

Lie Groups of Partial Differential Equations
and Their Application to the
Multidimensional Screening Problems

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Abstract

In this paper I described group theoretic methods that can be used for analyzing the boundary problems, which arise when the Hamiltonian method is applied to solve the relaxed problem for the multidimensional screening problem. This technique can provide some useful insights into the structure of solutions and some times may help to arrive at particular solutions.

1 INTRODUCTION

In many industries the price paid by the customers is not strictly proportional to the quantity purchased. Examples include railroad tariffs, electricity tariffs, and rental rates for durable goods and space. All these cases fall into a general category of *nonlinear tariffs*. The major justification for the nonlinear pricing is the existence of private information on the side of consumers. Often nonlinear tariffs specify the payment as a function of a variety of characteristics. For example, the railroad tariffs specify charges based on weight, volume, and distance of each shipment. Different customers may value each of these characteristics differently, hence the customer's type will not in general be captured by a one-dimensional characteristic and a problem of multi-dimensional screening arises.

The general formulation of the problem of multi-dimensional screening is due to Armstrong (1996) and Wilson (1993), and goes as follows. Consider a multi-product monopoly producing n goods (or a good with n quality dimensions) with a convex cost function. The preferences of a consumer over these goods can be parameterized by an m -dimensional vector. Types of consumers are distributed according to a density function $f(\cdot)$ defined over a convex open bounded set $\Omega \subset R^m$. Assume that $f(\cdot)$ is continuously

differentiable on Ω and can be extended by continuity on its closure. The monopolist is interested in maximizing profits by choosing a tariff, which is a function from the set of bundles of goods to the real line. The tariff determines how much a consumer will pay for a particular bundle of goods.

Finding the solution often involves solving a boundary problem for a system of nonlinear partial differential equations (see, Basov (2001, 2002)). Though no general methods for solving such problems exist, the problem can be considerably simplified and even solved explicitly if it possesses some symmetry.

In this paper I demonstrate how the theory of Lie groups of partial differential equations can be applied to the multidimensional screening problems. I also give a brief outline of the theory. For a detailed exposition see, for example, Cantwell (2002).

The paper is organized as follows. In Section 2 I formulate the monopolist's problem and illustrate by an example how symmetry considerations allow to arrive at solutions of particular problems. The example shows that application of symmetry considerations can be executed in two steps: finding the symmetry group of the problem and finding the invariants of this group. In Section 3 I describe a regular way to find a symmetry group of a system

of partial differential equations. In Section 4 the linear partial differential equation, which holds for the invariants of a group is derived. In Section 5 I revisit the example of Section 2 and solve a new example. Section 6 concludes.

2 THE MONOPOLIST'S PROBLEM AND AN EXAMPLE

In this Section I will formulate the monopolist's problem and illustrate by an example how symmetry considerations may help to arrive at the solution of a multidimensional screening problem. Consider a multi-product monopoly producing n goods. Consumers have preferences over the bundles of these goods that are parameterized by an m -dimensional column vectors. The types of consumers are distributed according to a density $f(\cdot)$ function, on the set $\Omega \subset R^m$. The set Ω is assumed to be open, bounded, and convex. Furthermore, in this paper I will assume that $\Omega = \times_{i=1}^m (a_i, b_i)$ and f is continuous and strictly positive on a convex open subset of Ω . The utility of a consumer of type $\alpha \in \Omega$, when she consumes a bundle $x \in X \subset R_+^n$ and makes a payment of t , is given by:

$$U(\alpha, x, t) = \sum_{i=1}^m \alpha_i v_i(x) - t, \quad (1)$$

where each of the functions $v_i(\cdot)$ is increasing and continuously differentiable, and satisfies a Lipschitz condition in x on X . For a given tariff $t : X \rightarrow R$, the firm's profits are given by:

$$\pi = \int [t(x(\alpha)) - c(x(\alpha))] f(\alpha) d\alpha \quad (2)$$

where $c(\cdot)$ is the cost of production and $x(\alpha)$ is the bundle purchased by all type- α consumers. The firm is interested in choosing a tariff $t(\cdot)$ to maximize its profits.

Given such a tariff, let

$$s(\alpha) = \max_{x \in X} (U(\alpha, x, t(x))). \quad (3)$$

Thus, $s(\alpha)$ is the surplus of a consumer of type α who chooses a bundle $x \in X$ that maximizes her utility. One can solve (1), (3) to get:

$$t(\alpha) = \sum_{i=1}^m \alpha_i v_i(x(\alpha)) - s(\alpha). \quad (4)$$

It is possible to show that $s(\cdot)$ is continuous, convex (and, hence, almost everywhere differentiable), and satisfies the envelope conditions:

$$\frac{\partial s}{\partial \alpha_i}(\alpha) = v_i(x(\alpha)), \quad i = 1, \dots, m. \quad (5)$$

For a proof see Armstrong (1996).

Conditions (4) and (5) show that the monopolist can be assumed to choose the consumer surplus $s(\alpha)$ subject to the envelope and convexity constraints. The usual practice is to drop the convexity constraint and than to check whether the solution satisfies it. The monopolist's problem with the convexity constraint dropped is called relaxed. problem.

Example 1. Let the utility of a consumer be given by:

$$U = \alpha_1 x - \frac{1}{\gamma}(\alpha_2 + c)x^\gamma - t,$$

where α_1 and α_2 are distributed independently and uniformly on $(0, a) \times (0, b)$ (i. e. $a_1 = 0$, $a_2 = a$, $b_1 = c$, $b_2 = b + c$), $\gamma > 1$ and $c > b/2$. The cost of production is zero. In the case $\gamma = 2$, $a = b = c = 1$ this problem was first considered by Laffont, Maskin, and Rochet (1987) and revisited by Basov

(2001). The envelope conditions (5) imply that

$$s_1 = x, \quad s_2 = -\frac{x^\gamma}{\gamma}.$$

Basov (2001) showed that if $s(\cdot)$ solves the monopolists problem then there exists $\mu \in H^1(\Omega)$ such that the following system holds:

$$\left\{ \begin{array}{l} s_2 + \frac{1}{\gamma}s_1^\gamma = 0 \\ \mu_1 s_1^{\gamma-1} + (\gamma - 1)\mu s_1^{\gamma-2}s_{11} + \mu_2 = 3 \\ \mu = c \text{ at } \alpha_2 = 0, \mu = b + c \text{ at } \alpha_2 = b, \mu s_1^{\gamma-1} = a \text{ at } \alpha_1 = a \end{array} \right. \quad . \quad (6)$$

Note that system (6) is invariant under the following transformation

$$\left\{ \begin{array}{l} \tilde{\alpha}_1 = \beta\alpha_1, \tilde{a} = \beta a \\ \tilde{\alpha}_2 = \beta^\gamma\alpha_2, \tilde{b} = \beta^\gamma b, \tilde{c} = \beta^\gamma c \\ \tilde{\mu} = \beta^\gamma\mu \end{array} \right. \quad . \quad (7)$$

This implies that if $s(\alpha_1, \alpha_2; a, b, c)$ is a solution to (6) so is $\tilde{s}(\alpha_1, \alpha_2; a, b, c) = s(\beta\alpha_1, \beta^\gamma\alpha_2; \beta a, \beta^\gamma b, \beta^\gamma c)$. Since the solution to the monopolist's problem should be unique (see, Basov (2001)) it should depend on the invariants of the transformation (7).

Let us introduce new variables

$$\begin{cases} \xi = (\alpha_1 - \delta_1 a)^\gamma / (\alpha_2 - \delta_2 b) \\ \zeta = (\alpha_2 - \delta_2 b) / a^\gamma \end{cases}, \quad (8)$$

where δ_1 and δ_2 are homogenous of zero degree functions of a , b , and c but do not depend on α_1 and α_2 .

Note that both ξ and ζ are invariant under transformation (7). In this variables the first equation of system (6) will have a form

$$\xi^{\gamma-1} s_\xi ((\gamma s_\xi)^{\gamma-1} - \xi^{2-\gamma}) + \zeta s_\zeta = 0. \quad (9)$$

Note that for s_ξ to remain finite as $\xi \rightarrow 0$ it should be that $s_\zeta = 0$. Therefore, the non-trivial non-singular solutions of (9) are given by

$$s(\xi, \zeta) = \frac{\gamma-1}{\gamma} \xi^{\frac{1}{\gamma-1}} + C, \quad (10)$$

where C is an arbitrary constant. Since surplus function (10) satisfies the envelope conditions and is convex, it is implementable (see, Rochet (1987)).

Using (11) and the definition of ξ one calculate that

$$x = \left(\frac{\alpha_1 - \delta_1 a}{\alpha_2 - \delta_2 b} \right)^{\frac{1}{\gamma-1}}. \quad (11)$$

Substituting this into (6) we will get the following boundary problem for μ :

$$\begin{cases} (\alpha_1 - \delta_1 a)\mu_1 + (\alpha_2 - \delta_2 b)\mu_2 = 3(\alpha_2 - \delta_2 b) - \mu \\ \mu = c \text{ at } \alpha_2 = 0, \mu = b + c \text{ at } \alpha_2 = b, \mu x^{\gamma-1} = a \text{ at } \alpha_1 = a \end{cases} \quad (12)$$

The boundary conditions should be satisfied as equalities almost everywhere on the intersection of the exterior boundary with the participation region. First, note that a continuous μ could not satisfy these conditions. Indeed, consider point $(a, 0)$. The boundary conditions imply that $x(a, 0) = (a/c)^{1/(\gamma-1)}$. Therefore, there is no distortion at the bottom right point. If μ were continuous then $x^{1-\gamma}(a, b) = 1/\mu = (a/(b+c))^{1/(\gamma-1)}$, which is the efficient level. But the incentive compatibility constraint between types $(a, 0)$ and (a, b) implies that it should be biased downwards.

The reason for the discontinuity of μ is non-smoothness of the boundary of set Ω . Since $x(\cdot, \cdot)$ is continuous inside the participation the solution to (12) should be sought separately in two regions separated by an isoquant

passing through the point (a, b) .

The general solution of the partial differential equation for μ has a form

$$\mu = \frac{3}{2}(\alpha_2 - \delta_2 b) + \frac{\phi\left(\frac{\alpha_1 - \delta_1 a}{\alpha_2 - \delta_2 b}\right)}{\alpha_2 - \delta_2 b}, \quad (13)$$

where ϕ is arbitrary continuously differentiable function. At the neighborhood of point $(a, 0)$ the following boundary conditions should be satisfied:

$$\mu = c \text{ at } \alpha_2 = 0, \quad \mu x^{\gamma-1} = a \text{ at } \alpha_1 = a. \quad (14)$$

The first boundary condition implies that $\phi = 0$ and

$$\mu = c + \frac{3}{2}\alpha_2, \quad \delta_2 = -\frac{2c}{3b}.$$

Using the second boundary condition and (11) one obtains:

$$\delta_1 = \frac{1}{3}. \quad (15)$$

Therefore,

$$x = \left(\frac{\alpha_1 - \frac{a}{3}}{\alpha_2 + \frac{2c}{3}}\right)^{\frac{1}{\gamma-1}} \quad (16)$$

Note that $\mu(a, b) = c + \frac{3}{2}b \neq b + c$, therefore this solution cannot be extended to the region containing the upper boundary. The solution in that region is given by (12) subject to

$$\mu = c \text{ at } \alpha_2 = 0, \quad \mu = b + c \text{ at } \alpha_2 = b. \quad (17)$$

It is straightforward to show that (12) and (17) imply

$$\left\{ \begin{array}{l} \delta_2 b = \frac{b-2c}{4} \\ \phi = -\frac{1}{32}(3b+2c)(b-2c) \end{array} \right. . \quad (18)$$

Therefore,

$$x = \left(\frac{4(\alpha_1 - \delta_1 a)}{4\alpha_2 - b + 2c} \right)^{\frac{1}{\gamma-1}}. \quad (19)$$

Using continuity of $x(\cdot, \cdot)$ at point (a, b) one obtains

$$\delta_1 = \frac{1}{2}. \quad (20)$$

and

$$\left\{ \begin{array}{l} x = 0 \text{ if } \alpha_1 \leq \frac{a}{2} \\ x = \left(\frac{4\alpha_1 - 2a}{4\alpha_2 - b + 2c}\right)^{\frac{1}{\gamma-1}} \text{ if } \alpha_1 \geq \frac{a}{2} \text{ and } (3b + 2c)\alpha_1 - 2a\alpha_2 \leq 2ac + ab \\ x = \left(\frac{3\alpha_1 - a}{3\alpha_2 + 2c}\right)^{\frac{1}{\gamma-1}} \text{ if } \alpha_1 \geq \frac{a}{2} \text{ and } (3b + 2c)\alpha_1 - 2a\alpha_2 \geq 2ac + ab \end{array} \right. \cdot \quad (21)$$

The optimal tariff is determined by

$$t(x) = \max_{\alpha} \left(\alpha_1 x - \frac{1}{\gamma} (\alpha_2 + c) x^{\gamma} - s(\alpha) \right), \quad (22)$$

where $s(\cdot)$ is given by (10) with δ_1 and δ_2 given above and C determined from $t(0) = 0$. Therefore,

$$t(x) = \left\{ \begin{array}{l} \frac{a}{2}x - \frac{1}{\gamma} \left(\frac{c}{2} + \frac{b}{4} \right) x^{\gamma}, \text{ if } x < \left(\frac{2a}{3b+2c} \right)^{\frac{1}{\gamma-1}} \\ \frac{a}{6} \left(\frac{2a}{3b+2c} \right)^{\frac{1}{\gamma-1}} - \frac{1}{\gamma} \left(\frac{c}{6} + \frac{b}{4} \right) \left(\frac{2a}{3b+2c} \right)^{\frac{\gamma}{\gamma-1}} + \frac{a}{3}x - \frac{c}{3\gamma} x^{\gamma}, \text{ if } x \geq \left(\frac{2a}{3b+2c} \right)^{\frac{1}{\gamma-1}} \end{array} \right. \cdot \quad (23)$$

If $\gamma = 2$ and $a = b = c = 1$ this solution coincides with one obtained by Laffont, Maskin, and Rochet (1987). Note that as $\gamma \rightarrow \infty$ the tariff becomes piecewise affine

$$t(x) = \left\{ \begin{array}{l} \frac{a}{2}x \text{ if } x < 1 \\ \frac{a}{6} + \frac{a}{3}x \text{ if } x \geq 1. \end{array} \right. \cdot \quad (24)$$

3 CALCULATING A SYMMETRY GROUP FOR A BOUNDARY PROBLEM

In the previous Section I used symmetry considerations to arrive at the solution of a multidimensional screening problem. Arriving at the solution involved going through the following steps. First, it is necessary to find an invariance group of the problem. In the example above the group is given by transformations (6). Second, find all independent invariants of the group. In the above example they are ξ and ζ . Third, rewrite the problem in terms of group invariants and attempt to solve it. If $m > n$ it may happen that the number of independent invariants of group is bigger than the number of the instruments. In that case one the solution will depend only on n invariants. One might try to guess from economic considerations which invariants will enter into the solution. Such a guess, if correct, can considerably simplify the calculations. However, while one can describe a regular procedure for the first two steps, guessing the right set of invariants is largely an art.

In this Section I describe a regular procedure for finding a symmetry group of the problem. Next Section deals with finding the invariants of the group. I will give only the basic outline of the theory and will omit all proofs

and lengthy derivations. For the details, see Cantwell (2002). First, let us defining the notion of a group.

Definition 1 *A set G together with a binary operation $m : G \times G \rightarrow G$ is called a group if the following properties hold:*

- a). (Associativity) $m(g_1, m(g_2, g_3)) = m(m(g_1, g_2), g_3) \forall g_1, g_2, g_3 \in G$
- b). (Identity) $\exists e \in G : m(g, e) = g \forall g \in G$
- c). (Inverse) $\forall g \in G \exists \tilde{g} \in G : m(g, \tilde{g}) = e. \tilde{g} \equiv g^{-1}.$

Operation $m(\cdot, \cdot)$ is usually called multiplication and denoted by \cdot , so $m(g_1, g_2) = g_1 \cdot g_2$, e is called the identity element and $\tilde{g} \equiv g^{-1}$ is called the inverse of g (it is straightforward to prove the identity element is unique and that each element has a unique inverse).

Definition 2 *Let (G, m) be a group and let $H \subset G$. If (H, m) is a group on its own write it is called a subgroup of (G, m) .*

To check that (H, m) is a subgroup of (G, m) one has to verify that $m(h_1, h_2) \in H$ for any $h_1, h_2 \in H$. If there is no confusion about operation m one usually refers to group (G, m) simply as group G .

Intuitively, Lie group consists of elements which can be represented as values of an analytical function of some set of real variables. In this paper we will be interested only in the so-called one parametric Lie groups.

Definition 3 Let $\Xi \subset R^m$ be an open set and $\tau \in R$. Assume that function $F : R^m \times R \rightarrow \Xi$ is infinitely differentiable in α and analytic in τ . Consider set G of coordinate transformations

$$g^\tau : \{\alpha = F(\tilde{\alpha}, \tau)\}. \quad (25)$$

with together with a binary operation m defined by

$$m(g^{\tau_1}, g^{\tau_2}) : \{\alpha = F(F(\tilde{\alpha}, \tau_1), \tau_2)\}. \quad (26)$$

If (G, m) is a group it is called a one-parametric Lie group. Parameter τ is usually chose in such a way that $g^0 = e$.

Clearly, a set that contains one element, call it e , together with $m : G \times G \rightarrow G$ defined by $m(e, e) = e$ is a group. We will call such group trivial. A group with more than one element is called non-trivial.

Proposition 1 Let (G, m) be a group and (H_1, m) and (H_2, m) its subgroups. Let $H = H_1 \cap H_2$. Then (H, m) is a subgroup of (G, m) . Moreover, it is a subgroup of (H_1, m) and (H_2, m) .

Again, if there is no confusion about the nature of the group multiplication, we will simply phrase Proposition 1 as: An intersection of two subgroups

is a subgroup.

The proof of this proposition is trivial and is omitted. I will use it below to construct the symmetry group for a system of partial differential equations (PDEs). The link between a linear multidimensional screening problem and a boundary value problem for a system of PDEs was established by Rochet and Chone (1998) and Basov (2001) (see, also Basov (2002) for the generalization of the results for the non-linear case). Even if the screening problem is linear the resulting system of partial differential equation will typically be non-linear (recall the example in the previous Section). No general technique for solving the boundary value problem for a system of nonlinear PDEs is available. However, if there exists a non-trivial group of transformation, which covers both dependent and independent variables, and the parameters of the model (in the previous example the parameters are a , b , and c) that leaves the boundary problem invariant, then this often can be used to arrive at the explicit solution.

The group of a PDE can be calculated in a systematic way. I will restrict attention to the first and second order PDEs, since these arise in screening

problems. Consider a PDE:

$$\Phi(\alpha, u, \nabla u, D^2 u) = 0, \quad (27)$$

where $\alpha \in R^m$, $u : R^m \rightarrow R$ is twice continuously differentiable, ∇u is gradient of u , $D^2 u$ is the symmetric tensor of its second derivatives and $\Phi : R^{m^2/2+5m/2+1} \rightarrow R$ is a continuously differentiable function, and transformation of the independent and dependent variables:

$$\begin{cases} \tilde{\alpha}_i = F_i(\alpha, u; \tau) \\ \tilde{u} = G(\alpha, u; \tau) \end{cases}, \quad (28)$$

where functions F_i and G are infinitely differentiable in α and u and analytic in τ , and $F_i(\alpha, u; 0) = \alpha_i$, $G(\alpha, u; 0) = u$. Let us define

$$\begin{cases} \theta^j = \frac{\partial F_j}{\partial \tau}(\alpha, u; 0) \\ \chi = \frac{\partial G}{\partial \tau}(\alpha, u; 0) \end{cases}. \quad (29)$$

Then one can write up to $O(\tau)$ terms:

$$\begin{cases} \tilde{\alpha}_i = \alpha_i + \tau\theta_i(\alpha, u) \\ \tilde{u} = u + \tau\chi(\alpha, u) \end{cases} . \quad (30)$$

Expression (30) is known as the infinitesimal form of (28).

Note that the transformations (28) form a one-parametric Lie group (representation (28) is called a finite form) if we define product of two transformations to be their composition, that is define m by (26).

Definition 4 *A subgroup of group (28) which leaves equation (27) invariant is called its symmetry group.*

To calculate the symmetry group of equation (27) one has first to extend group (28) to cover the transformations of the first and second derivatives of u . In doing so, one arrives at the so-called twice-extended group:

$$\begin{cases} \tilde{\alpha}_i = \alpha_i + \tau\theta^i(\alpha, u) \\ \tilde{u} = u + \tau\chi(\alpha, u) \\ \tilde{u}_i = u_i + \tau\chi_{\{i\}}(\alpha, u, \nabla u) \\ \tilde{u}_{ij} = u_{ij} + \tau\chi_{\{ij\}}(\alpha, u, \nabla u, D^2u) \end{cases} ., \quad (31)$$

where

$$\left\{ \begin{array}{l} u_i = \partial u / \partial \alpha_i, u_{ij} = \partial^2 u / \partial \alpha_i \partial \alpha_j \\ \chi_{\{i\}} = D_i \chi - \sum_{j=1}^m u_j D_i \theta^j \\ \chi_{\{ij\}} = D_i \chi_j - \sum_{k=1}^m u_{jk} D_i \theta^k \end{array} \right. , \quad (32)$$

and the total differentiation operator D_i is defined by

$$D_i \omega(\alpha, u) = \frac{\partial \omega}{\partial \alpha_i} + u_i \frac{\partial \omega}{\partial u}. \quad (33)$$

The invariance group of equation (27) can be found from the condition

$$\sum_{j=1}^m \theta^j \frac{\partial \Phi}{\partial \alpha_j} + \chi \frac{\partial \Phi}{\partial u} + \sum_{k=1}^m \chi_{\{i\}} \frac{\partial \Phi}{\partial u_i} + \sum_{k=1}^m \chi_{\{ij\}} \frac{\partial \Phi}{\partial u_{ij}} = 0, \quad (34)$$

which should hold on the surface $\Phi(\alpha, u, \nabla u, D^2 u) = 0$. Carrying out explicit calculations will result in a system of partial differential equations for functions (θ^i, χ) . Since we have to find a symmetry group, we will be usually interested in a particular finite parametric set of solutions to the system. Cantwell (2002) contains a software that can deal with the problem. In Section 5 we will illustrate this approach on some examples.

If one has to deal with the system of PDEs

$$\Phi_i(\alpha, u, \nabla u, D^2u) = 0, \quad i = 1, \dots, p \quad (35)$$

the above technique can be used to calculate the symmetry groups H_i of each of the equations. Then

$$H = \cap_{i=1}^p H_i \quad (36)$$

will be the symmetry group of the system.

Note that since the solution to a system of PDEs is typically not unique the fact that the system possesses a symmetry group does not mean that each solution will be invariant with respect to it. It will rather mean that the transformations of the group will take a solution into a solution. If one deals with a boundary value problem, the symmetry group of the equation is typically not the symmetry group of the problem. This, however, can be remedied if one allows transformations not only of the dependent and independent variables, but also parameters of the model. One has to supplement (29) with

$$\begin{cases} \tilde{a}_i = F_i(\alpha_{-i}, a_i, u) \\ \tilde{b}_i = F_i(\alpha_{-i}, b_i, u) \end{cases}$$

and modify (30) accordingly. If the boundary problem is invariant under this extended group of transformations then, since the solution to the boundary value problem is usually unique, it will possess the symmetry of the problem, i. e. will depend on variables and parameters only through the invariants of the symmetry group. In the next Section I will provide a regular method to find the invariants of a group.

4 FINDING THE INVARIANTS OF A LIE GROUP

Consider a one-parametric Lie group of transformations given by (28), whose infinitesimal form is given by (30). The main idea behind calculating the invariance group is to calculate its infinitesimal form and then to integrate to obtain the finite form. I will not try to justify this approach here. An interested reader should see Cantwell (2002).

Definition 5 *A continuously differentiable function $\Upsilon : R^{m^2/2+5m/2+1} \rightarrow R$ is called an invariant of group (28) if $\forall \tau > 0$*

$$\Upsilon(\tilde{\alpha}, \tilde{u}, \widetilde{\nabla u}, \widetilde{D^2 u}) = \Upsilon(\alpha, u, \nabla u, D^2 u). \quad (37)$$

Here $\widetilde{\nabla}u$ and \widetilde{D}^2u are calculated using twice extended group (31). Expression on the left hand side of (37) can be viewed as a function of the group parameter τ and the invariance condition can be read to say that it does not depend on τ , therefore

$$\frac{d\Upsilon}{d\tau} = 0 \quad (38)$$

or, taking the full derivative of (37) and using 31)

$$\sum_{j=1}^m \theta^j \frac{\partial \Upsilon}{\partial \alpha_j} + \chi \frac{\partial \Upsilon}{\partial u} + \sum_{k=1}^m \chi_{\{i\}} \frac{\partial \Upsilon}{\partial u_i} + \sum_{k=1}^m \chi_{\{ij\}} \frac{\partial \Upsilon}{\partial u_{ij}} = 0. \quad (39)$$

If the transformation affects not only the coordinates α but also vectors of parameters a and b (as in Example 1), they should be treated as additional arguments in Υ . Note that (39) is a linear homogenous partial differential equation. Often such an equation can be solved explicitly. We see that the problem of finding the invariants of a group is easier than finding the symmetry group. However, while for the last problem we usually are interested in finding a solution, for this problem we are usually interested in finding all independent invariants.

Consider an important case when $\chi = 0$ (pure coordinate transformation) and suppose we are interested in finding and invariant a function of α that

is invariant with respect to (30). Then (39) reduces to

$$\sum_{j=1}^m \theta^j \frac{\partial \Upsilon}{\partial \alpha_j} = 0. \quad (40)$$

This is the case that arises in the screening applications. The role of Υ is played by the consumer surplus function.

5 APPLICATIONS OF THE DEVELOPED TECHNIQUE

In this Section I give examples of applications of the developed technique.

Example 1 (revisited). Let us start with calculating the symmetry group of boundary problem (6). First, consider the first equation of the system

$$s_2 + \frac{1}{\gamma} s_1^\gamma = 0. \quad (41)$$

Our objective is to calculate pure coordinate transformations ($\chi = 0$) that leave equation (41) invariant. The invariance equation (34) for this case takes the form:

$$\chi_{\{1\}} s_1^{\gamma-1} + \chi_{\{2\}} = 0, \quad (42)$$

where

$$\begin{cases} \chi_{\{1\}} = -s_1\theta_1^1 - s_2\theta_1^2, \\ \chi_{\{2\}} = -s_1\theta_2^1 - s_2\theta_2^2 \end{cases} . \quad (43)$$

Substituting (43) into (42) and taking into account (41) one obtains:

$$\left(\theta_1^1 - \frac{1}{\gamma}\theta_2^2\right)s_1^\gamma - \frac{1}{\gamma}s_1^{2\gamma-1}\theta_1^2 + s_1\theta_2^1 = 0. \quad (44)$$

Since (44) should equal to zero identically for any function s_1 for which (39) has a solution coefficients before different powers of s_1 should vanish simultaneously. Therefore,

$$\begin{cases} \theta_1^1 - \frac{1}{\gamma}\theta_2^2 = 0 \\ \theta_1^2 = 0 \\ \theta_2^1 = 0 \end{cases} . \quad (45)$$

The last two equations of system (45) imply that θ^i depends only on α_i . Now, since θ_1^1 depends only on α_1 , while θ_2^2 depends only on α_2 , the first of the equation of system (45) implies that both of these derivatives are constant

and finally the solution is given by:

$$\begin{cases} \theta^1 = A(\alpha_1 - \alpha_1^*) \\ \theta^2 = A\gamma(\alpha_1 - \alpha_2^*) \end{cases}, \quad (46)$$

where A , α_1^* , and α_2^* are arbitrary constants. The finite form of the symmetry group of equation (41) has the form

$$\begin{cases} \widetilde{\alpha}_1 = \alpha_1^* + (\alpha_1 - \alpha_1^*) \exp(A\tau) \\ \widetilde{\alpha}_2 = \alpha_2^* + (\alpha_2 - \alpha_2^*) \exp(A\gamma\tau) \\ \widetilde{s} = s \end{cases}. \quad (47)$$

Introducing β by

$$\beta = \exp(A\tau) \quad (48)$$

and putting $\alpha_1^* = \alpha_2^* = 0$ one can recognize the coordinate transformation (7). Now it is straightforward to check that in order for the second equation of system (6) and the boundary conditions to be invariant, function μ and parameters a , b , and c should transform according to (7).

So far, we have established that the boundary problem (6) is invariant with respect to transformations (7). Since these transformations leave the

surplus function unchanged ($\tilde{s} = s$), it should be under (7). To find the most general form of such an invariant, consider equation (39), where a, b, c are treated as additional coordinates. Therefore,

$$\alpha_1 \frac{\partial s}{\partial \alpha_1} + \gamma \alpha_2 \frac{\partial s}{\partial \alpha_2} + a \frac{\partial s}{\partial a} + \gamma b \frac{\partial s}{\partial b} + \gamma c \frac{\partial s}{\partial c} = 0. \quad (49)$$

To find the general solution start with writing the system of characteristics:

$$\frac{d\alpha_1}{\alpha_1} = \frac{d\alpha_2}{\gamma \alpha_2} = \frac{da}{a} = \frac{db}{\gamma b} = \frac{dc}{\gamma c} = \frac{ds}{0}. \quad (50)$$

Five independent first integrals of system (50) are

$$\left\{ \begin{array}{l} b/c = C_1 \\ b/a^\gamma = C_2 \\ (\alpha_1 - \delta_1(b/c, b/a^\gamma))^\gamma / (\alpha_2 - \delta_2(b/c, b/a^\gamma)) = C_3 \\ (\alpha_2 - \delta_2(b/c, b/a^\gamma)) / a^\gamma = C_4 \\ s = C_5 \end{array} \right. \cdot \quad (51)$$

Therefore, the general solution of equation (49) has a form

$$s = s\left(\frac{(\alpha_1 - \delta_1(b/c, b/a^\gamma))^\gamma}{\alpha_2 - \delta_2(b/c, b/a^\gamma)}, \frac{\alpha_2 - \delta_2(b/c, b/a^\gamma)}{a^\gamma}, \frac{b}{c}, \frac{b}{a^\gamma}\right). \quad (52)$$

Introducing ξ and ζ by (8) and omitting the parametric dependence one can write $s = s(\xi, \zeta)$, which is the change of variables that lead to equation (9).

Example 2. Let the individual's utility be given by:

$$u(\alpha, x, t) = \alpha_1 x_1 + \alpha_2 x_2 + \sqrt{\alpha_1 \alpha_2} x_3 - t$$

and the cost of production is

$$c(x) = \frac{1}{2}(x_1^2 + x_2^2 + \kappa x_3^2).$$

The set of possible types is given by

$$\Omega = \{\alpha \in R_+^2 : \alpha_1 + \alpha_2 < b\}, \tag{53}$$

which is an open convex set of $(0, b) \times (0, b)$. The distribution of types is given by a density function:

$$f(\alpha_1, \alpha_2) = \frac{\exp(-\alpha_1 - \alpha_2)}{1 - (b + 1) \exp(-b)}.$$

The value of the outside option is type independent and normalized to be

zero. It can be shown (Basov, (2002)) that if the surplus function solves the relaxed screening problem it should satisfy the following system

$$\sum_{i=1}^2 \left(\frac{\partial(x_i(\alpha, \nabla s) - \alpha_i)}{\partial \alpha_i} - x_i(\alpha, \nabla s) + \alpha_i \right) = 1 \quad (54)$$

$$\left\{ \begin{array}{l} \sum_{i=1}^2 \left(\frac{\partial(x_i(\alpha, \nabla s) - \alpha_i)}{\partial \alpha_i} - x_i(\alpha, \nabla s) + \alpha_i \right) = 1 \\ \sum_{i=1}^2 x_i(\alpha, \nabla s) = b \text{ for } \alpha_1 + \alpha_2 = b \end{array} \right. , \quad (55)$$

where

$$\begin{aligned} x_1(\alpha, \nabla s) &= \frac{\alpha_1((4\kappa\alpha_2 + \alpha_1)s_1 - \alpha_2s_2)}{\alpha_1^2 + \alpha_2^2 + 4\kappa\alpha_1\alpha_2} \\ x_2(\alpha, \nabla s) &= \frac{\alpha_2((4\kappa\alpha_1 + \alpha_2)s_2 - \alpha_1s_1)}{\alpha_1^2 + \alpha_2^2 + 4\kappa\alpha_1\alpha_2}. \end{aligned}$$

Calculating the symmetry group of the system (54) may seem a daunting task. Notice, however, that this group should take the boundary $\alpha_1 + \alpha_2 = b$ into itself. The most general transformation that does it has a form

$$\left\{ \begin{array}{l} \widetilde{\alpha}_1 = \alpha_1 - \tau\theta(\alpha) \\ \widetilde{\alpha}_2 = \alpha_2 + \tau\theta(\alpha) \end{array} \right. . \quad (56)$$

Surplus function invariant with respect to transformations (56) solves

$$\theta \frac{\partial s}{\partial \alpha_1} - \theta \frac{\partial s}{\partial \alpha_2} = 0. \quad (57)$$

Assuming $\theta \neq 0$ one finds

$$s = \varphi(\alpha_1 + \alpha_2), \quad (58)$$

where φ is arbitrary differentiable function. Substituting (58) into (54) one can see that the system has a solution of this form if and only if $\kappa = 1/2$. In this case the solution is given by

$$\left\{ \begin{array}{l} \varphi(z) = \frac{z^2}{2} - z - \ln z + \int_1^z \frac{\exp(t-b)}{t} dt + C \\ C + \varphi(z^*) = 0 \\ \varphi'(z^*) = 0. \end{array} \right. \quad (59)$$

Using envelope conditions (5) one can find the allocation

$$\left\{ \begin{array}{l} x_1 = \alpha_1 \left(1 - \frac{\alpha_1 + \alpha_2 + 1}{(\alpha_1 + \alpha_2)^2}\right) \\ x_2 = \alpha_2 \left(1 - \frac{\alpha_1 + \alpha_2 + 1}{(\alpha_1 + \alpha_2)^2}\right) \\ x_3 = 2\sqrt{\alpha_1 \alpha_2} \left(1 - \frac{\alpha_1 + \alpha_2 + 1}{(\alpha_1 + \alpha_2)^2}\right) \end{array} \right. . \quad (60)$$

Note that for $\kappa = 1/2$ the optimal allocation can be found using two other techniques developed by Armstrong (1996): using integration by rays and showing that the optimal tariff is cost based (see, Basov (2002)). Basov (2002) also showed that allocation (60) is implementable.

For $\kappa \neq 1/2$ the solution cannot be found in the form (58), which implies that $\theta = 0$ and the symmetry group of the problem (54) is trivial. In this case the only way to solve system (54) is by numerical integration. Knowing solution (60) is, however, useful even in this case, since implementability of (60) implies that the numerical solution for (54) is also implementable for κ sufficiently close to $1/2$.¹

6 DISCUSSION AND CONCLUSIONS

In this paper I described group theoretic methods that can be used for analyzing the boundary problems, which arise when the Hamiltonian method is applied to solve the relaxed problem for the multidimensional screening problem. This technique can provide some useful insights into the structure of solutions and some times may help to arrive at particular solutions.

¹See Basov (2002) for the implementability condition for nonlinear problems and its economic discussion.

In this paper I dealt mainly with the relaxed problem (though the explicit solutions obtained in both examples are implementable). It is well known (see, Carlier (2002) and Basov (2002)) that the implementability constraint can be formulated as a generalized convexity condition for surplus. There exists now simple characterization of the set of generalized convex functions for arbitrary utility. One might, however, hope to obtain such a characterization for a class of generalized convex functions symmetric with respect to a particular group. This, if achieved, can allow to characterize the solution of the complete problem in a closed form.

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