Rational Overconfidence and Excess Volatility in General Equilibrium

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Abstract. Rational beliefs are expectations which though consistent with empirical observations, may deviate from the true underlying probability measure under which data is generated. This probability measure is not necessarily stationary, but is required to fulfill a weaker condition, called WAMS. In the first part of this article we provide results on, as well as a decomposition of, WAMS measures and use this to demonstrate that an agent that adopts a non-stationary rational belief is rationally overconfident. Next we turn to defining various classes of WAMS stochastic processes which are suitable for equilibrium analysis. The most important class consists of Markov processes on a continuous state space which do not have time invariant transitions and are not stationary. To apply the theory to models of general equilibrium, we introduce the concept of a sunspot rational beliefs structure which can be considered as the exogenously specified part of a state process with rational beliefs. A part of the state space will be a set of sunspot variables which are not correlated with fundamentals but will, in equilibrium, affect prices and other endogenous variables. In contrast with the traditional approach, we do not assume that the true distribution of sunspots and fundamentals is known but only that agents hold rational, but diverse beliefs about these variables. Equilibria with sunspot rational belief structures have some desirable properties like anonymity and conditional rationality. In the final part of this work we provide a simple example of the use of our results. We consider an infinite state space model where agents make production decision before knowing prices. Under rational beliefs, unlike under rational expectations, mistakes persist even though all agents make forecasts that are statistically consistent with the equilibrium process. Due to the correlation of subjective beliefs brought about by the sunspots, the equilibrium exhibits excess price volatility.

JEL classification number: C16, C62, C65, D50, D51, D84.
Key words: Rational Beliefs, WAMS Markov processes, Continuous state space, Rational beliefs structures, Anonymity, Rational overconfidence, Volatility

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1 Introduction

It has been observed that even very sophisticated economic agents form expectations which differ significantly from each other (see f.i. Kurz, 2002 and Taylor, 1995). In many cases these diverse expectations cannot be explained by diversity of information. The other possible source of diversity of expectations among rational agents is different priors or different models of the underlying reality. The assumption that all agents have rational expectations, which is today predominant in all branches of economic theory, is based on the postulate that empirical observations eventually lead all agents to have the same underlying model. Thus in the rational expectations framework only the first source of diverse opinions is left. This is unsatisfactory for several reasons. Not only does it not square well with what we observe about agents' expectations. At a more fundamental level, it also ignores that (i) agents may be irrational and not use the available data in a correct way (ii) even if agents are rational, convergence of the posteriors may be slow (i.e. learning takes time) (iii) convergence may never happen in a non-stationary environment. In the rational beliefs framework (originally defined in Kurz, 1994a) the focus is on the last aspect of the learning process. A rather mild form of non-stationary named stability is postulated under which an empirical distribution of observable variables can still be derived from data, but where, unlike in the stationary case, this empirical distribution does not uniquely identify the true underlying probability distribution of the data. Agents form beliefs which are rational in the sense that these beliefs could have been generated by the empirical distribution that everyone observes. However, since the identification of the true distribution is not perfect, there is still left room for diverse opinions.

Despite the strong assumptions that the economic environment is stable and that agents form rational beliefs which are consistent with the empirical distribution, the behavior of the economy way be very different from what would be the case, if agents had rational expectations. As a consequence, the assumption that agents have rational beliefs may be used to explain a host of phenomena that have been declared 'puzzles' in the framework of rational expectations (see f.i. Kurz and Beltratti, 1997, Kurz and Motolese, 2001). The assumption may also be used to enhance our understanding of the role of money and monetary policy (Kurz, 2002, Motolese, 2003) or such phenomena as speculation (Wu and Guo, 2003). Finally, it may become a vehicle for studying public policy problems which have hitherto been left unnoticed and which cannot even be posed in rational expectations models (Nielsen, 2003).

The theory of rational beliefs is also connected to a growing literature of economics based on studies of patterns of behavior and reasoning among individuals. One of the repeated observations of this literature is that many economic agents are showing overconfidence in their own abilities (see f.i. Camerer, 1999 and Daniel, Hirschleifer and Subrahmanyam, 1998). In the context of expectations we argue that when agents employ non-stationary rational beliefs, they exhibit overconfidence since they assume an ability to make predictions that are more precise than what the empirical distribution would suggest. While the many empirical studies and experiments of the behavioral economics literature has convincingly demonstrated that agents' beliefs are diverse (and hence mutually inconsistent), there seems as yet not to be agreement within this literature about what kind of beliefs we should expect agents to hold. In particular, there is often a weak link between what agents are assumed to believe and what they observe[1]. In contrast, the theory of rational beliefs proposes, as a priori requirements on agents' expectations, that they are consistent with empirical observations while retaining the possibility that they may not be correct. Thus overconfidence is in our framework always rational that is, based on a subjective model that cannot be rejected by data.

When agents have rational beliefs, in general these beliefs are diverse and consequently not all agents if any, use the correct belief as a basis for their decisions. Many of the just cited articles using the theory of rational beliefs employ another important aspect of that theory, namely that rational beliefs tend to be more volatile than rational expectations. If the volatility of the individual beliefs are somehow correlated across agents (below we postulate that such correlation may be realized via sunspots) this volatility shows up in macro data, for instance in prices. This 'endogenous' volatility is added to the volatility stemming from aggregate exogenous shocks and the end result is an economy with excess volatility as compared to the situation where only exogenous shocks were the source of randomness.

As with rational expectations, the rational beliefs theory postulates that everything learnable has already been learned. This is one important assumption that makes both of these models of expectations well suited for general equilibrium analysis. A framework for proving existence of general equilibrium in models where agents have diverse rational beliefs was formulated and put to use in Nielsen(1994) (see also Nielsen, 1996). This framework, which has been used in numerous studies, sometimes in modified versions (see f.i., Kurz and Schneider, 1996, Kurz and Wu, 1996 and Nielsen, 2003) has two components. One is a class of stable but non-stationary stochastic processes (SIDS or SSM processes as defined later). The other component is a so called Rational Beliefs Structure (RBS). In a rational beliefs equilibrium (RBE) (defined in Kurz, 1994b) agents have expectations about endogenous and exogenous variables which are consistent with the observed behavior of equilibrium values and these equilibrium values are in turn a consequence of the actions of agents based on their rational beliefs. Rather than formulating the RBE as a fixed point in a space of beliefs (stochastic processes) it is convenient to formulate a structure of fundamentals, that is exogenous variables and rational beliefs (about fundamentals and beliefs), and then show the existence of a equilibrium point for a function between fundamentals and endogenous variables. In the resulting equilibrium agents have rational beliefs about exogenous as well as endogenous variables.

This idea is extended in two directions. In Nielsen(1994) both SIDS and SSM processes were considered.

[1]As an example, see Scheinkmann and Xiong(2003) who like us, assume that agents have the same information but interpret it differently (i.e. in terms of different models).

The latter are Markov chains with non-stationary transitions. However, their stability properties were only proven for the case where the state space is countable. Here we extend the stability result to the case where the state space is a measurable subset of a Euclidean space. The RBS of Nielsen(1996) had a conceptual weakness. Because it is formulated for a finite number of agents each agent is necessarily 'big' in such a framework, causing the fluctuations in his belief to show up in macro data like prices. Firstly, this dependence between own belief and prices should be detected empirically and this makes the assumption that agents take prices 'as given' less plausible. Secondly but highly related, when agents form expectations about future prices, which in an RBS are formulated as beliefs about future beliefs, they end up forming beliefs about their own future beliefs. The present work can be seen as a justification for these simplifying assumptions made in the earlier work. Here we work explicitly with a continuum of agents, so every individual is small (anonymous). Furthermore, the aggregate fluctuations generated by beliefs is a result of agents subjectively conditioning their beliefs on sunspots. This is done in a way such that the rationality of their beliefs is preserved. Also, in the RBS formulated here the future beliefs of any individual agent contains no information about future fluctuations in endogenous variables like prices. Another important aspect of this RBS is, that not only do agents condition their beliefs on the publicly observed sunspots, they also form rational beliefs about these sunspots. The dependence of prices on sunspots is in this way an imposed condition, and consequently sunspot influence on prices may well arise in models where, if agents had rational expectations, this would not be possible.

The present work considers most theoretical issues of relevance to rational beliefs. We present new results on its mathematical foundation and on stochastic processes applicable to general equilibrium models with rational beliefs as well as an example of how to apply the developed theory in a general equilibrium setting. As a consequence, this paper is self-contained and can be read without any prior acquaintance with the literature on rational beliefs. Section 2, to follow, starts out with a study of WAMS (or stable) probability measures and, based on this, provides an elucidation and interpretation of the concept of rational beliefs. We then, in Section 3, turn to presenting various results on the existence of classes of stochastic processes which are stable but non-stationary. Most importantly we extend the concept of SSM processes to a continuous state space. Section 4 is devoted to a discussion of how we may formally model the assumption that individual agents are anonymous, but still in the aggregate, cause fluctuations in endogenous variables via their beliefs. The concept of Sunspot Rational Belief Structures with types is introduced and discussed by means of some simple examples and then formally defined. Also we discuss the issue of conditional stability and rationality, an issue that becomes relevant in the context of Markovian beliefs. We show that structural independence (defined in Nielsen, 1994) is (almost) sufficient for rational beliefs to be conditionally rational. Section 5 applies the developed framework to a simple general equilibrium model in the tradition of the cobweb model, where agents make decisions about output before knowing market prices. Section 6 concludes. In the appendix we have collected certain proofs that we deemed to be similar to existing ones in the literature.

2 WAMS measures and rational beliefs

2.1 Preliminaries

The generic set of state variables is denoted S, a (Borel measurable) subset of $\Re^K$. For any (Borel measurable) set Y (in a topological space) we denote by $\mathcal{B}(Y)$ the Borel σ-algebra for Y. $T : S^\infty \to S^\infty$ is the shift transformation i.e. $T(s_1, s_2, \ldots) = (s_2, s_3, \ldots)$ and letting ν be a probability measure on $(S^\infty, \mathcal{B}(S^\infty))$, $(S^\infty, \mathcal{B}(S^\infty), \nu, T)$ becomes a dynamical system. We often denote a sequence in S^∞ by $\langle s \rangle$. For a given measure, ν, $E_\nu(\cdot)$ denotes the expectation operator under ν. By δ_s we denote the Dirac measure at $s \in S$. A set $A \in \mathcal{B}(S^\infty)$ is said to be invariant (T-invariant) if $T^{-1}A = A$. We denote by $\mathcal{I}$ the set of invariant sets in $\mathcal{B}(S^\infty)$. The dynamical system $(S^\infty, \mathcal{B}(S^\infty), \nu, T)$ is said to be ergodic if any invariant set has either measure 1 or measure 0. Finally, let $\mathcal{C}(S^\infty)$ be the cylinders in $\mathcal{B}(S^\infty)$ and let $\mathcal{C}^d(S^\infty)$ be the d-dimensional cylinders (in general, $\mathcal{C}(Y^\infty)$ is the set of cylinders in Y^∞). By $\mathcal{N}$, we denote the set of natural numbers and $1_B, B \in \mathcal{B}(S^\infty)$ is the indicator function, defined on S^∞. Of the following definitions that of AMS is from Gray and Kieffer(1980) while the other two are from Kurz(1994a):

Definition 1 *Stability*

The dynamical system $(S^\infty, \mathcal{B}(S^\infty), \nu, T)$ as well as the measure ν are said to be stable if for all cylinders $C \in \mathcal{C}(S^\infty)$:

$$\lim_{J\to\infty} \frac{1}{J} \sum_{j=0}^{J-1} 1_C(T^j\langle s \rangle) \text{ exists for } \nu\text{-a.a. } \langle s \rangle \in S^\infty \quad \blacksquare$$

Definition 2 *WAMS and AMS*

The dynamical system $(S^\infty, \mathcal{B}(S^\infty), \nu, T)$ as well as the measure ν are said to be Weakly Asymptotic Mean Stationary (WAMS) if for all cylinders C, we have $\lim_{J\to\infty} \frac{1}{J}\sum_{j=0}^{J-1} \nu(T^{-j}C)$ exists. If the convergence is for all measurable sets the system is said to be AMS (Asymptotic Mean Stationary) $\blacksquare$

One can show that $(S^\infty, \mathcal{B}(S^\infty), \nu, T)$ is stable if and only if it is WAMS, (Proposition 2 of Kurz, 1994a). One of the implications is straight forward: If the system is stable then for any cylinder C we have $\frac{1}{J}\sum_{j=0}^{J-1} \nu(T^{-j}C) = \int \frac{1}{J}\sum_{j=0}^{J-1} 1_C(T^j\langle s \rangle)\nu(d\langle s \rangle)$ which converges by Lebesgue's bounded convergence theorem. The other implication is proved by using a suitably adjusted proof of Birkhoff's ergodic theorem.

If $(S^\infty, \mathcal{B}(S^\infty), \nu, T)$ is WAMS there is an *associated stationary measure* $\overline{\nu}$ s.t. $\forall C \in \mathcal{C}(S^\infty)$: $\lim_{J\to\infty} \frac{1}{J}\sum_{j=0}^{J-1} \nu(T^{-j}C) = \overline{\nu}(C)$ (Proposition 3, Kurz, 1994). This can easily be established by noting that for each d, the set valued function $\overline{\nu}_d$ defined on the σ-algebra $\mathcal{C}^d(S^\infty)$ by $\overline{\nu}_d(C) = \lim_{J\to\infty} \frac{1}{J}\sum_{j=0}^{J-1} \nu(T^{-j}C)$ is a probability measure (a consequence of the Vitali-Hahn-Saks Theorem) and that the sequence $\{\overline{\nu}_d\}_{d=1}^\infty$ is consistent such that the Kolmogorov Extension Theorem implies the existence of a probability measure $\bar{\nu}$ on $\mathcal{B}(S^\infty)$ (which when restricted to $\mathcal{C}^d(S^\infty)$ is equal to $\bar{\nu}_d$). We always use a bar over a stable measure to denote the associated stationary measure. Note, that when the system is stationary, $\bar{\nu} = \nu$.

The following example is worth having in mind, when one wants to understand the notion of stability and prove results involving it.

Example 1

A sequence of probability measures $\{\mu_n\}$ and a probability measure μ, all on (the Borel subsets of) $[0,1)$, and a countable generating field, $\mathcal{F}$, s.t. $\mu_n(F) \to \mu(F)$, $\forall F \in \mathcal{F}$, but for some $A \in \sigma(\mathcal{F}) = \mathcal{B}([0,1))$, $\mu_n(A)$ does not converge to $\mu(A)$.

Let $\mu_n = \delta_{\frac{1}{n}}$ and let $\mu = \delta_0$. Note that then μ_n converges to μ in the topology of weak convergence. Finally, let $\mathcal{F} = \{\cup_{n=1}^N [\underline{q}_n, \bar{q}_n) : N \in \mathcal{N}, \underline{q}_n, \bar{q}_n$ are rationals in $[0,1)\}$. Then note that $\mu_n(\{0\}) = 0$ does not converge to $\mu(\{0\}) = 1$ ■

2.2 Decomposition of WAMS probability measures

Lemma 1

Suppose S is countable ($= \{s_1, s_2, \ldots\}$) and ν is WAMS. Let

$$A_N^d = \left[\times_{n=1}^d \{s_N, s_{N+1}, s_{N+2}, \ldots\}\right] \times S^\infty, d \in \mathcal{N}, N \in \mathcal{N}$$

Then for ν a.a. $\langle s \rangle$ we have that

$$lim_{J\to\infty} \frac{1}{J} \sum_{j=0}^{J-1} 1_{A_N^d}(T^j \langle s \rangle)$$

is well defined for all N and d and, calling this limit $a_N^d(\langle s \rangle)$, that for all d, $a_N(\langle s \rangle) \to 0$ as $N \to \infty$.

Proof: Since ν is WAMS, for ν a.a. $\langle s \rangle$ the limit is well defined for all N and d. Suppose, for some d that the rest of the lemma did not hold. There would be a set $B \in \mathcal{B}(S^\infty)$ with $\nu(B) > 0$ and $\epsilon > 0$ s.t. for $\langle s \rangle \in B$ we have $a_N^d(\langle s \rangle) > \epsilon, \forall N$.

$$\bar{\nu}(A_N^d) = \int_{S^\infty} \limsup \frac{1}{J} \sum_{j=0}^{J-1} 1_{A_N^d}(T^j \langle s \rangle) \nu(d\langle s \rangle) \geq \int_B \limsup \frac{1}{J} \sum_{j=0}^{J-1} 1_{A_N^d}(T^j \langle s \rangle) \nu(ds) \geq \nu(B)\epsilon, \forall N$$

But $\bar{\nu}(A_N^d) \to 0$ as $N \to \infty$, a contradiction ■

The following proposition is an extension of Proposition 2 of Nielsen(1996).

Proposition 1

Suppose S is countable $(= \{s_1, s_2, \ldots\}$ and that ν is WAMS. Then for ν a.a. $\langle s \rangle \in S^\infty$ there is a stationary probability measure, $P_{\langle s \rangle}$ on $(S^\infty, \mathcal{B}(S^\infty))$ s.t.

$$\frac{1}{J}\sum_{j=0}^{J-1} 1_C(T^j\langle s \rangle) \to P_{\langle s \rangle}(C), \forall C \in \mathcal{C}(S^\infty) \tag{1}$$

Proof: Let the field $\tilde{\mathcal{C}}^d \subset \mathcal{C}^d(S^\infty)$ consist of $\emptyset$, S^∞ as well as all d-dimensional cylinders of the form $C^d \times S^\infty$ where $C^d \subset S^d$ is finite or cofinite. We have $\sigma(\tilde{\mathcal{C}}^d) = \mathcal{C}^d(S^\infty)$ and, since $\tilde{\mathcal{C}}^d$ is countable, that for ν a.a. $\langle s \rangle$ there is a function $\tilde{P}_{\langle s \rangle} : \cup_{d=1}^\infty \tilde{\mathcal{C}}^d \to [0, 1]$ s.t.

$$\frac{1}{J}\sum_{j=0}^{J-1} 1_C(T^j\langle s \rangle) \to \tilde{P}_{\langle s \rangle}(C), \forall C \in \cup_{d=1}^\infty \tilde{\mathcal{C}}^d \tag{2}$$

We also have for ν a.a $\langle s \rangle$ that

$$\{C_n\} \downarrow \emptyset \text{ in } \tilde{\mathcal{C}}^d \Rightarrow \lim_{n\to\infty} \limsup_{J\to\infty} \frac{1}{J}\sum_{j=0}^{J-1} 1_{C_n}(T^j\langle s \rangle) = 0 \tag{3}$$

The last claim is a consequence of Lemma 1 since if $\{C_n\} \downarrow \emptyset$ with $C_n \neq \emptyset, \forall n$ we have for all N that $\exists \overline{n}$ s.t. $n > \overline{n} \Rightarrow C_n \subset A_N^d$. It follows that for ν a.a. $\langle s \rangle$, $\tilde{P}_{\langle s \rangle}(C_n) \to 0$ if $\{C_n\} \downarrow \emptyset$ in $\tilde{\mathcal{C}}^d$. Because of this and since $\tilde{P}_{\langle s \rangle}$ is obviously finitely additive it is then also countably additive on $\tilde{\mathcal{C}}^d$. Thus there is for each d a unique extension of $\tilde{P}_{\langle s \rangle}$ restricted to $\tilde{\mathcal{C}}^d$ to a probability measure, $P^d_{\langle s \rangle}$ on $\mathcal{C}^d(S^\infty)$ (Caratheodory's Extension Theorem). Since the resulting sequence, $\{P^d_{\langle s \rangle}\}_{d=1}^\infty$ is consistent we have by Kolmogorov's extension theorem that there is a probability measure $P_{\langle s \rangle}$ on $(S^\infty, \mathcal{B}(S^\infty))$ that is consistent with $\{P^d_{\langle s \rangle}\}_{d=1}^\infty$.

Let $C \in \mathcal{C}(S^\infty)$. There are $\overline{C}_n \downarrow C$ and $\underline{C}_n \uparrow C$, with $\overline{C}_n, \underline{C}_n \in \cup_{d=1}^\infty \tilde{\mathcal{C}}^d, \forall n$. Thus

$$P_{\langle s \rangle}(\overline{C}_n) \geq \limsup \frac{1}{J}\sum_{j=0}^{J-1} 1_C(T^j\langle s \rangle) \geq \liminf \frac{1}{J}\sum_{j=0}^{J-1} 1_C(T^j\langle s \rangle) \geq P_{\langle s \rangle}(\underline{C}_n), \forall n \tag{4}$$

and by continuity from above and below in the limit, as $n \to \infty$ all inequalities in (4) hold as equalities ■
For this countable case we call a sequence $\langle s \rangle$, for which there is convergence to $P\langle s \rangle$ for all cylinders C, *ν-typical.*

When the state space is not countable there will in general not be convergence of the empirical frequency for all cylinders. In order to derive a probability measure, $P_{\langle s \rangle}$ which, like in the countable case, is based on the empirical frequencies of a countable set of cylinders we introduce the notion of a so called *standard field*[2].

[2]In Gray(1987) standard fields are uses to generate an empirical distribution for AMS measures. The concept of standard fields originates with Christensen(1974).

Definition 3 *Standard Field*

A field $\mathcal{F} \subset \mathcal{B}(S^\infty)$. is said to be standard if there is a sequence $\{\mathcal{F}_n\}$ of finite fields s.t.

(i) $\mathcal{F}_n \uparrow \mathcal{F}$

(ii) If $\{A_n\}$ is a decreasing sequence of sets, where, for each n, A_n is a non-empty atom of $\mathcal{F}_n$ then $\cap_{n=1}^{\infty} A_n \neq \emptyset$ ■

Definition 4 *Standard measurable space*

A measurable space $(\Omega, \mathcal{B})$ is said to be standard if there is a standard field, $\mathcal{F}$ that generates $\mathcal{B}$ (i.e. $\sigma(\mathcal{F}) = \mathcal{B}$) ■

Definition 5 *Countable extension property for a field*

A field $\mathcal{F}$ in $\mathcal{B}$ is said to have the countable extension property if every non-negative, finitely additive set function, P defined on $\mathcal{F}$ with $P(\Omega) = 1$ is also countably additive ■

The following theorems are respectively Theorem 2.6.1 and Theorem 3.3.1 of Gray(1987)

Theorem 1

A field has the countable extension property if and only if it is standard ■

Theorem 2

If Ω is a complete, separable, metric space then $(\Omega, \mathcal{B}(\Omega))$ is standard ■

It is a direct consequence of Lemma 2.3.1 of Gray(1987), that if S is a complete, separable, metric space then there is a countable standard field $\mathcal{F} \subset \mathcal{C}(S^\infty)$ such that $\sigma(\mathcal{F}) = \mathcal{B}(S^\infty)$. Thus when $S = \Re^N$ and $(S^\infty, \mathcal{B}(S^\infty), T, \nu)$ is WAMS, the empirical distribution for a given realization $\langle s \rangle$ is derived by looking at the frequencies of members of a countable standard generating field, $\mathcal{F}$. Specifically, we have for ν a.a. $\langle s \rangle$ that there is a set function $\tilde{P}_{\langle s \rangle}$ defined on $\mathcal{F}$ with $\tilde{P}_{\langle s \rangle}(S^\infty) = 1$, which is non-negative and finitely additive and s.t.

$$lim_{J \to \infty} \frac{1}{J} \sum_{j=0}^{J-1} 1_C(T^j \langle s \rangle) = \tilde{P}_{\langle s \rangle}(C), \forall C \in \mathcal{F} \tag{5}$$

Because $\mathcal{F}$ is standard, $\tilde{P}_{\langle s \rangle}$ is also countably additive and thus has a unique extension to a probability measure $P_{\langle s \rangle}$ on $\mathcal{B}(S^\infty)$. Associated with a realization $\langle s \rangle$ is then *the* empirical distribution $P_{\langle s \rangle}$. The use of the definite article here is justified by proposition 2 and its corollary to follow. Define for $C \in \mathcal{B}(S^\infty)$, $\mathcal{A}_C = \{\langle s \rangle \in S^\infty : \frac{1}{J} \sum_{j=0}^{J-1} 1_C(T^j \langle s \rangle)$ converges as $J \to \infty\}$.

Lemma 2

Suppose ν is WAMS and that for some $B \in \mathcal{B}(S^\infty)$ we have $\nu(B) > 0$. Define $\hat{\nu}$ on $\mathcal{B}(S^\infty)$ by $\hat{\nu}(A) = \frac{\nu(A \cap B)}{\nu(B)}$. Then $\hat{\nu}$ is WAMS.

Proof: $\hat{\nu}$ is WAMS iff $\hat{\nu}(A_C) = 1, \forall C \in \mathcal{C}(S^\infty)$. But since $\nu(A_C) = 1$, we have $\frac{\nu(A_C \cap B)}{\nu(B)} = 1$ ■

Proposition 2

Suppose ν is WAMS and let $\mathcal{F}$ be a standard generating field consisting of cylinders. Let $P_{\langle s\rangle}$ be the empirical distribution associated with $\mathcal{F}$ and $\langle s\rangle$. If $C \in \mathcal{C}(S^\infty)$ then for ν a.a. $\langle s\rangle \in S^\infty : \frac{1}{J}\sum_{j=0}^{J-1} 1_C(T^j\langle s\rangle) \to P_{\langle s\rangle}(C)$.

Proof: Suppose not. There is then (w.l.o.g.) $\epsilon > 0$ and $B \in \mathcal{B}(S^\infty)$ s.t. $\nu(B) > 0$ and such that for $\langle s\rangle \in B$ we have

$$\lim_{J\to\infty} \frac{1}{J}\sum_{j=0}^{J-1} 1_C(T^j\langle s\rangle) - P_{\langle s\rangle}(C) > \epsilon \tag{6}$$

Let $\hat{\nu}$ on $\mathcal{B}(S^\infty)$ be defined by $\hat{\nu}(A) = \frac{\nu(A\cap B)}{\nu(B)}$ and by Lemma 2, $\hat{\nu}$ is WAMS. Consequently,

$$\int \limsup_{J\to\infty} \frac{1}{J}\sum_{j=0}^{J-1} 1_C(T^j\langle s\rangle)\hat{\nu}(d\langle s\rangle) = \overline{\hat{\nu}}(C), \forall C \in \mathcal{C}(S^\infty)$$

Also, $\overline{P} \equiv \int_B P_{\langle s\rangle}\hat{\nu}(d\langle s\rangle)$ is a probability measure ($\overline{P}(S^\infty) = 1$, $\overline{P}$ is obviously finitely additive and if $A_n \downarrow \emptyset$ then since $P_{\langle s\rangle}(A_n) \to 0$ for all $\langle s\rangle$, we have $\overline{P}(A_n) \to 0$). Finally note that $\overline{P}$ and $\overline{\hat{\nu}}$ agree for all $F \in \mathcal{F}$, which implies that they are identical and this is in contradiction with (6) ■

Corollary 1

If ν is WAMS and if $\tilde{\mathcal{C}}$ is a countable set of cylinders, then there is for each $\langle s\rangle \in S^\infty$ a probability measure $P_{\langle s\rangle}$ such that for ν a.a. $\langle s\rangle$:

$$\frac{1}{J}\sum_{j=0}^{J-1} 1_C(T^j\langle s\rangle) \to P_{\langle s\rangle}(C), \forall C \in \tilde{\mathcal{C}} \ \blacksquare$$

We next study the measures $P_{\langle s\rangle}$ in more detail. First we show that they are ergodic.

Lemma 3

Suppose that for $C \in \mathcal{B}(S^\infty)$ and ν a.a. $\langle s\rangle$

$$\frac{1}{J}\sum_{j=0}^{J-1} 1_C(T^j\langle s\rangle) \to \beta. \ \text{ Then } \frac{1}{J}\sum_{j=0}^{J-1} \nu((T^{-j}C)\cap C) \to \beta\nu(C)$$

Proof: $\frac{1}{J}\sum_{j=0}^{J-1} \nu((T^{-j}C)\cap C) = \frac{1}{J}\sum_{j=0}^{J-1}\int_C 1_C(T^j\langle s\rangle)\nu(d\langle s\rangle) = \int_C \frac{1}{J}\sum_{j=0}^{J-1} 1_C(T^j\langle s\rangle)\nu(d\langle s\rangle) \to \nu(C)\beta$ ■

Proposition 3

Suppose that

$$\frac{1}{J}\sum_{j=0}^{J-1} 1_C(T^j\langle s\rangle) \to P_{\langle s\rangle}(C), \forall C \in \tilde{\mathcal{C}}$$

where $\tilde{\mathcal{C}}$ is a generating field with the property that $C \in \tilde{\mathcal{C}} \Rightarrow T^{-1}C \in \tilde{\mathcal{C}}$. Then $P_{\langle s\rangle}$ is stationary and ergodic.

Proof:

$$\frac{1}{J}\sum_{j=0}^{J-1} 1_{T^{-1}C}(T^j\langle s\rangle) = \frac{1}{J}\sum_{j=1}^{J} 1_C(T^j\langle s\rangle) \to P_{\langle s\rangle}(C), \forall C \in \tilde{\mathcal{C}}$$

so we have $P_{\langle s\rangle}(T^{-1}C) = P_{\langle s\rangle}(C), \forall C \in \tilde{\mathcal{C}}$ which implies stationarity of $P_{\langle s\rangle}$.
By Lemma 6.7.4 of Gray(1987), $P_{\langle s\rangle}$ is ergodic if

$$\frac{1}{J}\sum_{j=0}^{J-1} P_{\langle s\rangle}(T^{-j}C \cap C) \to P_{\langle s\rangle}(C)^2, \forall C \in \tilde{\mathcal{C}} \tag{7}$$

$\frac{1}{J}\sum_{j=0}^{J-1} P_{\langle s\rangle}(T^{-j}C \cap C) = \frac{1}{J}\sum_{j=0}^{J-1} \lim_{K\to\infty} \frac{1}{K}\sum_{k=0}^{K-1} \delta_{\langle s\rangle}(T^{-k}((T^{-j}C \cap C))$. By Lemma 3 we have

$$\frac{1}{J}\sum_{j=0}^{J-1} \delta_{\langle s\rangle}(T^{-k}((T^{-j}C)\cap C)) = \frac{1}{J}\sum_{j=0}^{J-1} \delta_{\langle s\rangle}[(T^{-j}(T^{-k}C))\cap(T^{-k}C)] \to P_{\langle s\rangle}(T^{-k}C)\delta_{\langle s\rangle}(T^{-k}C) = P_{\langle s\rangle}(C)\delta_{\langle s\rangle}(T^{-k}C)$$

But $\lim_{K\to\infty}\sum_{k=1}^{K-1} \frac{1}{K} P_{\langle s\rangle}(C)\delta_{\langle s\rangle}(T^{-k}C) = P_{\langle s\rangle}(C)^2$ so (7) holds ■
Note, that the assumption that $C \in \tilde{\mathcal{C}} \Rightarrow T^{-1}C \in \tilde{\mathcal{C}}$ is innocuous by Proposition 2.

Remark 1
The preceding propositions provide a strengthening of Proposition 4 of Kurz(1994), since we can use them to conclude that for any WAMS system $(S^\infty, \mathcal{B}(S^\infty), \mu, T)$ we have for μ a.a. $\langle s\rangle$ an associated ergodic stationary system $(S^\infty, \mathcal{B}(S^\infty), T, P_{\langle s\rangle})$ that is derived from the empirical frequencies generated by $\langle s\rangle$ ■

The following proposition shows the equivalent of Birkhoff's Ergodic Theorem for WAMS probability measures.

Proposition 4
Let ν be WAMS and $C \in \mathcal{C}(S^\infty)$. There is a version of the conditional probability, $\overline{\nu}(C|\mathcal{I})$, s.t. for ν a.a. $\langle s\rangle$:

$$\frac{1}{J}\sum_{j=0}^{J-1} 1_C(T^j\langle s\rangle) \to \overline{\nu}(C|\mathcal{I})(\langle s\rangle)$$

Proof: Define the function $P(C)$ on S^∞ by:
$P(C)(\langle s\rangle) = \lim_{J\to\infty} \frac{1}{J}\sum_{j=0}^{J-1} 1_C(T^j\langle s\rangle)$ if this limit exists, $P(C)(\langle s\rangle) = \overline{\nu}(C)$ else.
Then for ν a.a. $\langle s\rangle$: $\frac{1}{J}\sum_{j=0}^{J-1} 1_C(T^j\langle s\rangle) \to P(C)(\langle s\rangle)$. Also for any interval $(a,b]$ with rational endpoints (there are countably many such) $\{\langle s\rangle : P(C)(\langle s\rangle) \in (a,b]\} \in \mathcal{I}$, so $P(C)$ is measurable $\mathcal{I}$. By Birkhoff's Ergodic Theorem, for $\overline{\nu}$ a.a. $\langle s\rangle$: $\frac{1}{J}\sum_{j=0}^{J-1} 1_C(T^j\langle s\rangle) \to \overline{\nu}(C|\mathcal{I})(\langle s\rangle)$. We conclude that $P(C)$ is a version of $\overline{\nu}(C|\mathcal{I})(\langle s\rangle)$ ■
We next report a result on the relationship between ν and $\overline{\nu}$.

Proposition 5
If ν is WAMS and ergodic, $\overline{\nu}$ is ergodic.
Proof: By Kurz(1994) Corollary to Proposition 3, if ν is ergodic we have $\lim_{J\to\infty} \frac{1}{J}\sum_{j=0}^{J} 1_C(T^j\langle s\rangle) = \overline{\nu}(C)$ for ν a.a. $\langle s\rangle$. In other words, $P_{\langle s\rangle} = \overline{\nu}$ for ν a.a. $\langle s\rangle$. But $P_{\langle s\rangle}$ is ergodic ■

An implication in the other direction does not hold (Example 2, below). To see this (and for other purposes) it is useful to consider the smallest invariant sets in S^∞.

Definition 6

$$\text{Let } \langle s\rangle \in S^\infty. \text{ Define } I_{\langle s\rangle} \equiv \cup_{j=0}^{\infty} \cup_{k=0}^{\infty} T^{-k}(T^j\langle s\rangle) \blacksquare$$

Lemma 4

$I_{\langle s\rangle}$ is invariant.

Proof: For every j, $T^{-1}(\cup_{k=0}^{\infty} T^{-k}(T^j\langle s\rangle)) = \cup_{k=1}^{\infty} T^{-k}(T^j\langle s\rangle) \subset \cup_{k=0}^{\infty} T^{-k}(T^j\langle s\rangle) \subset I_{\langle s\rangle}$. So $T^{-1}I_{\langle s\rangle} \subset I_{\langle s\rangle}$

That $I_{\langle s\rangle} \subset T^{-1}I_{\langle s\rangle}$ follows from the fact that for each j, $T^j\langle s\rangle \in T^{-1}(T^{j+1}\langle s\rangle) \subset T^{-1}I_{\langle s\rangle}$ $\blacksquare$

Let S be countable and let $\mathcal{P}$ be the set of stationary and ergodic probability measures on $(S^\infty, \mathcal{B}(S^\infty))$. Suppressing reference to S we define for any $P \in \mathcal{P}$

$$K_P \equiv \{\langle s\rangle \in S^\infty : \frac{1}{J}\sum_{j=0}^{J-1} 1_C(T^j\langle s\rangle) \to P(C) \text{ as } J \to \infty, \forall C \in \mathcal{C}(S^\infty)\}$$

Then let $\mathcal{I}_{\mathcal{P}} = \sigma(\{K_P : P \in \mathcal{P}\})$. Since $K_P \in \mathcal{I}$, $\mathcal{I}_{\mathcal{P}} \subset \mathcal{I}$. When S is not countable we define for any stationary and ergodic probability measure P on S^∞ and for any countable (generating) set $\tilde{\mathcal{C}}$ of cylinders,

$$K_P(\tilde{\mathcal{C}}) \equiv \{\langle s\rangle \in S^\infty : \frac{1}{J}\sum_{j=0}^{J-1} 1_C(T^j\langle s\rangle) \to P(C) \text{ as } J \to \infty, \forall C \in \tilde{\mathcal{C}}\}$$

$K_P(\tilde{\mathcal{C}})$ is invariant and defining $\mathcal{I}_{(\mathcal{P},\tilde{\mathcal{C}})} = \sigma(\{K_P(\tilde{\mathcal{C}}) : P \in \mathcal{P}\})$ we have $\mathcal{I}_{(\mathcal{P},\mathcal{C})} \subset \mathcal{I}$.

Remark 2

(i) Suppose that for some $\langle s\rangle \in S^\infty$, that $T^j\langle s\rangle = T^k\langle s\rangle$ for some $k > j$ and that $T^j\langle s\rangle \neq T^h\langle s\rangle$ for all $j < h < k$. So $T^{j+m}\langle s\rangle = T^{k+m}\langle s\rangle$, $\forall m \geq 0$. In particular we have for $n = 1$ that

$$T^{j+m}\langle s\rangle = T^{j+n(k-j)+m}, \forall m \geq 0 \tag{8}$$

But if (8) holds for n, it also holds for $n+1$, since $T^{j+m}\langle s\rangle = T^{j+n(k-j)+m}\langle s\rangle = T^{k+n(k-j)+m}\langle s\rangle = T^{j+(n+1)(k-j)+m}\langle s\rangle$. As a consequence, the empirical measure $P_{\langle s\rangle}$ generated by $\langle s\rangle$ has a finite support and

$$P_{\langle s\rangle}(\{s_{j+m}, s_{j+m+1}, \ldots\}) = \frac{1}{k-j} \text{ for } m = 1, 2, \ldots, k-j$$

This implies, on the other hand, that if P is any stationary and ergodic probability measure with a marginal on S that either has a non-finite support or else is non-deterministic then we have for P a.a. $\langle s\rangle$ that

$$T^j\langle s\rangle \neq T^k\langle s\rangle \text{ for } j \neq k \tag{9}$$

(ii) If (9) holds for $\langle \overline{s}\rangle$ there is $B \in \mathcal{B}(S^\infty)$ s.t. $\limsup \frac{1}{J}\sum_{j=0}^{J-1} 1_B(T^j\langle \overline{s}\rangle) > \liminf \frac{1}{J}\sum_{j=0}^{J-1} 1_B(T^j\langle \overline{s}\rangle)$. B can be constructed as follows:

$$B = \cup_{k=1}^{\infty} \cup_{j=10^{2k-1}}^{10^{2k}} \{T^j\langle \overline{s}\rangle\}$$

(iii) In the countable case, if $(S^\infty, \mathcal{B}(S^\infty), T, \nu)$ is WAMS, then for ν a.a. $\langle s\rangle$, $\delta_{\langle s\rangle}$ is WAMS (but in general not AMS as was just demonstrated). This is in particular the case if the dynamical system is actually stationary. Based on a particular realization $\langle s\rangle$ there is no way to see whether the dynamical system, that $\langle s\rangle$ is a realization of, is stationary, AMS or WAMS. In all cases the empirical frequency will converge for all cylinders but not for all infinitely dimensional sets.
(iv) When S is not countable, in general, $\delta_{\langle s\rangle}$ is not WAMS for a.a. $\langle s\rangle$. To see this take a WAMS, ergodic probability measure μ for which $\overline{\mu}_1$, the marginal of $\overline{\mu}$ on S does not have a countable support. $\overline{\mu}_1$ has at most countably many atoms, let $A \subset S$ be the set consisting of them. Let $\mathcal{J} \subset \mathcal{C}^1(S^\infty)$ be the countable set of one-dimensional cylinders with rational endpoints and let $H = \{\langle s\rangle : \frac{1}{J}\sum_{j=0}^{J-1} 1_E(T^j\langle s\rangle) \to \overline{\mu}(E), \forall E \in \mathcal{J}\}$, so $\overline{\mu}(H) = 1$. Now let $F \subset S \setminus A$ be finite. There is for all $\epsilon > 0$ a finite collection $\{E_n\}$ in $\mathcal{J}$ s.t. $\cup E_n \subset S \setminus F$ and $P(\cup E_n) > 1 - \epsilon$. Thus $\limsup_{J\to\infty} \frac{1}{J}\sum_{j=0}^{J-1} 1_{F\times S^\infty}(T^j\langle s\rangle) \leq \epsilon$ and consequently, since ϵ was arbitrary, $\lim_{J\to\infty} \frac{1}{J}\sum_{j=0}^{J-1} 1_{F\times S^\infty}(T^j\langle s\rangle) = 0$ for $\langle s\rangle \in H$.
We can use this observation to construct for ν a.a. $\langle s\rangle$ a cylinder C such that $\frac{1}{J}\sum_{j=0}^{J-1} 1_C(T^j\langle s\rangle)$ does not converge. Let $J_1 \in \mathcal{N}$ be given and let $C_1 = \{s_1, s_2, \ldots, s_{J_1}\} \cup A$. There is $J_2 > J_1$ s.t.

$$\frac{1}{J_2}\sum_{j=0}^{J_2-1} 1_{C_1\times S^\infty}(T^j\langle s\rangle) < \frac{1}{4}(1 - \overline{\mu}_1(A))$$

There is then $J_3 > J_2$ s.t. if $C_3 = C_1 \cup \{s_{J_2+1}, \ldots, s_{J_3}\}$ then

$$\frac{1}{J_3}\sum_{j=0}^{J_3-1} 1_{C_3\times S^\infty}(T^j\langle s\rangle) > \frac{3}{4}(1 - \overline{\mu}_1(A))$$

We can then pick $J_4 > J_3$ s.t.

$$\frac{1}{J_4}\sum_{j=0}^{J_4-1} 1_{C_3\times S^\infty}(T^j\langle s\rangle) < \frac{1}{4}(1 - \overline{\mu}_1(A))$$

and so on. Letting $C = \cup C_j \times S^\infty$ we have the desired non convergence.
Note that also for the uncountable case are we unable to detect whether the underlying dynamical system is stationary, AMS or just WAMS, since in all cases an empirical distribution can be derived, but there are cylinders for which the empirical frequency does not converge.
(v) For any stationary and ergodic probability measure P, $P(K_P) = 1$, $P(K_P(\tilde{\mathcal{C}})) = 1$ in the countable and uncountable case respectively (see proof of Proposition 5). ■

Example 2 *ν is not ergodic, but $\overline{\nu}$ is.*

Let S be countable and let $\langle s'\rangle \in K_P$ where P is non-trivial, stationary and ergodic. By the previous remark there is $B \in \mathcal{B}(S^\infty)$ s.t. $\frac{1}{J}\sum_{j=0}^{J-1} 1_B(T^j\langle s'\rangle)$ does not converge. However, for P a.a. $\langle s\rangle$ we have that

$\frac{1}{J}\sum_{j=0}^{J-1} 1_B(T^j\langle s\rangle)$ does converge. So there is $\langle s''\rangle \in K_P$ s.t. $I_{\langle s'\rangle} \neq I_{\langle s''\rangle}$. Then let $\nu' = \lambda\delta_{\langle s'\rangle} + (1-\lambda)\delta_{\langle s''\rangle}$ with $\lambda \in (0,1)$. Since $\nu'(I_{\langle s'\rangle}) = \lambda$, ν' is not ergodic but $\overline{\nu}' = P$ is ergodic[3] ■

Remark 3

Under the conditions of Example 2, we have $\delta_{\langle s'\rangle}(I_{\langle s'\rangle}) = 1$. But, since $P(T^{-k}(T^j\langle s'\rangle)) = 0, \forall k, j$, we have $P(I_{\langle s'\rangle}) = 0$. This observation should be compared with Lemma 6.3.1 of Gray(1987) which states that if $(S^\infty, \mathcal{B}(S^\infty), T, \nu)$ is AMS then $\nu(I) = \overline{\nu}(I)$, $\forall I \in \mathcal{I}$. ■

Lemma 5

Let P be stationary and ergodic. If ν is a stationary measure with $\nu(K_P) = 1$ (in the uncountable case, $\nu(K_P(\tilde{\mathcal{C}}) = 1$,where $\tilde{\mathcal{C}}$ is a countable generating field), then $\nu = P$.

Proof: For ν a.a. $\langle s\rangle$: $\frac{1}{J}\sum_{j=0}^{J-1} 1_C(T^j\langle s\rangle) \to \nu(C|\mathcal{I})\langle s\rangle$ (Birkhoff's Ergodic Theorem). However we also have, by definition of K_P ($K_P(\tilde{\mathcal{C}})$) that for all $\langle s\rangle \in K_P$ ($\in K_P(\mathcal{C})$) that $\frac{1}{J}\sum_{j=0}^{J-1} 1_C(T^j\langle s\rangle) \to P(C)$. Hence for all cylinders C (all cylinders $C \in \tilde{\mathcal{C}}$) we have $\nu(C|\mathcal{I})\langle s\rangle = P(C)$ for ν a.a. $\langle s\rangle$ and the result follows ■

Lemma 6

Suppose $(S^\infty, \mathcal{B}(S^\infty), T, \nu)$ is WAMS and that $\nu \ll \overline{\nu}$. Then ν is AMS.

Proof: For $B \in \mathcal{B}(S^\infty)$ we have that for $\overline{\nu}$ a.a. $\langle s\rangle$, $\frac{1}{J}\sum_{j=0}^{J-1} 1_B(T^j\langle s\rangle)$ converges. But then there is also convergence for ν a.a. $\langle s\rangle$ ■

Lemma 7

Suppose $\{\nu_\alpha\}_{\alpha\in A}$ is a collection of WAMS probability measures on $(S^\infty, \mathcal{B}(S^\infty))$ and let ρ be a probability measure on A. Then $\nu = \int_A \nu_\alpha \rho(d\alpha)$ is a WAMS probability measure.

Proof: For a cylinder C, we have $\nu_\alpha(\mathcal{A}_C) = 1, \forall\alpha \in A$, hence $\nu(\mathcal{A}_C) = 1$ also ■

With the help of the results stated in this section it becomes easier to understand the relationship between stationary, AMS and WAMS probability measures. Let us first consider the ergodic case. Concentrating on the case where S is countable, a stationary ergodic probability measure P has support on K_P. If μ is AMS with support on K_P then if $\mu(I) = 1$ for some $I \in \mathcal{I}$, $P(I) = 1$. We may still have $P(B) = 0$ and $\mu(B) = 1$ for some measurable $B \subset K_P$. But then (Corollary 6.3.2 of Gray, 1987)

$$\lim_{j\to\infty} \mu(T^{-j}B) = 0 \tag{10}$$

If ν is ergodic, WAMS but not AMS with support on K_P then there is (Lemma 6) some invariant set $I \subset K_P$ s.t $\mu(I) = P(I) = 0$ while $\nu(I) = 1$, in particular (10) does not hold. In this sense, we say that the support

[3] In the following section we define probability measure (SIDS measures and SSM measures) that are WAMS, not AMS and ergodic with support on some $K_P(\tilde{\mathcal{C}})$. Convex combinations of these will then be non-ergodic.

of ν is smaller than that of μ and P. This can be phrased in another way: For any σ-algebra $\mathcal{G} \subset \mathcal{B}(S^\infty)$, we have that $P(\cdot|\mathcal{G})\langle s\rangle$ is a WAMS probability measure for P a.a. $\langle s\rangle$. The same kind of interpretation applies to the case where S is not countable.

In the non-ergodic case, and again considering the case where S is countable, a stationary probability measure P can be interpreted as a combination of ergodic stationary measures namely $P(\cdot|\mathcal{I}_\mathcal{P})$ (by Lemma 5, since these are all stationary and with support on $K_{P'}$ for some P') and similarly if ν is WAMS it is a combination of WAMS measures, namely $\nu(\cdot|\mathcal{I}_\mathcal{P})$ and a combination of ergodic WAMS measures, namely $\nu(\cdot|\mathcal{I})$ (this follows from Remark 2,(iii)). When S is not countable, $P(\cdot|\mathcal{I}_\mathcal{P})$ is still stationary, however whether $\nu(\cdot|\mathcal{I}_\mathcal{P})$ is WAMS, is an open question to this author. Certainly, $\nu(\cdot|\mathcal{I})$ is in general not (Remark 2). In any event, ν has a much "smaller" support than P when $P = \bar{\nu}$ and this is the crucial observation that we use when interpreting rational beliefs, as defined below (Definition 7).

Below, we usually assume that the true probability measure ν and hence $\bar{\nu}$ is ergodic, implying that $\nu(K_{P'}) = \bar{\nu}(K_{P'}) = 1$, for some $P' \in \mathcal{P}$. This is without loss of generality, since it just means that we concentrate on that $K_{P'}$ to which the realization $\langle s\rangle$ belongs. In that case we have for any $C \in \mathcal{C}(S^\infty)$: $\lim_{J\to\infty} \frac{1}{J}\sum_{j=0}^{J-1} 1_C(T^j\langle s\rangle) = \bar{\nu}(C)$ for ν-a.a. $\langle s\rangle$ (Kurz, 1994 Corollary to Proposition 3).

2.3 Rational beliefs

Whether the true probability measure ν is ergodic or not, agents are assumed to know the limit empirical frequencies for many cylinders C and based on this information they get to know $\bar{\nu}$ (in the ergodic case, else $\bar{\nu}(\cdot|\mathcal{I})\langle s\rangle$). The assumption that all agents know $\bar{\nu}$ is entirely the same as that of rational expectations models (f.i. in Lucas, 1978). This can, in an environment where the relevant observable variables on which agents form expectations are stable, and where consequently the empirical distribution exists, be justified as an approximation. Thus we suggest, in line with many interpretations of the rational expectations hypothesis that the knowledge about $\bar{\nu}$ is the result of a learning process which is not explicitly modelled. However, in models like Lucas(1978) one more assumption, which is not even explicated, is made, namely that not only is the true process of observable variables stationary but agents know this. Contrary to the assumption just discussed, this one cannot be justified as something agents will learn in the limit. As Remark 2 showed, there is no way to discern whether a system is stationary and not just stable[4]. When agents realize that the system may not be stationary, they are faced with different possible interpretations of what they see in terms of underlying statistical models. To know that some unknown dynamical system $(S^\infty, \mathcal{B}(S^\infty), \nu, T)$ generates $\bar{\nu}$ is not the same as knowing ν . There are many possible stable dynamical systems that will generate the

[4]Of course, since in practice we only have a finite number of observations, we can never be absolutely sure, whether there is convergence of the empirical frequency of a particular set B. However, note that if B is an infinite dimensional set, the empirical frequency is not even defined when the number of observations is finite.

same stationary measure.

Definition 7 *Weakly Rational Belief.*

A stable probability measure μ on $(\Omega, \mathcal{B})$ is said to be a Weakly Rational Belief for the stable dynamical system $(S^{\infty}, \mathcal{B}(S^{\infty}), \nu, T)$ if $\bar{\mu} = \bar{\nu}$ ■

This definition is a slight modification of the definition of a rational belief given in Kurz(1994a) and in the following we will not differentiate between the two notions and simply refer to weakly rational beliefs as rational beliefs.

The true realization is $\langle \overline{s} \rangle$ and what we basically assume, is that the agent knows to which of the subsets K_P (or $K_P(\tilde{C})$ in the uncountable case), $\langle \overline{s} \rangle$ belongs, say $K_{P'}$. Any distribution μ, with $\mu(K_{P'}) = 1$ is then a rational belief.

Remark 4

(i) To understand this reasoning, consider the possibility (assuming S countable) that the agent chooses as his belief, $\delta_{\langle s' \rangle}$ where $\langle s' \rangle \in K_{P'}$. This is a very extreme rational belief, since it means that the agent is absolutely certain about what is going to happen in the future. If the agent observes that $(\overline{s}_1, \overline{s}_2, \ldots, \overline{s}_t) \neq (s'_1, s'_2, \ldots, s'_t)$ for some date t, he concludes that his belief was wrong. However, he also knows that both sequences $\{s'_q\}_{q=t+1}^{\infty}$ and $\{\overline{s}_q\}_{q=t+1}^{\infty}$ belong to $\{\{s_q\}_{q=t+1}^{\infty} : (\overline{s}_1, \ldots, \overline{s}_t, s_{t+1}, s_{t+2}, \ldots) \in K_{P'}\}$. Thus there is no logical reason for him to think that $\{s'_q\}_{q=t+1}^{\infty}$ and $\{\overline{s}_q\}_{q=t+1}^{\infty}$ would be different in the future, i.e. no logical reason for him to change his belief as far as the future is concerned. The same reasoning would be applicable, would he have a more diffuse belief on $K_{P'}$ like an SIDS process as defined below (Definition 8). Finally, note that as time progresses it will be confirmed to the agent that $\langle \overline{s} \rangle \in K_{P'}$, that is no new learning takes place. If, in contrast the agent's belief was wrong in the sense that he thought $\langle \overline{s} \rangle \in K_{P''}$ $(P' \neq P'')$, and had chosen some μ with support on $K_{P''}$ then, as the empirical frequencies would converge to P', he would learn that his belief was mistaken and would, if he is rational, change it.

(ii) Infinity plays a crucial role in Definition 7. An agent may have any belief in the first n periods of his life and still be rational in the long run[5]. This may be considered a problem, however it resembles the heuristics used to explain the rational expectations assumption according to which agents cannot continue to have mistaken beliefs, i.e. cannot be wrong in the long run. Furthermore, any statement about the properties of a rational belief equilibrium is about what happens in the long run, typically based on the properties of the empirical distribution of the equilibrium process.

(iii) If an agent picks a WAMS belief μ (for instance the belief $\delta_{\langle s' \rangle}$ considered above) which is different from P' he demonstrates *rational overconfidence*. This is most evident when this belief is not AMS (as is the case

[5]However, the concept of rational beliefs only define what beliefs agents *can* have. Which particular rational beliefs they *actually* have is seen from the perspective of the theory an empirical issue.

for the beliefs studied in this paper) since then the support of μ, $H_\mu \subset K_{P'}$ has zero probability under P'. Despite, that there are no empirical reasons to think that the actual realization, $\langle \overline{s} \rangle$ belongs to H_μ, the agent is confident that it does. In other words, while his belief is consistent with observations, he thinks he knows more than what can safely be extracted from these observations. In particular, he holds that his ability to predict outperforms that of other agents (who employ other rational beliefs)[6]. On the other hand, adopting P' (the empirical distribution) as one's belief is a conservative strategy, since it implies that no subjective model is used to interpret the data that has been observed

(iv) We would not claim that in reality agents know the empirical distribution perfectly well. Learning certainly takes place all the time. But part of the problem that individual agents are faced with in reality is, that they cannot assume that other agents use the same learning model as they do. The diversity of models and the resulting uncertainty about the expectations of other agents is central to the theory of rational beliefs (see Section 5 for more on this). To model how learning takes place in general equilibrium is an important but also difficult task. Guidolin and Timmermann(2003) show how this may be accomplished, but one shortcoming of their approach is the assumption of homogeneous learning models and common knowledge about this homogeneity. One can only hope that the methods and insights from both the learning approach and from the literature on rational beliefs will eventually be incorporated into one general model ■

3 SSM processes

SSM-processes were introduced in Nielsen(1994) for the case of a countable state space. They are a class of Markov processes which are stable but not necessarily time homogeneous. The principle is the same as for SIDS processes, in fact, a subset of SIDS processes, which we called SIDS(i.i.d.) processes[7], are SSM processes. The distribution of an SSM process is generated by an initial distribution and a sequence of transition probabilities, the last being a typical realization of an i.i.d. process taking values in a countable set of transition probabilities. To compare, let us formally define SIDS measures (introduced in Nielsen, 1994 and 1996).

Definition 8 *SIDS measure.*

An SIDS measure ν on $(S^\infty, \mathcal{B}(S^\infty))$ is a probability measure s.t.

(a) $\nu = \otimes_{t=1}^{\infty} P_t$ where $\forall t, P_t$ is a probability measure on $(S, \mathcal{B}(S))$.

[6]The concept of rational overconfidence could, in principle, be employed in other contexts than that of rational beliefs. The ingredients would be a set of probability distributions, one conservative, the rest "bold" where all these are consistent with the empirical distribution (empirical observations). Overconfident agents would then choose a bold distribution, while cautious agents would choose the conservative distribution.

[7]Meaning that that the generating measure $Q = \times_{j=1}^{\infty} q$ where q is a probability measure on S.

(b) Let $\mathcal{P} = \cup_t \{P_T\}$ and write it as $\mathcal{P} = \{P^1, P^2, \ldots\}$. Let $Z : \mathcal{P} \to \mathcal{N}$ with $Z(P^k) = k$. Then there exists an ergodic and stationary probability measure Q on $(N^\infty, \mathcal{B}(N^\infty))$ s.t.

$$\forall C \in \mathcal{B}(N^\infty) : \frac{1}{J} \sum_{j=0}^{J-1} 1_C(T^j(\{Z(P_t)\}_{t=1}^\infty)) \to Q(C) \text{ as } J \to \infty \tag{11}$$

Q is then called the *generating* measure for ν ■

An SIDS measure is ergodic and stable (see Nielsen, 1996). The stationary measure $\bar{\nu}$ associated with $(S^\infty, \mathcal{B}(S^\infty), \nu, T)$ is described by

$$\forall D \in \mathcal{N}, \forall C \in \mathcal{C}^D(S^\infty) : \bar{\nu}(C) = \sum_{n^{(D)} \in \mathcal{N}^D} \left(\otimes_{t=1}^D P^{n_t} \right) (C_D) Q(\{n^{(D)}\} \times \mathcal{N}^\infty) \tag{12}$$

where C_D is the projection of C on S^D and $n^{(D)}$ denotes a vector in $\mathcal{N}^D$. We also have that for Q-a.a. $\langle n \rangle$ that $\otimes_{t=1}^\infty Z^{-1}(n_t)$ is an SIDS measure with the same associated stationary distribution as ν.

We provide a formal definition of SSM processes which is compatible with an uncountable state space. Whenever we have a sequence of transition probabilities $\{\Pi_t\}_{t=1}^\infty$ on a state space S and an initial distribution μ, we let $\gamma(\mu, \{\Pi_t\}_{t=1}^\infty)$ denote the induced measure on $(S^\infty, \mathcal{B}(S^\infty))$. If the Π_t's are all the same, equal to Π, we simply write $\gamma(\mu, \Pi)$. If $\gamma(\mu, \{\Pi_t\}_{t=1}^\infty)$ is WAMS, the associated stationary measure is denoted $\bar{\gamma}(\mu, \{\Pi_t\}_{t=1}^\infty)$.

Definition 9 *SSM Processes and Generating Measure*

An SSM measure on the state space S is a probability measure μ on $(S, \mathcal{B}(S))$ and a sequence $\{\Pi_t\}_{t=1}^\infty$ where for each t, $\Pi_t : S \times \mathcal{B}(S) \to \Re$ is a transition probability s.t.

(i) The Markov process with distribution $\gamma(\mu, \{\Pi_t\})$ is WAMS.

(ii) There is a probability measure q on $\mathcal{P} \equiv \cup_{t=1}^\infty \{\Pi_t\}$ s.t. for all cylinders C in $\mathcal{P}^\infty$ we have that, letting $Q = \otimes_{t=1}^\infty q$, $\frac{1}{J} \sum_{j=0}^{J-1} 1_C(T^j \{\Pi_t\}_{t=1}^\infty) \to Q(C)$ as $J \to \infty$. Q is then said to be the *generating measure* for μ. The associated canonical stochastic process is then called an SSM process ■

We write the countable set $\mathcal{P}$ defined in condition (ii) as $\mathcal{P} = \{\Pi^1, \Pi^2, \ldots\}$. The following result was then proved in Nielsen(1994):

Proposition 6

Suppose that the state space S is countable. Let $\mathcal{P}$ be a countable set of transition probabilities (matrices) on S and q a probability measure on $\mathcal{P}$. Suppose that there is an invariant ergodic measure, $\bar{\mu}$ for $\bar{\Pi} = \sum_{i=1}^\infty q(\{\Pi^i\}) \Pi^i$. Let $\mu \ll \bar{\mu}$. Then for Q a.a. $\{\Pi_t\}_{t=1}^\infty$, $\gamma(\mu, \{\Pi_t\})$ is an SSM process ■

Another route to this result uses a Conditional Stability Theorem like the one introduced in Kurz and Schneider(1996). We state and prove a slightly different version of that theorem.

Proposition 7

Let S be countable and Y a topological space. Suppose that $(S^\infty \times Y^\infty, \mathcal{B}(S^\infty \times Y^\infty), T, \nu)$ is a stable and ergodic dynamical system. Let the sub algebra $\mathcal{G} \ \subset \mathcal{B}(S^\infty \times Y^\infty)$ consist of sets of the form $G = S^\infty \times B$, where $B \in \mathcal{B}(Y^\infty)$ and suppose that $\nu(\cdot|\mathcal{G})(\cdot)$ is a regular conditional probability. For given $\langle s, y\rangle \in S^\infty \times Y^\infty$ let $\nu_S(\cdot|\mathcal{G})(\langle s, y\rangle)$ be the marginal of $\nu(\cdot|\mathcal{G})(\cdot)(\langle s, y\rangle)$ on S^∞. (a) For ν a.e. $\langle s, y\rangle$, $\nu_s(\cdot|\mathcal{G})(\langle s, y\rangle)$ is WAMS with associated stationary measure equal to $\bar{\nu}_S$ (the marginal of $\bar{\nu}$ on S^∞). (b) If Y is countable we have for ν a.a. $\langle s, y\rangle$ that $\nu(\cdot|\mathcal{G})\langle s, y\rangle$ is WAMS with associated stationary measure $\bar{\nu}$.

Proof: Let

$$K_S = \{\langle s\rangle : \frac{1}{J}\sum_{j=0}^{J-1} 1_C(T^j\langle s\rangle) \to \bar{\nu}(C), \forall C \in \mathcal{C}(S^\infty)\}$$

and

$$K = \{\langle s, y\rangle : \frac{1}{J}\sum_{j=0}^{J-1} 1_C(T^j\langle s, y\rangle) \to \bar{\nu}(C), \forall C \in \mathcal{C}((S \times Y)^\infty)\}$$

Then by Proposition 1, $\nu_S(K_S) = 1$ (since ν_S is stable) and if Y is countable, $\nu(K) = 1$. Now suppose there were a set $B \in \mathcal{B}(S^\infty \times Y^\infty)$ s.t. $\nu(B) > 0$ and s.t. for all $\langle s, y\rangle \in B : \nu(K_S \times Y^\infty|\mathcal{G})(\langle s, y\rangle)$ $=\nu_S(K_S|\mathcal{G})(\langle s, y\rangle) < 1$. Since ν is stable and thus its marginal on S^∞ is as well, $1 = \nu(K_S \times Y^\infty) = \int_B \nu(K_S \times Y^\infty|\mathcal{G})(\langle s, y\rangle)d\nu(\langle s, y\rangle) + \int_{(S^\infty \times Y^\infty)\setminus B} \nu(K_S \times Y^\infty|\mathcal{G})(\langle s, y\rangle)d\nu(\langle s, y\rangle)$ and since $\nu(K_S \times Y^\infty|\mathcal{G})(\langle s, y\rangle) \leq 1$ uniformly, this would give a contradiction. Thus $\nu_S(K_S|\mathcal{G})(\langle s, y\rangle) = 1$ for ν a.a. $\langle s, y\rangle$ which means that $\nu_S(\cdot|\mathcal{G})(\langle s, y\rangle)$ is stable with associated stationary measure $\bar{\nu}_S$. Using K in stead of K_S, the same kind of reasoning leads to (b) when Y is countable ■

Remark 5 *How to obtain Proposition 6 from Proposition 7.*

Let $Y = \mathcal{P}$ (from (ii) of Definition 9). Let the Markov transition on $S \times Y$ be $\Pi((s, \Pi^j), A \times \{\Pi^i\}) = \Pi^j(s, A)q(\Pi^i)$ for all $A \subset S$ (with a unique extension to all sets in $\mathcal{B}((S \times Y)^\infty)$). Let the initial distribution be $\mu \otimes q$. Then $\gamma(\mu \otimes q, \Pi)$ is stationary and ergodic and for Q a.a. $\{\Pi_t\}$ in $\mathcal{P}^\infty$ we have $\gamma(\mu, \{\Pi_t\})$ is stable and ergodic with associated stationary distribution equal to $\gamma(\bar{\mu}, \bar{\Pi})$, which is the marginal of $\gamma(\bar{\mu} \otimes q, \bar{\Pi})$ on S^∞ (see Proposition 10 for details) ■

As we shall demonstrate shortly, the result is not correct for S uncountable even when Y is finite. However the conditions stated in Proposition 7 have their own relevance. In fact, in Kurz(1997) a situation is studied, where the fluctuations in endogenous variables are partly the consequence of a series of regimes which never repeat themselves. This may be interpreted as a situation where the set Y of generating variables is uncountable. We now turn to demonstrating the limitations of the conditional stability approach. This is done by showing that if S is uncountable, the conclusion of Proposition 7 need not hold.

Example 3

Let $Y = \{1, 2\}$ and let $\gamma = \otimes_{t=1}^{\infty}\gamma^1$ where $\gamma^1(\{0\}) = \gamma^1(\{1\}) = \frac{1}{2}$. Then $(Y^\infty, \mathcal{B}(Y^\infty), T, \gamma)$ is a stationary system and by the Ergodic Theorem we have for all $B \in \mathcal{B}(Y^\infty)$ that for γ-a.a. $\langle y\rangle \in Y^\infty$, $\frac{1}{J}\sum_{j=1}^{J-1} 1_B(T^j\langle y\rangle) \rightarrow \gamma(B)$. We also have for γ-a.a. $\langle y\rangle \in Y^\infty$ that :

$$\text{(i) } \frac{1}{J}\sum_{j=0}^{J-1} 1_C(T^j\langle y\rangle) \rightarrow \gamma(C) \text{ for all } C \in \mathcal{C}(Y^\infty) \text{ (Proposition 1)}$$

$$\text{(ii) } \exists B \in \mathcal{B}(Y^\infty) \text{ s.t. } \frac{1}{J}\sum_{j=0}^{J-1} 1_B(T^j\langle y\rangle) \text{ does not converge (Remark 2)}$$

Let $S = [0, 1]$ and let $E \in \mathcal{B}(Y^\infty)$ be the set of $\langle y\rangle$ s.t. either $y_n = 1, \forall n$ or $\forall k, \exists n > k$, s.t. $y_n = 0$. Then the binary expansion $\tau(\langle y\rangle) = \sum_{n=1}^{\infty} \frac{y_n}{2^n}$ is a one-to-one map from Y^∞ onto S and is continuous (in the topology of pointwise convergence on Y^∞). Now define the injective map $f : E \rightarrow S^\infty$ by $f(\langle y\rangle)_n = \tau(T^{n-1}\langle y\rangle)$. Also define the transition probability $\Pi : E \rightarrow \mathcal{P}(S^\infty)$ by $\Pi(\langle y\rangle) = \delta_{f(\langle y\rangle)}$ (the Dirac measure at $f(\langle y\rangle)$). To establish that Π is a transition probability we need to show that it is measurable. We show continuity. Let $\langle y\rangle^k \rightarrow \langle y\rangle$ pointwise in E. So for any n, $T^n\langle y\rangle^k \rightarrow T^n\langle y\rangle$ pointwise that is $f(\langle y\rangle^k)_{n+1} \rightarrow f(\langle y\rangle)_{n+1}$ for all n. Finally, we know that when $\langle s\rangle^k \rightarrow \langle s\rangle$ we have $\delta_{\langle s\rangle^k} \rightarrow \delta_{\langle s\rangle}$ (in the topology of weak convergence). So Π is indeed a transition probability. Let γ_E be the restriction of γ to E and note that since $\gamma(E) = 1$, γ_E is a probability measure. Then γ_E together with Π induces a probability measure ν on $(S^\infty \times E, \mathcal{B}(S^\infty \times E))$ (defined by $\nu(A \times B) = \int_B \gamma_E(d\langle y\rangle)\Pi(\langle y\rangle, A)$) s.t. if we let $D = \{\langle s, y\rangle \in S^\infty \times E : \langle s\rangle = f(\langle y\rangle)\}$ then $\nu(D) = 1$. For γ_E a.a. $\langle y\rangle$ we have that the probability measure $\Pi(\langle y\rangle)$ is not stable since we can construct a cylinder $C \in \mathcal{C}(S^\infty)$ s.t. $f(\langle y\rangle)$ is not stable relative to C, that is $\frac{1}{J}\sum_{j=0}^{J-1} 1_C(T^j f(\langle y\rangle))$ does not converge. This is done as follows. We use that for γ-a.a. $\langle y\rangle$ we have that $T^j\langle y\rangle \neq T^{j+n}\langle y\rangle$ for all $n > 0$, which means that $(T^j f(\langle y\rangle))_1 \neq (T^{j+n} f(\langle y\rangle))_1$ for $n > 0$ (Remark 2). Then let $K \subset [0, 1]$ be defined as follows: $K = \cup_{n=1}^{\infty} \cup_{m=10^{2n}}^{10^{2n+1}} \{(T^m\langle s\rangle)_1\}$ where $\langle s\rangle = f(\langle y\rangle)$ and let $C = K \times S^\infty$. Then $f(\langle y\rangle)$ is not stable relative to C. Note that C is the smallest 1-dimensional cylinder containing $\cup_{n=1}^{\infty} \cup_{m=10^{2n}}^{10^{2n+1}} \{T^m\langle s\rangle\}$. The preimage of this last set under f is not a cylinder, which explains how this example works. The idea is to let the infinite dimensional information contained in the sequence $\langle y\rangle$ be transformed into finite dimensional information as in $f(\langle y\rangle)_1 = \tau(\langle y\rangle)$. To complete the example, we show that ν is AMS and ergodic. So let $F \in \mathcal{B}(S^\infty \times E)$ and let

$$K_F = \left\{ \langle s, y\rangle \in S^\infty \times E : \frac{1}{J}\sum_{j=0}^{J-1} 1_F(T^j\langle s, y\rangle) \text{ converge} \right\}$$

Then to show that $\nu(K_F) = 1$ we need only show that $\nu(K_F \cap D) = 1$. We show

$$\langle s, y\rangle \in K_F \cap D \text{ iff } \langle s, y\rangle \in K_{F\cap D} \cap D \tag{13}$$

First note that $\langle s,y\rangle \in D \Leftrightarrow T^j\langle s,y\rangle \in D, \forall j \in \mathcal{N}_0$. It follows that if $\langle s,y\rangle \in D$ then $T^j\langle s,y\rangle \in F \Rightarrow T^j\langle s,y\rangle \in F\cap D$ while, of course, $T^j\langle s,y\rangle \in F\cap D \Rightarrow T^j\langle s,y\rangle \in F$. Thus if $\langle s,y\rangle \in D$ then

$$\frac{1}{J}\sum_{j=0}^{J-1} 1_F(T^j\langle s,y\rangle) = \frac{1}{J}\sum_{j=0}^{J-1} 1_{F\cap D}(T^j\langle s,y\rangle)$$

from which (13) follows. Suppose $T^j\langle y\rangle \in \mathrm{Proj}_Y(F\cap D)$. Then $\exists\langle s\rangle \in S^\infty$ s.t. $(\langle s\rangle, T^j\langle y\rangle) \in F\cap D$. But if $(\langle s\rangle, T^j\langle y\rangle) \in D$ then $\langle s\rangle = f(T^j\langle y\rangle)$. So $T^j\langle f(\langle y\rangle), y\rangle = (f(T^j\langle y\rangle), T^j\langle y\rangle) \in F\cap D$. We conclude that

$$T^j\langle f(\langle y\rangle), y\rangle \in F\cap D \Leftrightarrow T^j\langle y\rangle \in \mathrm{Proj}_{Y^\infty}(F\cap D)$$

So

$$\frac{1}{J}\sum_{j=0}^{J-1} 1_{F\cap D}(T^j\langle f(\langle y\rangle), y\rangle) = \frac{1}{J}\sum_{j=0}^{J-1} 1_{\mathrm{Proj}_{Y^\infty}(F\cap D)}(T^j\langle y\rangle)$$

which converges for γ_E-a.a.$\langle y\rangle$ and thus for ν-a.a. $\langle s,y\rangle \in D$.

Finally, consider an invariant set, I. Then the projection, I_E on E is also invariant, so either $\gamma_E(I_E) = 1$ or $\gamma_E(I_E) = 0$. In the first case, $I = \{\langle f(\langle y\rangle), y\rangle : \langle y\rangle \in I_E\}$, so $\nu(I) = 1$.

Remark a: ν_{S^∞}, the marginal of ν on S^∞ is the distribution of a Markov chain $\{X_t\}$ with initial distribution being the uniform distribution on $[0,1]$ and deterministic transition π where, letting $h(s) = \tau(T(\tau^{-1}(s)))$, $\pi(s, \{h(s)\}) = 1$. Suppose we consider the sub σ-algebra $\mathcal{H} \subset \mathcal{B}(S^\infty)$ generated by all sets of the form $A_1 \times A_2 \times \ldots \times A_N \times S^\infty$ where $A_n \in \{[0,\frac{1}{2}), [\frac{1}{2},1]\}, \forall n$ s.t. $1 \le n \le N$. Thus $\mathcal{H}$ is defined by a finite partition at any date. None the less, $\nu_{S^\infty}(\cdot|\mathcal{H})\langle s\rangle = \Pi(f^{-1}(\langle s\rangle))$ is not WAMS. Thus even when we condition a stationary measure on a filtration consisting of finite σ-algebras, the result is not necessarily a WAMS measure when the state space is not countable.

Remark b: There is in fact no probability measure, μ with support on $f(E)$ ($= K_{\nu_{S^\infty}}$) that is WAMS but not AMS. For if that were the case there would, for some non-cylinder set $B \in \mathcal{B}(S^\infty)\cap f(E)$, be a set $A \in \mathcal{B}(S^\infty)\cap f(E)$ with $\mu(A) \ge 0$ s.t. for $\langle s\rangle \in A$, $\frac{1}{J}\sum_{j=0}^{J-1} 1_B(T^j(\langle s\rangle)$ would not converge. But since there is a one-to-one onto function $H : [0,1] \to f(E)$, letting $B_1 = H^{-1}(B)$ we would have $\frac{1}{J}\sum_{j=0}^{J-1} 1_B(T^j\langle s\rangle) = \frac{1}{J}\sum_{j=0}^{J-1} 1_{B_1\times S^\infty}(T^j\langle s\rangle), \forall J$, i.e. there would on A not be convergence for the cylinder $B_1 \times S^\infty$ either, implying that μ is not WAMS.

Remark c: The chain $\{X_t\}$ is not positive recurrent: For any $s \in [0,1]$ let $A = [0,1] \setminus \{s, h(s), h^2(s), \ldots\}$ and $\nu_{S^\infty}(A \text{ i.o. } |s) = 0$. Note that the chain is then not Harris (as defined in Definition 10 below).

Remark d: The chain is not indecomposable either: Let $s \neq s'$ be irrationals and let $E = \{s, h(s), h^2(s), \ldots\}$ and $E' = \{s', h(s'), h^2(s'), \ldots\}$. Then $E \cap E' = \emptyset$ and $\pi(s,E) = 1, \forall s \in E$ and $\pi(s', E') = 1, \forall s' \in E'$ ■

We next turn to study SSM processes when the state space is not countable. To follow are two preliminary results.

Proposition 8

(a) If $\gamma(\mu, \{\Pi_t\})$ is WAMS and if $\tilde{\mu} \ll \mu$ then $\gamma(\tilde{\mu}, \{\Pi_t\})$ is WAMS.

(b) If $\gamma(\mu, \Pi)$ is WAMS it is AMS.

Proof: (a) The condition implies that $\gamma(\tilde{\mu}, \{\Pi_t\}) \ll \gamma(\mu, \{\Pi_t\})$. If we take any cylinder $C \in \mathcal{C}(S^\infty)$ and let

$$\mathcal{A}_C = \left\{ \langle s \rangle : \frac{1}{N} \sum_{j=0}^{N-1} 1_c(T^j \langle s \rangle) \text{ converges } \right\}$$

then $\gamma(\mu, \{\Pi_t\})(\mathcal{A}_C) = 1$ implying that $\gamma(\tilde{\mu}, \{\Pi_t\})(\mathcal{A}_C) = 1$.

(b) Let $\bar{\gamma}$ be the stationary measure associated with $\gamma(\mu, \Pi)$. Define the probability measures $P_N, N \in \mathcal{N}$ on $(S, \mathcal{B}(S))$ by

$$P_N(A) = \frac{1}{J} \sum_{j=0}^{J-1} \int_S \mu(ds) \Pi^j(s, A) = \frac{1}{J} \sum_{j=0}^{J-1} \gamma(\mu, \Pi)(T^{-j}(A \times S^\infty)) \text{ for any } A \in \mathcal{B}(S)$$

Then $P_N(A) \to \bar{\gamma}_1(A)$ where $\bar{\gamma}_1$ is the marginal of $\bar{\gamma}$ on the first coordinate. Let $B \in \mathcal{B}(S^\infty)$ and note that

$$\frac{1}{J} \sum_{j=0}^{J-1} \gamma(\mu, \Pi)(T^{-j} B) = \frac{1}{J} \sum_{j=0}^{J-1} \int_S \mu(ds) \int_S \Pi^j(s, ds') \gamma(\delta_{s'}, \Pi)(B)$$

$\gamma(\delta_{s'}, \Pi)(B)$ is as a function of s' measurable (Proposition V.2.1, Neveu, 1965) so the last expression makes sense. This expression is in turn equal to $\int_S P_N(ds) \gamma(\delta_{s'}, \Pi)(B)$ which converges to $\int_S \bar{\gamma}_1(ds) \gamma(\delta_{s'}, \Pi)(B)$ ■

Remark 6

$\bar{\gamma}(\mu, \Pi)(B) = \int_S \bar{\gamma}_1(ds) \gamma(\delta_s, \Pi)(B)$. So $\bar{\gamma}_1$ is an invariant measure for Π and a necessary and sufficient condition for an initial distribution μ to exist s.t. $\Gamma(\mu, \Pi)$ is AMS is that Π has an invariant distribution ■

We are, however, interested in processes which are WAMS but not AMS. The reason is that asymptotic mean stationarity is too strong a condition to model beliefs which are in the long run diverse. Recall that if μ is AMS then (Gray,1987 corollary 6.3.2) $\bar{\mu}$ asymptotically dominates μ in the sense that $\bar{\mu}(B) = 0 \Rightarrow \lim_{j \to \infty} \mu(T^{-j} B) = 0$. Thus eventually, any significant statistical deviation from $\bar{\mu}$ has probability zero.

Proposition 9

Suppose $(S^\infty \times Y^\infty, \mathcal{B}(S^\infty \times Y^\infty), T, \nu)$ is a stable and ergodic dynamical system and that for ν-a.a. $\langle s, y \rangle$:

$$\frac{1}{J} \sum_{j=0}^{J-1} \nu(T^{-j} C | \mathcal{G}) \langle s, y \rangle \text{ converges for all } C \in \mathcal{C}(S^\infty \times Y^\infty) \tag{14}$$

where $\mathcal{G} \subset \mathcal{B}(S^\infty \times Y^\infty)$ is a sub σ-algebra. Then for ν-a.a. $\langle s, y \rangle$:

$$\frac{1}{J} \sum_{j=0}^{J-1} \nu(T^{-j} C | \mathcal{G}) \langle s, y \rangle \to \bar{\nu}(C), \text{ for all } C \in \mathcal{C}(S^\infty \times Y^\infty)$$

Proof: Let $\tilde{A}$ be the set where (14) holds. There is a set of probability measures $\nu_{\langle s,y\rangle}$ s.t. for $\langle s,y\rangle \in \tilde{A}$:

$$\frac{1}{J}\sum_{j=0}^{J-1} \nu(T^{-j}C|\mathcal{G})\langle s,y\rangle \to \nu_{\langle s,y\rangle}(C), \forall C \in \mathcal{C}(S^\infty \times Y^\infty)$$

This follows from the Vitali-Hahn-Saks Theorem. Let $\tilde{\mathcal{C}} \subset \mathcal{B}(S^\infty \times Y^\infty)$ be a countable generating set of cylinders for $\mathcal{B}(S^\infty \times Y^\infty)$ and

$$B^* = \left\{ \langle s,y\rangle : \frac{1}{J}\sum_{j=0}^{J-1} 1_c(T^j\langle s,y\rangle) \to \overline{\nu}(C), \forall C \in \tilde{\mathcal{C}} \right\}$$

Then, because of stability, $\mu(B^*) = 1$. Since

$$\nu(B^*) = \int_{S^\infty \times Y^\infty} \nu(B^*|\mathcal{G})(\langle s,y\rangle)\nu(d\langle s,y\rangle)$$

it follows that $\nu(B^*|\mathcal{G})(\langle s,y\rangle) = 1$ for $\langle s,y\rangle \in \tilde{C}$ for some $\tilde{C}$ s.t. $\nu(\tilde{C}) = 1$. Consider a $\langle s',y'\rangle \in \tilde{A} \cap \tilde{C}$. For any $C \in \tilde{\mathcal{C}}$:

$$\frac{1}{J}\sum_{j=0}^{J-1} \nu(T^{-j}C|\mathcal{G})(\langle s',y'\rangle) = \frac{1}{J}\sum_{j=0}^{J-1} \int_{B^*} 1_C(T^j\langle s,y\rangle)\nu(d\langle s,y\rangle|\mathcal{G})(\langle s',y'\rangle) =$$

$$\int_{B^*} \frac{1}{J}\sum_{j=0}^{J-1}(T^j\langle s,y\rangle)\nu(d\langle s,y\rangle|\mathcal{G})(\langle s',y'\rangle \to \overline{\nu}(C)$$

- the convergence following from Lebesgue's bounded convergence theorem. So for $\langle s,y\rangle \in \tilde{A} \cap \tilde{C} : \nu_{\langle s,y\rangle}(C) = \overline{\nu}(C), \forall C \in \tilde{\mathcal{C}}$, implying that $\nu_{\langle s,y\rangle}(B) = \overline{\nu}(B), \forall B \in \mathcal{B}(S^\infty \times Y^\infty)$ ■

We use the concept of a Harris Chain to prove that there is a large class of Markov processes on a state space S in $\Re^k$ which are stable but non-stationary. The following definition is taken from Durrett(1991):

Definition 10 *Harris Chain.*

A time homogenous Markov chain $\{Z_n\}$ taking values in S and with transition Π is a Harris chain if there are sets $A, B \in \mathcal{B}(S)$ and a function $k : S \times S \to \Re$, an $\epsilon > 0$, and a probability measure ρ with $\rho(B) = 1$ s.t. :

(i) $k(s,s') > \epsilon$ for $(s,s') \in A \times B$

(ii) If we let $\tau_A = \inf\{n \geq 0 : Z_n \in A\}$ then $P(\tau_A < \infty | Z_1 = s) > 0, \forall s \in S$

(iii) For $s \in A$ and $C \in \mathcal{B}(S)$, with $C \subset B$, we have that $\Pi(s,C) \geq \int_C k(s,s')\rho(ds')$ ■

Proposition 10

Suppose that $\bar{\Pi}$ on S is the transition probability of a Harris chain with ergodic and invariant distribution $\bar{\mu}$. Suppose furthermore that $\{\Pi_1, \Pi_2, \Pi_3, \ldots\}$ is a countable set of transition probabilities on S and $q_1, q_2, q_3, \ldots$ are non-negative numbers s.t.

(a) $\sum_{i=1}^{\infty} q_i \Pi_i = \bar{\Pi}$.

(b) $\exists c > 0$, s.t. $\forall i \in \mathcal{N}, \forall s \in S, \forall F \in \mathcal{B}(S) : c\bar{\Pi}(s, F) \leq \Pi_i(s, F)$.

Then if we let q be a probability measure on $\mathcal{N}$ with $q(\{i\}) = q_i$ and let $Q = \otimes_{t=1}^{\infty} q$ we have for Q a.a. $\langle i \rangle$ and for any $\tilde{\mu} \ll \bar{\mu}$ that

$$\gamma(\tilde{\mu}, \{\Pi_{i_t}\}_{t=1}^{\infty}) \otimes \delta_{\langle i \rangle}$$

is a stable probability measure on $((S \times \mathcal{N})^{\infty}, \mathcal{B}((S \times \mathcal{N})^{\infty}))$ and that $\gamma(\tilde{\mu}, \{\Pi_{i_t}\}_{t=1}^{\infty})$ is an SSM measure with associated stationary measure $\gamma(\bar{\mu}, \bar{\Pi})$.

Proof: Everywhere below we use regular versions of conditional probabilities. We consider the time homogenous stationary and ergodic Markov process $\{X_t, Y_t\}_{t=1}^{\infty}$ on $(S \times \mathcal{N})^{\infty}$ with stationary distribution P defined by: (X_1, Y_1) has distribution $\bar{\mu} \otimes q$ and

$$P(X_t \in F, Y_t \in G | X_{t-1} = s, Y_{t-1} = i, X_{t-2} = s_{t-2}, Y_{t-2} = i_{t-2}, \ldots, X_1 = s_1, Y_1 = i_1\} = q(G)\Pi_i(s, F) \quad (15)$$

with unique extension to all of $\mathcal{B}(S \times \mathcal{N})$ (To see that this process is indeed ergodic notice that if we let $\mathcal{N}^*$ be the support of q, $\int_{E \times \mathcal{N}^*} \bar{\mu} \otimes q(d(s,i)) \cdot q(\mathcal{N}^*)\Pi_i(s, E) = \int_E \bar{\mu}(ds)\bar{\Pi}(s, E) < 1$ unless $\bar{\mu}(E) = 1$).

Let $P_{S^{\infty}}$ be the marginal of P on S^{∞}. Then for any $F \in \mathcal{B}(S)$:

$$P_{S^{\infty}}(X_t \in F | X_{t-1} = s_{t-1}, X_{t-2} = s_{t-2}, \ldots, X_1 = s_1) = \bar{\Pi}(s_{t-1}, F) \quad (16)$$

Note, that if $\{Y_t\}$ were a Markov chain say, this would not be correct. For a proof of (16) see Appendix A. Let $\mathcal{G} = \{S^{\infty}, \emptyset\} \times \mathcal{B}(\mathcal{N}^{\infty})$. We want to show the following:

$$\text{for } P-\text{a.a.}\langle s, i \rangle : \forall C \in \mathcal{C}((S \times \mathcal{N})^{\infty}) : \frac{1}{J} \sum_{j=0}^{J-1} P(T^{-j}C | \mathcal{G})\langle s, i \rangle \text{ converges} \quad (17)$$

To this end we define a transformed (canonical) Markov process $\{\hat{X}_t, \hat{Y}_t\}$ on an enlarged state space, $((\hat{S} \times \mathcal{N})^{\infty}, \mathcal{B}(\hat{S})^{\infty} \times \mathcal{B}(\mathcal{N}^{\infty}))$. Here $\hat{S} \equiv S \cup \{\alpha\}$, α being an extra member added to S, and $\mathcal{B}(\hat{S}) = \mathcal{B}(S) \cup \{B \cup \{\alpha\} : B \in \mathcal{B}(S)\}$. Let $\hat{\mu}$ be an initial distribution on $\mathcal{B}(\hat{S})$ with $\hat{\mu}(\{\alpha\}) = \hat{\epsilon}\bar{\mu}(A)$ and $\hat{\mu}(F) = \bar{\mu}(F) - \rho(F)\hat{\epsilon}\bar{\mu}(A)$ for $F \in \mathcal{B}(S)$ (that $\hat{\mu}$ is in fact a probability measure is shown in Appendix A). The stationary distribution of $\{\hat{X}_t, \hat{Y}_t\}$ is denoted $\hat{P}$.

We replace k with $\hat{k} = ck$ and ϵ with $\hat{\epsilon} = c\epsilon$ and define in line with Durrett, the transformed transition probabilities $\hat{\Pi}_i, i = 1, 2, \ldots$ on $\hat{S} \times \mathcal{B}(\hat{S})$.

(a) If $s \in S \setminus A : \hat{\Pi}_i(s, C) = \Pi_i(s, C)$, for $C \in \mathcal{B}(S)$.

(b) If $s \in A: \hat{\Pi}_i(s,\{\alpha\}) = \hat{\epsilon}$ and $\hat{\Pi}_i(s,C) = \Pi_i(s,C) - \hat{\epsilon}\rho(C), C \in \mathcal{B}(S)$.

(c) $\hat{\Pi}_i(\alpha, D) = \int \hat{\Pi}_i(s,D)\rho(ds), D \in \mathcal{B}(\hat{S})$.

Then $\hat{P}((\hat{X}_1,\hat{Y}_1) \in G) = \hat{\mu} \otimes q(G)$ and $\hat{\Pi}_i(s,F)q(G) =$

$$\hat{P}((\hat{X}_t,\hat{Y}_t) \in F \times G | (\hat{X}_{t-1},\hat{Y}_{t-1}) = (s,i), (\hat{X}_{t-2},\hat{Y}_{t-2}) = (s_{t-2},i_{t-2}), \ldots, (\hat{X}_1,\hat{Y}_1) = (s_1,i_1))$$

If we let $\bar{\hat{\Pi}} = \sum_{i=1}^{\infty} q_i \hat{\Pi}_i$, $\hat{\mu} = \hat{\mu}\bar{\hat{\Pi}}$ (see Appendix A) so, since $\int(\hat{\mu}\otimes q)(d(s,i))\hat{\Pi}_i(s,F)q(G) = q(G)\int \hat{\mu}(ds)\bar{\hat{\Pi}}(s,F)$ the process is stationary and in fact also ergodic (also shown in Appendix A). Note that this implies that α is positive recurrent under $\hat{P}$, since $\hat{\mu}(\{\alpha\}) > 0$.

We next define the function $g: \hat{S} \times \mathcal{N} \to \{0,1\} \times \mathcal{N}$ by $g(s,i) = (1_\alpha(s), i)$ and let $G: (\hat{S} \times \mathcal{N})^\infty \to (\{0,1\} \times \mathcal{N})^\infty$ denote $\prod_{t=1}^{\infty} g$. Note that the induced measure $\hat{P} \circ G^{-1}$ on $\mathcal{B}((\{0,1\} \times \mathcal{N})^\infty)$ is stationary and ergodic. Let

$$\mathcal{E}^* = \left\{ \langle z,i \rangle \in (\{0,1\} \times \mathcal{N})^\infty : \frac{1}{J}\sum_{j=0}^{J-1} 1_C(T^j\langle z,i\rangle) \to \hat{P} \circ G^{-1}(C), \forall C \in \mathcal{C}((\{0,1\} \times \mathcal{N})^\infty) \right\}$$

Because of the stationarity of $\hat{P} \circ G^{-1}$ we have that $\hat{P} \circ G^{-1}(\mathcal{E}^*) = 1$ (a consequence of Proposition 1). Let $\mathcal{E} = G^{-1}(\mathcal{E}^*)$ and $\mathcal{D} = G^{-1}(\mathcal{B}(\{0,1\} \times \mathcal{N})^\infty)$. We now want to construct a useful representation of $\hat{P}(\cdot|\mathcal{D})$. Let $F \times (\hat{S} \times \mathcal{N})^\infty \in \mathcal{C}^k((\hat{S} \times \mathcal{N})^\infty)$ be given. For any $L \geq k+1$, let

$$\mathcal{A}_L = \{[(z_1,y_1),\ldots,(z_L,y_L)] \in (\{0,\alpha\} \times \mathcal{N})^L : z_1 = \alpha, z_{L-k+1} = \alpha, z_l \neq \alpha, 1 < l < L-k+1\}$$

For any $[(z_1,y_1),\ldots,(z_L,y_L)] \in \cup_{L \geq k+1}\mathcal{A}_L$ and any $0 \leq j \leq L-k-1$, let $K([(z_1,y_1),\ldots,(z_L,y_L)];j) =$

$$\frac{\hat{P}\left(\langle s,i\rangle : z_l = \alpha \Leftrightarrow s_l = \alpha, i_l = y_l, 1 \leq l \leq L, [(s_{j+1},i_{j+1}),\ldots,(s_{j+1+k},i_{j+1+k})] \in F\right)}{\hat{P}\left(\langle s,i\rangle : z_l = \alpha \Leftrightarrow s_l = \alpha, i_l = y_l, 1 \leq l \leq L,\right)}$$

whenever the denominator is positive, 0 else. Consider a given $\langle s',i'\rangle \in (\hat{S} \times \mathcal{N})^\infty$ and $j \geq 0$. If there are $\underline{t} \leq j+1$ s.t. $s'_{\underline{t}} = \alpha$ and $\bar{t} \geq j+2$ s.t. $s_{\bar{t}} = \alpha$ and s.t. $s'_t \neq \alpha$ for $\underline{t} < t < \bar{t}$, then if $\hat{P}(T^{-j}C|\mathcal{D})\langle s',i'\rangle > 0$,

$$\hat{P}\left(T^{-j}C|\mathcal{D}\right)\langle s',i'\rangle =$$

$$\frac{\hat{P}\left(\langle s,i\rangle : s_t = \alpha \Leftrightarrow s'_t = \alpha, i_t = i'_t, t \leq \bar{t}+k-1, [(s_{j+1},i_{j+1}),\ldots,(s_{j+1+k},i_{j+1+k})] \in F\right)}{\hat{P}\left(\langle s,i\rangle : s_t = \alpha \Leftrightarrow s'_t = \alpha, i_t = i'_t, t \leq \bar{t}+k-1\right)} =$$

$$\frac{\hat{P}\left(\langle s,i\rangle : s_t = \alpha \Leftrightarrow s'_t = \alpha, i_t = i'_t, \underline{t} \leq t \leq \bar{t}+k-1, [(s_{j+1},i_{j+1}),\ldots,(s_{j+1+k},i_{j+1+k})] \in F\right)}{\hat{P}\left(\langle s,i\rangle : s_t = \alpha \Leftrightarrow s'_t = \alpha, i_t = i'_t, \underline{t} \leq t \leq \bar{t}+k-1\right)} =$$

$$\frac{\hat{P}\left(\langle s,i\rangle : s_t = \alpha \Leftrightarrow s'_{t+\underline{t}-1} = \alpha, i_t = i'_{t+\underline{t}-1}, 1 \leq t \leq \bar{t}-\underline{t}+k, [(s_{j-\underline{t}+2},i_{j-\underline{t}+2}),..,(s_{j-\underline{t}+k+2},i_{j-\underline{t}+k+2})] \in F\right)}{\hat{P}\left(\langle s,i\rangle : s_t = \alpha \Leftrightarrow s'_{t+\underline{t}-1} = \alpha, i_t = i'_{t+\underline{t}-1}, 1 \leq t \leq \bar{t}-\underline{t}+k\right)} \quad (18)$$

where the second equality follows since the process is Markovian and the third is a consequence of stationarity. Finally, (18) is equal to $K([(1_\alpha(s'_{\underline{t}})\alpha, i'_{\underline{t}}), \ldots, (1_\alpha(s'_{\bar{t}+k-1})\alpha, i'_{\bar{t}+k-1})]; j - \underline{t} + 1)$. If we for any $L \geq k+1$ and any $[z, i]^L \in \mathcal{A}_L$ let

$$K([(z_1, i_1), \ldots, (z_L, i_L)]) = \sum_{j=0}^{L-k-1} K([(z_1, i_1), \ldots, (z_L, i_L)]; j)$$

we have

$$\sum_{j=\underline{t}-1}^{\bar{t}-2} \hat{P}(T^{-j}C|\mathcal{D})\langle s', i'\rangle = K([(1_\alpha(s'_{\underline{t}})\alpha, i'_{\underline{t}}), \ldots, (1_\alpha(s'_{\bar{t}+k-1})\alpha, i'_{\bar{t}+k-1})])$$

$$\text{It follows that } \frac{1}{J}\sum_{j=0}^{J-1} \hat{P}(T^{-j}C|\mathcal{D})\langle s', i'\rangle = \frac{1}{J}\sum_{j=0}^{J-1}\sum_{L=k}^{\infty}\sum_{(z,i)^L\in\mathcal{A}_L} M(T^j\langle s', i'\rangle; (z,i)^L)K((z,i)^L)$$

$$= \frac{1}{J}\sum_{j=0}^{\inf\{t:s'_t=\alpha\}-1} \hat{P}(T^{-j}C|\mathcal{D})\langle s', i'\rangle - \frac{1}{J}\sum_{j=J}^{\inf\{t>J+1:s'_t=\alpha\}} \hat{P}(T^{-j}C|\mathcal{D})\langle s', i'\rangle$$

where $M(T\langle s, i\rangle; (z,i)^L) = 1$, if $[(s_1, i_1), \ldots, (s_L, i_L)] = (z,i)^L$, $= 0$ else. For all $\langle s', i'\rangle \in \mathcal{E}$, the first part of this sum converges as $J \to \infty$ while the second and third parts tend to 0.

We have shown that there is a version of $\hat{P}(\cdot|\mathcal{D})(\cdot)$ s.t. for $\hat{P}$-a.a. $\langle s', i'\rangle$: (that is for all $\langle s', i'\rangle \in \mathcal{E}$) $\frac{1}{J}\sum_{j=0}^{J-1} \hat{P}(T^{-j}C|\mathcal{D})\langle s', i'\rangle$ converges for all k-dimensional cylinders in $\mathcal{C}(\hat{S}^\infty \times \mathcal{N}^\infty)$. Letting $\mathcal{G}_\alpha = \{\hat{S}^\infty, \emptyset\} \times \mathcal{B}(\mathcal{N}^\infty)$ we use Lebesgue's bounded convergence theorem to conclude from this that for $\hat{P}$-a.a. $\langle s', i'\rangle$:

$$\frac{1}{J}\sum_{j=0}^{J-1} \hat{P}(T^{-j}C|\mathcal{G}_\alpha)\langle s', i'\rangle$$

converges for all k-dimensional cylinders in $\mathcal{C}(\hat{S}^\infty \times \mathcal{N}^\infty)$. Let $\langle s', i'\rangle \in \mathcal{E}$.

$$\frac{1}{J}\sum_{j=0}^{J-1} \hat{P}(T^{-j}C|\mathcal{G}_\alpha)\langle s', i'\rangle = \int_{(\hat{S}\times\mathcal{N})^\infty} \left[\frac{1}{J}\sum_{j=0}^{J-1} \hat{P}(T^{-j}C|\mathcal{D})\langle s, i\rangle\right] \hat{P}(d\langle s, i\rangle|\mathcal{G}_\alpha)\langle s', i'\rangle$$

Since $\left|\frac{1}{J}\sum_{j=0}^{J-1} \hat{P}(T^{-j}C|\mathcal{D})\langle s, i\rangle\right| \leq 1$, $\forall\langle s, i\rangle$ and converges for $\hat{P}(\cdot|\mathcal{G}_\alpha)\langle s', i'\rangle$-a.a. $\langle s, i\rangle$ (namely on $\mathcal{E}$) we have that

$$\frac{1}{J}\sum_{j=0}^{J-1} \hat{P}(T^{-j}C|\mathcal{G}_\alpha)\langle s', i'\rangle \to \int_{(\hat{S}\times\mathcal{N})^\infty} (\lim_{J\to\infty} \frac{1}{J}\sum_{j=0}^{J-1} \hat{P}(T^{-j}C|\mathcal{D})\langle s, i\rangle)\hat{P}(d\langle \hat{s}, i\rangle|\mathcal{G}_\alpha)\langle s', i'\rangle$$

In the final step of the proof we show (17) using the convergence result just established. Let v' be a transition probability on $\hat{S}$ with $v(s, \{s\}) = 1, s \in S$ and $v(\alpha, C) = \rho(C)$ for $C \in \mathcal{B}(\hat{S})$ then $\mu = \hat{\mu}v$. (see Appendix A). We also have for all i that $v\hat{\Pi}_i = \hat{\Pi}_i$ and $\hat{\Pi}_i v = \Pi_i$. The proof is a repetition of that of Lemma 6.1 from Chapter 5 of Durrett(1991), see Appendix A for details. Consider $\langle \hat{s}', i'\rangle \in \mathcal{E}$. Let μ_j, respectively $\hat{\mu}_j$ be the marginals on the j'th coordinate of S^∞, respectively $\hat{S}^\infty$ of $P(\cdot|\mathcal{G})\langle s, i'\rangle$ and $\hat{P}(\cdot|\mathcal{G}_\alpha)\langle \hat{s}, i'\rangle$ respectively. We then have $\mu_j = \hat{\mu}_j v'$ This follows since if $A \in \mathcal{B}(S)$ then

$$\mu_j(A) = \int_S \mu(ds_1)\int_S \Pi_{i'_1}(s_1, ds_2)\cdots\int_S \Pi_{i'_{j-1}}(s_{j-1}, A) = \int_S \hat{\mu}v'(ds_1)\int_S \hat{\Pi}_{i'_1}v'(s_1, ds_2)\cdots\int_S \hat{\Pi}_{i'_{j-1}}v'(s_{j-1}, A) =$$

$$\int_S \int_{\hat S} \hat\mu(d\hat s_1) v'(\hat s_1, ds_1) \int_S \int_{\hat S} \hat\Pi_{i'_1}(s_1, d\hat s_2) v'(\hat s_2, ds_2) \cdots \int_{\hat S} \hat\Pi_{i'_{j-1}}(s_{j-1}, d\hat s_j) v'(\hat s_j, A) =$$

$$\int_{\hat S} \hat\mu(d\hat s_1) \int_{\hat S} \hat\Pi_{i'_1}(\hat s_1, d\hat s_2) \cdots \int_{\hat S} \hat\Pi_{1'_{j-1}}(\hat s_{j-1}, d\hat s_j) v'(\hat s_j, A) = \int_{\hat S} \hat\mu_j(d\hat s_j) v'(\hat s_j, A)$$

Then let the transition probability ν on $\hat S \times \mathcal{N}$ be defined by $\nu((s,i), A \times \{j\}) = \nu'(s, A)\delta_{\{i\}}(\{j\}), \forall A \in \mathcal{B}(\hat S)$.

Let $C = F \times (S \times \mathcal{N})^\infty \in \mathcal{C}^k((S \times \mathcal{N})^\infty)$. We study $\frac{1}{J}\sum_{j=0}^{J-1} P(T^{-j}C|\mathcal{G})\langle s, i'\rangle$. We have

$$P(T^{-j}C|\mathcal{G})\langle s, i'\rangle = \Big[\int_{S\times\mathcal{N}} \mu_{j+1} \otimes \delta_{i'_{j+1}}(d(s_{j+1}, y_{j+1})) \int_S \Pi_{i'_{j+1}}(s_{j+1}, ds_{j+2}) \int_{\mathcal{N}} \delta_{i'_{j+2}}(dy_{j+2}) \dots$$

$$\int_S \Pi_{i'_{j+k-1}}(s_{j+k-1}, ds_{j+k}) \int_{\mathcal{N}} \delta_{i'_{j+k}}(dy_{j+k})\Big] \cdot 1_F((s_{j+1}, y_{j+1}), \dots, (s_{j+k}, y_{j+k})) =$$

$$\Big[\int_{\hat S\times\mathcal{N}} \hat\mu_{j+1} \otimes \delta_{i'_{j+1}}(d(s,i)) \int_{S\times\mathcal{N}} \nu((s,i), d(s_{j+1}, y_{j+1})) \int_S \Pi_{i'_{j+1}}(s_{j+1}, ds_{j+2}) \int_{\mathcal{N}} \delta_{i'_{j+2}}(dy_{j+2}) \dots$$

$$\dots \int_S \Pi_{i'_{j+k-1}}(s_{j+k-1}, ds_{j+k}) \int_{\mathcal{N}} \delta_{i'_{j+k}}(dy_{j+k})\Big] \cdot 1_F((s_{j+1}, y_{j+1}), \dots, (s_{j+k}, y_{j+k})) \tag{19}$$

Letting

$$H[s_1, y_1, \dots, y_k] = \int_{S\times\mathcal{N}} \nu((s_1, y_1), d(s,y)) \int_S \Pi_{y_1}(s, ds_2) \dots \int_S \Pi_{y_{k-1}}(s_{k-1}, ds_k) 1_F((s_1, y_1), \dots, (s_k, y_k))$$

which is a bounded and measurable function, (19) can be rewritten as

$$\int_{\hat S} \hat\mu \otimes \delta_{i'_{j+1}}(d(\hat s_{j+1}, y_{j+1})) H(\hat s_{j+1}, i'_{j+1}, \dots, i'_{j+k}) = E_{\hat P}(H(\hat X_{j+1}, \hat Y_{j+1}, \dots, \hat Y_{j+k})|\mathcal{G}_\alpha)\langle \hat s, i\rangle$$

In conclusion,

$$\frac{1}{J}\sum_{j=0}^{J-1} P(T^{-j}C|\mathcal{G})\langle s, i'\rangle = \frac{1}{J}\sum_{j=0}^{J-1} E_{\hat P}(H(\hat X_{j+1}, \hat Y_{j+1}, \dots, \hat Y_{j+k})|\mathcal{G}_\alpha)\langle \hat s, i'\rangle \tag{20}$$

which converges because of the convergence of $\frac{1}{J}\sum_{j=0}^{J-1} \hat P(T^{-j}C|\mathcal{G}_\alpha)\langle s', i'\rangle$ for all cylinders.

We have shown, for all k, all $C \in \mathcal{C}^K(S^\infty)$ and for all $\langle s, i\rangle \in \mathcal{E}$ that (17) holds. From Proposition 9 it follows, that the convergence is in fact to $P(C)$ for Q a.a. $\langle i\rangle$.
The marginal of $P(\cdot|\mathcal{G})\langle s, i'\rangle$ is $\gamma(\bar\mu, \{\Pi_{i'_t}\})$ as follows from (15), thus $P(\cdot|\mathcal{G})\langle s, i'\rangle = \gamma(\bar\mu, \{\Pi_{i'_t}\}) \otimes \delta_{\langle i'\rangle}$. From (16) it follows that $\bar\gamma(\bar\mu, \{\Pi_{i'_t}\}) = \gamma(\bar\mu, \bar\Pi)$. By Proposition 8, these features are not changed if we replace $\bar\mu$ with $\tilde\mu$ ■

Remark 7

It should be noted that we could weaken somewhat the requirement that $c\bar\Pi(s, F) \le \Pi_i(s, F), \forall F \in \mathcal{S}$, since it need only hold for $s \in A$ and $F \subset B$ ■

4 Sun-spot rational belief structures, anonymity and structural independence

4.1 Sunspot rational belief structures

The purpose of this section is to motivate and define the concept of a Sunspot Rational Belief Structure (SRBS). A SRBS defines a stochastic process of exogenous variables $\{X_t\}$ and a stochastic process of sunspots $\{D_T\}$. The latter are sunspots in the sense that their *empirical* distribution is independent of the *empirical* distribution of the exogenous variables. However, according to the subjective rational beliefs held by agents and described in the SRBS, there need not be independence[8].

In the first example discussed here we stay in the framework of a rational belief structure as defined in Nielsen(1996). There (and in other papers using the concept of a rational belief structure) it was assumed that there are M agents, where agent m has a sequence $\{B_{mt}\}_{t=1}^{\infty}$ of either one-period beliefs for the SIDS case or transition probabilities (i.e. one-period conditional beliefs) for the SSM case and furthermore, that there is an exogenous stochastic process $\{X_t\}_{t=1}^{\infty}$. In many cases (f.i. the one commodity stochastic OLG model), the equilibrium price function then has the form $p_t = P(X_t, B_{1t}, \ldots, B_{Mt}\}$. In the next example we assume that M is equal to 2 and that each agent only has 2 possible one-period (conditional) beliefs and furthermore, that X_t only takes two values. This means that there will at most be 8 observed equilibrium prices for the economy. One problem we encounter with this framework is that there is an *obvious* correlation between the beliefs of the individual agent and the prices. In previous work (i.e. Kurz and Schneider, 1996, Kurz and Wu, 1996, and Nielsen, 1996) it was assumed that agents do not explore this connection and, comparing it to the assumption of competitive behavior in GE models, it was stated that one should expect this phenomena to disappear when the number of agents becomes larger. More specifically it was assumed that no agent consider his own belief part of the data when deriving the stationary measure. If he did so, he would discover the connection between his beliefs and prices, and knowing his own future beliefs he would be able to predict more accurately future prices. As we shall see below, in rational beliefs equilibria where beliefs (on fundamentals) are described by an SRBS, no agent can discover any correlation between his own beliefs and endogenous variables even at infinity. None the less, sunspots act as sources of correlated fluctuations in beliefs that in turn create fluctuations in aggregate variables like prices. This is much the same effect as was obtained when rational belief structures were used (i.e. without anonymity). Thus we may conclude that the results obtained hitherto (without anonymity) can be interpreted in terms of models with many anonymous

[8] In this sense, sunspots are *not* extrinsic, however we use the term 'sunspots' because of their stated independence from exogenous fundamentals and because they function as coordination devices for the beliefs of individual agents. In the original article on sunspots, Cass and Shell(1983), the possibility that agents may have subjective expectations about sunspots was considered (see p. 208).

agents as it was (implicitly) claimed and that in this sense, the issue of anonymity is mostly of a technical nature. However, as the following example demonstrates, in the case where agents have Markovian beliefs, so that they condition their forecasts on present observations there is potentially a separate issue, since *realized* expectations may then be quite arbitrary.

Example 4

Suppose first that there is an exogenous process which has 8 different states and that the stationary measure is Markovian with a transition matrix that looks as follows:

$$\begin{pmatrix} \frac{1}{8} & \cdot & \cdot & \frac{1}{8} \\ \frac{1}{8} & \cdot & \cdot & \frac{1}{8} \\ \cdot & \cdot & \cdot & \cdot \\ \frac{1}{8} & \cdot & \cdot & \frac{1}{8} \end{pmatrix}$$

i.e. the stationary distribution is i.i.d. with every state having the same probability. Now suppose the belief of some agent is SSM with two transition matrices each occurring with frequency $\frac{1}{2}$. The two transition matrices are as follows:

$$B^1 = \begin{pmatrix} \frac{1}{4} & 0 & \frac{1}{4} & 0 & \frac{1}{4} & 0 & \frac{1}{4} & 0 \\ \frac{1}{4} & 0 & \frac{1}{4} & 0 & \frac{1}{4} & 0 & \frac{1}{4} & 0 \\ \frac{1}{4} & 0 & \frac{1}{4} & 0 & \frac{1}{4} & 0 & \frac{1}{4} & 0 \\ \frac{1}{4} & 0 & \frac{1}{4} & 0 & \frac{1}{4} & 0 & \frac{1}{4} & 0 \\ 0 & \frac{1}{4} & 0 & \frac{1}{4} & 0 & \frac{1}{4} & 0 & \frac{1}{4} \\ 0 & \frac{1}{4} & 0 & \frac{1}{4} & 0 & \frac{1}{4} & 0 & \frac{1}{4} \\ 0 & \frac{1}{4} & 0 & \frac{1}{4} & 0 & \frac{1}{4} & 0 & \frac{1}{4} \\ 0 & \frac{1}{4} & 0 & \frac{1}{4} & 0 & \frac{1}{4} & 0 & \frac{1}{4} \end{pmatrix} \text{ and } B^2 = \begin{pmatrix} 0 & \frac{1}{4} & 0 & \frac{1}{4} & 0 & \frac{1}{4} & 0 & \frac{1}{4} \\ 0 & \frac{1}{4} & 0 & \frac{1}{4} & 0 & \frac{1}{4} & 0 & \frac{1}{4} \\ 0 & \frac{1}{4} & 0 & \frac{1}{4} & 0 & \frac{1}{4} & 0 & \frac{1}{4} \\ 0 & \frac{1}{4} & 0 & \frac{1}{4} & 0 & \frac{1}{4} & 0 & \frac{1}{4} \\ \frac{1}{4} & 0 & \frac{1}{4} & 0 & \frac{1}{4} & 0 & \frac{1}{4} & 0 \\ \frac{1}{4} & 0 & \frac{1}{4} & 0 & \frac{1}{4} & 0 & \frac{1}{4} & 0 \\ \frac{1}{4} & 0 & \frac{1}{4} & 0 & \frac{1}{4} & 0 & \frac{1}{4} & 0 \\ \frac{1}{4} & 0 & \frac{1}{4} & 0 & \frac{1}{4} & 0 & \frac{1}{4} & 0 \end{pmatrix}$$

If an agent has a Markovian beliefs, at each date, he conditions his forecast on the realized observed variables at that date. Thus, his realized belief depends both on his subjective belief and on the true (unknown) distribution (both probability measures on S^∞). If there is no particulary bad[9] correlation between the realization of the variable and the sequence of generating variables, then the forecasts conditional on the observed variable will on average be correct. This means that if for instance one looks at those dates where state number 1 is observed, then the average forecast on those dates will be equal to the average empirical distribution on dates following state number 1.

Next consider a different interpretation of these beliefs. Suppose that the 8 different states describe fundamentals in terms of a two agent situation (where each agent has two possible transition matrices) and two

[9]Specifically, if we have structural independence as defined below (see also Example 6).

exogenous states i.e. the fundamental states are $s^1 = (B^1, B^1, X^1)$, $s^2 = (B^1, B^1, X^2)$, $s^3 = (B^1, B^2, X^1)$,...., $s^7 = (B^2, B^2, X^1)$, $s^8 = (B^2, B^2, X^2)$. Each agent m uses each transition matrix $B^j, j = 1, 2$ half of the dates. This would imply that the second half of B^1 and the first half of B^2 would never be used by agent 1: Whenever he uses matrix B^1, one of the first four states (which are the states where he uses B^1) is the present realized state, and he will be conditioning on one of these four states (thus not using the second half of B^1). Whenever he uses matrix B^2, one of the last four states is the present realized state, which he is then conditioning on, and consequently he is not using the first half of B^2).

So although according to his rational belief as well as the (empirical) stationary measure, the state X^2 happens half of the dates, and although this is confirmed by his observations, his *conditional* belief for the *next* period will always be that state has 0 probability. Note, that at any date, his belief predicts that the frequency of X^2 will be $\frac{1}{2}$ in the future (but that X^2 will not happen the next date) $\blacksquare$

That such a case can arise in the context of rational beliefs may or may not be considered a problem. The crux of the theory is after all that agents are mistaken, and it is not ruled out that it can be discovered that their beliefs have *hitherto* performed badly. One should also note that the above phenomenon only arises because agents implicitly condition on their own beliefs. If they for instance only conditioned on exogenous variables we would be back in the framework of the first interpretation of the above set-up and there would be no problem. The next example is used to explain the idea behind SRBS and to see how we get anonymity and avoid the problem from Example 4 (see also Kurz, 1998 for another approach to dealing with these issues).

Example 5

Define the sunspot state to be either D^1 or D^2 and let the exogenous state be X^1 or X^2 as before. There are 8 agents in the economy (we will later assume that there is a continuum) and they are all identical in terms of endowments and preferences and differ only in terms of the *timing* but not *frequency* of their (conditional) beliefs. At any date (no matter the state), $3/4 \times 1/2 = 3/8$ of the agents have belief B^1, $3/4 \times 1/2 = 3/8$ of them have belief B^2, $1/4 \times 1/2 = 1/8$ of them have belief B^3 and $1/4 \times 1/2 = 1/8$ of them have B^4. Which of the agents have the different beliefs differs. The state space of fundamentals is : $\{(D^1, X^1), (D^1, X^2), (D^2, X^1), (D^2, X^2)\}$. The beliefs on this state space are as follows:

$$B^1 = \begin{pmatrix} \frac{1}{3} & \frac{1}{6} & \frac{1}{3} & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{3} & \frac{1}{6} & \frac{1}{3} \\ \frac{1}{12} & \frac{5}{12} & \frac{1}{12} & \frac{5}{12} \\ \frac{1}{8} & \frac{1}{8} & \frac{3}{8} & \frac{3}{8} \end{pmatrix} \quad B^2 = \begin{pmatrix} \frac{1}{3} & \frac{1}{6} & \frac{1}{3} & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{3} & \frac{1}{6} & \frac{1}{3} \\ \frac{5}{12} & \frac{1}{12} & \frac{5}{12} & \frac{1}{12} \\ \frac{3}{8} & \frac{3}{8} & \frac{1}{8} & \frac{1}{8} \end{pmatrix} \quad B^3 = \begin{pmatrix} 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ \frac{1}{12} & \frac{5}{12} & \frac{1}{12} & \frac{5}{12} \\ \frac{1}{8} & \frac{1}{8} & \frac{3}{8} & \frac{3}{8} \end{pmatrix} \quad B^4 = \begin{pmatrix} 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ \frac{5}{12} & \frac{1}{12} & \frac{5}{12} & \frac{1}{12} \\ \frac{3}{8} & \frac{3}{8} & \frac{1}{8} & \frac{1}{8} \end{pmatrix}$$

The stationary transition matrix is

$$\bar{B} = \begin{pmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \end{pmatrix} = \frac{3}{8}B^1 + \frac{3}{8}B^2 + \frac{1}{8}B^3 + \frac{1}{8}B^4$$

When the state is (D^1, X^1), 3/4 of the agents have the probability distribution $(\frac{1}{3}, \frac{1}{6}, \frac{1}{3}, \frac{1}{6})$ over next period's state and 1/4 of them have $(0, \frac{1}{2}, 0, \frac{1}{2})$. When the state is (D^2, X^1), 1/2 of the agents have the belief $(\frac{1}{12}, \frac{5}{12}, \frac{1}{12}, \frac{5}{12})$ and the rest have $(\frac{5}{12}, \frac{1}{12}, \frac{5}{12}, \frac{1}{12})$ over the next period's state. When the state is (D^1, X^2), 3/4 of the agents have the probability distribution $(\frac{1}{6}, \frac{1}{3}, \frac{1}{6}, \frac{1}{3})$ over next period's state and 1/4 of them have $(\frac{1}{2}, 0, \frac{1}{2}, 0)$. When the state is (D^2, X^2), 1/2 of the agents have the belief $(\frac{1}{8}, \frac{1}{8}, \frac{3}{8}, \frac{3}{8})$ and the other half have $(\frac{3}{8}, \frac{3}{8}, \frac{1}{8}, \frac{1}{8})$ over the next period's state. We should check whether there will be endogenous uncertainty, that is whether the prices in, say the states (D^1, X^1) and (D^2, X^1) are different. Note that in state (D^1, X^1), 3/4 of the agents put probability 2/3 on X^1 and 1/4 of them put probability 0 on X^1. On the other hand, in state (D^2, X^1), 1/2 of the agents put probability 2/12 on X^1 and 1/2 of them put probability 10/12 on X^1. This means that agents believe that the current value of the sunspot affects the probability of the exogenous variables next period. Since all the agents are the same, in a model with risk aversion (where beliefs matter), agents will behave differently in the two states if the prices in those states are the same, so the prices cannot be the same. A similar argument shows that the two states (D^1, X^2) and (D^2, X^2) must be different in terms of prices. Thus the four states are different in terms of observed variables: prices and X. In this sense we have *excess volatility*, that is the price volatility is not only determined by exogenous shocks but also by changes in agents' beliefs. ■

In the example just presented, the agents who are all identical (except for the *timing* of the sequence of conditional beliefs) have no way to discern a statistical connection between the states they observe and their individual beliefs. This is so, because when the individual agent uses the matrix B^n, at 1/4 of the times the state is (D^1, X^1), at 1/4 of the times it is (D^1, X^2) and so on. Thus the individual agent who has an SSM belief with four one-period conditional beliefs , B^1, B^2, B^3, and B^4 has no way to convert his knowledge about his own (or his dynasty's) future belief into a knowledge about future prices. For this idea to work it is important that the endogenous variables that may arise, do only depend on D and X and not on who has which beliefs in any endogenous state. That is why it was assumed that the eight agents were identical, i.e. were of the same type. Note, however that we can expand on this example and allow for many different *types* of agents as long as there are many of each type. We then require that at any date, the number of agents of a certain type that uses a certain (conditional) belief is the same. We now turn to the formal definition of a sunspot RBS.

Definition 11 *Sunspot Rational Belief Structure (SRBS).*

The Borel measurable set of sunspot states is $D \subset \Re^{K_D}$. The Borel measurable set of exogenous states is $X \subset \Re^{K_X}$. Let $S = X \times D$.

The Stationary Measure: $\left(\mu_X, \overline{\Pi}_X\right)$ represents the stationary measure for the exogenous variables where $\bar{\Pi}_X$ is a transition probability on X and μ_X is an invariant ergodic probability measure for $\bar{\Pi}_X$ having X as support. $\left(\mu_D, \overline{\Pi}_D\right)$ represents the stationary measure for the sunspot variables where $\bar{\Pi}_D$ is a transition probability on D and μ_D is an invariant ergodic probability measure for $\bar{\Pi}_D$ having D as support. Let $\overline{\Pi} = \overline{\Pi}_X \times \overline{\Pi}_D$ and $\mu = \mu_X \otimes \mu_D$. Here the transition probability $\overline{\Pi}_X \times \overline{\Pi}_D$ is defined by $\overline{\Pi}_X \times \overline{\Pi}_D((x,d), A_X \times A_D) = \overline{\Pi}_X(x, A_X)\overline{\Pi}_D(d, A_D)$ for $A_X \in \mathcal{B}(X)$ and $A_D \in \mathcal{B}(D)$. The stationary measure on $(S^\infty, \mathcal{B}(S^\infty))$ then is $\gamma(\mu, \overline{\Pi})$.

True Distribution: $B_0 = \gamma\left(\mu_0, \{\Pi_{0t}\}_{t=1}^{\infty}\right)$ is the distribution of an SSM process on S with associated stationary measure being equal to $\gamma\left(\mu, \overline{\Pi}\right)$.

Types of agents: There are $K \leq \infty$ types of agents in the economy. For each type k there is a continuum of agents represented by the interval $(0,1]$. Type k is having an SSM belief using the set $\{\Pi_{kl}\}_{l=1}^{\infty}$ of transition probabilities on S, each having frequency $q_{kl} \geq 0$ s.t. $\sum_{l=1}^{\infty} q_{kl}\Pi_{kl} = \bar{\Pi}$. We let $Q_k = \otimes_{t=1}^{\infty} q_k$ where q_k is the probability measure on $\mathcal{N}$ with $q_k(\{l\}) = q_{kl}$.

Beliefs of type k agents: Consider an agent $m \in (0,1]$ of type k. We represent his belief B_{km} by a sequence of transition probabilities as follows:

Divide the interval $[0,1]$ into the countable set of intervals, I_{kj}, $j = 1, 2..$ where $I_{kj} = \left(\sum_{l=1}^{j} q_{kl} - q_{kj}, \sum_{l=1}^{j} q_{kl}\right]$ and define the transition $T_k : (0,1] \to (0,1]$ as follows:

$$T_k(x) = \frac{1}{q_{kj}} x - \frac{\sum_{l=1}^{j} q_{kj} - q_{kj}}{q_{kj}} \text{ if } x \in I_{kl}$$

Then define the function $d_{k,1} : (0,1] \to \mathcal{N}$ by $d_{k,1}(m) = j$ if $m \in I_{kj}$ and define for $t > 1$ $d_{k,t} : (0,1] \to \mathcal{N}$ by $d_{k,t}(m) = d_{k,1}(T_k^{t-1} m)$. Then the belief of agent m is

$$B_{km} = \gamma\left(\mu_k, \left\{\Pi_{kd_{k,t}(m)}\right\}_{t=1}^{\infty}\right)$$

if (i) B_{km} is stable and (ii) $\overline{B}_{km} = \gamma(\mu, \overline{\Pi})$, else it is $\gamma\left(\mu, \overline{\Pi}\right)$ ■

It should be noted that the SRBS is constructed so as to let prices affect beliefs. If in equilibrium each fundamental state is associated with a different price, we can assume that agents observe and condition on prices and exogenous variables rather than on the variables in D. In this particular respect the construction is similar to that of Kurz and Schneider(1996), who also allow for an influence of prices on beliefs. Another interpretation of a SRBS would suggest that different types of agents may not observe and condition their beliefs on the same sunspot. However, since the sunspot that a particular type observes will typically influence prices, other agents will indirectly, by conditioning on prices, condition their beliefs on that sunspot. In this sense beliefs about sunspots are contagious. Finally it should be noted that if all agents use the empirical distribution as their belief, this is equivalent to them adopting stationary rational expectations.

4.2 Structural Independence

The concept of structural independence was introduced in Nielsen(1994) as a condition that guarantees that systems of stable beliefs are jointly stable and also as a natural notion of incorrectness of rational beliefs. In the context of SSM beliefs, structural independence between a belief and the true process turns out also to guarantee that the realized sequence of conditional beliefs (as a function of a realized sequence of states) of an individual agent is rational. Without any assumptions about the relationship between beliefs and the true distribution this is not necessarily the case (see Example 6 below). Consider a particular agent, m and his rational SSM belief $\gamma(\mu_m, \{\Pi_{mt}\})$ on the sequence space S^∞. The realized sequence, $\langle s_t \rangle$ is governed by the probability measure $\gamma(\mu_0, \{\Pi_{0t}\})$. At any date, t, the realized belief of the agent, after s_t has been observed, is $\gamma(\delta_{s_t}, \{\Pi_{ms}\}_{s=t}^\infty)$. Let $\{\mathcal{F}_t\}_{t=0}^\infty$ be the natural filtration (induced by information at date t). It is a consequence of Chuang(1997), Theorem 1 that $\forall C \in \mathcal{C}(S^\infty)$:

$$\lim_{J\to\infty} \frac{1}{J} \sum_{j=0}^{J-1} \gamma(\mu_m, \{\Pi_{mt}\}_{t=1}^\infty)(T^{-j}C|\mathcal{F}_j)\langle s \rangle = \bar{\gamma}(\mu_m, \{\Pi_{mt}\}_{t=1}^\infty)(C) \text{ for } \gamma(\mu_m, \{\Pi_{mt}\}_{t=1}^\infty) \text{ a.a. } \langle s \rangle \tag{21}$$

However, we are interested in whether there is convergence for $\gamma(\mu, \{\Pi_{0t}\})$ a.a. $\langle s \rangle$:

Definition 12 *Conditional WAMS and Conditional Rationality.*
Let μ be a probability measure on $(S^\infty, \mathcal{B}(S^\infty))$ and let ν be a stable probability measure on the same space. Then μ is conditionally WAMS relative to ν if

$$\forall C \in \mathcal{C}(S^\infty) : \frac{1}{J} \sum_{j=0}^{J-1} \mu(T^{-j}C|\mathcal{F}_j)\langle s \rangle \text{ converges for } \nu\text{-a.a.}\langle s \rangle \tag{22}$$

If the convergence in (22) is to $\bar{\nu}(\cdot|\mathcal{I})\langle s \rangle$, we say that μ is conditionally rational relative to ν ■

Example 6 *Absence of conditional rationality.*
Let $S = \{0, 1\}$ and the belief an SSM measure involving two matrices, Π^0 and Π^1 each with frequency $\frac{1}{2}$ (i.e. $q_0 = q_1 = \frac{1}{2}$).

$$\Pi^0(0, \{0\}) = \Pi^0(1, \{1\}) = 1$$
$$\Pi^1(0, \{1\}) = \Pi^1(1, \{0\}) = 1$$

The empirical distribution is i.i.d. with probability of 0 being $\frac{1}{2}$. Let $\{\Pi_t\}$ be a Q-typical realization defining, together with the belief that $X_1 = 0$ with probability 1, a WAMS rational belief. Now construct a sequence $\{s_t\}$ as follows:

$$\text{If } \Pi_t = \Pi^0, s_t = 0$$
$$\text{else } s_t = 1$$

Then $\Pi_t(s_t, \{0\}) = 1, \forall t$ so the belief is not conditionally rational relative to $\delta_{\langle s \rangle}$. Moreover, the associated stationary measure of $\delta_{\langle s \rangle}$ is equal to the empirical distribution.
Note that we do not have structural independence between the belief and the true measure (which is an SSM measure). Also, by modifying the true sequence in a suitable way, we get a belief which is not conditionally WAMS ■

If the belief of an agent is not conditionally WAMS relative to the true measure, then we could have that his realized actions, as they depend on his realized beliefs, are non-stable. However, conditional WAMS is not sufficient to ensures stability of the resulting action as the following example demonstrates.

Example 7 *Belief is stable but resulting action is not.*

Suppose an agent is betting on a fair coin with outcome, x_t at date t, in $\{0, 1\}$. Let the belief of the agent be defined as follows. $T^* \subset \mathcal{N}$ fulfills

$$\frac{1}{J} \sum_{t=0}^{J-1} 1_{T^*}(t) \text{ does not converge as } J \to \infty$$

The belief is: $\mu(x_t = 1) = 1/2 + 1/t$ if $t \in T^*$, $1/2 - 1/t$, else. If the agent wins he receives \$1 else he pays \$1 and if he is risk neutral, he will bet on 1 at all dates in T^*, and on 0 at all other dates. The belief is WAMS, while the actions are not stable. Neither conditional WAMS nor WAMS is sufficient for stability of an equilibrium process ■

The stability of the actions of an individual agent may also become an issue if he is infinitely lived, even when his belief is WAMS. Suppose his SIDS belief, $\otimes_{t=1}^{\infty} \Pi_t$ in non-stationary. Then we have for $t \neq t'$, that $\otimes_{s=t}^{\infty} \Pi_s$ $\neq \otimes_{s=t'}^{\infty} \Pi_t$ (Remark 2). So the countable sequence of beliefs about the infinite future never repeats itself and in that particular sense this sequence is "non-stable". If the actions of this agent are sensitive to beliefs into the distant future, possibly his actions may then become be non-stable. If the agent discounts future utility, this may however not be the case. The question about induced stability of actions is important, but we will not consider it further here since, as is implied by Proposition 11 to follow, for the systems of beliefs we study, stability of actions is always the case for short-lived agents. The following definition is an extension from Nielsen(1994).

Definition 13 *Structural Independence.*

Two SSM measures $\gamma(\mu^1, \{\Pi_t^1\}_{t=1}^{\infty})$ and $\gamma(\mu^2, \{\Pi_t^2\}_{t=1}^{\infty})$ (generated by $Q_i = \otimes_{t=1}^{\infty} q_i, i = 1, 2$ respectively) are said to be structurally independent if the sequence $\{\Pi_t^1, \Pi_t^2\}_{t=1}^{\infty}$ is $Q_1 \otimes Q_2$-typical ■

Proposition 11

Let q be a probability measure on $\mathcal{P} = \{\Pi^1, \Pi^2, \ldots\}$, a countable set of transition probabilities on S and let q_B be a probability measure on $\mathcal{P}_B = \{\Pi^{B1}, \Pi^{B2}, \ldots\}$, a countable set of transition probabilities on S, s.t. $\sum_i q(\{i\})\Pi^i = \sum_i q_B(\{i\})\Pi^{Bi} = \overline{\Pi}$ and $\overline{\Pi}$ has an ergodic, invariant measure $\bar{\mu}$. Then letting $Q = \otimes_{t=1}^{\infty} q$ and $Q_B = \otimes_{t=1}^{\infty} q_B$ we have:
(a) For the case where S is countable,
(b) For the general case, if the conditions of Proposition 10 hold,
that for $Q \otimes Q_B$ a.a. $\langle \Pi, \Pi^B \rangle \in \mathcal{P}^{\infty} \times \mathcal{P}_B^{\infty}$, $\gamma(\bar{\mu}, \{\Pi_t^B\}_{t=1}^{\infty})$ is an SSM measure which is conditionally rational relative to the SSM measure $\gamma(\bar{\mu}, \{\Pi_t\}_{t=1}^{\infty})$.

Proof: (a) Construct the probability measure ν on $S \times \mathcal{P} \times \mathcal{P}_B$ as follows: The initial distribution is $\bar{\mu} \otimes q \otimes q_B$ and the transition probability M is defined by:

$$M((s, \Pi, \Pi^B), \{\tilde{s}\} \times \{\tilde{\Pi}\} \times \{\tilde{\Pi}^B\}) = \Pi(s, \{\tilde{s}\}) q(\{\tilde{\Pi}\}) q_B(\{\tilde{\Pi}^B\})$$

ν is ergodic and stationary. Let $\mathcal{G} = \sigma[\{S^{\infty}, \emptyset\} \times \mathcal{B}(\mathcal{P}^{\infty}) \times \mathcal{B}(\mathcal{P}_B^{\infty})]$. From Proposition 7 we have (using the notation of that proposition) that for $Q \otimes Q_B$ a.a. $\langle \tilde{\Pi}, \tilde{\Pi}^B \rangle$ and any $\tilde{s}$ that $\nu(\cdot|\mathcal{G})\langle \tilde{s}, \tilde{\Pi}, \tilde{\Pi}^B \rangle$ is WAMS with associated stationary measure being ν. Thus if $F : (S \times \mathcal{P} \times \mathcal{P}^B)^L \rightarrow \Re$ is integrable we have for $\nu(\cdot|\mathcal{G})\langle \tilde{s}, \tilde{\Pi}, \tilde{\Pi}^B \rangle$ a.a. $\langle s, \Pi, \Pi^B \rangle$ that

$$\frac{1}{J} \sum_{J=0}^{J-1} F((s_{j+1}, \Pi_{j+1}, \Pi_{j+1}^B), \ldots, (s_{j+L}, \Pi_{j+L}, \Pi_{j+L}^B)) \rightarrow E_{\nu} F \text{ as } J \rightarrow \infty$$

For a cylinder $C = C_{L+1} \times S^{\infty}$ in $\mathcal{C}^{L+1}(S^{\infty})$ we let

$$F(s, \Pi_1^B, \ldots \Pi_L^B) = \int_S \Pi_1^B(s, ds_2') \int_S \Pi_2^B(s_2', ds_3') \cdots \int_S \Pi_L^B(s_L', ds_{L+1}') 1_{C_{L+1}}(s, s_2' \cdots s_{L+1}') \tag{23}$$

Note that $F(s_{j+1}, \Pi_{j+1}^B, \ldots, \Pi_{j+L}^B) = \gamma(\bar{\mu}, \{\Pi_t^B\})(T^{-j}C|\mathcal{F}_j)\langle s \rangle$ for $j = 0, 1, \ldots$. Then

$$E_{\nu} F = \int_S \bar{\mu}(ds_1) \int_S \overline{\Pi}(s_1, ds_2) \cdots \int_S \overline{\Pi}(s_L, ds_{L+1}) 1_{C_{L+1}}(s_1, \cdots s_{L+1}) = \gamma(\bar{\mu}, \overline{\Pi})(C)$$

(b) The argument is basically the same as in (a). Let $h : \mathcal{P} \times \mathcal{P}_B \rightarrow \mathcal{N}$ be one-to-one and onto and define g_1 and g_2 on $\mathcal{N}$ by $h^{-1}(n) = (g_1(n), g_2(n))$. Let $\tilde{Q}$ on $\mathcal{N}$ be defined by $\tilde{Q}\{n\} = Q \otimes Q_B(\{(g_1(n), g_2(n))\})$ and let $\tilde{\mathcal{P}} = \{\tilde{\Pi}_1, \tilde{\Pi}_2, \ldots\}$ be defined by $\tilde{\Pi}_j = g_1(j)$. Then by Proposition 10 we have for $\tilde{Q}$ a.a. $\langle \bar{n} \rangle$ that $\gamma(\bar{\mu}, \{\tilde{\Pi}_{\bar{n}_t}\}_{t=1}^{\infty}) \otimes \delta_{\langle \bar{n} \rangle}$ is WAMS on $((S \times \mathcal{N})^{\infty}, \mathcal{B}((S \times \mathcal{N})^{\infty}))$. In other words, we have that $\gamma(\bar{\mu}, \{\tilde{\Pi}_{\bar{n}_t}\}_{t=1}^{\infty}) \otimes \delta_{\{g_1(\bar{n}_t)\}} \otimes \delta_{\{g_2(\bar{n}_t)\}}$ is WAMS on $((S \times \mathcal{P} \times \mathcal{P}_B)^{\infty}, \mathcal{B}((S \times \mathcal{P} \times \mathcal{P}_B)^{\infty}))$. As before, using the definition in (23), we have for $\gamma(\bar{\mu}, \{\tilde{\Pi}_{\bar{n}_t}\}_{t=1}^{\infty})$ a.a. $\langle s \rangle$ that $\frac{1}{J} \sum_{j=1}^{J} F(s_j, g_2(\bar{n}_j), \ldots, g_2(\bar{n}_{j+L})) \rightarrow \gamma(\bar{\mu}, \overline{\Pi})(C)$ (by the rationality of beliefs) ■

Remark 8

The proposition can be interpreted as stating that for "almost all" true distributions and rational beliefs which are structurally independent, we have conditional rationality of the belief. On the other hand, if an agent happens to have a correct belief, we do have conditional rationality but not structural independence ■

Remark 9

Note that when we have an SRBS (if relevant, fulfilling the conditions of Proposition 10), it is the case that (Lebesgue) almost all agents have rational beliefs which are conditionally rational and which induce actions that are stable when agents are short lived (or myopic), i.e. their actions only depend on their forecasts for a limited number of future periods ■

5 An application

To illustrate how the concept of a Sunspot Rational Belief Structure may be applied, we present a general equilibrium, continuous state space version of the model of Muth(1961). In the context of rational beliefs, partial equilibrium versions of this model were studied in Kurz(1994) (where the existence of an equilibrium with homogenous beliefs was proved) and in Nielsen(1996) (where an example with diverse beliefs and a finite state space was provided). In these kind of models, whether rational expectations or rational beliefs are imposed, producers decide on their output before they know prices. In our version of the model, there is a continuum of agents that interpret sunspots differently over time in such a way that these interpretations are consistent with the observed average independence between sunspots and fundamentals of the economy. Despite this empirical independence, the sunspots do influence real economic activity, since they influence beliefs and hence production decisions (and in the following period prices of all commodities)[10]. This would be so, whether agents observe the current production decisions (i.e. the current beliefs) of other agents or not. However, we assume that they do not, i.e. contrary to the rational expectations case we do not assume common knowledge of beliefs. The consequence is a miscoordination similar to what was found in the original cobweb model, but now founded on a rigorous theory about expectations. The model we present does not strive for maximal generality, but is constructed to exemplify how the theoretical concepts and results, we presented may apply to general equilibrium models.

There are N commodities and K types of agents, both finite numbers. Each type of agent, $k = 1, 2, \ldots, K$ at any date uses one of L_k possible short term beliefs (that is transition probabilities). We assume that X is an interval in $\Re$ and that D is finite. Time is discrete and runs from 1 to ∞. In odd periods (denoted

[10] As a consequence, even if the exogenous process is stationary, the equilibrium process need not be.

$\mathcal{N}_o$) agents make production decisions based on their expectations about prices and exogenous shocks the following period. In even periods, production is realized, commodities are exchanged at market clearing prices and consumption takes place. In odd period t an agent of type k receives the signal $d_t^k \in D_k$ and observes the exogenous shock x_t, but he does not observe the signal of other agents (there is asymmetric information). As a consequence an individual tries to make forecasts not only about future exogenous shocks but also about how much other agents are currently deciding to produce (that is about the current beliefs of other types as parametrized by d). Let $D = \times_{k=1}^K D_k$ and $d_t = (d_t^1, \ldots, d_t^K)$. Mainly in order to facilitate the presentation of the model we assume that the beliefs have the following format:

$$\Pi_{kl}((x, d^k), A_x \times A_d) = \pi_{kl}((x, d^K), A_x)\mu_D(A_d), \forall k, l, (x, d) \in X \times D, A_x \in \mathcal{B}(X), A_d \subset D \tag{24}$$

where π_{kl} maps $X \times D_k$ into the set of probability measures on $(X, \mathcal{B}(X))$. We also assume that μ_X on X is equivalent to Lebesgues measure and that

$$\pi_{kl}((x, d^k), \cdot) \approx \mu_X \approx \overline{\Pi}_X(x, \cdot), \forall k, l, (x, d) \tag{25}$$

At any $t \in \mathcal{N}_0$, an agent of belief-type (k, l) decides on input of labor, $\alpha \in [0, A^k]$ in producing $F_k(\alpha)$ of commodity $n(k) \in \{1, 2, \ldots, N\}$. F_k is C^2, increasing and concave, $F_k(0) = 0$. We assume that, $\forall n \in \{1, 2, \ldots, N\}, \exists k \in \{1, 2, \ldots, K\}$ s.t. $n(k) = n$.

Each agent consumes leisure and the N consumption goods. The consumption set (for any state $(d, x, \overline{x})$) of an agent of type k is $\Re_{++}^N \times [0, A^k]$. The preferences over consumption depend on the exogenous shocks. Thus, we define a utility function $u_k : \Re_{++}^N \times [0, A^k] \times X^2 \to \Re$ for each even period. Given stochastic consumption of consumption goods and leisure, $C : D \times X^2 \to \Re_{++}^N$, $l : X \times D \to [0, A^k]$ and given a belief B on $(X \times D)^\infty$ the expected utility of an agent of type k is:

$$E_B\{\sum_{t \in \mathcal{N}_0} \rho^t u_k[C(d_t, x_t, x_{t+1}), l(d_t, x_t), x_t, x_{t+1}]\}$$

where $\rho \in (0, 1)$ is the discount factor. We assume that u_k is smooth and that for all $(x, x') \in X^2$: $u_k(\cdot, \cdot, x, x')$ is strictly increasing, strictly concave, with indifference curves that are uniformly bounded away from $\partial\Re_+^{N+1}$.

Let Δ^N be the N-dimensional simplex and $\overset{\circ}{\Delta}^N$ its interior. In equilibrium there will be a measurable price function $p : D \times X \times X \to \overset{\circ}{\Delta}^N$ (we normalize prices state by state). At odd date t, an agent of belief-type (k, l) with $\overline{d}^k$, $\overline{x}$ and p given is faced with the following problem:

Problem of belief-type (k, l): Choose $\alpha \in [0, A^k]$ and $C : D \times X \to \Re_+^N$ to maximize

$$\int_X \sum_{d \in D} u_k[C(d, x), A^k - \alpha, x, \overline{x}]\mu_D(d|\overline{d}^k)\pi_{kl}((\overline{x}, \overline{d}^k), dx) \tag{26}$$

subject to

$$\sum_{n=1}^N C_n(d, x)p_n(d, x, \overline{x}) = p_{n(k)}(d, x, \overline{x})F_k(\alpha) \text{ for } \mu_D(\cdot|\overline{d}^k) \otimes \pi_{kl}((\overline{x}, \overline{d}^k), \cdot) \text{ a.a. } (d, x) \tag{27}$$

The first order conditions for this problem are:

$$\int_X \sum_{d\in D} \left[-u'_{kn+1}(C(d,x), A^k - \alpha, x, \overline{x}) + \lambda(d,x) p_{n(k)}(d,x,\overline{x}) F'_k(\alpha)\right] \mu_D(d|\overline{d}^k)\pi_{kl}((\overline{x},\overline{d}^d), dx) = 0 \quad (28)$$

$$u'_{kn}(C(d,x), A^k - \alpha, x, \overline{x}) - \lambda(d,x)p_n(d,x,\overline{x}) = 0, 1 \le n \le N, \text{ for } \mu_D(\cdot|\overline{d}^k) \otimes \pi_{kl}((\overline{x},\overline{d}^k),\cdot) \text{ a.a. } (d,x) \quad (29)$$

One consequence of these first order conditions is:

$$\int_X \sum_{d\in D} \left[\frac{-1}{F'_k(\alpha)} u'_{kn+1}(C(d,x), A^k - \alpha, x, \overline{x}) + u'_{kn(k)}(C(d,x), A^k - \alpha, x, \overline{x})\right] \mu_D(d|\overline{d}^k)\pi_{kl}((\overline{x},\overline{d}^k), dx) = 0 \quad (30)$$

We are now ready to present the definition of equilibrium:

Definition 14 *Equilibrium*

A price function p, labor supply: $\alpha_{kl} : D \times X \to [0, A^k], \forall k, l$, measurable, and consumption, $C_{kl} : D \times X^2 \to \Re^N_{++}, \forall k, l$, measurable s.t. $\forall \overline{d} \in D$ and μ_X a.a. $\overline{x}$:

(a) $C_{kl}(\cdot,\cdot,\overline{x}) : D \times X \to \Re^N_{++}$ and $\alpha_{kl}(\overline{d},\overline{x})$ solve belief-type (k,l)'s problem given $(\overline{d}^k,\overline{x})$ and p.

(b) $\sum_{k=1}^K \sum_{l=1}^{L_k} F_k(\alpha_{kl}(\overline{d},\overline{x}))q_{kl}1_{n(k)}(n) = \sum_{k=1}^K \sum_{l=1}^{L_k} C_{kln}(\overline{d},x,\overline{x})q_{kl}$ for $n = 1, 2, \ldots, n$, for μ_X a.a. x ■

Recall that q_{kl} is the weight of the continuum of agents of belief-type (k,l). A particular agent i is of a certain type k, but which belief (in terms of π_{kl}) he uses varies over time. However, any agent of belief-type (k,l) acts the same way in this model no matter the date or the past. Consequently, we need only consider actions of the different belief-types and not of the individual agents.

Sketch of proof of existence of equilibrium

We sketch the proof here, further details are provided in Appendix B. The equilibrium is found as a fixed point of a correspondence from the set of production decisions into that same set. This correspondence is defined for a given $\overline{x}$. For every array of production decisions, $\{\alpha_{kl}\} \gg 0$ there is a set of measurable price functions p s.t. for every (d,x), $p(d,x,\overline{x}) \in \overset{\circ}{\Delta}{}^N$ is an equilibrium for the even date sub-economy with belief-type (k,l) having utility function $u_k(\cdot, \alpha_{kl}, x, \overline{x})$ and initial endowment of commodity $n(k)$ in the amount of α_{kl}. On the other hand, for every price function $p : D \times X \to \overset{\circ}{\Delta}{}^N$ there is an optimal supply for an agent of belief-type (k,l), $\alpha_{kl} : D \to \Re$. Composing the correspondence from production decisions into (measurable) price functions with the function from price functions into production decisions, we obtain the desired correspondence ■

Using the results obtained previously, let us summarize the features that this model exhibits. Each agent holds a rational belief about the stochastic process of equilibrium prices (remember, that presenting the sunspot RBS in terms of beliefs on primitives is only a modelling device and that in equilibrium the beliefs of each agent can be transformed into beliefs about prices, using the functional relationship between (d,x) and p) - rational in the sense that each belief would, if it were correct, generate the same empirical distribution of exogenous variables and prices as does the true (unknown) distribution. These beliefs are in terms of

(Markovian) WAMS probability measures on a continuous state space and moreover, they are for almost all agents conditionally rational, that is the sequence of realized conditional beliefs also conforms to the empirical (stationary) measure.

Recalling that the basic motivation of rational beliefs is to describe an environment where something but not everything is learnable, the model thus demonstrates how mistakes and increased price volatility arise in such environments. The fact that not everything is learnable means that there are many possible models that fits the observed behavior of the economy. We argued that using the stationary measure is a conservative strategy and that the use of any other rational belief is a sign of overconfidence on the part of the agent. Overconfident agents continue to believe that they can forecast the future prices better than they actually did (when they look back at their performance). This overconfidence is however rational precisely in the sense that the belief employed is consistent with observations.

An important assumption made was that agents have asymmetric information about the sunspot. In other words, agents of one type are at odd dates uncertain about what beliefs agents of other types hold[11]. This is one way of modelling what we consider to be a fact, namely that market participants (whether they be producers or financial firms) only get to know the beliefs and hence actions of other participants with a delay[12]. This together with the assumption of some sort of adaptive expectations were the fundamental assumptions underlying the cobweb model. Muth's contribution sought to confront the assumption made about expectations, not the assumption made about the timing of information. In the context of the cobweb model, the aim of the rational beliefs literature is precisely to provide a more solid foundation, in terms of the modelling of expectations, for the miscoordination of market actions that the original cobweb model sought to illustrate. In particular, in contrast with the original model, there is in the rational beliefs version no singular easily recognizable sequence of price movements that, one could argue, all rational agents ought to identify and adopt as their own belief. By introducing sunspots as quasi-public coordination devices, we retain the assumption of anonymity, an important assumption in general equilibrium theory, but allow for individual beliefs to have an impact on aggregate variables like prices. These sunspots, acting as coordinating devices for beliefs, allow us to capture two important facts about the market participants: (i) that there are correlated movements in their subjective expectations and (ii) that it is prohibitive costly if not outright impossible to get to know these expectations for each and every one of them.

From the perspective of social welfare, we argue that there are two distinct (although both consequences of the diversity of beliefs) sources of inefficiency in the equilibrium presented here. Firstly, and less contro-

[11] We assumed though, that they were using the empirical distribution μ_D in forming expectations about other's beliefs - an assumption that could be relaxed at the expense of a more complicated notation. In that case agents would have subjective beliefs about the current distribution of other agents' beliefs.

[12] The importance attributed to aggregate (and thus imprecise) market indicators, including various measurements of consumer sentiments, by both the private and public sector, demonstrates that this information is not readily available.

versially, the fact that agents do not observe each other's subjective beliefs or actions at odd dates introduces an added element of uncertainty which is social rather than exogenous. This social uncertainty has as consequence that prices at even dates will exhibit excess volatility, that is there are more price states than would be present if agents had rational expectations. Because this uncertainty is social it is also *endogenous* that is, it is a product of the particular economic institution. To take an example, if a social planner dictated the production level to each agent, said miscoordination and excess volatility would not be present (although other types of miscoordination might then be present in stead).

Secondly, we argue that even if there were full information about the beliefs (or production decisions) of all agents, there would still be an inefficiency due to the fact that individual agents use mutually inconsistent beliefs in their optimization problem. There are at any date $\sum_k L_k$ different short term beliefs, of which at most one of them is correct. The inconsistency of these rational beliefs results in a social inefficiency[13]: Most agents, if not all, make suboptimal decisions. Obviously, this position is at variance with the use of Pareto ranking as a criterion for evaluating social incomes. Rather we propose to employ the so called ex post optimality criterion, according to which a single belief should be used when evaluating the outcome of an equilibrium for a particular economic institution. The concept of ex-post optimality is presented in Hammond(1981) and employed in the context of rational beliefs in Nielsen(2003) where a fuller discussion of the nature of this inefficiency, and how it can be remedied is provided.

6 Conclusion

With the new framework presented here we have considerably enlarged the scope for applying rational beliefs to various general equilibrium frameworks. Specifically it is now possible to study dynamic models with an Markovian empirical distribution on a continuous state space where agents are not only assumed to be, but are in fact, anonymous. Such applications can be carried out along the lines developed in Nielsen(1996), where the existence of a general equilibrium is established by showing the existence of a fixed point for a translation on a set of exogenous and endogenous variables. By imposing from the outset, that beliefs are rational relative to the fundamentals of the economy, we ensure that in equilibrium, these beliefs are rational relative to all variables. Compared to finding an equilibrium in a space of beliefs and distributions of endogenous variables this considerably simplifies the problem at hand. The growing literature on behavioral economics demonstrates the need to go beyond the rational expectations assumption, which has been the source of too many "puzzles" - or, if one prefers, empirical refutations. However, the theoretical foundation of behavioral economics still seems underdeveloped. We have argued that non-stationary rational beliefs exhibit

[13]This is the case whether markets are complete or not, however incompleteness of markets would probably accentuate the problem.

rational overconfidence and in this way the theory captures a recurring concept in behavioral economics. The rationality requirement imposed, on beliefs then provides a clear delineation between which beliefs can be considered acceptable to the theory and which cannot. Especially in finance there has been a very active search for models that go beyond rational expectations and behavioral economics has has come to the fore here. Since finance models often work with a continuous state space, we expect that SSM processes on a continuous state space will be particularly useful in this context.

Appendix A

(1) Proof of (16):

We let $x^t = (x_t, x_{t-1}, \ldots, x_1)$ and for any measure ρ on a space of sequences, ρ_t, is the restriction to the first t coordinates. Let $A = A_{t-1} \times S^\infty \in \mathcal{A}_{t-1}$ (where $\mathcal{A}_{t-1}$ is the σ algebra generated by $(X_{t-1}, \ldots, X_1)$. We then have

$$\int_A P_{S^\infty}(\{X_t \in F\}|\mathcal{A}_{t-1})(\langle s\rangle)P_{S^\infty}(d\langle s\rangle) = \int_{A_{t-1}\times Y^{t-1}} P_t\left(\{X_t \in F\}\,|(s^{t-1}, y^{t-1})\right) P_t(d(s^{t-1}, y^{t-1}))$$
$$= \int_{A_{t-1}\times Y^{t-1}} \Pi_{y_{t-1}}(s_{t-1}, F)P_t(d(s^{t-1}, y^{t-1})) = \int_{A_{t-1}\times Y^{t-2}} \int_Y \Pi_{y_{t-1}}(s_{t-1}, F)q(dy_{t-1})P_t(d(s^{t-1}, y^{t-2})) =$$
$$\int_{A_{t-1}\times Y^{t-2}} \bar{\Pi}(s_{t-1}, F)P_{t-1}(d(s^{t-1}, y^{t-2})) = \int_{A_{t-1}} \bar{\Pi}(s_{t-1}, F)P_\infty(d\langle s\rangle) \blacksquare$$

(2) $\hat{\mu}$ is a probability measure:

We only need to show that $\hat{\mu}$ is non-negative. For this it is sufficient to consider any measurable $F \subset B$.

$$\mu(F) = \int_S \bar{\Pi}(s, F)\bar{\mu}(ds) \geq \int_A \bar{\Pi}(s, F)\bar{\mu}(ds) \geq \int_A \int_F \hat{\epsilon}\rho(ds')\bar{\mu}(ds) = \bar{\mu}(A)\hat{\epsilon}\rho(F)\blacksquare \tag{31}$$

(3) Stationarity of $\gamma(\hat{\mu}, \bar{\bar{\Pi}})$:

$$\text{(i)} \int_{\hat{S}} \bar{\bar{\Pi}}(\hat{s}, \{\alpha\})\hat{\mu}(d\hat{s}) = \int_A \hat{\epsilon}\hat{\mu}(d\hat{s}) + \bar{\bar{\Pi}}(\alpha, \{\alpha\}(\hat{\mu}(\{\alpha\}) = \hat{\epsilon}\hat{\mu}(A) + \int_S \bar{\bar{\Pi}}(s, \{\alpha\})\rho(ds)\hat{\mu}(\{\alpha\})$$
$$= \hat{\epsilon}[\bar{\mu}(A) - \rho(A)\hat{\epsilon}\bar{\mu}(A)] + \rho(A)\hat{\epsilon}^2\bar{\mu}(A) = \hat{\epsilon}\bar{\mu}(A) \equiv \hat{\mu}(\{\alpha\}).$$

$$\text{(ii) For } F \in \mathcal{B}(S) : \int_{\hat{S}} \bar{\bar{\Pi}}(\hat{s}, F)\hat{\mu}(d\hat{s}) = \int_S \bar{\bar{\Pi}}(\hat{s}, F)\hat{\mu}(d\hat{s}) + \bar{\bar{\Pi}}(\alpha, F)\hat{\mu}(\{\alpha\}) = \int_S \bar{\bar{\Pi}}(s, F)d\left[\bar{\mu}(s) - \rho(s)\hat{\epsilon}\bar{\mu}(A)\right]$$
$$+ \int_S \bar{\bar{\Pi}}(s, F)\rho(ds)\hat{\epsilon}\bar{\mu}(A) = \int_S \bar{\bar{\Pi}}(s, F)d\bar{\mu}(s) = \int_S \bar{\Pi}(s, F)\bar{\mu}(ds) - \hat{\epsilon}\rho(F)\bar{\mu}(A) = \bar{\mu}(F) - \hat{\epsilon}\rho(F)\bar{\mu}(A) \blacksquare$$

(4) Ergodicity of $\gamma(\hat{\mu}, \bar{\bar{\Pi}})$:

Note first that the inequality in (31) is strict if $\bar{\mu}(F) > 0$ or $\rho(A) > 0$. So if $\bar{\mu}(F) > 0$, $\hat{\mu}(F) > 0$. And if $\bar{\mu}(F) = 0$ then $\hat{\mu}(F) = 0$. So if we let $\hat{\mu}_S$ be $\hat{\mu}$ restricted to S, $\hat{\mu}_S \approx \bar{\mu}$.

We also have for all $s \in S$ that $\bar{\hat{\Pi}}(s, A) = 0 \Leftrightarrow \bar{\Pi}(s, A) = 0$

Next note that if E is $\bar{\hat{\Pi}}$-invariant and $\hat{\mu}(E) > 0$ then $\alpha \in E$. Else, E would also be $\bar{\Pi}$-invariant and we would have $\bar{\mu}(E) = 1$. But since $\bar{\hat{\Pi}}(s, \{\alpha\}) > 0$, $\forall s \in A$ and $\hat{\mu}(A) > 0$ this would give a contradiction.

Thus if E is $\bar{\hat{\Pi}}$-invariant and $0 < \hat{\mu}(e) < 1$, there is measurable $K \subset \hat{S} \setminus E$ s.t. $\hat{\mu}(K) > 0$ and $\bar{\hat{\Pi}}(s, K) = 0$ for $\hat{\mu}$ a.a. $s \in E$. But then $1 > \bar{\mu}(K) > 0$ and $E \setminus K$ is $\bar{\Pi}$-invariant, contradicting the ergodicity of $(\bar{\mu}, \bar{\Pi})$ ■

(5) $\mu = \hat{\mu}\nu$:

$$\int_{\hat{S}} \nu(s, F) d\hat{\mu}(s) = \int_{S} \nu(s, F) d[\mu(s) - \rho(s)\hat{\epsilon}\mu(A)] + \rho(F)\hat{\mu}(\{\alpha\}) = \mu(F) - \rho(F)\hat{\epsilon}\mu(A) + \rho(F)\hat{\epsilon}\mu(A) = \mu(F) \blacksquare$$

(6) $\nu\hat{\Pi}_i = \hat{\Pi}_i$:

$\nu\hat{\Pi}_i(s, C) = \int_{\hat{S}} \nu(s, ds')\hat{\Pi}_i(s', C) = \hat{\Pi}_i(s, C)$ for $s \neq \alpha$, $\int_{\hat{S}} \rho(ds')\hat{\Pi}_i(s', C) = \hat{\Pi}_i(\alpha, C)$, else■

(7) $\hat{\Pi}_i\nu = \Pi_i$:

Let $s \in S$, $C \in \mathcal{B}(S)$. $\hat{\Pi}_i\nu(s, C) = \int_{\hat{S}} \hat{\Pi}_i(s, ds')\nu(s', C)$. If $s \in S \setminus A$, this is equal to $\int_S \Pi_i(s, ds')\nu(s', C) = \Pi_i(s, C)$. If $s \in A$, it is equal to $\int_S [\Pi_i(s, ds') - \hat{\epsilon}\rho(ds')]\nu(s', C) + \hat{\epsilon}\nu(\alpha, C) = \int_S \Pi_i(s, ds')1_C(s') - \int_S \hat{\epsilon}\rho(ds')1_C(s') + \hat{\epsilon}\rho(C) = \Pi_i(s, C)$ ■

Appendix B

Proof of Equilibrium

Let for $q \in \mathcal{N}$ with $q > \min_k \frac{1}{A^k}$, $A_k^q = [\frac{1}{q}, A^k]$, $A^q = \times_{k=1}^{K} \times_{l=1}^{L_k} A_k^q$ and $A_D^q = \times_{d \in D} A^q$. For the rest of the proof we fix an $\overline{x} \in X$. Define for $\{\alpha_{kl}\} \in A^q$ and $x \in X$ the exchange economy $\epsilon(\{\alpha_{kl}\}, x)$ by:

Belief-type (k, l) has preferences on $\Re_{++}^N$ represented by the utility function $u_k(\cdot, \alpha_{kl}, x, \overline{x}) : \Re_{++}^N \to \Re$ and initial endowment α_{kl} of commodity $n(k)$. We have at least one equilibrium price $p \in \overset{\circ}{\Delta}$ for this economy [14].

Let

$$\mathcal{P} = \{p \in (\Delta^N)^X : p \text{ is measurable } \mathcal{B}(X)\}$$

and

$$\tilde{\mathcal{P}} = \{p \in \mathcal{P} : p(x) \in \overset{\circ}{\Delta}^N, \forall x \in X\}$$

In the following we use the topology of pointwise convergence on $(\Delta^N)^X$. Using lemma 1 on p. 55 of Hildenbrand(1974), one establishes (see below for details) that the correspondence

$$\Psi^q : A_D^q \Rightarrow \tilde{\mathcal{P}}_D \equiv \times_{d \in D} \tilde{\mathcal{P}}$$

defined by

$$\Psi^q(\alpha) = \{p \in \tilde{\mathcal{P}}_D : \forall (x, d), p(d, x) \text{ is an equilibrium price for } \epsilon(\{\alpha_{kl}(d)\}, x)\}$$

has non-empty values.

[14] This follows from the properties of the excess demand function, see f.i. Lemma 1 of Hildenbrand(1974), p.150.

For every $p \in \tilde{\mathcal{P}}_D$ we have for each $d \in D$ and each (k,l) a solution $\alpha_{kl}(d)$ to the restricted problem of an agent of belief-type (k,l), where we in his problem replace the condition $\alpha \in [0, A^k]$ with $\alpha \in A^q_k$. By strict concavity of u, this solution is unique. Thus we have defined, for each q, a function

$$g^q : \tilde{\mathcal{P}}_D \to A^q_D$$

Lemma 8 Ψ^q *is upper hemi continuous (u.h.c.).*

Proof: Let $\alpha^r \to \overline{\alpha}$ in A^q_D and $p^r \to \overline{p}$ in $\tilde{\mathcal{P}}_D$ s.t. $p^r \in \Psi^q(\alpha^r), \forall r$. So for given (d,x), $p^r(x,d)$ is an equilibrium for the economy $\epsilon(\{\alpha^r_{kl}(d)\}, x), \forall r$ and $(\{\alpha^r_{kl}(d)\}, x, p^r(d,x)) \to (\{\overline{\alpha}_{kl}(d)\}, x, \overline{p}(d,x))$. Because of continuity of u, $\overline{p}(d,x)$ is then an equilibrium price for the economy $\epsilon(\{\overline{\alpha}_{kl}(d)\}, x)$ ■

Lemma 9 g^q *is continuous.*

Proof: Let $p^r \to \overline{p} \in \tilde{\mathcal{P}}_D$ pointwise. Consider a particular belief-type (k,l) and a state $\overline{d} \in D$. For each r we then have a unique solution, $(\alpha^r_{kl}(\overline{d}), c^r_{kl})$ to the restricted problem of belief-type (k,l). $\{\alpha^r_{kl}(\overline{d})\}_r$ has a convergent subsequence, w.l.o.g. itself, converging to some $\overline{\alpha}_{kl}(\overline{d})$. If we consider (27) and (29) for given $\overline{d}$ and x, it follows from the smoothness of u that there are $\overline{c}(\overline{d},x)$ and $\overline{\lambda}(\overline{d},x)$ s.t. $c^r(\overline{d},x) \to \overline{c}(\overline{d},x)$ and $\lambda^r(\overline{d},x) \to \overline{\lambda}(\overline{d},x)$. It then follows from Lebesgue's bounded convergence theorem that also (28) holds in the limit, so that $\overline{\alpha}_{kl}(\overline{d})$ is indeed (part of) the solution to the belief-type's problem ■

It now follows from Hildenbrand(1974), Corollary to Proposition 1 (p. 22) that $\Phi^q \circ g^q$ is u.h.c. Kakutani's fixed point theorem gives us a sequence of fixed points $\{\alpha^q\}$ in $\times_{d\in D} \times_{k=1}^{K} \times_{l=1}^{L_k} [0, A^k]$ a compact set (as well as a sequence, $\{c^q\}$ of optimal consumptions). Hence, there is a cluster point, $\overline{\alpha}$ for $\{\alpha^q\}$. To show that $\overline{\alpha}$ is an equilibrium for the unrestricted economy, it is sufficient to show, for each (k,l) and d, that $\overline{\alpha}_{kl}(d) > 0$.

Suppose this was not the case for some (k,l) and $\overline{d}$. Rewriting (30) we have for each q,

$$\int_X \sum_{d\in D} \left[\left\{ \frac{-u'_{kn+1}(c^q_{kl}(d,x), A^k - \alpha^q_{kl}(d), x, \overline{x})}{u'_{kn(k)}(c^q_{kl}(d,x), A^k - \alpha^q_{kl}(d), x, \overline{x}) F'_k(\alpha^q_{kl}(d))} + 1 \right\} \cdot \right.$$
$$\left. u'_{kn(k)}(c^q_{kl}(d,x), A^k - \alpha^q_{kl}(d), x, \overline{x}) F'_k(\alpha^q_{kl}(d)) \right] \mu_D(d|\overline{d}^k)\pi_{kl}((\overline{d}^k, \overline{x}), dx) \leq 0 \qquad (32)$$

But if $\alpha^q_{kl}(\overline{d}) \to 0$ then $c^q_{kln(k)}(d,x) \to 0$ for $\mu_D(\cdot|\overline{d}_k) \otimes \pi_{kl}((\overline{d}_k, \overline{x}), \cdot)$ a.a. (d,x). Since we assumed that the indifference curves are uniformly bounded away from the boundary of $\Re^{N+1}_+$, this would imply that

$$\frac{-u'_{kn+1}(c^q_{kl}(d,x), A^k - \alpha^q_{kl}(d), x, \overline{x})}{u'_{kn(k)}(c^q_{kl}(d,x), A^k - \alpha^q_{kl}(d), x, \overline{x})} \to 0$$

and since $F'_k(\alpha^q)$ is bounded away from 0 this is incompatible with (32) ■

Ψ^q has non-empty values

Let E_{kl} be the excess demand of belief-type (k,l) as a function of p and x (and with $\alpha_{kl} > 0$). E_{kl} is continuous:

Let $(p^r, x^r, E^r) \to (\overline{p}, \overline{x}, \overline{E})$ with $\overline{p} \in \overset{\circ}{\Delta}{}^N$ and where $E^r = E_{kl}(p^r, x^r), \forall r$. Then the first order conditions hold for all r and, since u is smooth, they also hold in the limit.

Let $\Phi_{\{\alpha_{kl}\}} : X \rightrightarrows \overset{\circ}{\Delta}{}^N$ assign, to each $x \in X$, the set of equilibrium prices for the economy $\epsilon(\{\alpha_{kl}\}, x)$. Then $\Phi_{\{\alpha_{kl}\}}$ is closed valued since if $p^r \to \overline{p}$, $p^r \in \Phi_{\{\alpha_{kl}\}}(x), \forall r$ then, letting $E = \sum_{kl} E_{kl}$, $E(p^r) = 0, \forall r$ and so, also $E(\overline{p}) = 0$.

Let $F \subset \Delta^N$ be closed. We show that $B = \{x : \Phi_{\{\alpha_{kl}\}}(x) \cap F \neq \emptyset\}$ is also closed. So let $\{x^r\}$ be a sequence in B converging to $\overline{x}$ and let $p^r \in \Phi_{\{\alpha_{kl}\}}(x^r) \cap F, \forall r$. $\{p^r\}$ has a converging subsequence, $\{p^{r_n}\}$ converging to some $\overline{p}$. Since $\{p^r\}$ is a sequence of equilibrium prices it is bounded away from $\partial\Delta^N$, hence $\overline{p} \in \overset{\circ}{\Delta}{}^N$. We then have $E(p^{r_n}, x^{r_n}) = 0, \forall n$ and so by continuity of E also for $(\overline{p}, \overline{x})$. Since $p^{r_n} \in F, \forall n$, $\overline{p} \in F$, also and hence $\overline{x} \in B$.

It follows from Hildenbrand(1972) Lemma 1, p.55 that $\Phi_{\{\alpha_{kl}\}}$ has a measurable selection. Let $\{\alpha_{kl}(d)\} \in A^q$ be given for each $d \in D$. Then if $p \in \tilde{\mathcal{P}}_D$ is defined by $p_d : X \to \overset{\circ}{\Delta}{}^N$ is a measurable selection from $\Phi_{\{\alpha_{kl}(d)\}}$, we have $p \in \Psi^q(\alpha)$ ■

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