Joint Production Games with Mixed Sharing Rules

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Abstract

We study joint production games under a mixed output sharing rule in which part of the output (the mixing parameter) is shared in proportion to inputs and the rest according to exogenously determined shares. We show that this game has a Nash equilibrium which is unique. When the mixing parameter is set to the equilibrium elasticity of production (optimal mixing) and all players have identical preferences and the same exogenous shares, the corresponding equilibrium outcome is efficient. Furthermore, it is envy free when there are only two players and passes the unanimity test when the elasticity of production is constant. When there are many players and payoffs are evaluated to first order, all equilibrium outcomes are efficient and, by appropriate choice of the endogenous shares, all efficient solutions which respect voluntary participation can be generated. Furthermore, under equal shares the corresponding equilibrium is envy free and pass the unanimity and stand-alone tests.

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1 Introduction

Joint owners of a production technology with non-constant returns to scale and with controlled access face the problem of selecting a procedure for distributing the output. A natural procedure is proportional sharing in which each owner receives a share of the output equal to their share of the aggregate input. This sharing rule may be chosen on the basis of ethical principles or may simply be an inevitable consequence of uncontrolled access to a jointly owned resource. In the latter case, free entry leads to the familiar tragedy of the commons[11],[15]. More generally, this sharing rule results in over-production in equilibrium: a Pareto improvement on an interior solution can be achieved by proportionally reducing inputs[12], reflecting incomplete internalization of the externality. A polar opposite case is the equal shares procedure in which the output is divided equally amongst all the owners. Forcing equal consumption on all players makes this case formally equivalent to provision of public goods [14], [3]). Voluntary provision of such goods typically results in under-production, at least when the technology exhibits non-increasing returns to scale[10]. The equal sharing rule is readily generalized to exogenous sharing in which each owner receives an exogenously determined proportion of the output independent of her input. Once again, under-production is characteristic of equilibria under exogenous sharing. Yet another class of procedures uses voting over levels of input and output. For example, if inputs and outputs can be monitored and, say, are divided equally amongst all players, only the aggregate input remains to be determined. This can be achieved by asking each player to cast a vote for this aggregate level and then selecting one of the votes, for example the median, as the collective decision.

In an attempt to overcome the inefficiency and other undesirable features of such sharing rules, Moulin and Shenker introduced the serial cost sharing procedure[13], which may also be implemented via an output sharing variant. Serial output sharing has several desirable properties. Under non-increasing returns and convex, monotonic preferences, a Nash equilibrium of this sharing rule exists and is unique. Furthermore, this equilibrium passes the unanimity test (each player does at least as well as they could possibly do under an equal split of input and output) and the stand-alone test (each player is no better off than they would be if they had sole access to the technology). Typically the equilibrium allocation will not be efficient, though efficiency is ensured if all players have identical preferences or if the technology has constant returns to scale. Note though, that although no coalition can upset the equilibrium, all members of the coalition may benefit by pooling and redistributing inputs and outputs. The proportional and exogenous shares procedures are proof against such strategic manipulation since, in both cases, the total output received by any coalition is a function of the aggregate input of that coalition and does not otherwise depend on individual inputs.

The over-production which arises with proportional sharing is a consequence of the failure to penalize a player for the negative externality she imposes on the other players by increasing her input. Under-production with exogenous sharing arises from the inadequate incentive provided to a player who only receives a small proportion of the value of her additional input. This suggests that an intermediate procedure may do a better job of balancing incentives with internalizing the externality. More specifically, we study a mixture of the average and exogenous sharing rules in which the output is first split into two piles. Each owner receives a share of the first pile in proportion to her input and an exogenously determined share of the second pile[1]. We refer to the proportion of total output in the first pile as the mixing parameter. Considerations of continuity suggest that there will be a value of the mixing parameter for which the level of total output is efficient. A simple case of this procedure was analysed by Cauley, Cornes and Sandler[5]. In the sequel, we investigate this claim as well as the existence, uniqueness and properties of Nash equilibria of the game with mixed sharing and the choice of mixing parameter and exogenous shares. The next paragraph summarizes our results.

When preferences are monotonic and binormal[2], the technology exhibits nonincreasing returns to scale and a technical condition is satisfied, the game resulting from a mixed sharing rule has a Nash equilibrium which is unique. When all players have identical preferences, the equilibrium allocation is efficient provided exogenous shares are equal and the mixing parameter is equal to the equilibrium value of the elasticity of production. All subsequent results assume this value of the mixing parameter. Efficiency holds even when players' preferences differ, provided either returns to scale are constant or the exogenous shares are suitably chosen. With equal exogenous shares, a weakened form of no-envy holds: every player prefers her own share of the equilibrium allocation to the average input and output of her rivals. With the same assumption on exogenous shares, the equilibrium passes the unanimity test, if the elasticity of production is constant. It is possible for the stand-alone test to fail, though it is always passed by players whose share of the input exceeds their share of the output (net contributors).

These conclusions hold for any number of players, n. However, when n is large but preferences fall into a finite set of types and payoffs are evaluated to first order in $1/n$, the results are much tighter. Equilibrium payoffs are the same for all players, up to a multiplicative constant. This means that all players agree on their preferred value of the mixing parameter. Furthermore, the equilibrium allocation with this mixing parameter is efficient for any set of exogenous shares. Conversely, these shares can be chosen to realize any efficient allocation which respects voluntary participation. When the exogenous shares are all equal, the equilibrium allocation is also envy free and passes the unanimity test. The stand-alone test is also passed; indeed a stronger variant of the test is satisfied for net contributors.

To establish these results, we adopt a novel method of analysis which exploits the fact that the choice of inputs under a mixed sharing rule is an example of an 'aggregative' game; any player's payoff is a function only of her individual input and the sum of the inputs of all the players. For such games, the complications of handling n-dimensional best response functions, which besets the analysis of multi-player games, particularly when non-interior equilibria are admitted, can be circumvented. To do this we work with a 'share function' for each player. This function maps levels of aggregate input to the player's (possibly zero) preferred share of input that is consistent, in equilibrium, with the given level of aggregate input. Equilibria are then characterized by the consistency condition

[1] Sen [16] briefly discusses mixed sharing rules in which, as he puts it, a part of income is distributed according to 'needs' and the rest according to 'work'. He restricts attention to games involving identical players.

[2] Both input and output are normal goods.

that the sum of the share functions should equal unity in equilibrium. We show that under standard assumptions, each player has a well-defined share function which also satisfies other useful properties; notably it is strictly decreasing where positive, which rules out multiple equilibria. Share functions also underpin the proofs of the other properties listed above, which are related to comparative statics as well as providing a valuable tool for deriving the asymptotic type-payoffs which lead to the results for large-games.

Section 2 formally describes the mixed sharing rule, introduces our running assumptions and uses them to establish the existence and properties of share functions. These properties are used to prove existence and uniqueness of a Nash equilibrium. In Section 3, we discuss the properties of equilibrium allocations listed above for games which are not large. The two subsequent sections discuss large games. In Section 4.2, we describe the setup used for large games: a finite set of types each of which is represented by many players. The main results of this section give asymptotic results, characterizing the equilibria and offering formulae for the asymptotic aggregate payoffs for each type. The efficiency and other normative properties outlined above are restated in asymptotic form and proved in Section 5. Section 6 concludes.

2 Share functions, existence and uniqueness

Suppose that a set I of n players jointly use a technology which converts total input L into total output X via the production function $X = F(L)$. We consider an output-sharing rule that mixes proportional and exogenous sharing: if player i supplies input ℓ_i, she receives the output x_i, where

$$x_i = \gamma_i(\ell_i, L) = \left\{\lambda \frac{\ell_i}{L} + (1-\lambda)\theta_i\right\} X, \tag{1}$$

and $L = \sum_{j \in I} \ell_j$. Here λ is an exogenous *mixing parameter* satisfying $0 \leqq \lambda \leqq 1$ and the θ_i's are positive *exogenous weights* satisfying $\sum_{j \in I} \theta_j = 1$. An important special case is *equal shares*: $\theta_i = 1/n$ for all i, but the generalization to arbitrary θ_i allows us, for example, to single out some players for an enhanced share of the output. Alternatively, (1) can be viewed as the overall effect of an initial division according to the proportional sharing rule followed by the imposition of a redistributive tax with a rate of $1 - \lambda$.

Player i's preferences are represented by a utility function $u_i(x_i, \ell_i)$. If $L > 0$, player i's payoff to strategy profile $(\ell_1, \dots, \ell_n)$ is $u_i(x_i, \ell_i)$ where x_i is determined by (1). Otherwise it is $u_i(0, 0)$. We make the following assumptions:

A.1(Preferences) For all i, player i's utility function $u_i(x_i, \ell_i)$ is quasi-concave, locally non-satiable, increasing in x_i, decreasing in ℓ_i, continuous and continuously differentiable[3]. Both x_i and ℓ_i are normal[4].

[3] In her analysis of uniqueness and comparative statics of the game with proportional sharing, Watts (1996) does not assume differentiability of utility functions. In our proof of existence and uniqueness, we make this assumption for expository reasons, but the proofs go through *mutatis mutandis* in the nondifferentiable case if the MRS at (x_i, l_i) is interpreted as the slope of a supporting line to player i's upper preference set at (x_i, l_i).

[4] The assumption of convex preferences is redundant when inputs and outputs are normal. We make it explicit for expositional clarity.

A.2(Technology) The production function $F(L)$ is increasing, strictly concave, continuous and continuously differentiable for $L > 0$, and $F(0) = 0$.

A.3(Boundedness) For all i, there exists a value of $L > 0$ such that $u_i(F(L), L) \leqq u_i(0, 0)$.

The first two assumptions are standard. Our characterization of normality in A.1 follows Watts [18]. Specifically, the marginal rate of substitution of player i at (x_i, ℓ_i) is non-decreasing in both x_i and ℓ_i. Assumption A.3 excludes the indifference curve through the origin lying entirely below the graph of the production function when ℓ_i is measured along the horizontal axis and x_i along the vertical axis. This leaves two possibilities. The whole indifference curve may lie on or above the graph of the production function. If a player with such preferences had exclusive use of the technology, she would choose to supply no input. *A fortiori* this is the case if there are other players. Alternatively, the curves cross for some positive L and a monopoly owner of the resource would supply a positive but finite input. Sufficient conditions for A.3 are (a) $F'(L) \to 0$ as $L \to \infty$, or (b) the indifference curve through the origin becomes arbitrarily steep.

We write $f_i(x_i, \ell_i)$ for player i's marginal rate of substitution at (x_i, ℓ_i) and $\zeta_i(\sigma_i, L)$ for the value of $f_i(x_i, \ell_i)$ when

$$x_i = \{\lambda\sigma_i + (1 - \lambda)\theta_i\} F(L) \quad (2)$$

and $\ell_i = \sigma_i L$. Note that an increase in either σ_i or L cannot lead to a decrease in either x_i or ℓ_i. Hence, by Assumption A.1, the MRS cannot decrease.

Lemma 2.1 *Assume A.1. Player i's marginal rate of substitution: $\zeta_i(\sigma_i, L)$ is a non-decreasing function of σ_i for fixed $L > 0$ and of L for fixed σ_i.*

Similarly, player i's marginal rate of transformation of input into output can also be expressed as a function of σ_i and L. Holding all other players' input levels fixed and differentiating (2) with respect to ℓ_i, we obtain the following expression for the MRT of player i as a function of σ_i and L:

$$\frac{\partial x_i}{\partial \ell_i} = \{\lambda\sigma_i + (1 - \lambda)\theta_i\} F'(L) + \lambda[1 - \sigma_i]\frac{F(L)}{L} \equiv \tau_i(\sigma_i, L). \quad (3)$$

Concavity of F implies that, for fixed σ_i, an increase in L reduces τ_i. For fixed $L > 0$, increasing σ_i places more weight on $F'(L)$ and less on $F(L)/L$. Since the average exceeds the marginal product, the MRT must fall for $\lambda > 0$. Summarizing, we have:

Lemma 2.2 *Assume A.2 and $\lambda > 0$. Player i's marginal rate of transformation: $\tau_i(\sigma_i, L)$ is a strictly decreasing function of L for fixed σ_i and of σ_i for fixed $L > 0$.*

The first-order conditions for ℓ_i to be a best response to $L_{-i} = \sum_{j \in I, j \neq i} \ell_j$ can be written in terms of σ_i and L as:

$$\zeta_i(\sigma_i, L) \geqq \tau_i(\sigma_i, L) \quad (4)$$

$$\zeta_i(\sigma_i, L) = \tau_i(\sigma_i, L) \text{ if } \sigma_i > 0. \quad (5)$$

It is straightforward to verify that, if $X = F(L)$, then x_i satisfying (1) is an increasing, strictly concave function of ℓ_i for any $L_{-i} \geq 0$. Since u_i is also quasiconcave, conditions (4) and (5) are necessary and sufficient. It follows from Lemmas 2.1 and 2.2 that these conditions cannot have multiple solutions.

Corollary 2.1 *Assume A.1 and A.2. If $\lambda > 0$ and $L > 0$, there is at most one $\widehat{\sigma}_i$ satisfying (4) and (5).*

The case $\lambda = 0$ may be accommodated by a slight strengthening of our assumptions. In this case, τ_i is a constant function of σ_i for any L, but the proposition would still hold provided ζ_i were *strictly* increasing in σ_i. This would follow from a slightly stricter interpretation of normality in which MRSs were strictly increasing in x_i and ℓ_i. We shall refer to this by describing player i's preferences as *strictly normal*. Observe, however, that strict normality would rule out linear preferences.

For any $L > 0$ for which conditions (4) and (5) have a solution in σ_i, we write $s_i(L)$ for that solution. We refer to s_i as the *share function* of player i. This function is the foundation of all our subsequent analysis and the next proposition, proved in the Appendix, sets out the key properties of share functions.

Proposition 2.2 *Assume A.1, A.2 and A.3 and that preferences are strictly normal if $\lambda = 0$. For all i there is a continuous share function s_i which satisfies exactly one of the following:*

1. *$s_i(L) = 0$ for all $L > 0$;*
2. *there is $\underline{L}_i > 0$ such that $s_i(\underline{L}_i) = 1$ and $s_i(L)$ is positive and strictly decreasing for $L > \underline{L}_i$; furthermore, $s_i(L) \longrightarrow 0$ as $L \longrightarrow \infty$;*
3. *there are $\underline{L}_i > 0$ and $\overline{L}_i > \underline{L}_i$ such that $s_i(\underline{L}_i) = 1$ and $s_i(L)$ is positive and strictly decreasing for $\underline{L}_i < L < \overline{L}_i$; furthermore, $s_i(L) = 0$ for $L \geq \overline{L}_i$.*

The lower limit of the domain of the share function: $\underline{L}_i$ is the monopoly output of player i, modified to allow for exogenous sharing. More specifically, $\underline{L}_i$ maximizes $u_i([\lambda + (1-\lambda)\theta_i] F(L), L)$ with respect to L. If the maximizer is 0, Case 1 applies.

In Case 3, the share function reaches the axis at the value $\overline{L}_i$ which we call the *dropout value*. It follows from (4) and (5) that $\overline{L}_i$ satisfies

$$\zeta_i\left(0, \overline{L}_i\right) = \tau_i\left(0, \overline{L}_i\right). \tag{6}$$

It is convenient to set $\overline{L}_i = \infty$ to cover Case 2.

Example 2.3 *Suppose $u_i(x_i, l_i) = a_i x_i - l_i$, $F(L) = L^{1/2}$, $\lambda > 0$ and $\theta_i = 1/n$. We find*

$$\underline{L}_i = \left[\frac{1 - \lambda + n\lambda}{2n} a_i\right]^2$$

and, for $L \geq \underline{L}_i$*,*

$$s_i(L) = \max\left\{2 + \frac{(1-\lambda)}{\lambda n} - \frac{2}{\lambda a_i}\sqrt{L}, 0\right\}. \tag{7}$$

Note that $\overline{L}_i$ *is finite in this example and satisfies*

$$\overline{L}_i = \left[\frac{1-\lambda+2n\lambda}{2n}a_i\right]^2.$$

The main use of share functions is to determine and characterize equilibria by imposing the consistency requirement that shares must sum to unity in equilibrium. Let the *aggregate share function* $S(L) = \sum_{j=1}^{n} s_j(L)$. It is readily confirmed that $\widehat{L} > 0$ is an equilibrium value of L if and only if $S\left(\widehat{L}\right) = 1$ and the corresponding equilibrium satisfies $\widehat{\ell}_i = s_i\left(\widehat{L}\right)$ for each i. If Case 1 of Proposition 2.2 holds for all i, the null strategy profile is the unique equilibrium. Otherwise, Proposition 2.2 implies that $S(L) \geq 1$ for some L and $S(L) < 1$ for all large enough L. Since S is continuous and strictly decreasing where positive, $S\left(\widehat{L}\right) = 1$ for a unique $\widehat{L}$. This establishes the next theorem which extends the result of Watts [18].

Theorem 2.4 (Existence and Uniqueness) *Under the assumptions of Proposition 2.2, there is a unique Nash equilibrium.*

We can illustrate the use of share functions to obtain equilibria with an example.

Example 2.5 *Suppose* $n = 3$ *in Example 2.3 with* $\theta_i = 1/3$*,* $(a_1, a_2, a_3) = (30, 20, 15)$ *and* $\lambda = 1/2$*. The individual and aggregate share functions are drawn in Figure 1. The unique Nash equilibrium corresponds to the point N at which* $\widehat{L} = 121$ *and the corresponding strategy profile is* $(13/15, 2/15, 0)\,\widehat{L} = \left(104\tfrac{13}{15}, 16\tfrac{2}{15}, 0\right)$*. Note that* $\widehat{L}$ *exceeds the dropout value for player 3, so the latter provides no input.*

The next example shows that with an appropriate choice of mixing parameter, the equilibrium level of output is efficient.

Example 2.6 *Suppose* $a_i = 1$ *for all* n *players in Example 2.3. The equilibrium value of* L *can be found by solving* $s_i(L) = 1/n$*:*

$$\widehat{L} = \left[\left(1 - \frac{1}{n}\right)\lambda + \frac{1}{2n}\right]^2.$$

Substituting into the payoff function, we see each player has equilibrium payoff

$$\frac{1}{2n^2}\left(1 - \frac{1}{2n}\right) + \frac{1}{n}\left(1 - \frac{1}{n}\right)^2\left(\lambda - \lambda^2\right).$$

Note that equilibrium payoffs are maximized with respect to λ *at* $\lambda = 1/2$*, the elasticity of production. The unique efficient level of output can be found by maximizing total surplus:* $L - L^{1/2}$*. This gives* $L = 1/4$*, the value of* $\widehat{L}$ *when* $\lambda = 1/2$*.*

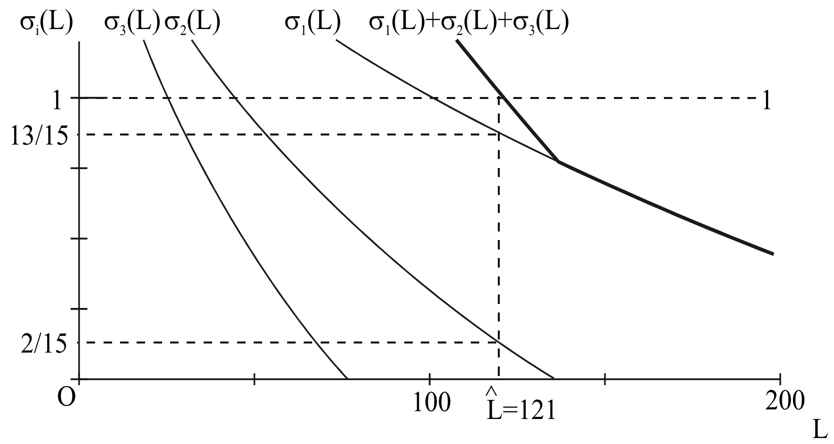

Figure 1:

The final example illustrates that efficiency of the equilibrium extends to asymmetric equilibria provided that the game is large and payoffs are evaluated to first order in $1/n$.

Example 2.7 *Suppose n is even, with $u_i = x_i - l_i$ for odd i and $u_i = 2x_i - l_i$ for even i. Using the result in Example 2.3, the equilibrium satisfies $s_i(L) = 0$ if i is odd and $s_i(L) = 2/n$ if i is even, resulting in*

$$\widehat{L} = \left[\left(2 - \frac{1}{n}\right)\lambda + \frac{1}{n}\right]^2.$$

When n is large, equilibrium payoffs are

$$2\frac{\lambda(1-\lambda)}{n} + O\left(\frac{1}{n^2}\right) \text{ if } i \text{ is odd}^5,$$
$$4\frac{\lambda(1-\lambda)}{n} + O\left(\frac{1}{n^2}\right) \text{ if } i \text{ is even.}$$

Note that equilibrium payoffs are maximized to first order by both types at $\lambda = 1/2$ at which $\widehat{L} = 1 + o(1/n)$. The efficient level of output can again be found by maximizing total surplus and gives $L = 1$ which is equal to $\widehat{L}$ to first order.

In the following sections, we develop the results in these examples.

[5] We write $a_n = O(b_n)$ if there exists a contant $k > 0$ such that $|a_n| \leq kb_n$ for all n. We write $a_n = o(b_n)$ as $n \to \infty$ if $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$. See Apostol ([1], p. 192) for a brief discussion of 'little oh' and 'big oh' functions.

3 Mixed sharing with optimal mixing

3.1 Efficiency

In this section, we explore several properties of the equilibrium of joint production games with mixed sharing rules commencing, in this subsection, with efficiency. We start with the observation that the equilibrium is efficient when returns to scale are constant[6] provided the mixing parameter is set to unity (proportional sharing). To see this, note that conditions (4) and (5) imply that the equilibrium MRS of each player is no less than the (constant) average product, with equality if that player supplies positive input (and therefore receives positive output). Given the convexity assumptions in A.1, these conditions are sufficient for efficiency. This result reflects the fact that under constant returns production imposes no externality. Under symmetry, however, efficiency of the equilibrium can be ensured even for strictly decreasing returns but then the mixing parameter must be less than one and exogenous shares must be equal. The next result, proved in the Appendix, gives the details.

Proposition 3.1 *Assume A.1 - A.3, identical preferences and equal shares. If the mixing parameter is equal to the equilibrium elasticity of production, the equilibrium allocation is efficient.*

Cauley, Cornes and Sandler [5] observe that, if $\lambda = 1$, equilibrium entails overproduction whereas, if $\lambda = 0$, there will be underproduction. Assuming continuity, they conclude that there must be of a value of λ for which the equilibrium is efficient. Proposition 3.1 refines this result by identifying the required value of the mixing parameter, which we shall refer to as *optimal.*

If we drop the assumption of equal shares, identical preferences are not essential for an equilibrium allocation to be efficient. For example, when every player's utility function is quasilinear in input, the efficient level of aggregate input is unique. In this case any efficient allocation can be achieved as the equilibrium of a joint production game, provided the mixing parameter is optimal and exogenous shares are suitably chosen. The next result, proved in the Appendix, gives a formal statement.

Proposition 3.2 *Assume A.1 - A.3 and that preferences are quasilinear in income. Consider an efficient allocation in which player i receives output x_i^{e} and aggregate input is L^{e}. Then the exogenous shares can be chosen so that the equilibrium of the surplus sharing game with $\lambda = \eta\left(L^{\mathrm{e}}\right)$ satisfies $\widehat{x}_i = x_i^{\mathrm{e}}$ for $i = 1, \ldots, n$ and $\sum_{i=1}^{n} \widehat{\ell}_i = L^{\mathrm{e}}$.*

3.2 Envy free equilibria

The acceptability of an allocation may be enhanced if it is envy free: no player prefers a rival's input/output combination to her own. Mixed sharing equilibria need not be envy free. In the equilibrium of Example 2.5, player 3 supplies no input and the utility of player 1 would rise from $35\frac{1}{3}$ to $36\frac{2}{3}$, were she to receive the output of player 3 and not be required to supply any input. Note that this is a well-behaved example: preferences are linear, elasticity of production

[6] Linear production functions do not satisfy A.2 but it is straightforward to verify that the analysis of the previous section carries through provided preferences are strictly normal.

is constant, weights are equal and the mixing parameter is optimal. However, a weaker condition is satisfied: all players prefer their equilibrium input and output to an equal division of the aggregate input and output were divided. This result requires optimal mixing and equal weights. It is formally stated below and proved in the Appendix.

Proposition 3.3 *Let* $\left(\widehat{x}_i, \widehat{\ell}_i\right)_{i=1}^n$ *denote the equilibrium of a joint production game which satisfies A.1 - A.3 and has equal exogenous weights. If* $\widehat{L}$ *denotes aggregate input and* $\lambda = \eta\left(\widehat{L}\right)$*, then*

$$u_i\left(\widehat{x}_i, \widehat{\ell}_i\right) \geq u_i\left(\frac{F\left(\widehat{L}\right)}{n}, \frac{\widehat{L}}{n}\right) \text{ for } i = 1, \ldots, n. \tag{8}$$

The inequality is strict if player i*'s preferences are strictly normal and* $\widehat{\ell}_i \neq \widehat{L}/n$ *for some* i*.*

Note that the equal sharing of input can be thought of as a weighted combination of the input of player i (with weight $1/n$) and the average input of her rivals (with weight $(n-1)/n$) and similarly for output. It follows that a risk averse player would reject the opportunity to swap her share of the allocation with that of a randomly chosen rival. When there are only two players, these observations amount to lack of envy.

Corollary 3.4 *Assume A.1 - A.3, equal shares, optimal mixing and* $n = 2$*. Then the equilibrium allocation is envy free.*

3.3 Unanimity test

Player i's *unanimity lower bound* is the highest payoff she could receive under equal sharing of input and output:

$$\max_{L \geq 0} u_i\left(\frac{F(L)}{n}, \frac{L}{n}\right). \tag{9}$$

An allocation in which every player's payoff is as least as great this bound is said to pass the *unanimity test.* The equilibrium outcome of a sharing rule which passes the test will (weakly) Pareto dominate any procedure which uses some social decision rule to choose the aggregate input and divides input and output equally.

The unanimity test is more stringent than no envy on average as expressed in Proposition 3.3. Note, however, that when all players have identical preferences these bounds agree. This follows from Proposition 3.1, for then the outcome is efficient and therefore each player's payoff achieves the unanimity bound. Even when preferences differ, under mixed sharing, with the optimal mixing and equal shares, each player's payoff exceeds her unanimity lower bound, provided the elasticity of production is constant.

Proposition 3.5 *Assume A.1 and* F *proportional to* L^α*. The equilibrium of the mixed sharing game with* $\lambda = \alpha$ *and equal exogenous weights passes the unanimity test.*

The proposition is proved in the Appendix. A natural extension in which the unanimity lower bound (9) is replaced by

$$\max_{L\geq 0} u_i\left(\theta_i F\left(L\right), \theta_i L\right),$$

where $(\theta_1, \ldots, \theta_n)$ is the vector of exogenous shares is also valid, provided mixing is optimal and elasticity of production is constant.

3.4 Stand-alone test

The stand-alone test is a formalization of the ethical principle that players should not benefit from the negative externality they impose on their rivals. In particular, the *stand-alone test* requires that no player do better in equilibrium than if they had sole use of the technology. Formally, the equilibrium payoff of player i is bounded above by

$$u_i^{\mathrm{M}} = \max_{L\geq 0} u_i\left(F\left(L\right), L\right).$$

Under mixed sharing rules, it is possible for some (but not all) players to fail the stand-alone test for any positive exogenous shares and any value of the mixing parameter less than unity. This is most easily seen in the case of an unbounded production function and a player i with finite dropout point $\overline{L}_i$. If $L \geq \overline{L}_i$, then $s_i\left(L\right) = 0$ and the player i has payoff

$$u_i\left(\left\{\left(1-\lambda\right)\theta_i\right\} F\left(L\right), 0\right).$$

The unboundedness of F means that it is always possible to choose L sufficiently large that the payoff of player i exceeds u_i^{M}. By suitable choice of the preferences of the other players, we can construct an equilibrium in which the stand-alone test fails for player i.

If $\widehat{L}$, the equilibrium aggregate output of player i satisfies $\underline{L}_i \leq \widehat{L} < \overline{L}_i$, the share function of player i is positive and her payoff is

$$\begin{aligned} & u_i\left(\left\{\lambda s_i\left(\widehat{L}\right) + \left(1-\lambda\right)\theta_i\right\} F\left(\widehat{L}\right), s_i\left(\widehat{L}\right)\widehat{L}\right) \\ \leq\ & u_i\left(\left\{\lambda + \left(1-\lambda\right)\frac{\theta_i}{s_i\left(\widehat{L}\right)}\right\} F\left(s_i\left(\widehat{L}\right)\widehat{L}\right), s_i\left(\widehat{L}\right)\widehat{L}\right). \end{aligned}$$

The right hand side is a consequence of utility increasing in output together with the inequality

$$s_i\left(\widehat{L}\right) F\left(\widehat{L}\right) \leq F\left(s_i\left(\widehat{L}\right)\widehat{L}\right),$$

which follows from Assumption A.2. If $s_i\left(\widehat{L}\right) \geq \theta_i$, we call player i a *net contributor* and in such a case the term in braces is no greater than unity so the right hand side is bounded above by u_i^{M}. We conclude that net contributors pass the stand-alone test.

In the next two sections, we show that when the game is large, equilibrium allocations are efficient, envy free and satisfy the unanimity and stand-alone test, at least when payoffs are evaluated to first order in $1/n$.

4 Payoffs in large games

Results for large games are often sharper for smaller games. There are two reasons for this. Firstly, strategic effects are weakened in such games[7] and, secondly, input and output are small, permitting the use of a linear approximation to the utility function. We shall analyze output sharing games as the number of players tends to infinity using share functions as the vital analytical tool.

To avoid over-complicating the exposition, we reinforce A.1-A.3 by adding some further mild assumptions. Only the second is essential to what follows.

A.1* Assumption A.1 holds and, for all i, player i has strictly positive marginal rate of substitution at the origin.

A.2* Assumption A.2 holds and $F(L)/L \longrightarrow 0$ as $L \longrightarrow \infty$.

A.3* Assumption A.3 holds and there is an $L > 0$ for which $F(L)/L$ exceeds the marginal rate of substitution at the origin of at least one player.

The additional requirement in A.1* rules out the possibility that the slope of the indifference curve through the origin falls to zero as it reaches it. This assumption allows us to approximate preferences in the neighborhood of the origin by linear preferences which continue to satisfy A.1.

Under assumption A.2*, individual shares of outputs become small as the number of players becomes large. The assumption holds, in particular, when a bounded resource is exploited but can also be valid when F is unbounded above. Note that the marginal product, bounded above by the average product, also vanishes asymptotically.

For simplicity of exposition, it is convenient to rule out null equilibria and the additional assumption in A.3* does this. It holds, for example, under constant elasticity of production.

4.1 Large symmetric games

We first analyze games with equal shares, in which all players have the same preferences. Theorem 2.4 shows that there is a unique equilibrium value of aggregate input for each n, which we denote L^n and our first aim is to characterize the limit of $\left\{\widehat{L}^n\right\}$ as $n \longrightarrow \infty$. If $\widehat{L}^n > 0$, then $s\left(\widehat{L}^n\right) = 1/n$, and the equality of the MRS and MRT can be written:

$$f\left(\frac{F\left(\widehat{L}^n\right)}{n}, \frac{\widehat{L}^n}{n}\right) = \frac{1}{n}F'\left(\widehat{L}^n\right) + \lambda\left(1 - \frac{1}{n}\right)\frac{F\left(\widehat{L}^n\right)}{\widehat{L}^n}, \tag{10}$$

recalling $f(x, \ell)$ denotes the MRS evaluated at (x, ℓ). In the proof of the next lemma, given in the Appendix, we show that $\widehat{L}^n \longrightarrow \widetilde{L}$, say. Taking the limit in (10) shows that

$$f(0,0) = \frac{\lambda F\left(\widetilde{L}\right)}{\widetilde{L}}. \tag{11}$$

[7] Though not necessarily eliminated. See Cornes and Hartley [2002] for analysis of this point for rent-seeking contests.

Note that the right hand side of (11) is decreasing in $\widetilde{L}$ and approaches zero for large $\widetilde{L}$ by A.2*. Hence, (11) has a unique solution for $\lambda \in (\underline{\lambda}, 1]$, where $\underline{\lambda}$ is defined by:

$$\underline{\lambda} = f(0,0)\left[\sup_{L>0}\left\{\frac{F(L)}{L}\right\}\right]^{-1} < 1. \tag{12}$$

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If $\lambda > 0$ satisfies $\lambda \leq \underline{\lambda}$, we can show that $\widehat{L}^n \longrightarrow 0$ and the limiting results still holds, provided we define $\widetilde{L} = 0$ for all such λ.

Lemma 4.1 *Assume A.1*-A.3* and that all players are identical. Then* $\widehat{L}^n \longrightarrow \widetilde{L}$ *as* $n \longrightarrow \infty$, *where* $\widetilde{L}$ *is the unique solution of (11) for* $\underline{\lambda} < \lambda \leq 1$ *and* $\widetilde{L} = 0$ *if* $0 < \lambda \leq \underline{\lambda}$.

It can also be shown that $\widetilde{L}$ is the limit of dropout values of the n-player share function, which we write $\overline{L}^n$. The equation for the dropout value (6) becomes

$$f\left(\frac{(1-\lambda)F\left(\overline{L}^n\right)}{n}, 0\right) = \frac{1-\lambda}{n}F'\left(\overline{L}^n\right) + \lambda\frac{F\left(\overline{L}^n\right)}{\overline{L}^n}. \tag{13}$$

The proof of Lemma 4.1 is readily modified to show that the sequence $\left\{\overline{L}^n\right\}$ is finite and has a limit. Taking limits in (13) shows that $\overline{L}^n \longrightarrow \widetilde{L}$. The following lemma summarizes these observations.

Lemma 4.2 *Under the assumptions of Lemma 4.1,* $\overline{L}^n \longrightarrow \widetilde{L}$ *as* $n \longrightarrow \infty$.

Under A.2*, the average product falls to zero, so it is not surprising that individual payoffs approach the reservation value, $u(0,0)$, in the limit. However, the aggregate excess payoff (over the reservation value) has a finite limit:

$$\begin{aligned} n&\left[u\left(\left\{\frac{\lambda}{n} + \frac{(1-\lambda)}{n}\right\}F\left(\widehat{L}^n\right), \frac{1}{n}\widehat{L}^n\right) - u(0,0)\right] \\ &\longrightarrow \frac{\partial u(0,0)}{\partial x}\left[F\left(\widetilde{L}\right) - \widetilde{L}f(0,0)\right] \\ &= \frac{\partial u(0,0)}{\partial x}(1-\lambda)F\left(\widetilde{L}\right) \text{ as } n \longrightarrow \infty, \end{aligned} \tag{14}$$

where we use (11) to obtain the final line when $\lambda \in (\underline{\lambda}, 1]$. Under proportional sharing ($\lambda = 1$) the limit is zero: as the number of players increases, the aggregate surplus is fully competed away. When the mixing parameter is less than unity, the aggregate benefit of that portion of the output that is shared proportionally vanishes; what remains is the exogenously shared part and, if $\widetilde{L} > 0$, this has a positive limit even in a large game.

[8] The inequality is a consequence of A.3*. Note that $\underline{\lambda} = 0$ is possible, if the average product is unbounded e.g. constant elasticity of production.

4.2 Large asymmetric games

To analyze the case when players differ, we envisage an infinite sequence of potential players each of whom falls into one of T distinct types. All players of the same type have identical preferences and we write $u_{(t)}$ for the utility function of players of type $t\,(= 1, \ldots, T)$ and extend the convention of enclosing type subscripts in parentheses, where they could be confused with individual players, to share functions, marginal rates of substitution etc. Let $\mathcal{G}^n$ denote the game played by the first n members of the sequence and let $n_t(n)$ denote the number of players of type t in $\mathcal{G}^n$. We require that the proportion of players of each type has a positive limit.

A.4 For all $t = 1, \ldots, T$, as $n \longrightarrow \infty$, we have $n_t(n)/n \longrightarrow \nu_t > 0$.

To determine the exogenous weights in $\mathcal{G}^n$, we choose a non-negative *type-weight* μ_t for each type t to satisfy $\sum_{t=1}^{T} \mu_t = 1$ and set $\theta_i = \mu_t / n_t(n)$ if player i is of type t in $\mathcal{G}^n$. This is the most general set of weights which treats players of the same type symmetrically.

We can use (12) to define $\underline{\lambda}_t$ for each type by substituting $f_{(t)}$ for f_i. Assumption A.3* guarantees that $\underline{\lambda}_t < 1$ for at least one type. For such types, if $\lambda \in (\underline{\lambda}_t, 1]$ there will be a level of input which satisfies (11) and we use $\widetilde{L}_t$ to denote this value; it is the unique solution of

$$f_{(t)}(0,0) = \lambda \frac{F\left(\widetilde{L}_t\right)}{\widetilde{L}_t}. \tag{15}$$

We also write $\widetilde{L}_t = 0$ for any type t for which $\lambda \leq \underline{\lambda}_t$.

All players of the same type t have the same share function which also depends, through the exogenous weights, on n. We use $s^n_{(t)}$ to denote this share function and note that, by the same arguments as for the symmetric case, it reaches the axis at a finite value: $\overline{L}^n_{(t)}$, the dropout value of type t. If $\lambda < \underline{\lambda}_t$, the share function is identically zero for positive L for all large enough n. For such a share function, we set $\overline{L}^n_t = 0$. Lemma 4.2 implies that $\overline{L}^n_{(t)} \longrightarrow \widetilde{L}_t$ as $n \longrightarrow \infty$. This observation allows us to establish that, if $\widetilde{L}_{t'} < \widetilde{L}_t$, players of type t' cease to participate once the number of players of type t becomes sufficiently large.

Lemma 4.3 *Assume A.1*-A.3*, A.4 and $\lambda > 0$. If $\widetilde{L}_{t'} < \widetilde{L}_t$, players of type t' supply no input in the equilibrium of $\mathcal{G}^n$ for all large enough n.*

The proof is based on a simple idea. Suppose two types in some game were to have distinct dropout values, independent of the number of players of each type. Once there are enough players of the type with larger dropout value, the aggregate share function at the lower dropout value exceeds one, so players with the latter dropout value supply no input in equilibrium. The proof of Lemma 4.3 is a little more delicate because of the need to take account of the dependence of dropout values on the number of players and details are given in the Appendix.

Let $\mathcal{T}$ denote the set of types which maximize $\widetilde{L}_t$ and $\overline{\mathcal{T}}$ its complement in $\{1, \ldots, T\}$. We can assume, without loss of generality, that $T \in \mathcal{T}$ which means that $\widetilde{L}_t = \widetilde{L}_T$ for $t \in \mathcal{T}$ and $\widetilde{L}_t < \widetilde{L}_T$ for $t \in \overline{\mathcal{T}}$. By equation (15), this is equivalent to assuming that $f_{(t)}(0,0) = f_{(T)}(0,0)$ for $t \in \mathcal{T}$ and $f_{(t)}(0,0) > f_{(T)}(0,0)$ for $t \in \overline{\mathcal{T}}$. Lemma 4.3 shows that only players of types in $\mathcal{T}$ participate in the limit in $\mathcal{G}^n$ and the next lemma shows that aggregate input of these types approaches $\widetilde{L}_T$.

Lemma 4.4 *Assume A.1*-A.3*, A.4 and $\lambda \in (0,1]$. If $\widehat{L}^n$ denotes aggregate equilibrium input in $\mathcal{G}^n$, then $\widehat{L}^n \longrightarrow \widetilde{L}_T$ as $n \longrightarrow \infty$.*

This result, which is proved in the Appendix, generalizes Lemma 4.1. We would like to exploit it to generalize the expression (14) for the limiting aggregate excess payoffs of each type. To do this, we need the limiting value of the aggregate input supplied by each type in $\mathcal{T}$. Unfortunately, the existence of this limit is not an immediate corollary of Lemma 4.4. We need the following result, which is proved in the Appendix.

Lemma 4.5 *Assume A.1*-A.3*, A.4 and $\lambda \in (0,1]$. For each $t \in \mathcal{T}$, there is a $\widetilde{\sigma}_t \in [0,1]$ satisfying $\sum_{t \in \mathcal{T}} \widetilde{\sigma}_t = 1$, such that $n_t(n)\, s^n_{(t)}\left(\widehat{L}^n\right) \longrightarrow \widetilde{\sigma}_t$ as $n \longrightarrow \infty$.*

This lemma asserts the existence of limiting aggregate shares, which allows us to deduce the limiting equilibrium aggregate allocation. The aggregate equilibrium input and output of type t in $\mathcal{G}^n$, which we write $\widehat{L}^n_t$ and $\widehat{X}^n_t$, respectively, have finite limits. In particular, for $t \in \mathcal{T}$, we have

$$\widehat{L}^n_t = n_t s^n_{(t)}\left(\widehat{L}^n\right)\widehat{L}^n \longrightarrow \widetilde{\sigma}_t \widetilde{L}_T. \tag{16}$$

If $t \in \overline{\mathcal{T}}$, Proposition 4.3 implies $\widehat{L}^n_t = 0$ for all large enough n so (16) continues to hold provided we define

$$\widetilde{\sigma}_t = 0 \text{ for all } t \in \overline{\mathcal{T}}.$$

as $n \longrightarrow \infty$ using Lemma 4.5. For outputs, we have

$$\widehat{X}^n_t = n_t \left\{ \lambda s^n_{(t)}\left(\widehat{L}^n\right) + \frac{(1-\lambda)\mu_t}{n_t} \right\} F\left(\widehat{L}^n\right)$$

and, letting $n \longrightarrow \infty$ yields:

$$\widehat{X}_t = \left\{ \lambda \widetilde{\sigma}_t + (1-\lambda)\mu_t \right\} F\left(\widetilde{L}_T\right) 0.$$

Since players of the same type have the same share function, all such players enjoy the same input and output in equilibrium; we call such allocations *type-symmetric*. We will use $\{X_t, L_t\}_{t=1}^T$ to denote a *type-allocation*, where L_t and X_t are the aggregate input supplied and output received by players of type t. For an individual i of type t in $\mathcal{G}^n$, we have $\ell_i = L_t/n_t$ and $x_i = X_t/n_t$. For any such allocation, we write $\psi_t(X_t, L_t)$ for the limiting aggregate excess payoff of players of type t:

$$\begin{aligned} \psi_t(X_t, L_t) &= \lim_{n \longrightarrow \infty} n_t(n) \left[u_{(t)}\left(\frac{L_t}{n_t(n)}, \frac{X_t}{n_t(n)} \right) - u_{(t)}(0,0) \right] \\ &= \frac{\partial u_{(t)}(0,0)}{\partial x} \left[X_t - L_t f_{(t)}(0,0) \right]. \end{aligned} \tag{17}$$

This notation is used to write limiting payoffs in the following proposition, which also summarizes the previous discussion. The final assertion is proved in the Appendix.

Proposition 4.1 *Assume A.1*-A.3*, A.4 and $\lambda \in (0,1]$. The limiting equilibrium type-allocation is given by*

$$\left(\widehat{X}_t, \widehat{L}_t\right) = \left(\{\lambda \widetilde{\sigma}_t + (1-\lambda)\mu_t\} F\left(\widetilde{L}_T\right), \widetilde{\sigma}_t \widetilde{L}_T\right).$$

The limiting equilibrium aggregate excess payoff of players of each type t in $\mathcal{G}^n$ is

$$\psi_t\left(\widehat{X}_t, \widehat{L}_t\right) = \frac{\partial u_{(t)}(0,0)}{\partial x} \mu_t \phi(\lambda),$$

where

$$\phi(\lambda) = (1-\lambda) F\left(\widetilde{L}_T\right) \tag{18}$$

and $\widetilde{L}_T$ is the solution of (15) for players of type T.

The proposition shows that the limiting equilibrium type-allocation and excess payoffs are approximations, to first order in $1/n$, of the equilibrium allocation and payoffs in $\mathcal{G}^n$ for large n. The proposition also has an interesting consequence for the choice of mixing parameter in large games for, to first order in $1/n$, every player will agree on their most preferred value of λ, namely the value that maximizes ϕ. Note that this is true even though individual preferences and/or exogenous shares may differ. The universally preferred value can be found by using (15) and (18) to obtain

$$\frac{d\phi}{d\lambda} = F\left(\widetilde{L}_T\right)\left\{\frac{\eta\left(\widetilde{L}_T\right)[1-\lambda]}{\lambda\left[1-\eta\left(\widetilde{L}_T\right)\right]} - 1\right\}.$$

We may deduce that $\lambda = \eta\left(\widetilde{L}_T\right)$ is the unique stationary point of ϕ. Furthermore, since $d\phi/d\lambda$ is positive [negative] if $\lambda <[>] \eta\left(\widetilde{L}_T\right)$, the stationary point is a maximum. The unanimously preferred value of the mixing parameter in a large game is the limiting equilibrium value of the elasticity of production. As above, we refer to the case $\lambda = \eta\left(\widetilde{L}_T\right)$ as *optimal mixing.*

Under optimal mixing 15, with $t = T$, becomes

$$f_{(T)}(0,0) = F'\left(\widetilde{L}_T\right).$$

The following lemma, which will prove useful in the sequel, records an immediate consequence of this equality.

Lemma 4.6 *Assume A.1*-A.3*, A.4 and optimal mixing. Then $\psi_t(F(L), L)$ is maximised at $L = \widetilde{L}_T$.*

In the next section, we will establish that optimal mixing leads to an efficient, envy-free limiting equilibrium, which also passes the unanimity and stand-alone tests.

5 Asymptotic equilibria under optimal mixing

5.1 Efficiency

We will say that a type-symmetric type allocation $\{X_t, L_t\}_{t=1}^T$ is *asymptotically efficient* if there is no other feasible allocation $\{X'_t, L'_t\}_{t=1}^T$ satisfying $\psi_t(X'_t, L'_t) \geq \psi_t(X_t, L_t)$ for $t = 1, \ldots, T$ with at least one strict inequality. Convexity of preferences implies that, if a type-symmetric allocation is dominated by some other allocation, it is dominated by a type-symmetric allocation namely that obtained by giving all players the average input and output for their type. This means we only need consider re-allocations between types. In particular, we can conclude from the limiting results in the preceding section that an allocation obtained from an asymptotically efficient type allocation by sharing input and outputs equally amongst players of the same type is efficient to first order in $1/n$. The next theorem, proved in the Appendix, establishes that optimal mixing implies asymptotic efficiency.

Theorem 5.1 *Assume A.1*-A3*, A.4 and optimal mixing. The limiting equilibrium type-allocation:* $\left\{\widehat{X}_t, \widehat{L}_t\right\}_{t=1}^T$ *is asymptotically efficient.*

The limiting aggregate output $\widetilde{L}_T$ is independent of the exogenous weights $\{\mu_t\}_{t=1}^T$. However Proposition 4.1 shows that the choice of weights does affect the payoffs of particular types. We can use the freedom to choose these weights to effect redistribution. Indeed, we can prove a converse of the preceding result: any type-symmetric efficient allocation which respects voluntary participation can be realized asymptotically as an equilibrium with a mixed sharing rule with an appropriate choice of weights. Voluntary participation requires $\psi_t(X_t, L_t) \geq 0$ for $t = 1, \ldots, T$. By "realization", we mean that any such allocation is payoff-equivalent to an equilibrium of $\mathcal{G}^n$. This result is formalized in the next theorem and proved in the Appendix.

Theorem 5.2 *Assume A.1*-A3*, A.4 and optimal mixing. Let* $\{X^{\mathrm{e}}_t, L^{\mathrm{e}}_t\}_{t=1}^T$ *be an asymptotically efficient type-symmetric allocation satisfying voluntary participation. Then there is a set of type-weights for which the limiting equilibrium type-allocation:* $\left\{\widehat{X}_t, \widehat{L}_t\right\}_{t=1}^T$ *satisfies*

$$\psi_{(t)}\left(\widehat{X}_t, \widehat{L}_t\right) = \psi_{(t)}\left(X^{\mathrm{e}}_t, L^{\mathrm{e}}_t\right) \text{ for } t = 1, \ldots, T. \tag{19}$$

Note that players of types in $\overline{\mathcal{T}}$ supply no input so the output they receive in $\mathcal{G}^n$ is exactly that in the efficient allocation. It follows that, if $\mathcal{T}$ is a singleton, the equilibrium of $\mathcal{G}^n$ can achieve any efficient allocation respecting voluntary participation (and not just one which is payoff equivalent to that allocation).

5.2 Envy freeness

Theorems 5.1 and 5.2 show that, under optimal mixing, all and every efficient allocation dominating the null allocation is a Nash equilibrium for suitable choice of exogenous shares. In this subsection, we show that, under equal shares the equilibrium is also envy free to first order in $1/n$. This recalls the results of, for example, Varian [17] and Champsaur and Laroque [6] on the equivalence of efficient, envy free allocations and competitive equilibria with equal initial endowments. Consider a type-allocation $\{X_t, L_t\}_{t=1}^{T}$ in which players of the same type receive equal allocations. If a player of type t were to receive the output and supply the input of a player of type t' in this allocation, her payoff would be

$$u_{(t)}(0,0) + \frac{\widetilde{u}}{n} + \xi_n$$

where

$$\widetilde{u} = \lim_{n \longrightarrow \infty} n \left[u_{(t)} \left(\frac{X_{t'}}{n_{t'}(n)}, \frac{L_{t'}}{n_{t'}(n)} \right) - u_{(t)}(0,0) \right] = \frac{\psi_t (X_{t'}, L_{t'})}{\nu_{t'}}$$

and $n\xi_n \longrightarrow 0$ as $n \longrightarrow \infty$. We call the allocation *asymptotically envy-free* if

$$\frac{\psi_t (X_t, L_t)}{\nu_t} \geq \frac{\psi_t (X_{t'}, L_{t'})}{\nu_{t'}} \text{ for all } t, t' = 1, \ldots, T. \tag{20}$$

Under a generalisation of equal exogenous weights, the equilibrium of a large game is envy free. The natural asymptotic version of equal weights is *proportional type-weights*: $\mu_t = \nu_t$ for all t. The next result, proved in the Appendix, formalizes this result.

Proposition 5.3 *Assume A.1*-A3*, A.4, optimal mixing and proportional type weights. Then the limiting equilibrium type-allocation is asymptotically envy free.*

We can apply this proposition to $\mathcal{G}^n$ for large n and equal weights. All players of the same type will receive the same output and supply the same input in equilibrium and so no player will envy a rival of the same type. Envy is only possible between players of different types and, by the preceding proposition, this is ruled out under equal weight to first order in $1/n$: the equilibrium allocation is envy free to first order.

5.3 Asymptotic unanimity test

The equilibrium of $\mathcal{G}^n$ under equal weights and optimal mixing has other desirable properties. In Subsection 3.3, we established that the equilibrium satisfies the unanimity test for production functions with constant elasticity. For large games, an asymptotic version of the test holds without additional restrictions on the production function. This involves a bound computed by supposing that all n players in $\mathcal{G}^n$ have access to a copy of the technology, all supply the same level of input and the output is split equally. If aggregate input is L, limiting aggregate excess payoff is

$$\lim_{n \longrightarrow \infty} n_t(n) \left[u_{(t)} \left(\frac{F(L)}{n}, \frac{L}{n} \right) - u_{(t)}(0,0) \right] = \nu_t \psi_t (F(L), L).$$

We define the *asymptotic unanimity lower bound* for type t to be the maximal limiting aggregate excess payoff of players of this type:

$$\nu_t \max_L \psi_t (F(L), L). \tag{21}$$

The next proposition states that, under equal shares and optimal mixing, the equilibrium satisfies the asymptotic unanimity test. The proof is in the Appendix.

Proposition 5.4 *Assume A.1*-A.3*, A.4, optimal mixing and proportional type weights. Then the asymptotic equilibrium payoff is not less than the asymptotic unanimity lower bound.*

The proposition allows us to conclude that, in the equilibrium of $\mathcal{G}^n$ with equal shares and large n, every player is at least as well off as when players cooperatively run a copy of the technology and split the output equally (to first order in $1/n$).

5.4 Stand-alone tests

The stand-alone test requires that no player do better than when they have sole access to the technology. For large n the equilibrium of $\mathcal{G}^n$ satisfies this test trivially: excess payoffs in equilibrium approach zero as $n \longrightarrow \infty$ whereas a player can always achieve a postive excess payoff with sole access to the technology. A more demanding version of the test applies it to types, assuming all and only players of type t have access to the technology and that the output and input are split equally amongst the players of that type. We therefore define the *asymptotic stand-alone payoff* of type t to be

$$\max_L \psi_t \left(F\left(L\right), L\right) = \frac{\partial u_{(t)}\left(0,0\right)}{\partial x} \max_L \left[F\left(L\right) - Lf_{(t)}\left(0,0\right)\right].$$

If $t \in \mathcal{T}$, we have $f_{(t)}\left(0,0\right) = f_{(T)}\left(0,0\right)$ and so the right hand side, using Lemma 4.6 and (15), is

$$\frac{\partial u_{(t)}\left(0,0\right)}{\partial x} \left(1-\lambda\right) F\left(\widetilde{L}_T\right) \geq \frac{\partial u_{(t)}\left(0,0\right)}{\partial x} \mu_t \left(1-\lambda\right) F\left(\widetilde{L}_T\right).$$

By Proposition 4.1, the right hand side is the aggregate equilibrium payoff to type t. We conclude that types in $\mathcal{T}$ do not benefit from the negative externality they impose; the asymptotic stand-alone test is satisfied for these types. This does not necessarily extend to the remaining types since $f_{(t)}\left(0,0\right) > f_{(T)}\left(0,0\right)$ for $t \in \overline{\mathcal{T}}$. The benefit to type t from leaving production to more efficient types may outweigh the fact that players of type t do not receive all the output. Of course, if their type weight is sufficiently small, the stand-alone test will be satisfied.

When $\mathcal{T}$ consists of a single type, we can additionally assert that, in large games, $\eta\left(\widetilde{L}_T\right)$ is the unique value of the mixing parameter for which the equilibrium is Pareto efficient. To see this, we simply note that, to first order in $1/n$, all players have payoffs proportional to $F\left(L\right) - Lf_{(T)}\left(0,0\right)$ which is uniquely maximized at the solution of $F'\left(L\right) = f_{(T)}\left(0,0\right)$. This is solved at $L = \widetilde{L}_{(T)}$ only if $\lambda = \eta\left(\widetilde{L}_T\right)$.

6 Conclusion

When the mixing parameter is set optimally and the game is "small", equilibrium under mixed sharing and equal shares has a number of desirable properties.

The equilibrium allocation is efficient when returns to scale are constant or preferences are identical, envy free for two players, passes the unanimity test under constant elasticity of production and passes the stand-alone test for net contributors. When the game is "large", there is finite set of distinct types and payoffs for types are taken to be asymptotic aggregate excess payoffs, the sets of efficient allocations respecting voluntary participation and of equilibrium allocations with optimal mixing are identical. Indeed, by varying the exogenous weights, the whole set can be mapped out. If, in addition, exogenous weights are equal, asymptotic equilibria are envy free and pass the unanimity test. The stand-alone test is also satisfied, indeed a stronger type-specific version holds for types which are net contributors.

Furthermore, for large games, equilibrium payoffs for each type are proportional to the same function of the mixing parameter. This means that players of all types prefer the (optimal) value of this parameter and suggests the use of a more elaborate procedure in which the choice is endogenous. For example, consider a two stage mechanism in which, in the first stage, players vote for their preferred value of the mixing parameter and, in the second stage, play a joint production game with mixing parameter set to the median vote. In section 4, we showed that second stage payoffs are single-peaked in the mixing parameter with peak equal to the optimal value. Hence, voting for the optimal value is a dominant strategy for every player in the first stage [4].

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7 Appendix: proofs

7.1 Proof of Proposition 2.2

Proposition 2.2 is a consequence of the following pair of lemmas, the first of which implies that, if the domain of s_i is non-empty, it must be a semi-infinite interval and that s_i is non-increasing, and indeed strictly decreasing where positive.

Lemma 7.1 *Assume A.1 and A.2 and that preferences are strictly normal if* $\lambda = 0$. *If* $L^1 > L^0 > 0$ *and* $s_i(L^0)$ *exists, then* $s_i(L^1)$ *exists and* $s_i(L^1) \leqq s_i(L^0)$. *The latter inequality is strict if* $s_i(L^0) > 0$.

Proof. From Lemmas 2.1 and 2.2 and the first-order conditions, we have

$$\zeta_i(s_i(L^0), L^1) \geqq \zeta_i(s_i(L^0), L^0) \geqq \tau_i(s_i(L^0), L^0) > \tau_i(s_i(L^0), L^1).$$

Hence, either (a) $\zeta_i(\sigma_i, L^1) > \tau_i(\sigma_i, L^1)$ for all $\sigma_i \in (0, 1)$, or (b) $\zeta_i(\widehat{\sigma}_i, L^1) = \tau_i(\widehat{\sigma}_i, L^1)$ for some $\widehat{\sigma}_i$. In case (a), $s_i(L^1) = 0 \leqq s_i(L^0)$ and the inequality is strict if $s_i(L^0) > 0$. In case (b), since ζ_i is non-decreasing in σ_i and τ_i is strictly decreasing in σ_i, we have $s_i(L^1) < s_i(L^0)$. Note that (b) can only occur if $s_i(L^0) > 0$. Again, if $\lambda = 0$, Lemma 7.1 remains valid when the normality requirement in A.1 is strengthened to strict normality. ■

Our second lemma rules out pathological behavior such as an empty domain, downward jumps or a strictly positive limit as $L \to \infty$.

Lemma 7.2 *Assume A.1, A.2 and A.3 and that preferences are strictly normal if* $\lambda = 0$. *Either* $s_i(L) = 0$ *for all* $L > 0$, *or, for any* $\sigma_i \in (0, 1]$, *there is a value of* L *satisfying* $s_i(L) = \sigma_i$.

Proof of Lemma 7.2. Suppose $\theta_i > 0$. We write $F'(0)$ for the limit of $F'(L)$ as $L \longrightarrow 0$. When $F'(L)$ is unbounded, we take $F'(0) = \infty$ and make the usual arithmetic and comparative assumptions. From A.2, for any $L > 0$, there is an L^0 satisfying $0 < L^0 < L$ and $F'\left(L^0\right) = F(L)/L$. Hence, $F(L)/L \longrightarrow F'(0)$ as $L \longrightarrow 0$ and, taking limits in (3) shows that

$$\tau_i(\sigma_i, L) \longrightarrow \{\lambda + (1-\lambda)\theta_i\} F'(0) \text{ as } L \longrightarrow 0.$$

We can focus on the case where

$$\{\lambda + (1-\lambda)\theta_i\} F'(0) > f_i(0,0) = \zeta_i(\sigma_i, 0), \tag{22}$$

for, if (22) did not hold, A.1 and A.2 would imply that the indifference curve through the origin would lie below the graph of F. In this case, player i would never participate actively: $s_i(L) = 0$ for all $L > 0$.

We will prove that (5) holds for some $L > 0$ by demonstrating the existence of L^* such that

$$\tau_i(\sigma_i, L^*) \leq \zeta_i(\sigma_i, L^*) \tag{23}$$

and appealing to continuity of τ_i and ζ_i in L. Let $\sigma^* = \min\{\sigma_i, \theta_i\}$ and define

$$\phi_i^*(L) = \sigma^* F\left(\frac{L}{\sigma^*}\right)$$

For any $L > 0$, A.2 implies $\phi_i^*(L) < F(L)$; the graph of ϕ_i^* lies below that of F. By (22), $F'(0) > f_i(0,0)$ and A.3 implies that the graph of F crosses the indifference curve through the origin from above. Since $\phi_i^{*\prime}(0) = F'(0)$, we may deduce that the graph of ϕ_i^* also crosses this indifference curve through the origin from above. If $L = L'$ at the crossing point, the gradient of a line from the origin to the crossing point does not exceed the slope of this indifference curve at the crossing point:

$$\frac{\phi_i^*(L')}{L'} \leq f_i(\phi_i^*(L'), L').$$

Write $L^* = L'/\sigma^*$, so that

$$\frac{F(L^*)}{L^*} \leq f_i(\sigma^* F(L^*), \sigma^* L^*) \leq \zeta_i(\sigma_i, L^*).$$

The second inequality uses A.1, the expression (2) and the inequalities $\sigma^* \leq \lambda\sigma_i + (1-\lambda)\theta_i$ and $\sigma^* \leq \sigma_i$. Noting that $F(L^*)/L^*$ is an upper bound to $\tau_i(\sigma_i, L^*)$ establishes (23) and completes the proof for positive θ_i.

If $\theta_i = 0$ and $\lambda > 0$, the same proof is valid with $\sigma^* = \lambda\sigma_i$. If $\lambda = \theta_i = 0$, player i receives no output and therefore supplies no input in equilibrium: $s_i(L) = 0$ for all $L > 0$. ■

These two lemmas ensure that s_i is (a) continuous[9], (b) strictly decreasing where positive, (c) approaches or equals 0 as $L \longrightarrow \infty$ which completes the proof of Proposition 2.2.

[9] A discontinuous, non-increasing function would have to exhibit downward jumps, ruled out by Lemma 7.2.

7.2 Proof of Proposition 3.1

If the share function is zero for all $L > 0$, the only equilibrium is null and the corresponding allocation is vacuously efficient. Otherwise, Theorem 2.4 allows us to conclude that there is a unique value of the aggregate equilibrium input, $\widehat{L}$, which satisfies $s_i\left(\widehat{L}\right) = 1/n$ for every player i. The interior first order condition (5) can therefore be written

$$\left[\frac{\lambda}{\eta\left(\widehat{L}\right)}\left(1-\frac{1}{n}\right)+\frac{1}{n}\right]F'\left(\widehat{L}\right) = f_i\left(\frac{\widehat{X}}{n},\frac{\widehat{L}}{n}\right), \tag{24}$$

where $\widehat{X}$ denotes the aggregate output.

Any allocation which maximizes the sum of utilities: $\sum_{i=1}^{n} u\left(x_i, \ell_i\right)$ subject to the feasibility conditions: $\sum_{i=1}^{n} x_i = F\left(\sum_{i=1}^{n} \ell_i\right)$ and $\left(x_i, \ell_i\right) \geq (0,0)$ for all i will be efficient. This problem has a unique maximum which satisfies $\left(x_i, \ell_i\right) = \left(x^{\mathrm{f}}, \ell^{\mathrm{f}}\right)$ for all i (say), where

$$f\left(x^{\mathrm{f}}, \ell^{\mathrm{f}}\right) = F'\left(n\ell^{\mathrm{f}}\right).$$

If we set $\lambda = \eta\left(\widehat{L}\right)$, we may deduce that $x^{\mathrm{f}} = \widehat{X}/n$ and $\ell^{\mathrm{f}} = \widehat{L}/n$ is efficient and this completes the proof.

7.3 Proof of Proposition 3.2

If $u_i\left(x_i, \ell_i\right) = v_i\left(x_i\right) - \ell_i$ for $i = 1, \ldots, n$, the feasible allocation $\left\{\left(x_i, \ell_i\right)\right\}_{i=1}^{n}$ is efficient if and only if it maximizes the aggregate surplus. Necessary and sufficient conditions for this are

$$v_i'\left(x_i\right) F'\left(L\right) \leq 1 \text{ for } i = 1, \ldots, n, \tag{25}$$

with equality if $x_i > 0$. Note that any set of individual inputs summing to L will be efficient, provided outputs satisfy (25). With such preferences, conditions (4) and (5) can be written as

$$\left[\frac{\lambda}{\eta\left(L\right)}\left(1-\frac{\ell_i}{L}\right)+\frac{x_i}{X}\right] v_i'\left(x_i\right) F'\left(L\right) \leq 1 \text{ for } i = 1, \ldots, n, \tag{26}$$

with equality if $\ell_i > 0$.

If L^{e} and x_i^{e} denote the efficient values of L and x_i and the mixing parameter satisfies $\lambda = \eta\left(L^{\mathrm{e}}\right)$, then (25) and (26) are equivalent provided individual inputs satisfy $\ell_i = x_i^{\mathrm{e}} L^{\mathrm{e}}/F\left(L^{\mathrm{e}}\right)$ for $i = 1, \ldots, n$. From (1), this holds for $\theta_i = x_i^{\mathrm{e}}/F\left(L^{\mathrm{e}}\right)$.

7.4 Proof of Proposition 3.3

For each $i = 1, \ldots, n$, define the function

$$u_i^s\left(\sigma\right) = u_i\left(\left\{\lambda\sigma_i + \frac{1-\lambda}{n}\right\} F\left(\widehat{L}\right), \sigma_i \widehat{L}\right)$$

for $0 \leq \sigma_i \leq 1$ and consider a player i for whom $\widehat{\sigma}_i \leq 1/n$, where $\widehat{\sigma}_i = \widehat{\ell}_i/\widehat{L}$. If $\widetilde{\sigma}_i$ maximizes u_i^s, we shall show that $\widetilde{\sigma}_i \leq \widehat{\sigma}_i$. Quasiconcavity of u_i and therefore of u_i^s then allows us to conclude that $u_i^s(\widehat{\sigma}_i) \geq u_i^s(1/n)$, which is simply a rewriting of (8). (Alternatively, if $\widehat{\sigma}_i > 1/n$ a proof along identical lines gives $\widetilde{\sigma}_i \geq \widehat{\sigma}_i$ and so $u_i^s(\widehat{\sigma}_i) \geq u_i^s(1/n)$, again. We omit details of this case.)

Necessary and sufficient first order conditions for maximizing u_i^s when $0 \leq \widetilde{\sigma}_i < 1$ can be written

$$\lambda \frac{F\left(\widehat{L}\right)}{\widehat{L}} \leq \zeta_i\left(\widetilde{\sigma}_i, \widehat{L}\right), \tag{27}$$

with equality if $\widetilde{\sigma}_i > 0$.

Firstly, suppose $\widehat{\sigma}_i > 0$. With optimal mixing and equal shares, (5) can be written

$$\left\{1 + (1-\lambda)\left[\frac{1}{n} - \widehat{\sigma}_i\right]\right\} \lambda \frac{F\left(\widehat{L}\right)}{\widehat{L}} = \zeta_i\left(\widehat{\sigma}_i, \widehat{L}\right).$$

We have assumed $\widehat{\sigma}_i \leq 1/n$, so the term in braces exceeds unity and $\widetilde{\sigma}_i \leq \widehat{\sigma}_i$ follows from the fact that $\zeta_i\left(\sigma, \widehat{L}\right)$ is non-decreasing in σ (Lemma 2.1). If player i's preferences are strictly normal, ζ_i is strictly increasing in σ which justifies the final claim in the proposition.

In the case $\widehat{\sigma}_i = 0$, (4) can be written

$$\left\{1 + \frac{1-\lambda}{n}\right\} \lambda \frac{F\left(\widehat{L}\right)}{\widehat{L}} \leq \zeta_i\left(0, \widehat{L}\right) \leq \zeta_i\left(\widetilde{\sigma}_i, \widehat{L}\right),$$

where the last inequality is justified by Lemma 2.1. It follows that (27) can never be satisfied with equality and therefore $\widetilde{\sigma}_i = 0 \leq \widehat{\sigma}_i$ as required. To justify the final claim in the proposition in this case, note that we have established that $u_i^s(0) \geq u_i^s(\sigma)$ for all $\sigma \in [0,1]$. Were this inequality to hold with equality for some $\sigma^* > 0$, quasiconcavity of u_i^s would imply that $u_i^s(\sigma) = u_i^s(0)$ for $0 \leq \sigma \leq \sigma^*$ which would contradict strictly normal preferences. We may conclude $u_i^s(\widehat{\sigma}_i) = u_i^s(0) > u_i^s(1/n)$ as required.

7.5 Proof of Proposition 3.5

Note that for the production function in the proposition both A.2 and A.3 are satisfied. Under the assumptions of the proposition, the payoff to player i in an equilibrium in which the value of aggregate equilibrium input is L is:

$$u_i\left(\left\{\alpha s_i(L) + \frac{1-\alpha}{n}\right\} L^\alpha, s_i(L)\,L\right) \equiv u_i^e(L).$$

The proposition is proved by showing that u_i^e is minimized at $\widetilde{L}_i$, the value of L that achieves the maximum in the definition of the unanimity bound (9). The necessary and sufficient first order conditions for the maximization in (9) can be written:

$$\alpha \widetilde{L}_i^{\alpha-1} = \zeta_i\left(\frac{1}{n}, \widetilde{L}_i\right). \tag{28}$$

When $\underline{L}_i < L < \overline{L}_i$ the share function s_i is positive and, by (3) and (5), satisfies

$$\left\{\alpha s_i(L) + \frac{1-\alpha}{n}\right\}\alpha L^{\alpha-1} + \alpha\left[1 - s_i(L)\right]L^{\alpha-1} = \zeta_i\left(s_i(L), L\right). \qquad (29)$$

Comparing (28) and (29) shows that $s_i\left(\widetilde{L}_i\right) = 1/n$.

Now s_i is differentiable, by the implicit function theorem, and therefore so is u_i^e. After some manipulation, the derivative can be written as:

$$\begin{aligned}\frac{du_i^e}{dL} &= \frac{\partial u_i}{\partial x}\left[1 - s_i(L) - s_i'(L)L\right]\left\{\zeta_i\left(s_i(L), L\right) - \alpha L^{\alpha-1}\right\} \\ &= \alpha(1-\alpha)\frac{\partial u_i}{\partial x}\left[1 - s_i(L) - s_i'(L)L\right]\left\{\frac{1}{n} - s_i(L)\right\}.\end{aligned}$$

For this range of L we have $s_i(L) < 1$ and $s_i'(L) \leq 0$ by Proposition 2.2 and it follows that u_i^e is minimized where $s_i(L) = 1/n$ i.e. at $L = \widetilde{L}_i$. Since s_i is decreasing, we may deduce that the slope of u_i^e has the sign of $L - \widetilde{L}_i$. We may conclude that u_i^e is minimized at $\widetilde{L}_i$, as claimed.

7.6 Completion of proof of Lemma 4.1

Under Assumption A.1, $f(x, \ell) \geqq f(0, 0)$ for all $(x, \ell) \geqq (0, 0)$ and under A.2, $F' < F/L$. When $\widehat{L}^n > 0$ for all n, (10) implies $f(0, 0) \leq (1 + \lambda)F\left(\widehat{L}^n\right)/\widehat{L}^n$ for all n. This means that the sequence $\{L^n\}_{n=1}^{\infty}$ is bounded. That it is convergent follows from the fact that all convergent subsequences have the same limit, the unique solution of (11).

Since all players are identical, $\widehat{L}^n = 0$ requires $s(L) = 0$ for all $L > 0$ and, from (4) and (5), this entails

$$f\left((1-\lambda)F(L), 0\right) \geq \frac{(1-\lambda)}{n}F'(L) + \lambda\frac{F(L)}{L} \text{ for all } L > 0. \qquad (30)$$

Observe that the LHS is non-decreasing in L (by Assumption A.1) and the RHS is strictly decreasing in L (by Assumption A.2). Further, both average and marginal products have the same supremum, so (30) is equivalent to

$$\lambda + \frac{1-\lambda}{n} \leq \underline{\lambda} \qquad (31)$$

using (12). If $\lambda \geq \underline{\lambda}$, we deduce that $L^n > 0$ for all n and the argument in the first paragraph can be applied. If $\lambda < \underline{\lambda}$, (31) holds for all large enough n and we may conclude that $L^n = 0$ for such n so $L^n \longrightarrow \widetilde{L} = 0$. Note that in the knife-edge case, $\lambda = \underline{\lambda}$, we have $L^n > 0$ for all n but $L^n \longrightarrow 0 = \widetilde{L}$.

7.7 Proof of Lemma 4.3

As the number of players of type t grows without limit, the exogenous weight of each player of that type falls to zero. It is convenient to define a 'limiting' share function in which this weight is set to zero. We will write $\widetilde{s}_{(t)}$ for this function and note that, for any $L > 0$, it satisfies the first order conditions for players of type t with zero exogenous weight:

$$f_{(t)}\left(\lambda\sigma F\left(L\right),\sigma L\right) \geqslant \left\{\lambda\sigma F'\left(L\right) + \lambda\left[1-\sigma\right]\frac{F\left(L\right)}{L}\right\} \tag{32}$$

with equality if $\sigma > 0$, where $\sigma = \widetilde{s}_{(t)}\left(L\right)$. Lemmas 2.1 and 2.2 still apply to the left hand and right hand sides of (32), respectively. This allows us to conclude that $\widetilde{s}_{(t)}$ (i) is the unique σ satisfying (32) with equality if σ is positive, (ii) has the properties of a share function set out in Section 2 and, (iii) has dropout value $\widetilde{L}_t$. Assertion (iii) follows by noting that the dropout value of L satisfies (32) with equality if $\sigma > 0$ and noting that this gives the definition of $\widetilde{L}_t$. The next lemma shows that $\widetilde{s}_t$ is the pointwise limit of type-t share functions.

Lemma 7.3 *Let $s^n_{(t)}$ denote the share function of players of type t with exogenous weight $\mu_t/n_t\left(n\right)$. If $L < \widetilde{L}_t$, then $s^n_{(t)}\left(L\right) \longrightarrow \widetilde{s}_t\left(L\right)$ as $n \longrightarrow \infty$.*

Proof. Fix $L < \widetilde{L}_t$ and define

$$\begin{aligned}\nabla^n &= f_{(t)}\left(\left\{\lambda s^n_{(t)}\left(L\right) + \frac{(1-\lambda)\,\mu_t}{n_t\left(n\right)}\right\}F\left(L\right), s^n_{(t)}\left(L\right)L\right) \\ &\quad -\left\{\lambda s^n_{(t)}\left(L\right) + \frac{(1-\lambda)\,\mu_t}{n_t\left(n\right)}\right\}F'\left(L\right) - \lambda\left[1-s^n_{(t)}\left(L\right)\right]\frac{F\left(L\right)}{L}.\end{aligned}$$

Conditions (4) and (5) mean that $\nabla^n \geq 0$ and $s^n_{(t)}\left(L\right)\nabla^n = 0$. Note that, if a convergent subsequence of $\left\{s^n_{(t)}\left(L\right)\right\}_{n=1}^{\infty}$ has limit σ, the limit of ∇^n on this subsequence is the difference between the left hand and right hand sides of (32). Since this limit is non-negative and equal to zero when $\sigma > 0$, we may deduce that $\sigma = \widetilde{s}_t\left(L\right)$. Since the sequence $\left\{s^n_{(t)}\left(L\right)\right\}_{n=1}^{\infty}$ is bounded (between 0 and 1), the conclusion of the lemma follows. ■

Completion of Proof of Lemma 4.3. Define $L^* = \left(\widetilde{L}_{t'} + \widetilde{L}_t\right)/2$. Note that $\widetilde{s}_t\left(L^*\right) > 0$ which means that there is a positive integer $\overline{n}_1$ such that $n_t\left(n\right)\widetilde{s}_t\left(L^*\right) \geqq 2$ for $n > \overline{n}_1$. The preceding lemma implies that there is a positive integer $\overline{n}_2$ such that $s^n_{(t)}\left(L^*\right) > \widetilde{s}_t\left(L^*\right)/2$ for all $n > \overline{n}_2$. Furthermore, convergence of type-t' dropout values to $\widetilde{L}_{t'}$ (Lemma 4.2) implies that there is a positive integer $\overline{n}_3$ such that $L^n_{t'} < L^*$ for all $n > \overline{n}_3$. Hence, if $n > \max\left\{\overline{n}_1, \overline{n}_2, \overline{n}_3\right\}$, we may conclude that $n_t\left(n\right)s^n_{(t)}\left(L^n_{t'}\right) > 1$, since $s^n_{(t)}$ is decreasing (Proposition 2.2). Hence, the equilibrium value of L in $\mathcal{G}^n$ exceeds $L^n_{t'}$, the dropout value of players of type t', no player of that type supplies positive input in $\mathcal{G}^n$. ■

7.8 Proof of Lemma 4.4

The first order conditions for type t can be written:

$$f_{(t)}\left(\left\{\lambda s^n_{(t)}\left(\widehat{L}^n\right) + \frac{(1-\lambda)\,\mu_t}{n_t}\right\}F\left(\widehat{L}^n\right), s^n_{(t)}\left(\widehat{L}^n\right)\widehat{L}^n\right) \geq \tag{33}$$

$$\left\{\lambda s^n_{(t)}\left(\widehat{L}^n\right) + \frac{(1-\lambda)\,\mu_t}{n_t}\right\}F'\left(\widehat{L}^n\right) + \lambda\left(1-s^n_{(t)}\left(\widehat{L}^n\right)\right)\frac{F\left(\widehat{L}^n\right)}{\widehat{L}^n}$$

with equality if $s^n_{(t)}(L^n) > 0$. By Lemma 4.3, we can choose n large enough to ensure that $s^n_{(t)}\left(\widehat{L}^n\right) = 0$ for $t \in \overline{\mathcal{T}}$. If, for such n, we multiply (33) by n_t, sum over $t \in \mathcal{T}$, use the equilibrium condition

$$\sum_{t=1}^{T} n_t(n)\, s^n_{(t)}\left(\widehat{L}^n\right) = 1 \tag{34}$$

and divide by $n_{\mathcal{T}}(n) = \sum_{t\in\mathcal{T}} n_t(n)$ to obtain

$$\sum_{t\in\mathcal{T}} \frac{n_t}{n_{\mathcal{T}}} f_{(t)}\left(\left\{\lambda s^n_{(t)}\left(\widehat{L}^n\right) + \frac{(1-\lambda)\mu_t}{n_t}\right\} F\left(\widehat{L}^n\right), s^n_{(t)}\left(\widehat{L}^n\right)\widehat{L}^n\right) =$$
$$\left\{\lambda + (1-\lambda)\sum_{t\in\mathcal{T}}\mu_t\right\}\frac{F'\left(\widehat{L}^n\right)}{n_{\mathcal{T}}} + \lambda\left(1 - \frac{1}{n_{\mathcal{T}}}\right)\frac{F\left(\widehat{L}^n\right)}{\widehat{L}^n}.$$

The assumption $v_t > 0$ for all t implies that $n_{\mathcal{T}}(n) \longrightarrow \infty$ as $n \longrightarrow \infty$ and, using (34), $s^n_{(t)}\left(\widehat{L}^n\right) \longrightarrow 0$. Taking limits on any subsequence of $\left\{\widehat{L}^n\right\}_{n=1}^{\infty}$ which converges to L^*, yields

$$f_{(T)}(0,0) = \lambda\frac{F(L^*)}{L^*},$$

where we have used the fact that $f_{(t)}(0,0) = f_{(T)}(0,0)$ for $t \in \mathcal{T}$. It follows that $L^* = \widetilde{L}_T$. A similar argument to the symmetric case can be used to demonstrate that the sequence $\left\{\widehat{L}^n\right\}_{n=1}^{\infty}$ is bounded and we can conclude that $\widehat{L}^n \longrightarrow \widetilde{L}_T$ as $n \longrightarrow \infty$.

7.9 Proof of Lemma 4.5

The first-order conditions (33) for $s^n_{(t)}(L^n)$ can be rewritten, using (15) as

$$\begin{aligned} & n_t\left[f_{(t)}\left(\left\{\lambda s^n_{(t)}\left(\widehat{L}^n\right) + \frac{(1-\lambda)\mu_t}{n_t}\right\}F\left(\widehat{L}^n\right), s^n_{(t)}\left(\widehat{L}^n\right)\widehat{L}^n\right) - f_{(t)}(0,0)\right] \\ \geq\; & \left\{\lambda n_t s^n_{(t)}\left(\widehat{L}^n\right) + (1-\lambda)\mu_t\right\}F'\left(\widehat{L}^n\right) \\ & +\lambda\left[n_t - n_t s^n_{(t)}\left(\widehat{L}^n\right)\right]\frac{F\left(\widehat{L}^n\right)}{\widehat{L}^n} - \lambda n_t \frac{F\left(\widetilde{L}_T\right)}{\widetilde{L}_T}, \end{aligned} \tag{35}$$

with equality if $s^n_{(t)}\left(\widehat{L}^n\right) > 0$. First observe that, for all $t \in \mathcal{T}$, the sequence $\left\{n_t(n)\, s^n_{(t)}\left(\widehat{L}^n\right)\right\}_{n=1}^{\infty}$ is bounded between 0 and 1 by the equilibrium condition (35). We claim that the sequence:

$$\left\{n_t(n)\left[\frac{F\left(\widehat{L}^n\right)}{\widehat{L}^n} - \frac{F\left(\widetilde{L}_T\right)}{\widetilde{L}_T}\right]\right\}_{n=1}^{\infty} \tag{36}$$

is also bounded. This can be justified by contradiction. If it were false, we could find a subsequence with infinite limit and then choose a type $t \in \mathcal{T}$ and a sub-subsequence on which $n_t(n)\, s^n_{(t)}\left(\widehat{L}^n\right)$ were also convergent and positive for all n. On this subsequence, (35) holds with equality and taking limits would lead to a contradiction: a finite value on the left hand side and an infinite one on the right hand side. (Recall our assumption that $u_{(t)}$ is twice continuously differentiable.)

We complete the proof by showing that on every subsequence on which (36) converges (to $\widetilde{AP}$, say) and $\left\{n_t s_{(t)}\left(\widehat{L}^n\right)\right\}$ converges for each t (to $\widetilde{\sigma}_t$, say) these limiting values are uniquely determined. To justify this claim, we first observe that $s_{(t)}(L^n)$ must be positive for all but a finite set of n. Taking limits on the subsequence in (35), we find that, for all $t \in \mathcal{T}$,

$$\text{either } \widetilde{AP} \leq \gamma_t \text{ or } \widetilde{AP} = \gamma_t + \beta_t\, \widetilde{\sigma}_t, \tag{37}$$

where

$$\begin{aligned} \gamma_t &= \frac{(1-\lambda)\,\mu_t}{\lambda}\left[F\left(\widetilde{L}_T\right) f_{(t)1}(0,0) - F'\left(\widetilde{L}_T\right)\right], \\ \beta_t &= F\left(\widetilde{L}_T\right) f_{(t)1}(0,0) + \lambda^{-1}\widetilde{L}_T f_{(t)2}(0,0) + \frac{F\left(\widetilde{L}_T\right)}{\widetilde{L}_T} - F'\left(\widetilde{L}_T\right). \end{aligned}$$

and $f_{(t)1}$ and $f_{(t)2}$ denote the derivatives of the MRS with respect to its first and second arguments. Note that the limit of the equilibrium requirement (34) gives:

$$\sum_{t \in \mathcal{T}} \widetilde{\sigma}_{(t)} = 1. \tag{38}$$

Under A.1, the MRS is non-decreasing in each argument, so both partial derivatives of $f_{(t)}$ are non-negative. Since the average product exceeds the marginal product, we conclude that $\beta_t > 0$ for all $t \in \mathcal{T}$. Using the ????? theorem of ???????? [ref ?????], we may deduce that there is a unique $\widetilde{AP}$, $\{\widetilde{\sigma}_t\}_{t \in \mathcal{T}}$ which solves (37) and (38).

[Alternative proof: (37) and (38) are the Kuhn-Tucker conditions for the optimization problem:

$$\begin{aligned} &\min \sum_{t \in \mathcal{T}} \left[\gamma_t x_t + \frac{1}{2}\beta_t x_t^2\right] \\ &\text{subject to } \sum_{t \in \mathcal{T}} x_t = 1,\ x_t \geq 0 \text{ for all } t \in \mathcal{T}. \end{aligned}$$

The objective function is a sum of strictly convex functions and therefore, itself, strictly convex. The feasible set is compact and convex. Such an optimization problem has a unique optimal solution.]

7.10 Completion of proof of Proposition 4.1

We can use Lemma 4.5 to deduce the following expression for the aggregate equilibrium payoff of players of types $t \in \mathcal{T}$:

$$\begin{aligned}
&\lim_{n \longrightarrow \infty} n_t(n) \left[u_{(t)} \left(\left\{ \lambda s_{(t)} \left(\widehat{L}^n \right) + \frac{(1-\lambda)\mu_t}{n_t(n)} \right\} F\left(\widehat{L}^n \right), s_{(t)} \left(\widehat{L}^n \right) \widehat{L}^n \right) \right. \\
&\qquad \left. - u_{(t)}(0,0) \right] \\
&\qquad = \frac{\partial u_{(t)}(0,0)}{\partial x} \left[\left\{ \lambda \widetilde{\sigma}_{(t)} + (1-\lambda)\mu_t \right\} F\left(\widetilde{L}_T \right) - \widetilde{\sigma}_{(t)} \widetilde{L}_T f_{(t)}(0,0) \right] \\
&\qquad = \psi_t \left(\widehat{X}_t, \widehat{L}_t \right) = \frac{\partial u_{(t)}(0,0)}{\partial x} (1-\lambda)\mu_t F\left(\widetilde{L}_T \right), \qquad (39)
\end{aligned}$$

where we have used (15) and (17) to obtain the final line, noting that $f_{(t)}(0,0) = f_{(T)}(0,0)$ for $t \in \mathcal{T}$.

Other types supply no input once n is large enough (Lemma 4.3), so their aggregate excess payoff is the limit as $n \longrightarrow \infty$ of

$$n_t(n) \left[u_{(t)} \left(\frac{(1-\lambda)\mu_t}{n_t(n)} F(L^n), 0 \right) - u_{(t)}(0,0) \right]$$

which yields the formula (18) immediately.

7.11 Proof of Theorem 5.1

We can establish efficiency by showing that the allocation $\left\{ \widehat{X}_t, \widehat{L}_t \right\}_{t=1}^{T}$ maximizes a positive weighted combination of the $\psi_t(X_t, L_t)$. Writing $\{\alpha_t\}_{t=1}^{T}$ for the weights, note that, if we set $\alpha_t = \left[\partial u_{(t)}(0,0) / \partial x \right]^{-1}$ for each t, then

$$\Psi \equiv \sum_{t=1}^{T} \alpha_t \psi_t (X_t, L_t) = F(L) - \sum_{t=1}^{T} L_t f_{(t)}(0,0).$$

We have written $L = \sum_{t=1}^{T} L_t$ and used the feasibility requirement $\sum_{t=1}^{T} X_t = F(L)$. Since Ψ is a concave function of $(L_1, \ldots, L_T)$, the first order conditions are necessary and sufficient for a maximum. For each t, these are

$$\frac{\partial \Psi}{\partial L_t} = F'(L) - f_{(t)}(0,0) \leq 0,$$

with equality if $L_t > 0$.

To show that limiting equilibrium inputs: $\left(\widehat{L}_1, \ldots, \widehat{L}_T \right)$ are efficient we use the fact that $\sum_{t=1}^{T} \widehat{L}_t = \widetilde{L}_T$. Hence, if $\lambda = \eta\left(\widetilde{L}_T \right)$, then (15) implies

$$\begin{aligned}
F'\left(\widetilde{L}_T \right) &= f_{(T)}(0,0) \leq f_{(t)}(0,0) \text{ for all } t, \\
F'\left(\widetilde{L}_T \right) &= f_{(T)}(0,0) = f_{(t)}(0,0) \text{ for } t \in \mathcal{T}.
\end{aligned}$$

The optimality conditions are satisfied since $\widehat{L}_t > 0$ if and only if $t \in \mathcal{T}$, by Lemma 4.3.

7.12 Proof of Theorem 5.2

The proof is in three steps. Firstly, we show that types in $\overline{\mathcal{T}}$ supply no input in an efficient type-symmetric allocation which respects voluntary participation. Secondly, we show that the aggregate input in such an allocation is $\widetilde{L}_T$. Finally, for any allocation with these properties, we display a set of type-weights for which the equilibrium allocation is limiting efficient.

(i) Suppose we had $L^{\mathrm{e}}_{t'} > 0$ for some $t' \in \overline{\mathcal{T}}$. Voluntary participation requires $X_{t'} > 0$. Consider a new allocation which is identical to the original allocation except that

$$\begin{aligned} L'_{t'} &= L^{\mathrm{e}}_{t'} - \varepsilon, \\ X'_{t'} &= X^{\mathrm{e}}_{t'} - \varepsilon f_{(t')}(0,0), \\ L'_T &= L^{\mathrm{e}}_T + \varepsilon, \\ X'_T &= X^{\mathrm{e}}_T + \varepsilon f_{(t')}(0,0). \end{aligned}$$

This is feasible for small enough $\varepsilon > 0$ and

$$\begin{aligned} \psi_t(X'_t, L'_t) &= \psi_t(X^{\mathrm{e}}_t, L^{\mathrm{e}}_t) \text{ for } t = 1, \dots, T-1, \\ \psi_T(X'_T, L'_T) &= \psi_T(X^{\mathrm{e}}_t, L^{\mathrm{e}}_t) + \frac{\partial u_{(T)}(0,0)}{\partial x}\varepsilon\left[f_{(t')}(0,0) - f_{(T)}(0,0)\right] \\ &> \psi_T(X^{\mathrm{e}}_t, L^{\mathrm{e}}_t). \end{aligned}$$

This is a Pareto improvement and contradicts the hypothesized efficiency of $\{X^{\mathrm{e}}_t, L^{\mathrm{e}}_t\}_{t=1}^T$.

(ii) Suppose we had

$$L^{\mathrm{e}} = \sum_{t=1}^{T} L^{\mathrm{e}}_t \neq \widetilde{L}_T.$$

We will show that this contradicts the presumed efficiency of $\{X^{\mathrm{e}}_t, L^{\mathrm{e}}_t\}$ by constructing a dominating allocation which is identical to the original allocation for types in $\overline{\mathcal{T}}$.

We first note that $L^{\mathrm{e}}_t = 0$ if $t \in \overline{\mathcal{T}}$ by (i), so $L^{\mathrm{e}} = \sum_{t\in\mathcal{T}} L^{\mathrm{e}}_t$. Now define

$$Y^{\mathrm{e}} = \sum_{t\in\mathcal{T}} X^{\mathrm{e}}_t - L^{\mathrm{e}} f_{(T)}(0,0).$$

Note that voluntary participation dictates a non-negative payoff; in particular $X^{\mathrm{e}}_t - L^{\mathrm{e}}_t f_{(T)}(0,0) \geq 0$ so Y^{e} is non-negative. For each $t \in \mathcal{T}$, define

$$\beta^{\mathrm{e}}_t = \begin{cases} \left[X^{\mathrm{e}}_t - L^{\mathrm{e}}_t f_{(T)}(0,0)\right]/Y^{\mathrm{e}} & \text{if } Y^{\mathrm{e}} > 0, \\ 1/\#\mathcal{T} & \text{if } Y^{\mathrm{e}} = 0, \end{cases}$$

where $\#\mathcal{T}$ denotes the cardinality of $\mathcal{T}$. In either case, we have

$$\beta^{\mathrm{e}}_t Y^{\mathrm{e}} = X^{\mathrm{e}}_t - L^{\mathrm{e}}_t f_{(T)}(0,0) \text{ for all } t \in \mathcal{T}.$$

For $t \in \overline{\mathcal{T}}$, set $(X''_t, L''_t) = (X^{\mathrm{e}}_t, L^{\mathrm{e}}_t)$ and, for $t \in \mathcal{T}$, set

$$\begin{aligned} L''_t &= \beta^{\mathrm{e}}_t \widetilde{L}_{(T)}, \text{ and} \\ X''_t &= \beta^{\mathrm{e}}_t \left[F\left(\widetilde{L}_T\right) - \sum_{t\in\overline{\mathcal{T}}} X^{\mathrm{e}}_t\right]. \end{aligned}$$

Note that the strictly concave function $F(L) - Lf_{(T)}(0,0)$ is uniquely maximized at $\widetilde{L}_T$ under optimal mixing and therefore

$$F\left(\widetilde{L}_T\right) - \widetilde{L}_T f_{(T)}(0,0) > F(L^{\mathrm{e}}) - L^{\mathrm{e}} f_{(T)}(0,0), \tag{40}$$

for $L \neq \widetilde{L}_T$. It follows that, if $t \in \mathcal{T}$,

$$\begin{aligned} X_t'' - L_t'' f_{(t)}(0,0) &= X_t'' - L_t'' f_{(T)}(0,0) \\ &= \beta_t^{\mathrm{e}} \left[F\left(\widetilde{L}_{(T)}\right) - \widetilde{L}_T f_{(T)}(0,0) - \sum_{t \in \overline{\mathcal{T}}} X_t^{\mathrm{e}} \right] \\ &> \beta_t^{\mathrm{e}} \left[F\left(\overline{L}\right) - L^{\mathrm{e}} f_{(T)}(0,0) - \sum_{t \in \overline{\mathcal{T}}} X_t^{\mathrm{e}} \right] \\ &= \beta_t^{\mathrm{e}} Y^{\mathrm{e}} = X_t^{\mathrm{e}} - L_t^{\mathrm{e}} f_{(T)}(0,0) \geq 0, \end{aligned}$$

where the strict inequality follows from (40) and the final line uses the feasibility condition $\sum_{t=1}^{T} X_t^{\mathrm{e}} = F(L^{\mathrm{e}})$. The final inequality is a consequence of voluntary participation and implies that $X_t \geq 0$, so the constructed allocation is feasible. It also dominates the original allocation since $\psi_t(X_t'', L_t'') > \psi_t(X_t^{\mathrm{e}}, L_t^{\mathrm{e}})$ for $t \in \mathcal{T}$ and $\psi_t(X_t'', L_t'') = \psi_t(X_t^{\mathrm{e}}, L_t^{\mathrm{e}})$ for $t \in \overline{\mathcal{T}}$; this is the promised contradiction.

(iii) First, note that,

$$(1-\lambda) F\left(\widetilde{L}_T\right) = F\left(\widetilde{L}_T\right) - \widetilde{L}_T f_{(T)}(0,0) > 0$$

and set

$$\mu_t = \frac{X_t^{\mathrm{e}} - L_t^{\mathrm{e}} f_{(t)}(0,0)}{F\left(\widetilde{L}_T\right) - \widetilde{L}_T f_{(T)}(0,0)} \text{ for } t = 1, \ldots, T.$$

It follows from voluntary participation, feasibility and the result in (ii) that all $\mu_t \geq 0$ and $\sum_{t=1}^{T} \mu_t = 1$. The equivalence (19) follows from Proposition 4.1.

7.13 Proof of Proposition 5.3

For any $t, t' = 1, \ldots, T$,

$$\begin{aligned} &\frac{\psi_t(X_{t'}, L_{t'})}{\nu_{t'}} \\ =\;& \frac{\partial u_{(t)}(0,0)}{\partial x} \frac{\{\lambda \widetilde{\sigma}_{t'} + (1-\lambda)\mu_{t'}\} F\left(\widetilde{L}_T\right) - \widetilde{\sigma}_{t'} \widetilde{L}_T f_{(t)}(0,0)}{\nu_{t'}} \\ \leq\;& \frac{\partial u_{(t)}(0,0)}{\partial x} \frac{\{\lambda \widetilde{\sigma}_t + (1-\lambda)\mu_{t'}\} F\left(\widetilde{L}_T\right) - \widetilde{\sigma}_t \widetilde{L}_T f_{(T)}(0,0)}{\nu_{t'}} \\ =\;& \frac{\partial u_{(t)}(0,0)}{\partial x} (1-\lambda) F\left(\widetilde{L}_T\right), \end{aligned}$$

using $f_{(t)}(0,0) \geq f_{(T)}(0,0)$ for all t and $\mu_{t'} = \nu_{t'}$ (proportional type weights). Furthermore, the inequality is actually an equality if $t \in \mathcal{T}$ (which implies $f_{(t)}(0,0) = f_{(T)}(0,0)$) or $t' \in \overline{\mathcal{T}}$ (which implies $\widetilde{\sigma}_{t'} = 0$). The inequalities (20) follow immediately.

7.14 Proof of Proposition 5.4

Recalling that $f_{(t)}(0,0) \geq f_{(T)}(0,0)$ for all t, the asymptotic unanimity lower bound (21) satisfies

$$\begin{aligned} \nu_t \max_L \psi_t(F(L), L) &= \nu_t \frac{\partial u_{(t)}(0,0)}{\partial x} \max_L \left\{F(L) - Lf_{(t)}(0,0)\right\} \\ &\leq \nu_t \frac{\partial u_{(t)}(0,0)}{\partial x} \max_L \left\{F(L) - Lf_{(T)}(0,0)\right\} \\ &= \nu_t \frac{\partial u_{(t)}(0,0)}{\partial x} \left\{F\left(\widetilde{L}_T\right) - \widetilde{L}_T f_{(T)}(0,0)\right\} \\ &= \nu_t \frac{\partial u_{(t)}(0,0)}{\partial x} (1-\lambda) F\left(\widetilde{L}_T\right), \end{aligned} \tag{41}$$

using Lemma 4.6 to justify the third line and Proposition 4.1 for final line. With proportional type weights ($\mu_t = \nu_t$), the right hand side of (41) is equal to $\psi_t\left(\widehat{X}_t, \widehat{L}_t\right)$ where $\left\{\widehat{X}_t, \widehat{L}_t\right\}_{t=1}^{T}$ is the limiting equilibrium type allocation. This proves the required result.