i^- (a) | a \in S_i\}.$$

Under this viewpoint, the definition of an open set needs only slight modification:

Definition 3.6 (Open Sets in the Perceptual Topology). *For $i \in I$, let $\langle X, \Vdash_i, S_i \rangle$ be agent i's perceptions. Let $D \subseteq X$. Then D is* open in the perceptual topology *iff, for every $x \in X$, if $x \in D$ then there is some $a \in S$ such that $a \in \Vdash_i (x)$ and $\Vdash_i^- (a) \subseteq D$.*

Thus, $D \subseteq X$ is open in the perceptual topology if every point in D can be perceived as a conception that can only be the perception of something in D. Formally, this says that the *interior* of $D \subseteq X$ is

$$\text{int } D \equiv \{x \in X | (\exists a \in S_i)(x \Vdash_i a \wedge (\forall y \in X)(y \Vdash_i a \rightarrow y \in D))\}.$$

In the topological reading, int D is the collection of points in D that have an open neighborhood contained in D. So this condition matches the definition.

[5] I first came across this interpretation in Valentini [51].

Observe that the last part of thedefinition,

$$(\forall y \in X)(y \Vdash_i a \to y \in D),$$

has already been discussed in the last section. Recall that

$$\Box D \equiv \{a \in S_i | (\forall y \in X)(y \Vdash_i a \to y \in D)\}.$$

So the last part of the definition of the interior of D just says that $a \in \Box D$. That shortens the definition of the interior of D to

$$\text{int } D \equiv \{x \in X | (\exists a \in S_i)(x \Vdash_i a \wedge a \in \Box D)\}.$$

Also, recall that

$$\text{ext } U \equiv \{x \in X | (\exists a \in S_i)(x \Vdash_i a \wedge a \in U)\}.$$

Letting $U = \Box D$ gives

$$\text{int } D \equiv \text{ext}\Box D.$$

That is, the interior operator on X is just

$$\text{int} \equiv \text{ext}\Box.$$

Definition 3.7 (Perceptual Interior). *Let $\langle X, \Vdash_i, S_i \rangle$ be agent i's perceptions. The* perceptual interior *operator on X is the composition*

$$\text{int} \equiv \text{ext}\Box$$

A closure operator is dual to an interior operator. Intuitively, replacing strong and weak images throughout the definition of an interior operator should define a closure operator. This suggest the following definition:

Definition 3.8 (Perceptual Closure). *Let $\langle X, \Vdash_i, S_i \rangle$ be agent i's perceptions. The* perceptual closure *operator on X is the composition*

$$\text{cl} \equiv \text{rest}\Diamond$$

In fact, this definition is correct. In general, the closure of a set in a topological space is the collection of points for which every open neighborhood has positive intersection with the set. For $D \subseteq X$, the closure of D is

$$\text{cl } D \equiv \{x \in X | (\forall a \in S_i)(x \Vdash_i a \to (\exists y \in X)(y \Vdash_i a \wedge y \in D))\}.$$

A real-world object x is in the perceptual closure of D iff it is perceptually inseparable from D, in the sense that every possible perception of x can be the perception of some member of D. This is related to the notion of nearness presented in Vîţă and Bridges [55].

The second part of the above condition says

$$(\exists y \in X)(y \Vdash_i a \wedge y \in D),$$

which is the definition of $a \in \Diamond D$. So this shortens the condition to

$$\text{cl } D \equiv \{x \in X | (\forall a \in S_i)(x \Vdash_i a \rightarrow a \in \Diamond D)\}.$$

Letting $U = \Diamond D$ and recalling the definition of restriction makes this

$$\text{cl } D \equiv \text{rest}\Diamond D$$

as anticipated.

It should be noted that not everything in int D is *necessarily* perceived as something in D. The definition only stipulates that anything in the perceptual interior of D *can* be perceived as something in D, but there still be conceptions in S_i that can be the perceptions of something in D as well as of something outside of D. Thus there is use for a stronger notion of interior, called the *stable perceptual interior* of $D \subseteq X$:

Definition 3.9 (Stable Perceptual Interior). *Let $\langle X, \Vdash_i, S_i \rangle$ be agent i's perceptions. Let $D \subseteq X$. The* stable perceptual interior *of D is given by*

$$\text{sint } D \equiv \text{rest}\Box D.$$

If a set D is equal to its interior, I will call it *open*. If it is equal to its stable interior, I will call it *stable open*. The intuition is that the stable interior of D is stable to the possible ways it can be perceived, while the interior may not be.[6]

Similarly, the perceptual closure of $D \subseteq X$ contains the real-world objects that can necessarily be perceived as something possibly in D. There is use here for a weaker notion, specifically, those objects that can possibly be perceived as something possibly in D.

Definition 3.10 (Perceptual Direct Connection). *Let $\langle X, \Vdash_i, S_i \rangle$ be agent i's perceptions. Let $D \subseteq X$. The* perceptual direct connection *of D is given by*

$$\text{con } D \equiv \text{ext}\Diamond D.$$

[6] The terminology is based on Prawitz's notion of stable logic; cf. [40].

Intuitively, the points in con D are connected to D because they share a conception with some conception that meets D.

This idea can be iterated: there are points that are directly connected to con D that are not connected to D. The following definition formalizes this notion:

Definition 3.11. *Let $\langle X, \Vdash_i, S_i\rangle$ be agent i's perceptions. Let $D \subseteq X$. Let $n \in \mathbb{N}$. The set of real-world objects* perceptually n-chain connected *to D is given by*

$$\mathrm{con}^n\ D,$$

i.e., by the n-fold composition of con *on D.*

It is possible for X not to be connected, in the sense that there may be $D, E \subseteq X$ such that $D = \mathrm{con}\ D$, $E = \mathrm{con}\ E$, and any perception of anything in D is necessarily not in E. I.e.,

$$(\forall a \in S)(a \in \Diamond D \to \neg(a \in \Diamond E)).$$

In such a case, nothing perceptually connected to D is perceptually connected to E. That is:

Definition 3.12 (Tolerance Class). *Let $\langle X, \Vdash_i, S_i\rangle$ be agent i's perceptions. Let $n \in \mathbb{N}$. Let $D \subseteq X$. Suppose* $\mathrm{con}^n D$ *is a fixed point of the operator* con*, i.e.,*

$$\mathrm{con}^n D = \mathrm{con}^{n+1} D.$$

Then D is the tolerance class *of D, written* scl D*.*

The tolerance class is thus the set of points that are eventually n-chain connected to D. If the tolerance class of every $E \subseteq D$ is $D \subseteq X$, then D is *perceptually connected*. This definition matches Schreider [45].

Two extreme cases are as follow:

Example 3.2 (Arrow-Debreu). *Suppose that*

$$(\forall x \in X)(\exists a \in S_i)(a \in \Box\{x\}),$$

i.e., every real-world object has a unique way of being perceived. Then every singleton $\{x\} = \mathrm{int}\ \{x\}$; i.e., the induced topology is the discrete *topology.*

Even in this case, the agent cannot necessarily perceive differences between every $x, y \in X$; it is only possible that such differences will be perceived. If, however, every conception can be perceived in at most one way, i.e., if

$$(\forall a \in S_i)(\forall x, y \in X)(x \Vdash_i a \wedge y \Vdash_i a \to x = y),$$

then every singleton is equal to its stable perceptual interior. If X is interpreted as the commodity space, this stable discrete topology is the Arrow-Debreu world.

Example 3.3 (Vickrey Weak Transparency). *Suppose that S_i is a singleton. Then, by Assumption 3.1, for every $x \in X$, $\Diamond\{x\} = S_i$, giving* $\text{cl}\,\{x\} = X$. *Also, for any $D \subsetneq X$, $\Box D = \varnothing$, so the only open set is X. I.e., the induced topology is the* indiscrete *topology. This is Vickrey's case of weak transparency.*

The finest perceptual topology would thus seem to be the Arrow-Debreu world, i.e., the stable discrete topology, where everything is necessarily perceived with infinite precision. The coarsest topology seems to be the Vickrey Weak Transparency case, where everything is perceived with zero precision.

In addition to being useful for defining the Arrow-Debreu world, the stable interior has some desirable properties, in particular when discussing complementation. In general, for $D \subsetneq X$, there may be some $x \in D$, $y \notin D$, and $a \in S_i$ with x and y both in $\Vdash^{-}(a)$. That is, it is not in general possible for an agent to determine whether a boundary point is inside or outside of a given set.

This may seem to have disastrous consequences—e.g., it is hard to see how agents who cannot find the boundary of their budget sets can consume on their budget lines—but the following definition and proposition limit the impact.

Definition 3.13. *For $D \subseteq X$, the* complement *of D is*

$$-D \equiv \{x \in X | \neg(x \in D)\}.$$

Similarly, for $U \subseteq S_i$, the complement of U is

$$-U \equiv \{a \in S_i | \neg(a \in U)\}.$$

Proposition 3.1 (Separation). *Let $D \subseteq X$. Fix a perceptual topology on X. Then*

$$\text{sint}\; -D = -\text{con}\; D.$$

I.e., the stable interior of the complement of D is the complement of the perceptual direct connection of D.

Proof. By definition,

$$\text{sint}\; -D \equiv \{x \in X | (\forall a \in S_i)(x \Vdash_i a \rightarrow (\forall y \in X)(y \Vdash_i a \rightarrow \neg y \in D))\}$$

$$= \{x \in X | (\forall a \in S_i)(x \Vdash_i a \to \neg(\exists y \in X)(y \Vdash_i a \wedge y \in D))\}$$
$$= \{x \in X | (\forall a \in S_i)(x \Vdash_i a \to \neg(a \in \Diamond D))\}$$
$$= \{x \in X | \neg(\exists a \in S_i)(x \Vdash_i a \to a \in \Diamond D)\}$$
$$= \{x \in X | \neg(x \in \text{ext}\Diamond D)\} = -\text{ext}\Diamond D = -\text{con } D.$$

□

The above definitions and results show that an agent's perceptions $\langle X, \Vdash_i, S_i \rangle$ can be used to define interior and closure operators on X which match the usual definitions. To show that $\langle X, \Vdash_i, S_i \rangle$ does in fact induce a topology, it remains to show the following:

1. The collections $\varnothing$ and X are open in the perceptual topology.
2. For any family D_α of open sets, $\bigcup_\alpha D_\alpha$ is open.
3. For any open sets D_1, D_2, $D_1 \cap D_2$ is open.

The following theorem says that these conditions hold.

Theorem 3.1 (Perceptual Topology). *Let $i \in I$, and let $\langle X, \Vdash_i, S_i \rangle$ be i's perceptions. For any $D \subseteq X$, call D* open *iff $D = \text{int } D$. Then the collection of open sets forms a topology; i.e., the perceptual topology meets the definition of a topology.*

The proof is through a sequence of lemmata.

Lemma 3.1. *Under the hypothesis of Theorem 3.1, $\varnothing$ and X are open.*

Proof. By definition, int $\varnothing = \text{ext}\Box\varnothing$. Since $\Box\varnothing = \{a \in S_i | (\forall x \in X)(x \Vdash_i a \to x \in \varnothing\}$, it follows that $\text{ext}\Box\varnothing = \varnothing$.

Similarly, the interior of X is ext $\Box X$, and $\Box X = \{a \in S_i | (\forall x \in X)(x \Vdash_i a \to x \in X)\}$. By Assumption 3.1, $\text{ext}\Box X = X$. □

Lemma 3.2. *Under the hypothesis of Theorem 3.1, the union of opens in the perceptual topology is open.*

Proof. Let $D = \bigcup_\alpha D_\alpha$, where each D_α is open. If $x \in D$, then $x \in D_\alpha$ for some α, so there is $a \in S_i$ with $x \Vdash_i a$, and every $y \in X$ with $y \Vdash_i a$ belongs to D_α, hence to D. □

Lemma 3.3. *Under the hypothesis of Theorem 3.1, the intersection of two opens is open.*

Proof. Let D, E be open. For $x \in D$, there are $a, b \in S_i$ such that every $y \in X$ perceptible as a belongs to D and every $z \in X$ perceptible as b belongs to E, and $\{a, b\} \subseteq \Vdash_i (x)$. By Assumption 3.2, there is $c \in S_i$ such that anything perceptible as c is perceptible as both a and b. Then every $y \in X$ with $y \Vdash_i c$ is in $D \bigcap E$, so every member of $D \bigcap E$ has a neighborhood contained in $D \bigcap E$. □

4 Reporting

4.1 Overview of the Reporting System

In this model, no agent sees the real-world; thus, even two agents with the identical perceptions[7] could not verify that they had the same understanding of the world. That is, since no one knows anyone else's conceptions, there is no way to make shared conceptions common knowledge.

A shared language is introduced as a way around this difficulty.[8] Intuitively, a shared language is a common vocabulary and syntax and a method for each agent to describe subjective conceptions in the common vocabulary. If agents can phrase things in accepted terms, trade becomes possible.

Reports go from the agent's conceptions to the common language. As with perceptions, there are gray areas in reports: this is because the terms in the reporting language cannot be guaranteed to be isomorphic to the agent's conceptions. Thus, the map from the agent's conceptions to the common language is not in general a function. Instead, there is a relation between the agent's conceptions and the common language.

Definition 4.1. *A* shared language *is a collection of triples $\langle S_i, R_i, T\rangle$, for each $i \in I$, where S_i is i's set of conceptions, T is the common vocabulary, and R_i is a binary relation between S_i and T, called the* reporting relation.

For a fixed agent $i \in I$, $\langle S_i, R_i, T\rangle$ is referred to as i's *reporting system* The relation R_i is read as "can be reported by i as." If $a \in S_i$ and $t \in T$, then aR_it means that i can report a as t. Following the pattern in

[7] *Identical* here is in meant in the sense of being isomorphic.

[8] The development and evolution of the language system are beyond the scope of this paper. These issues are treated extensively in the computer science literature—see especially Ahn [2, 3], Ahn and Borghuis [4], and Linder et al. [52]. In the psychology literature, Heit and Barsalou [24] discuss the development of what here is called the reporting rule; that is, they discuss how agents associate examples of terms in a common language with their private conceptions. Instead, the focus here is restricted to characterizing an exogenously given language.

the section on perceptions, I define two correspondences that are associated with the reporting relation. These are given by, for $a \in S_i$ and $t \in T$,

$$R_i(a) \equiv \{t \in T | aR_it\}$$

and

$$R_i^-(a) \equiv \{a \in S_i | aR_it\}.$$

If R_i^- is interpreted as a relation (defined by $aR_i^-t \equiv tR_ia$ for $a \in S_i$ and $t \in T$), then $\langle T, R_i^-, S_i \rangle$ is a reporting system, called i's *report interpretions*.

The shared language thus works exactly like the individual's perceptions. Moreover, the range of the agent's perceptual relation $\Vdash_i$ is the domain of the agent's reporting relation R_i. Linking real-world objects to the reporting system is then an exercise in composition. That is, agent i can report real-world object $x \in X$ as some term $t \in T$ iff

$$t \in R_i(\Vdash_i (x)). \tag{4}$$

Equivalently, i can report x as t iff

$$(\exists a \in S_i)(x \Vdash_i a \wedge aR_it). \tag{5}$$

If this condition holds, then the report t and the real-world object x are said to *meet* in S_i at a. This is denoted as

$$\Diamond x \between R_i^-(t).$$

Here $\Diamond x$ gives the ways the agent can perceive x, while $R_i^-(t)$ gives the conceptions i can report as t, or, equivalently, the interpretations that i assigns to t. So x is reportable as t if the agent's interpretation of t is consistent with a way the agent can perceive x.

4.2 The Reporting Topology

This section makes use of the symmetry of i's perceptions with i's reporting system. Just as i's perceptions induce a topology on i's conceptions, the reporting system induces a topology on i's conceptions.

Specifically, let $\langle S_i, R_i, T \rangle$ be i's reporting system. Let $W \subseteq T$ be a subset of the common vocabulary. From Definition 3.4, the weak image of W along R_i^- is

$$r_i^-(W) \equiv \{a \in S_i | (\exists t \in T)(aR_it \wedge t \in W)\},$$

and the strong image of W along R_i^- is

$$r_i^*(W) \equiv \{a \in S_i | (\forall t \in T)(aR_it \to t \in W)\}.$$

Here r_i^- is completely analogous to ext, and r_i^* is completely analogous to rest.

Conversely, let $U \subseteq S_i$. The weak image of U along R_i is then

$$r_i(U) \equiv \{t \in T | (\exists a \in S_i)(aR_it \wedge a \in U)\},$$

and the strong image of U along R_i is

$$r_i^{-*}(U) \equiv \{t \in T | (\forall a \in S_i)(aR_it \to a \in U)\}.$$

It is clear that r_i and r_i^{-*} are analogous to $\Diamond$ and $\Box$.

The following assumption guarantee that the shared language allows the agent to make reports:

Assumption 4.1. *Let $i \in I$, and let $\langle S_i, R_i, T\rangle$ be i's trade reporting system. For each $a \in S_i$, there is a $t \in T$ such that aR_it.*

Assumption 4.1 is a straightforward translation of Assumption 3.1 into the reporting setting.

An analogue of Assumption 3.2 states that the reporting language is consistent:

Assumption 4.2. *Let $i \in I$, and let $\langle S_i, R_i, T\rangle$ be i's reporting system. For each $a \in S_i$ and each $t, u \in T$, if i can report a as either t or u, then there is some $v \in T$ such that whatever i can report as v can be reported as t or u. That is,*

$$(\forall a \in S_i)(\forall t, u \in T)((aR_it \wedge aR_iu) \to$$
$$((\exists v \in T)(\forall b \in S_i)(bR_iv \to bR_it \wedge bR_iu))).$$

Thus, if i has two ways of reporting the same thing, then i can report something like "this looks like both t and u." Assumption 4.2 evidently means that the language is rich enough to support conjunction, and that i's reports are consistent with conjunction.

From here, the definitions are completely analogous to those on the perceptual topology. Formally,

Definition 4.2. *Let $i \in I$. Let $\langle S_i, R_i, T\rangle$ be i's trade reporting system. Let $U \subseteq S_i$. Let $W \subseteq T$. Then*

1. *The* reporting interior *of* U *is*

$$\mathrm{int}_T(U) \equiv r^-(r^{-*}(U)) \equiv \{a \in S_i | (\exists t \in T)(aR_i t \wedge (\forall b \in S_i)(bR_i t \to b \in U))\}$$

2. *The* reporting closure *of* U *is*

$$\mathrm{cl}_T(U) \equiv r^*(r(U)) \equiv \{a \in S_i | (\forall t \in T)(aR_i t \to (\exists b \in S_i)(bR_i t \wedge b \in U))\}$$

3. *The* reporting stable interior *of* U *is*

$$\mathrm{sint}_T(U) \equiv r^*(r^{-*}(U)) \equiv \{a \in S_i | (\forall t \in T)(aR_i t \to (\forall b \in S_i)(bR_i t \to b \in U))\}$$

4. *The* reporting direct connection *of* U *is*

$$\mathrm{con}_T(U) \equiv r^-(r(U)) \equiv \{a \in S_i | (\exists t \in T)(aR_i t \wedge (\exists b \in S_i)(bR_i t \wedge b \in U))\}$$

With these definitions and assumptions, I have the following theorem.

Theorem 4.1. *Let $i \in I$, and let $\langle S_i, R_i, T\rangle$ be i's reporting system. For any $U \subseteq S_i$, call U open iff $U = \mathrm{int}_T(U)$. Then the collection of open sets forms a topology.*

The proof is analogous to the proof with the perceptual topology.

5 Heterogeneous Faithfulness

5.1 Overview

Recall Diagram 2 from the discussion of the Demski-Sappington model presents fully revealing disclosure as a commutative square:

$$\begin{array}{ccc} S & \xrightarrow{\eta_t} & Y \\ {\scriptstyle h_t}\downarrow & & \downarrow{\scriptstyle \iota^-} \\ H_t & \xrightarrow{r} & T \end{array}$$

In their setting, the maps h_t, η_t, and r are all functions. The square is fully revealing if ι^- has an inverse ι, giving a unique interpretation to the market in Y of the firm's report in T. In other words, the reporting system is fully revealing for Demski and Sappington iff $r \circ h_t = \iota^- \circ \eta_t$. As discussed above, unless the model is weakly transparent in Vickrey's sense, fully-revealing disclosure requires faithfulness.

Closely related is the model of Kirschenheiter, where Diagram 3 is essentially a relabeling of the vertices in Diagram 2. Signals in Kirschenheiter's model go from the "true signals" X (analogous to the real-world objects here) to the partitions each individual has of X. This mapping for Kirschenheiter is a function, namely, a true signal x is perceived as the unique set it belongs to in $S_i \in \mathfrak{P}(X)$. Additionally, each agent in Kirschenheiter's model has a function that assigns members of the agent's information partition to reports in the shared language T. Information is hard for Kirschenheiter if the relabeled diagram commutes. That is, the fully transparent disclosure of Demski-Sappington is a special case of Kirschenheiter's information hardness.

The current model replaces the above functions with relations, and with their associated correspondences. Accordingly, the notion of faithfulness has to be modified: the reporting system cannot in general guarantee that the same real-world object will always be reported the same way. For a given real-world object x, there will be a collection $\Vdash_i (x)$ of possible ways an individual can perceive x, and for each of these, there will be a collection $r_i \circ \Vdash_i (x)$ of possible reports. The definition of faithfulness is hence as follows:

Definition 5.1. *Let $i, j \in I$. Let $\langle X, \Vdash_i, S_i \rangle$, $\langle X, \Vdash_j, S_j \rangle$ be i's and j's perceptions. Let $\langle S_i, R_i, T \rangle$, $\langle S_j, R_j, T \rangle$ be i's and j's reporting systems. Then the shared language is* heterogeneously faithful between i and j *iff the following diagram commutes:*

$$\begin{array}{ccc} X & \xrightarrow{\Vdash_i} & S_i \\ {\scriptstyle \Vdash_j}\downarrow & & \downarrow{\scriptstyle r_i} \\ S_j & \xrightarrow{r_j} & T \end{array} \tag{6}$$

That is, the shared language is heterogeneously faithful between i and j iff

$$r_i \circ \Vdash_i = r_j \circ \Vdash_j .$$

If the trade reporting system is heterogeneously faithful between every $i, j \in I$, then the shared language is said to be heterogeneously faithful.

The definition of heterogeneous faithfulness is symmetric, in the following sense:

Proposition 5.1. *Let $i, j \in I$. Let $\langle X, \Vdash_i, S_i \rangle$, $\langle X, \Vdash_j, S_j \rangle$ be i's and j's perceptions. Let $\langle S_i, R_i, T \rangle$, $\langle S_j, R_j, T \rangle$ be i's and j's reporting systems. Suppose the trade reporting system is heterogeneously faithful between i and j, i.e., $r_i \circ \Vdash_i = r_j \circ \Vdash_j$. Then the interpretation of the reports is also heterogeneously faithful, i.e., $\Vdash_i^- \circ r_i^- = \Vdash_j^- \circ r_j^-$.*

Intuitively, the above proposition says that if reporting is heterogeneously faithful, then so is the interpretation. Hence heterogeneous faithfulness does not require taking the sides of end-users or of reporting entities—what is meaningful for one is meaningful for the other.

Proof. The heterogeneous faithfulness between i and j means that, given $x \in X$ and $t \in T$, i can report x as t iff j can report x as t. By definition, i can report x as t iff

$$(\exists a \in S_i)(x \Vdash_i a \wedge a R_i t),$$

which means that

$$(\exists a \in S_i)(a \in r_i^-(t) \wedge x \in \Vdash_i^-(a)),$$

i.e., x is reportable as t iff t is interpretable as x. By an identical argument, if j can report x as t, then j can interpret t as x. So commutativity of the square in Equation 5.1 holds iff the following square commutes:

$$\begin{array}{ccc} X & \xleftarrow{\Vdash_i^-} & S_i \\ {\scriptstyle \Vdash_j^-}\uparrow & & \uparrow{\scriptstyle r_i^-} \\ S_j & \xleftarrow{r_j^-} & T \end{array}$$

□

The symmetry in the above proposition does not hold in the Demski-Sappington model. This is because a relation can always have an inverse relation defined, whereas the Demski-Sappington model works with functions, which are not necessarily invertible. However, heterogeneous faithfulness and Demski-Sappington fully revealing disclosure agree in the following sense:

Proposition 5.2. *Suppose that the income disclosure in the Demski-Sappington model is fully revealing. Then the reporting system is heterogeneously faithful between the firm and the market.*

Sketch. From Diagram 1, the disclosure is heterogeneously faithful iff there is an interpretation function ι from the report T to the market's possible signals (i.e., to the market's set of conceptions) Y such that, for every $s \in S$,

$$\eta_t(s) = \iota \circ r \circ h_t(s),$$

where $r \equiv \langle x_t, I_t \rangle$ is the cash flow-income disclosure pair. Let ι^- be the inverse (not necessarily a function) of ι. Then $\iota^-(y)$ is the set of reports $t \in T$ that the market can interpret as y. Since the disclosure by hypothesis is fully revealing, any $t \in \iota^-(y)$ is uniquely interpreted by the market as y. This means that $\iota^- \circ \eta_t = r \circ h_t$, which is the definition of heterogeneous faithfulness. □

A converse will not hold, as the following example illustrates. Suppose the collection of real-world objects is $\{0, 1, 2\}$, and that $S_i = S_j = X$. Suppose the reporting language is $T = \{0, 1\}$, and each individual reports according to $r_i(0) = r_j(0) = 0$, $r_i(1) = r_i(2) = r_j(1) = r_j(2) = 1$. Each individual's perceptual map is the identity function. The reporting system is heterogeneously faithful: i can report 0 iff j can report 0 (when the real-world object is 0), and i and j can both report 1 otherwise. However, a report of 1 is not invertible as a function: if either individual sees a report of 1, both 1 and 2 are possible interpretations.

Note that a trivial counter-example is also possible: just take $T = \varnothing$. Then every report that i can make has the same meaning as the same report by j, since there are no possible reports. However, the reporting system is not invertible in the signal, unless the set of possible signals is also degenerate.[9]

The reason that heterogeneous faithfulness is weaker than fully revealing disclosure is that the Demski-Sappington model requires unique identification, because all of their maps are functions. When perception is imperfect and conceptions are not common knowledge, the best we can hope for is consistency.

5.2 Remarks on Fullness

The fullness of the Demski-Sappington model captures the followign general sense of completeness:

C1 Every distinction a reporting entity can make has a distinct report in the shared langauge.

[9]Kirschenheiter gives a similar counter-example.

C2 Every distinction an end-user can make has a distinct report in the shared language.

That is, the reporting system is complete both for the senders and for the users of the reports.

To see that completeness holds in these senses, recall that the firm sees a cash flow and a signal. If the disclosure is fully revealing, then the report includes the cash flow and an income calculation that together suffice to inform the market of the signal. Hence, given a cash flow, the firm needs to be able to report a unique income calculations for each signal. This just says that different cash flow and signal pairs generate distinct reports; i.e., Demski-Sappington fully revealing disclosure implies **C1**. Moreover, since fully revealing disclosure identifies the signal, a fully revealing system must be complete in the sense of **C2** as well.

Completeness for the reporting entitiy also holds in Kirschenheiter's information hardness notion: **C1** follows from the fact that every mapping in Kirschenheiter's model is a total function. On the other hand, **C2** does not follow from information hardness, as is seen from considering the case where the common vocabulary is a singleton.

To understand completeness better, consider the following examples, based on Shin's model (cf. [46]):

Example 5.1 (Shin's model, N=1). *In Shin's model, suppose*

$$X = \{(0,0),(1,0),(1,1)\} = S_1 = S_2$$
$$\Vdash_1 ((0,0)) = \{(0,0)\}$$
$$\Vdash_1 ((1,0)) = \Vdash_2 ((1,1)) = \{(1,0),(1,1)\}$$
$$\Vdash_2 ((0,0)) = \Vdash_2 ((1,0)) = \{\{(0,0),(1,0)\}$$
$$\Vdash_2 ((1,1)) = \{(1,1)\}$$

*If the shaerd language is complete in the sense of **C1**, then the smallest common vocabulary is*

$$\bigcup_{x\in X} \Vdash_1 (x))\bigcup(\bigcup_{x\in X} \Vdash_2 (x))$$
$$= \{\{(0,0)\},\{(1,0),(1,1)\},\{(0,0),(1,0)\},\{(1,1)\}\}.$$

*If the shared language is complete in the sense of **C2**, then the smallest common vocabulary is*

$$(\bigcup_{x\in X} \Vdash_1 (x)) \times (\bigcup_{x\in X} \Vdash_2 (x)).$$

Any set with four elements can work equally well in the previous example. When X is slightly larger, completeness is no longer symmetrc:

Example 5.2 (Shin's model, N=2). *Suppose now that*

$$X = \{(0,0),(1,0),(1,1),(2,1),(2,2)\} = S_1 = S_2$$

$$\Vdash_1 ((0,0)) = \{(0,0)\}$$
$$\Vdash_1 ((1,0)) = \Vdash_2 ((1,1)) = \{(1,0),(1,1)\}$$
$$\Vdash_1 ((2,1)) = \Vdash_1 ((2,2)) = \{(2,1),(2,2)\}$$
$$\Vdash_2 ((0,0)) = \Vdash_2 ((1,0)) = \{\{(0,0),(1,0)\}$$
$$\Vdash_2 ((1,1)) = \Vdash_2 ((2,1) = \{(1,1),(2,1)\}$$
$$\Vdash_2 ((2,2)) = \{(2,2)\}$$

*If the shared language is complete in the sense of **C1**, then the smallest common vocabulary is*

$$(\bigcup_{x\in X} \Vdash_1 (x)) \bigcup (\bigcup_{x\in X} \Vdash_2 (x))$$
$$= \{\{(0,0)\},\{(1,0),(1,1)\},\{(0,0),(1,0)\},\{(1,1),(2,1)\},\{(2,2)\}\}.$$

*If the shared language is complete in the sense of **C2**, then the smallest common vocabulary is*

$$(\bigcup_{x\in X} \Vdash_1 (x)) \times (\bigcup_{x\in X} \Vdash_2 (x)).$$

The smallest set that makes the above model complete for the reporting entities has five elements, whereas nine elements are necessary to be complete for the end-user.

In the above examples, there were no cases where $\Vdash_1 (x) = \Vdash_2 (x)$. Since conceptions are private, even if two individuals have the same conceptions, the private, subjective nature of the S_i prevents them from verifying that their conceptions perfectly overlap. Accordingly, completeness in the sense of **C1** can require some repitition. This motivates the following:

Definition 5.2. *For each $i \in I$, let $\langle X, \Vdash_i, S_i\rangle$ be the perceptions of agent i. Let $\langle S_i, r_i, T\rangle$, be i's reporting system. The shared language is* complete for reporting entities *iff there is an injection from T to the disjoint union*

$$\sum_{i\in I}(\bigcup_{x\in X} \Vdash_i (x)) \equiv \{(i, \Vdash_i (x)) | i \in I, x \in X\},$$

and for each $t \in T$, for every $i, j \in I$, and for every $a \in S_i$ and $b \in S_j$,

$$aR_it \to \neg(bR_it)$$

The shared language is complete for the end users *iff there is an injection from T to the Cartesian product*

$$\prod_{i \in I}(\bigcup_{x \in X} \Vdash_i (x)),$$

and if, for each $i \in I$, $r_i^-(\cdot)$ is a projection map onto $(\bigcup_{x \in X} \Vdash_i (x))$.

The above definition shows in particular that completeness is not symmetric, as the disjoint union is not in general isomorphic to the Cartesian product. Since heterogeneous faithfulness is symmetric by Proposition 5.1, the following holds:

Proposition 5.3. *If the shared language is complete for the reporting entities, then it is not necessarily heterogeneously faithful.*

Proof. (Sketch) Follows directly from the definition of disjoint union. □

5.3 Heterogeneous Faithfulness as Continuity

As described in Section 3.4 and Section 4.2, the agent's perceptions $\langle X, \Vdash_i, S_i\rangle$ can be viewed as a topological space, with S_i forming a base of the topology induced by $\Vdash_i$ on X. By an entirely analogous argument, the agent's reporting system $\langle S_i, R_i, T\rangle$ is a topological space, with T a base of a topology induced by R_i on S_i.

Recall that the definition of heterogeneous faithfulness given in Section 5.1 says that the following diagram commutes:

$$\begin{array}{ccc} X & \xrightarrow{\Vdash_i} & S_i \\ {\scriptstyle \Vdash_j}\downarrow & & \downarrow{\scriptstyle r_i} \\ S_j & \xrightarrow{r_j} & T \end{array}$$

In other words, the map $\Vdash_j$ takes a topological space $\langle X, S_i\rangle$ (with the perceptual topology of i) to a topological space $\langle S_j, T\rangle$ (with the reporting topology. The map r_i is a relation between neighborhoods of these topological spaces. Commutativity of the diagram then says that $\Vdash_j$ takes open sets to open sets. Moreover, since relations always have inverse relations, commutativity also says that $\Vdash_j$ is a map whose inverse image takes open sets to open sets. In the special case where the correspondences are single-valued, the following holds:

Theorem 5.1 (Hardness Implies Continuity). *If information in a shared language is hard between two agents i and j, then $\Vdash_j$ is a continuous map from X with i's perceptual topology to S_j with the reporting topology.*

Proof. In the information hardness model, for every $x \in X$, there is a unique $a \in S_i$ such that $x \Vdash_i a$, because $S_i \in \mathfrak{P}(X)$. In other words, every $x \in X$ has a unique open neighborhood in i's perceptual topology.

Since $S_j \in \mathfrak{P}(X)$, there is a unique $b \in S_j$ such that $\Vdash_j (x) = b$. In other words, $\Vdash_j (\cdot)$ is a function. Reports are also single-valued, so there is exactly one $t \in T$ such that bR_jt. That means that there is a unique neighborhod of $\Vdash_j (x)$ in j's reporting topology.

By the hardness of information, j can report x as t iff i can report x as t. Thus, $a \in \{\widehat{a} \in S_i | \widehat{a} R_i t\}$, which means that $a \in r_i^-(t)$. Hence, every $x \in X$, the inverse image of every open neighborhood of $\Vdash_j (x)$ in j's reporting topology contains an open neighborhood of x in i's perceptual topology. Therefore, $\Vdash_j (\cdot)$ is continuous. □

The converse requires $\Vdash_i (\cdot)$ and $r_i(\cdot)$ to be single-valued for every $i \in I$. That is, information hardness does not apply to settings where perceptions can overlap.

When $\Vdash_i$ and R_i are relations, there are many possible definitions of continuity. For example, an order relation may be required to have closed upper and lower countour sets, as in Debreu [16]. Alternatively, one may consider a relation continuos if the associated correspondence is upper or lower hemicontinuous.

To preserve the intuition of continuous functions, I define continuity as follows: $X \xrightarrow{r} Y$ is continuous iff r carries open subsets of X to open subsets of Y. It is shown in Gebellato and Sambin [21] that this definition of continuity is necessary in order to guarantee that the composition of continuous relations is continuous. Formally:

Definition 5.3. *Let $\langle X, S\rangle$, $\langle Y, T\rangle$ be topological spaces, where S is a base of the topology on X and T is a base of the topology on Y. A binary relation R between X and Y is* continuous *iff for every $x \in X$, if $t \in T$ is an open neighborhood of some $y \in Y$ such that xRy, then there is an open neighborhood a of x such that $a \subseteq r^-(t)$.*

From this definition, it is almost immediate to conclude the following:

Theorem 5.2 (Transparency Is Continuity). *Let $i, j \in I$ be two agents. Let $\langle X, \Vdash_i, S_i\rangle$, $\langle X, \Vdash_j, S_j\rangle$ be i's and j's perceptions. Let $\langle S_i, R_i, T\rangle$, $\langle S_j, R_j, T\rangle$*

be i's and j's reporting systems. If the shared language is heterogeneously faithful between i and j, then $\Vdash_i$ is a continuous relation from X with i's perceptual topology to S_j, with j's reporting topology.

Conversely, if $\Vdash_j$ is a continuous relation between X with i's perceptual topology and S_j with j's reporting topology, then there exists a reporting relation for i such that the shared language is heterogeneously faithful between i and j.

Proof. The proof is essentially a comparison of definitions. Heterogeneous faitufulness means that $r_j \circ \Vdash_j = r_i \circ \Vdash_i$. The map $r_j(\cdot)$ carries members of S_j to their neighborhoods in j's reporting topology. The map $\Vdash_i$ carries members of X to their neighborhoods in i's perceptual topology. Therefore, $r_i(\cdot)$ carries open neighborhoods of any $x \in X$ in i's perceptual topology to open neighborhoods of $\Vdash_j (x)$ in j's reporting topology. That is, $\Vdash_j$ is a continuous relation.

Conversely, let $\Vdash_j$ be a continuous relation. Let r_i be the correspondence from S_i to T defined by

$$r_i(a) \equiv r_j \circ \Vdash_j \circ \Vdash_i^- (a).$$

Then r_i carries open neighborhoods of any $x \in X$ to open neighborhoods of $\Vdash_j (x)$; i.e., the resulting shared language is heterogeneously faithful. □

Heterogeneous faithfulness and continuity are therefore two ways of asserting the same thing. The reason that hardness is a special case of heterogeneous faithfulness is that the maps $\Vdash_i (\cdot)$ and $r_i(\cdot)$ must be single-valued for every agent in order for information to be hard. Viewing heterogoeneous faithfulness as continuity, this says that the perceptual and reporting topologies all need to be degenerate in the sense that every point has exactly one open neighborhood in the base of the topology.

5.4 Example

To illustrate the connection between perceptions and reporting, this section revisits the example from Shin [46] above.

Example 5.3 (Shin's model, N=1). *In Shin's model from [46], suppose*

$$X = \{(0,0),(1,0),(1,1)\} = S_1 = S_2$$

$$\Vdash_1 ((0,0)) = \{(0,0)\}$$

$$\Vdash_1 ((1,0)) = \Vdash_2 ((1,1)) = \{(1,0),(1,1)\}$$

$$\Vdash_2((0,0)) = \Vdash_2((1,0)) = \{\{(0,0),(1,0)\}$$
$$\Vdash_2((1,1)) = \{(1,1)\}$$

Observe that in Shin's model, both agents have exactly one way of perceiving each point in X. That is, every point has a unique neighborhood in the perceptual topology. Since both have neighborhoods with more than one point, and since each point belongs to exactly one neighborhood, the topology is not T-1.[10]

There is no hope of finding a non-trivial information structure that is hard. This is because the first agent perceives $(1,0)$ and $(1,1)$ as belonging to the same neighborhood, while the second agent perceives $(0,0)$ and $(1,0)$ as belonging to the same neighborhood. Since information hardness requires that every conception has exactly one possible report and that agents agree, the only possibility is that both agents report $(0,0)$, $(1,0)$, and $(1,1)$ the same way. Thus, information hardness requires the reporting language to be a singleton.

If the reporting maps are allowed to be relations, consider the following possibility:

$$T = \{\text{“not }(0,0)\text{”}, \text{“not }(1,1)\text{”}, \text{“unknown”}\}$$

$$r_1(\{(0,0)\}) = \{\text{“not }(1,1)\text{”}, \text{“unknown”}\}$$
$$r_1(\{(1,0),(1,1)\}) = \{\text{“not }(0,0)\text{”}, \text{“unknown”}\}$$
$$r_2(\{(1,1)\}) = \{\text{“not }(0,0)\text{”}, \text{“unknown”}\}$$
$$r_1(\{(0,0),(1,0)\}) = \{\text{“not }(1,1)\text{”}, \text{“unknown”}\}$$

If $x = (0,0)$ or $x = (1,1)$, the possible reports for each agent are the same. So clearly the reporting system is heterogeneously faithful at both these points. If $x = (1,0)$ and both agents report “unknown,” then there is agreement. If $x = (1,0)$ and agent 1 reports “not (0,0),” then agent 2 looks at $r_2^-(\text{“not }(0,0)\text{”}) = \{(1,1)\}$, which 2 knows is incorrect. Thus the reporting system can be made heterogeneously faithful at $(0,0)$ and $(1,1)$ in this way, but at the cost of losing heterogenous faithfulness at $(1,0)$. This breakdown occurs because the topology is not T-1.

[10] A topological space satisfies the T-1 separation axiom if, for every pair of distinct points, there is a neighborhood of one that does not contain the other. That is, if $x \neq x'$, then there is a neighborhood $a \in S_i$ such that $x \Vdash_i a \wedge \neg(x' \Vdash_i a)$. In this model, a T-1 space is one where different objects have at least the possibility of being perceived differently. For details on this axiom and continuity properties, see [36] or [53].

6 An Individual Consumer's Problem

Preferences are defined over the choices an agent faces as they subjectively appear, and not over choices as they objectively are. This departure from neoclassical consumer theory means that the usual definition of preference relations needs some modification. In particular, because conceptions can overlap, preferences in general are incomplete.

Additionally, the agent need not have experience with every conception in S_i (some may not even exist, and the agent may not know this), and may have a hard time determining what is preferrable among things that have not be experienced. The literature in psychology suggests that agents reason about conceptions by envisioning examples (see e.g. Heit and Barsalou [24]), so not having witnessed something may make it difficult to specify a preference.

To accomodate these requirements, I define preferences for agent $i \in I$ by four binary relations:

Definition 6.1. *Let $i \in I$. Let S_i be i's set of conceptions. Then agent i's* preferences *are four binary relations $(\succ_i, \not\succ_i, \sim_i, \not\sim_i)$ on $S_i \times S_i$.*

The relations are interpreted as follows: $(\forall a, b, c \in S_i)$,

1. $a \succ_i b$ is interpreted as i prefers a to b.
2. $a \not\succ_i b$ is interpreted as i does not prefer a to b.
3. $a \sim_i b$ is interpreted as i is indifferent between a and b.
4. $a \not\sim_i b$ is interpreted as i is not indifferent between a and b.

Incompleteness of preferences means it is not required that $a \succ_i b \vee a \not\succ_i b$, or that $a \sim_i b \vee a \not\sim_i b$.

I make the following assumptions about the four preference relations:

Axiom 6.1. *For each $i \in I$, the relations $(\succ_i, \not\succ_i, \sim_i, \not\succ_i)$ satisfy the following: $(\forall a, b, c \in S_i)$,*

1. *$a \not\succ_i b \rightarrow \neg(a \succ_i b)$ and $a \not\sim_i b \rightarrow \neg(a \sim_i b)$.*
2. *$a \succ_i b \rightarrow b \not\succ_i a$.*
3. *$a \succ_i b \rightarrow a \not\sim_i b$.*
4. *$\sim_i$ is an equivalence relation:*

- $a \sim_i a$,
- $a \sim_i b \rightarrow b \sim_i a$, *and*
- $a \sim_i b \wedge b \sim_i c \rightarrow a \sim_i b$.

5. $a \sim_i b \wedge b \succ_i c \rightarrow a \succ_i c$.
6. $a \succ_i b \wedge b \sim_i c \rightarrow a \succ_i c$.
7. $a \succ_i b \wedge b \succ_i c \rightarrow a \succ_i c$.
8. $a \not\succ_i b \wedge b \not\succ_i a \rightarrow a \sim_i b$.
9. $a \not\prec_i b \wedge b \not\succ_i a \rightarrow a \succ_i b$.
10. $a \not\succ_i b \wedge b \not\succ_i c \rightarrow a \not\succ_i c$.

The last item, which states that $\not\succ_i$ is transitive, can be weakened somewhat; e.g., one can require that the transitive closure of $\not\succ_i$ have empty intersection with $\succ_i$. Nonetheless, transitivity of $\not\succ_i$ seems natural, and adding this stronger than necessary assumption gives greater intuitive clarity.

Lemma 6.1. *Let $X = S_i$, and let $\Vdash_i = \sim_i$ or $\Vdash_i = \not\succ_i$. Assume there is at least one element of S_i. Then Assumptions 3.1 and 3.2 hold.*

Proof. The indifference relation is an equivalence relation by Axiom 6.1 (4). So trivially, for any $a \in S_i$, there exists $b \in S_i$ (namely a) such that $a \sim_i b$. So Assumption 3.1 holds for $\sim_i$. Similarly, by Axiom 6.1 (2), if $a \succ_i a$ then $a \not\succ_i a$, and by Axiom 6.1 (1), both cannot hold simultaneously. So it must be that $a \not\succ_i a$, giving Assumption 3.1 for $\succ_i$.

Suppose for $a, b, c \in S_i$, $a \sim_i b \wedge a \sim_i c$. By transitivity of an equivalence relation, for any $d \in S_i$, if $d \sim_i a$ then $d \sim_i b \wedge d \sim_i c$. So Assumption 3.2 holds for $\sim_i$. Similarly, for $\not\succ_i$, the transitivity in Axiom 6.1 (10) gives Assumption 3.2. □

The lemma suffices to prove the following:

Theorem 6.1 (Preference topologies). *If Axiom 6.1 holds, then the agent's indifference relation $\sim_i$ and no-better-than relation $\not\succ_i$ each induce a topology on S_i.*

Proof. Entirely analogous to the proof of Theorem 3.1. □

The definition of preferences suggests the following notion of Pareto-dominance:

Definition 6.2. *Let $a, a' \in \prod_{i \in I} S_i$ be perceived allocations. Then a* Pareto dominates *a' iff, for every $i \in I$, $a' \not\succ_i a$, and, for some $j \in I$, $a \succ_i a'$.*

As an illustration, the interior of some set $U \subseteq S_i$ in the indifference topology is

$$\text{int}_{\sim_i} U \equiv \{a \in S_i | (\exists b \in S_i)(a \sim_i b \wedge (\forall c \in S_i)(b \sim_i c \rightarrow c \in U))\}.$$

This is the set of conceptions to which something is known to be indifferent, and which are indifferent only to members of U. That is, the indifference-interior of U is the set of conceptions a which are witnessed to be indifferent to something and which can only be indifferent to members of U.

Since $\sim_i$ is symmetric, a can serve as its own witness. Thus, $\text{int}_{\sim_i} U$ becomes

$$\{a \in S_i | (\forall c \in S_i)(a \sim_i c \rightarrow c \in U)\},$$

i.e., this becomes a set that contains everything indifferent only to members of U. A dual argument shows that the indifference closure of U is a set containing something indifferent to something in U.

Thus the topological notions presented above have indifference as a special case, and provide notions related to the neoclassical indifference curves. An analogous argument shows that the relation $\not\succ_i$ induces a topology that is a gives notions related to neoclassical upper and lower contour sets.

These topologies are more than curiousities. In particular, consider the following:

Theorem 6.2. *Let $\langle S_i, \sim_i, S_i \rangle$ and $\langle S_i, \not\succ_i, S_i \rangle$ be reporting systems, where agent i reports internally what is indifferent or not preferred to a given conception. Let $\langle X, \Vdash_i, S_i \rangle$ be i's perceptions. Then indifference and weak preference topologies induce comparable relations on X if the trade reporting system is heterogeneously faithful.*

To be completed, but essentially from the commutative diagrams. □

7 Heterogeneous Faithfulness and Equilibrium

7.1 Optimality, Equilibrium, and Heterogeneous Faithfulness

Each individual reports desired trades in a common language T. Prices are defined on T, and not on X; that is, prices are stated in terms of the shared language.

In this setting, the individual's decision problem is modified as follows:

Definition 7.1. *Let $t' \in T$ be a member of the common vocabulary. Given prices p on T, the reports that can feasibly trade for t' are the members of*

$$I(p, t') \equiv \{t \in T | p \cdot t' \leq p \cdot t\}.$$

The set $I(p, t')$ is called the set of reports that suffice *for t'.*

Observe that the reporting language simplifies the expression of what is needed to afford a particular choice. The reason is that the reporting language is common knowledge, which means that prices and alternatives are the same for everyone.

Feasibility of the individual's budget problem is now stated as follows:

Definition 7.2. *Let $i \in I$; let $\langle S_i, R_i, T \rangle$ be i's reporting system; and let $a \in S_i$ be i's perceived endowment. Some $t' \in T$ is* budget feasible *for i iff*

$$(\exists t \in T)(a R_i t \wedge t \in I(p, t')),$$

i.e., if the set of possible reports of the perceived endowment meets the reports that suffice for t':

$$r_i(a) \between I(p, t').$$

Thus, feasibility is expressed by the following diagram:

$$\begin{array}{ccc} S_i & \xrightarrow{\mathrm{r_i}} & T \\ & & \big\downarrow \mathrm{I(p,\cdot)} \\ & & T \end{array}$$

Because reports do not map perfectly to the individual's conceptions, it will not in general be possible to choose a report that is both feasible and always interpreted as at least as good as any other feasible choice. The reason is that $r_i(\cdot)$ is not single-valued. Hence I consider a weaker notion of optimality. Intuitively, a chosen report is budget optimal if any other feasible choice has an interpretation that is not better than some interpretation of the chosen report. That is, any other budget feasible choice cannot be necessarily better.

I make this precise with the following definition:

Definition 7.3. *Let $i \in I$, and let $U \subseteq S_i$. Let $(\succ_i, \nsucceq_i, \sim_i, \nprec_i)$ be i's preferences. The conceptions* no better than *a are*

$$\widehat{W}(U) \equiv \{b \in S_i | (\exists a \in S_i)(b \nsucceq_i a \wedge a \in U)\}.$$

An optimal choice can now be defined as follows:

Definition 7.4. *Let $i \in I$; let $\langle S_i, R_i, T \rangle$ be i's reporting system; and let $a \in S_i$ be i's perceived endowment. Some $t' \in T$ is* budget optimal *for i iff t' is budget feasible, and, for any budget feasible $t'' \in T$,*

$$r_i^-(t'') \between \widehat{W}(r_i^-(t').$$

Thus, optimality is a commutative square:

$$\begin{array}{ccc} S_i & \xrightarrow{\mathrm{r_i}} & T \\ {\scriptstyle \widehat{\mathrm{W}}}\downarrow & & \downarrow{\scriptstyle \mathrm{I(p,\cdot)}} \\ S_i & \xrightarrow{\mathrm{r_i}} & T \end{array}$$

The following then holds:

Corollary 7.1. *An individual's budget problem has an optimal choice iff there the reporting system is heterogeneously faithful in the individual's no-better-than topology, i.e., in the topology induced by $\nsucceq_i$.*

Proof. Immediate from the definition of optimality and from Theorem 5.1. □

Since prices are defined on T, the definition of market clearing is the natural one. This motivates the following definition of competitive equilibrium:

Definition 7.5. *Let $\langle X, \Vdash_i, S_i \rangle_{i \in I}$ be the perceptions of the individuals in an economy. Let $\langle S_i, R_i, T \rangle_{i \in I}$ be the trade reproting system. For each $i \in I$, let a_i be individual i's perceived endowment. A* competitive equilibrium *is a map $p : T \longrightarrow \mathbb{R}$ and a collection $\{(t_i, t_i')\}_{i \in I}$ of pairs in $T \times T$ such that, for each $i \in I$, the following conditions hold:*

Feasibility *$a_i R_i t_i$ and $p \cdot t'_i \leq p \cdot t_i$;*

Optimality *For $b_i \in S_i$, if $(\exists t''_i \in T)(bR_i t''_i)$, then $(\exists c_i \in S_i)(c_i R_i t'_i \wedge b_i \not\succ_i c_i)$; and*

Market Clearing $\sum_{i \in I} t_i = \sum_{i \in I} t'_i$,

where the summations are in the group addition operation on T.

In the above definition, markets clear relative to the reporting language. Because the reporting language and the individuals' perceptions may be imprecise, not every welfare-improvit trade will be made. More precisely, the following holds:

Theorem 7.1. *Comptetitive equilibrium is not necessarily Pareto optimal.*

Proof. Consider the following counter-example: let $I = \{1, 2\}$, and suppose

$$X = \{x, y\} \qquad S_1 = \{a, b\} \qquad S_2 = \{c, d\} \qquad T = \{t, t'\}$$

$$\Vdash_1 (x) = \{a\} \qquad \Vdash_1 (y) = \{b\} \qquad \Vdash_2 (x) = \{c\} \qquad \Vdash_2 (y) = \{d\}$$

$$r_1(a) = \{t\} \qquad r_1(b) = \{t'\} \qquad r_2(c) = \{t'\} \qquad r_2(d) = \{t\}$$

$$\omega_1 = x \qquad \omega_2 = y \qquad b \succ_1 a \qquad c \succ_2 d$$

$$p(t) = 1 \qquad p(t') > 1.$$

Here individual 1 has perceived endowment of a, and would prefer b. Individual 2 has perceived endowment d, and would prefer c. The reporting system is not heterogenously faithful, and in particular both report t in this case. Here autarky is a competitive equilibrium, but both individuals would be strictly better off if they were to swap endowments, and doing so is feasible. □

The lack of heterogeneous faithfulness in the previous example causes the failure of competitive equilibrium to be optimal. However, even heterogeneously faithful systems can be Pareto dominated. This is because a proposed trade does not fully specify what will be sent or received.

Example 7.1. *Let $T = \{0\}$, with the usual definition of addition. Then T is a group, and every map to T is a group homomorphism; i.e., if $Y \xrightarrow{f} T$ and $y, y' \in Y$, then $f(y + y') = f(y) + f(y')$. Here the reporting system is heterogeneously faithful, as it is weakly transparent in Vickrey's sense. Irrespective of what endowments, perceptions, preferences, and prices are, and irrespective of whether any goods are exchanged, the market is in equilibrium.*

On the other hand, we have the following positive result.

Theorem 7.2. *Suppose that the shared language is heterogeneously faithful relative to the individuals' perceptions. Let $\omega_i \in X$ be individual i's endowment, and let $x_i \in X$ be what the individual receives in equilibrium. If*

$$\sum_{i \in I} \omega_i = \sum_{i \in I} x_i, \tag{7}$$

then for each i there are reports $t_i \in r_i \circ \Vdash_i (\omega_i)$ and $t_i' \in r_i \circ \Vdash_i (x_i)$ such that markets clear in T. Conversely, if markets clear in T, there are real-world objects in X corresponding to the reports such that Equation 7 holds.

Proof. (Sketch of proof) The mappings from $X \longrightarrow T$ are group homomorphisms on topological groups. Hence, they are invertible. □

Thus while heterogeneous faithfulness does not guarantee optimality of equilibrium, it does make equilibrium possible. Without heterogeneous faithfulness, the mapping between X and T generated by an individual's perceptual and reporting maps would not be continuous, hence the invertibility of the group homomorphisms would not be guaranteed. In other words, if equilibrium in the reporting language is to correspond to equilibrium in the real-world objects, then the equilibrium needs to be at a point of continuity, i.e., of heterogeneous faithfulness.

8 Concluding Remarks

This paper models the perception reporting under perceptual differences. Here perceptions are not uniquely defined, which in turn suggests that the ways in which information is reported can have gray areas.

Making perceptions and reporting relations rather than functions generates a rich mathematical structure. Specifically, perceptions induce a topology on the collection of real-world objects, with the agent's conceptions serving as a base for this topology. Standard topological notions such as that of open sets, closed sets, and connected sets can all be defined in terms of the agent's conceptions. Thus, the structure of the agent's perceptions can be discussed purely in terms of what the agent finds meaningful, without any need of mentioning the actual real-world objects independent of how they can be perceived.

Reporting is introduced as a way for agents with different conceptions to communicate with each other. Each agent has a method for reporting subjective perceptions in the agreed-upon reporting language. As with perceptions, reporting may be ambiguous: an agent may have several ways to report the same subjective conception, as the language may not perfectly fit what is in the agent's mind. Conversely, there may be distinctions the agent can make that the reporting language ignores.

The reporting system induces a topology on the agent's conceptions, just as perceptions induce a topology on the real-world objects. The reporting language serves as a base of the reporting topology, which means that open sets, closed sets, connected sets, and other topological notions can be described in terms of the reporting language.

This structure is used to define heterogeneous faithfulness, which I compare with the information hardness notion of Ijiri, as modeled by Kirschenheiter, and to the notion of fully-revealing disclosure modeled by Demski and Sappington. The reporting system is heterogeneously faithful if different agents seeing the same thing can report it in the same ways. An alternative definition would state that two agents seeing the same report would interpret it in ways corresponding to perceptions of the same real-world objects. It turns out that these two definitions are equivalent, though this is not true of information harndess or of fully-revealing disclosure.

Thus, heterogeneous faithfulness has the desirable property that a reporting system is faithful among reporting entities if and only if it is faithful among end-users. This property, corresponding to the accountant's notion of *neutrality*, does not hold of the related notion of completeness: reporting structures that are complete from the viewpoint of the reporting entities are not in general complete from the viewpoint of the end users.

The topological interpretation leads to an additional desirable property: heterogeneous faithfulness turns out to be identical to continuity. An information structure is heterogeneously faithful between two agents if and only if the first agent's perceptions form a continuous map from the real-world objects, endowed with the second agent's perceptual topology, to the first agent's conceptions, endowed with the reporting topology. Any refinement of a set of reporting rules that maintains continuity therefore has no impact on heterogeneous faithfulness.

The welfare effects of heterogeneous faithfulness are addressed in the context of a consumer's budget choice problem. Optimality of a consumer's budget problem is shown to depend on faithfulness between preferences and the reporting system. Finally, the existence of competitive equilibrium with natural properties is shown to depend on the equilibrium being at a point of heterogeneous faithfulness.

Nevertheless, competitive equilibria need not be Pareto optimal, even when reporting is heterogeneously faithful. The reporting system may coarsen the set of feasible trades to the point where mutually desirable trades cannot be communicated. An example shows that refining the reporting language can increase the number of points of heterogeneous faithfulness, even if the system as a whole is no longer heterogeneously faithful.

A natural direction for future work is to explore this trade-off in greater detail. The topological interpretation of reporting, perceptions, and preferences suggests that it may suffice for a system to be heterogeneously faithful on all but a topologically small set, such as a set of the first Baire category in the respective topologies. (Related ideas are discussed in Calude and Zamfirescu [15] in a number theoretic context, and in Calude [14] in the context of computational complexity theory.) Also, since many of the counter-examples failed in topologies that did not satisfy the T-1 separation axiom, future work may explore the connection between a positive notion of apartness and optimality, as in the work of Bridges and Vîţa [10, 9] and Bridges, Vîţa, and Schuster [8].

Many of the results presented here have a similar feel to those in Kanodia [30], who argues that the role of an accounting system is not to determine whether markets are efficient, but to determine what the set of possible efficient equilibria might be. Here the reporting system determines whether equilibrium or even optimization is feasible. Refining the reporting system to the point where continuity deteriorates does not just change the nature of equilibrium, but makes optimization impossible, in or out of equilibrium.

A Appendix: Constructivity and Conceptions

Because the agents's conceptions are mental creations, I restrict claims about the conceptions to those that are fully constructive. For the purposes of this paper, *fully constructive* claims must involve only predicative definitions and must be intutionistically valid. Roughly, this means that definitions cannot depend on what is being defined (no vicious circle), and that proofs of positive assertions cannot depend on the excluded middle.

The notion of predicative definition, due to Poincaré [38, 39] and Russell [41], is essentially non-circularity. A definition that presupposes the defined term to exist is *impredicative*; definitions that do not implicitly refer to themselves are *predicative*. Classical mathematics permits impredicative definitions (a standard example is the usual construction of the real numbers). The position here is that an agent's conceptions are the creations of the agent, and hence that the individual must be able to describe them in a non-circular way.

Predicativity, then, is constructivism with respect to definitions. But predicative reasoning need not be fully constructive. For example, it is not circular to define an integer x by $x = 0$ if some open question turns out to be true, and $x = 1$ if the question turns out to be false. Nevertheless, it is unreasonable to expect an agent to know the answer to any or all unsolved questions. Such a definition is non-constructive, not because of impredicativity, but because it relies on the principle of the excluded middle. That is, the integer x defined above is only well-defined if we expect every well-formed question to be solvable. Logic that omits the principle of the excluded middle from classical logic (as well as the related principle of double negation, which is derived as a consequence of the excluded middle) is called *intuitionistic* logic, originating in Brouwer's Ph. D. thesis [11] and specified fully by Heyting and Kolmogorov [26, 25, 32].

The usual explanation of intuitionistic logic is in terms of the "BHK-conditions" (for Brouwer-Heyting-Kolmogorov), which are as follows:

1. A proof of $A \wedge B$ is a proof of A and a proof of B.
2. A proof of $A \vee B$ is a proof of A or a proof of B.
3. A proof of $A \rightarrow B$ is a method of turning a proof of A into a proof of B.
4. A proof of $\perp$ (read as "falsehood") is impossible.
5. A proof of $(\forall x)A(x)$ is a method of turning any x into a proof of $A(x)$.
6. A proof of $(\exists x)A(x)$ is an agent x and a proof of $A(x)$.

The notation $\neg A$, read "not A," is a shorthand for $A \rightarrow \perp$. I.e., "not A" is read as "assuming A leads to a contradiction."

The above specification gives the idea behind intuitionistic logic. However, it is unsatisfactory as a definition, as it relies on what is understood by "a method" for transforming proofs, and also on what is understood by a proof. There are many satisfactory ways to define the rules of inference for intuitionistic logic; for details, see Negri and von Plato [37], Prawitz [40], Troelstra and van Dalen [49], or Martin-Löf [34, 35].[11]

[11] The foundation due to Martin-Löf has attracted considerable attention. It is both predicative and intuitionistic, and has natural interpretations in category theory. However, the original form of Martin-Löf's intuitionistic type theory was inconsistent. An introduction to Martin-Löf's corrected version, see Valentini [50]. An interesting historical background and philosophical discussion is in Sommaruga [47].

The discussion of constructivism here is necessarily cursory. For an overview, see Bishop and Bridges [5], Heyting [27, 28], or Spitters [48]. Detailed development, and contrasts with different forms of constructivism, are in Troelstra and van Dalen [49], Bridges and Richman [7], and Spitters [48].

In the setting of the model here, the fully constructive approach requires care in interpreting differences between sets and collections. All sets are collections, but for a collection to be a set, there must be a way to construct all of its canonical (i.e., definitional) elements, and it must be possible to determine whether two canonical elements are equal; see Bishop and Bridges or Martin-Löf for details.

The discussion throughout this paper refers to subsets of the collection of real-world objects, arbitrary subcollections of an agent's internal conceptions, relations between arbitrary collections, and so forth. These definitions are entirely consistent with a constructive approach; however, one needs to be careful in how subsets and relations are defined.

I do not discuss the foundations in detail in the paper; however, the definitions and proofs are deliberately chosen in order to be consistent with the approach in Sambin and Valentini [44]. A subset is not properly thought of as a set; instead, it is thought of as a predicate defined on some collection. Thus, if X is an arbitrary collection, $D(x)$ might be thought of as a statement like "x is an even number." Because the reasoning is constructive, it is not always the case that $D(x) = \top$ or $D(x) = \bot$; it is possible that the collection X or the predicate D is not constructively well-enough defined for the truth value of $D(x)$ to be meaningful.[12] The notation '$D \subseteq X$' is taken as a shorthand for D is a predicate on X, while '$x \in D$' is read as $x \in X$ and $D(x) = \top$. This is a slight abuse of notation, since '$\in$' is used in two senses. Sambin and Valentini introduce the symbol 'ϵ' to be used instead for this situation.

Generalizing from subsets to relations is entirely straightforward. Classically, a binary relation between two collections X and S is typically presented as a subset of $X \times S$. Alternatively, a binary relation can be presented as a 2-place predicate; logicians often favor this approach (see e.g. Boolos and Jeffrey [6]). Given the definition of subsets used here, the two approaches appear equivalent. Thus, a binary relation $\Vdash$ between

[12]Dummett [20] gives as an example the statement, "Hamlet wore a moustache." This is clearly well-formed as a proposition, but it need not have a fixed truth value. The character Hamlet is a mental creation, and is not necessarily specified to this level of detail. Brouwer gave similar examples in mathematics throughout his career; see e.g. [12], or [13] for details on the general structure of intuitionistic counter-examples.

arbitrary collections X and S is predicate on $X \times S$. If $x \in X$ and $a \in S$, then '$x \Vdash a$' is a shorthand for $\Vdash (x, a) = \top$, while $\neg(x \Vdash a)$ is a shorthand for $\Vdash (x, a) = \bot$. As with subsets, relations may not have constructively meaningful truth values. This means that $(x \Vdash a) \vee \neg(x \Vdash a)$ is not a valid rule of inference, which just means that the logic is indeed intuitionistic.

There are alternative approaches to constructive set theory; e.g., Aczel [1] also works in intuitionistic type theory. An thorough development of the history and general approaches is in Sommaruga [47]. Other approaches are discussed in Troelstra and van Dalen [49].

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