# Asymmetry, Loss Aversion and Forecasting

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## Abstract

Conditional volatility models, such as GARCH, have been used extensively in financial applications to capture predictable variation in the second moment of asset returns. However, with recent theoretical literature emphasising the loss averse nature of agents, this paper considers models which capture time variation in the second lower partial moment. Utility based evaluation is carried out on several approaches to modelling the conditional second order lower partial moment (or semi-variance), including distribution and regime based models. The findings show that when agents are loss averse,

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there are utility gains to be made from using models which explicitly capture this feature (rather than trying to approximate using symmetric volatility models). In general direct approaches to modelling the semi-variance are preferred to distribution based models. These results are relevant to risk management and help to link the theoretical discussion on loss aversion to emprical modelling.

*Keywords*: Asymmetry, loss aversion, semi-variance, volatility models.

*JEL Reference*: G10, C22.

# 1 Introduction

Recent theoretical and empirical advances in financial economics have emphasised that preferences and return processes exhibit asymmetry. In terms of preference asymmetry Campbell and Cochrane (1999) show how the behaviour of the aggregate stock market can be explained when agents are sensitive to consumption falling below a previous 'habit' level. Experimental evidence from Fishburn and Kochenberger (1979) or Kahneman and Tversky (1979) has long suggested an inherent asymmetry in preference structures albeit in a non-expected utility framework. Barberis et al. (2001) develop an asset pricing model incorporating the key features of Kahneman and Tversky's prospect theory. Sensitivity of agents to negative or below target outcomes is reviewed most recently in Rabin and Thaler (2001), however, it should be noted that even Markowitz (1959) was aware of the possibility that agents could be loss averse.

In terms of asset return outcomes, empirical research by Perez-Quiros and Timmermann (2000) has presented evidence to show that asymmetry in asset returns may be linked to the economic cycle, having argued that restricted access to credit at the end of a period of expansion or beginning of a recession may skew the payoffs for small capitalisation stocks. Negative skewness during downturns in the economy is also apparent in the commercial property market (Bond and Patel 2003) and emerging equity markets (Bekaert et al. 1998).

Given both the possibility of asymmetry in preferences and asymmetry in returns this paper examines how volatility forecasts can be developed in the presence of these characteristics. In particular, the measure of volatility chosen in this paper is a second order lower partial moment. This has the advantage that it is conceptually similar to the variance, however, unlike the variance, it measures squared deviations below a prespecified target rate or benchmark. Such a measure was used by Markowitz (1959) in his earlier work on the construction of portfolios for loss averse investors. Markowitz used the term semi-variance to describe the second order lower partial moment and that term will also be used in this paper[1]. However to distinguish the conditional semi-variance measure, which may be of more use in portfolio or risk management applications, from the unconditional, the models discussed in this paper are referred to as dynamic semi-variance models.

In order to assess the potential of dynamic semi-variance models in financial management, the models introduced in the following section are applied to data from three emerging market stock markets. In each case the models are estimated recursively over an evaluation horizon, forecasts of expected returns and semi-variance are generated and used to optimise a two asset portfolio (including equities and a risk-free asset) using a mean–semi-variance framework for an arbitrary set of utility function parameters. The semi-variance forecast and subsequent portfolio optimisation are performed for a one month ahead interval, although the evaluation interval extends to two years. The focus on a one period model

[1] Satchell and Sortino (2001) discuss the widespread use of semi-variance in financial applications.

comes about primarily because of the difficulty in developing multi-period forecasts for two of the semi-variance models. In this regard the paper differs from recent work by, for example, Balduzzi and Lynch (1999) or Campbell and Viceira (1999), which emphasize portfolio allocation using dynamic multi-period portfolio weights rather than myopic or one period weights. This is not the draw back that it may first seem. For example, Ang and Bekaert (1999) find that the multi-period portfolio weights in their application are very similar to myopic portfolio weights. Also, interest in a one period setting is still quite active, in part recognising the technical difficulties associated with multi-period portfolio allocation, see for instance the work by Pesaran and Timmermann (1995), and West, Edison and Cho (1993). Whilst also recognising doubts as to the relevance of multi-period decision making to what actually goes on in financial institutions (see Simon 1947). The current paper, however, does not incorporate the research in return predictability using fundamental and financial variables, such as dividend yield, treasury bills or gilt yield-equity yield ratio. Of course, incorporating such fundamental variables into the conditional mean equation would provide for an interesting extension of this work, but this is left for further research. The main contribution of this paper lies in evaluating the potential of time varying estimates of downside risk in financial management applications and providing a means of discriminating between the three models proposed, rather than trying to examine the issue of predictability in emerging market asset returns *per se*.

The issue of comparing the performance of the models is not without difficulty. Because the true conditional semi-variance of a series is never observed standard statistical measures of goodness of fit need to be used cautiously. Instead the alternatives centre on either an economic based criteria or a more advanced statistical procedure. The use of economic (including utility based) criteria in distinguishing between competing inputs into the asset allocation process is widespread and has been used to examine issues such as the performance of GARCH models (West, Edison and Cho 1993), non-linear trading rules (Satchell and Timmermann 1995) and return predictability (Kandel and Stambaugh 1996). In such a study attention usually focuses on the portfolio or utility outcomes for each model. Comparisons can then be made in terms of the final wealth of an agent and sometimes include a measure of certainty equivalence, such as how much they may have paid to have had access to competing models.

An alternative to the economic criteria approach is to use a statistical measure to discriminate between the models. As stated earlier, it is difficult to use, say, a standard mean-square error approach in comparing the models as the true semi-variance is not observed. Similar problems exist in comparing the statistical performance of ARCH models in that the true conditional population variance is also unobserved. The solution to this issue in the ARCH literature is usually to compare the closeness of the conditional variance estimates with the squared departures of the observed outcome from the conditional mean (though Andersen and Bollerslev 1998 have noted the problem of interpretation in such tests). In the case of semi-variance models this issue is further complicated because if the observation does not fall in the downside state then no observed outcome can be used to assess the conditional estimate of downside risk. Obviously, just because a downside outcome is not observed, or *ex post* it was observed at zero, does not mean that the population measure of downside risk in that period was zero[2]. Where a statistical procedure may be useful for comparing the models is the recent work of Diebold, Gunther and Tay (1998) who have proposed a methodology for comparing the fit of the entire conditional distribution of a series. Such a methodology could at least be used to determine which of the suggested conditional distributions more realistically describes the series. Then it could be assumed that the model which best fits the data must also provide the best measure of downside risk if the risk measure is estimated directly from the density function[3,4]. In the current study the economic based criteria is used to discriminate between the performance of the semi-variance models. Because this paper provides an illustration of the use of semi-variance in a financial management application it seems only natural to then examine the outcomes of

[2]Indeed, for most statistical processes of interest in finance it is generally taken that, if $x_{t+1}$is returns at time $t+1$,

$$E_t\left(x_{t+1}^2 \mid x_{t+1} < 0\right) > 0 \quad \text{a.s.}$$

[3]Diebold, Gunther and Tay (1998, p.867) state that *regardless of the loss function, we know that the correct density is weakly superior to all forecasts, which suggests that we evaluate forecasts by assessing whether the forecast density is correct.* Granger and Pesaran (2000a) arrive at a similar conclusion.

[4]Perhaps a weighted measure of goodness of fit which emphasises the downside section of the conditional distribution is appropriate in this instance.

the asset allocation decisions as a basis to distinguish between the models.

The following section briefly reviews the different models proposed for modelling conditional semi-variance. The optimisation of a two asset portfolio where preferences are described by a mean–semi-variance utility function is outlined in Section 3. Section 4 describes the data used in the evaluation and also the results obtained. The final section, Section 5, provides a conclusion to this paper.

## 2 Conditional Semi-variance Models

In this section the three conditional semi-variance models to be evaluated are described. Two of the models have been suggested as a means to capture the time series behaviour of the conditional expectation

$$sv_t\left(x_{t+1}\right) = \int_{-\infty}^{\tau} x_{t+1}^2 f\left(x_{t+1}|\Omega_t\right) dx_{t+1} \tag{1}$$

where the returns on a portfolio are represented by the series $\{x_t\}_{t=1}^{T}$, $\tau$ is a pre-defined target rate of return and $\Omega_t$ is the information set available up to the current period. The expression in equation (1) above will be recognised as the conditional lower partial moment of order two. The third model adopts a regime based approach in determining the expectation, conditional on a downside observation being observed. To signify the differences in approach between that mentioned above, the conditional expectation developed from the regime model is denoted as

$$\widetilde{sv}_t\left(x_{t+1}\right) = \int_{-\infty}^{\tau} x_{t+1}^2 f\left(x_{t+1}|\Omega_t, x_{t+1} < \tau\right) dx_{t+1}. \tag{2}$$

Finally a baseline model is introduced. This model consists of a simple linear projection of the lower partial moment constructed by using a random variable $s_t$, where $s_t$ is given by[5]

$$s_t = x_t^2 I_t \tag{3}$$

with the indicator variable, $I_t$, defined as

$$I_t \equiv \begin{cases} 1 & \text{if} \quad x_t \leq \tau \\ 0 & \text{if} \quad x_t > \tau, \end{cases} \tag{4}$$

and the conditional semi-variance as

$$sv_t\left(x_{t+1}\right) = E_t\left[s_{t+1}|\Omega_t\right]. \tag{5}$$

In each case the models outlined below, with the exception of the baseline model, are set within the ARCH class of models. While the issue of modelling the second lower partial moment is somewhat different from that of modelling the second moment, there is an obvious relationship and this will become evident when the models are described. A second reason for using the ARCH framework is that it is well known and widely recognised as well as being reasonably straightforward to implement using maximum likelihood techniques.

[5] The authors would like to thank Adrian Pagan for suggesting this as a benchmark.

## 2.1 Baseline Model

The baseline model is a linear projection of the semi-variance for the risky asset which is combined with a projection of the conditional mean to provide the conditional expectations used in determining the optimal portfolio weights. The model is set out, using $\Phi(L), \Theta(L)$ and $\Psi(L)$ to denote lag polynomials, as

$$\begin{aligned} x_{t+1} &= a_0 + \Phi(L) x_t + \Theta(L) e_{t+1} \\ s_{t+1} &= \alpha + \Psi(L) s_t + u_{t+1} \end{aligned}$$

where $s_{t+1}$ is given by equation (3), $\Psi(L)$ is a lag polynomial whose coefficients are non-negative and $e_t$ and $u_t$ are assumed to be uncorrelated *iid* error terms. Each equation is estimated separately by OLS. While more sophisticated estimation procedures could be employed in estimating the model, the purpose of the baseline model is to provide a benchmark which encapsulates the simplest technology available to the investor. Models of increasing sophistication are then estimated and the results compared to the benchmark in order to provide an indication of the value of the more complex approaches. The models discussed below will build on the benchmark model by considering joint estimation of the first two moments and also the inclusion of skewness (including time varying skewness) in the conditional density function. The first of these is the GARCH semivariance (GARCH-SV) model which considers the joint estimation of the first two conditional moments of the series.

## 2.2 GARCH-SV Model

The GARCH semi-variance (GARCH-SV) model is a regime based model which captures the volatility of a series when it is in a downside state. It is based on a self-exciting, threshold autoregressive ARCH (SETAR-ARCH) specification. In this way we considered it as a conditional measure (conditioned on a downside outcome) rather than an unconditional population measure of downside risk. The downside risk variable will record a value of zero if the model is not in the downside state, even though the expectation of the true population measure of downside risk is unlikely to be zero. The model presented in this paper is a GARCH-SV model (with differential impact), which allows for past lags of upside and downside risk to affect the current level of risk in a different manner.

$$
\begin{aligned}
x_{t+1} &= \left(\widetilde{sv}_{t+1}^{-}(x_{t+1})\right)^{1/2} z_{t+1} \qquad for \quad x_{t+1-d} < 0 \\
&= \left(\widetilde{sv}_{t+1}^{+}(x_{t+1})\right)^{1/2} z_{t+1} \qquad for \quad x_{t+1-d} \geq 0 \\
\widetilde{sv}^{-}(x_{t+1}) &= \alpha_0^{-} + \sum_{i=0}^{q} \alpha_{i+1}^{-} x_{t-i}^2 + \sum_{j=0}^{p} \beta_{j+1}^{-} s\tilde{v}^{-}(x_{t-j}) + \sum_{j=0}^{p} \phi_{j+1}^{-} s\tilde{v}^{+}(x_{t-j}) \quad \text{for } x_{t-d} < 0 \\
&= 0 \qquad \text{for } x_{t-d} \geq 0 \\
\widetilde{sv}^{+}(x_{t+1}) &= \alpha_0^{+} + \sum_{i=0}^{q} \alpha_{i+1}^{+} x_{t-i}^2 + \sum_{j=0}^{p} \beta_{j+1}^{+} s\tilde{v}^{-}(x_{t-j}) + \sum_{j=0}^{p} \phi_{j+1}^{+} s\tilde{v}^{+}(x_{t-j}) \quad \text{for } x_{t-d} \geq 0 \\
&= 0 \qquad \text{for } x_{t-d} < 0
\end{aligned}
$$

where $z_t \sim N(0,1)$ and $d$ is a delay parameter usually taken to equal one.

## 2.3 GARCH-Double Gamma Model

The first of the density based models uses the standard ARCH framework, and allows for the conditional density function of the innovation term to take the shape of a double gamma distribution (hereafter GARCH-DG model). This model extended, to a conditionally heteroscedastic framework, the earlier work of Knight, Satchell and Tran (1995), who suggested the density function as a means of capturing asymmetry in the returns of an asset. The model takes the form

$$
\begin{aligned}
x_{t+1} &= \sigma_{t+1} z_{t+1} \\
\sigma_{t+1}^2 &= \alpha_0 + \sum_{i=0}^{q} \alpha_{i+1} x_{t-i}^2 + \sum_{j=0}^{p} \beta_{j+1} \sigma_{t-j}^2
\end{aligned}
$$

with the conditional density function given by

$$
\begin{aligned}
f\left(z_t | \Omega_t, \lambda_1, \lambda_2, \alpha_1, \alpha_2\right) &= \frac{\lambda_1}{\Gamma\left(\alpha_1\right)} z_t^{(\alpha_1 - 1)} \exp\left(-\lambda_1 z_t\right) \qquad \text{for} \qquad z_t > 0 \\
&= \frac{\lambda_2}{\Gamma\left(\alpha_2\right)} \left(-z_t\right)^{(\alpha_2 - 1)} \exp\left(-\lambda_2 z_t\right) \qquad \text{for} \qquad z_t \leq 0.
\end{aligned}
$$

The moment restrictions

$$
\begin{aligned}
\lambda_1 &= \frac{p \alpha_1 \lambda_2}{(1-p)\,\alpha_2} \\
\lambda_2 &= \left\{ \alpha_2 (1-p) \left[ \frac{(\alpha_1 + 1)\,\alpha_2 (1-p)}{p \alpha_1} + (\alpha_2 + 1) \right] \right\}^{\frac{1}{2}}
\end{aligned}
$$

imposed to ensure $z_t \sim DG(0,1)$. Making use of the known form of the moments of the double gamma distribution, an analytical expression for the conditional semi-variance of the portfolio returns can be derived from the parameters of the density (under the assumption that $\tau = 0$ and $E_t [x_{t+1}] = 0$), such that

$$sv(x_{t+1}) = \sigma_{t+1}^2 (1-p) \frac{\alpha_2 (\alpha_2 + 1)}{\lambda_2^2} \tag{6}$$

where

$$p = \Pr[(z_t > 0) | \Omega_t].$$

One advantage of using the double gamma distribution is that it allows for the downside and upside of the conditional distribution to be modelled separately, and this is consistent with the risk modelling framework presented by Fishburn (1984). As discussed in Bond (2000); the model also allows for the semi-variance to be readily calculated from the estimated parameters of the distribution (as given by equation 6). A disadvantage of the double gamma density is the inability of the model to adequately capture probability mass around the origin. The model also imposes a bi-modal shape on the conditional density, and is best used in circumstances where the conditional mean is close to or equal to zero for the semi-variance estimate to hold. The form of the conditional asymmetry is also assumed to be constant at each point in time. One way of overcoming this last limitation is to consider a model with time varying skewness and this is discussed in the next section.

### 2.4 GARCH-Skewed t Model

In contrast to the previous model where an analytical expression for the semi-variance existed, the measure of semi-variance derived from the model we now discuss is calculated by numerical integration of the conditional density function. The density function chosen is the $t$ distribution, with time varying skewness, proposed by Hansen (1994) (hereafter referred to as GARCH-ST). Despite the need to employ numerical integration techniques to calculate the semi-variance, this model has a number of advantages over the double gamma model listed above. The most appealing feature is that the model can capture time varying skewness in the data, in contrast to the double-gamma model above, which implicitly assumes that conditional skewness is constant. Other advantages are that the distribution is uni-modal and non-zero when evaluated at zero. The semi-variance calculation is also unaffected by the inclusion of a conditional mean equation[6]. The model is

$$\begin{aligned} x_{t+1} &= \sigma_{t+1} z_{t+1} \\ \sigma_{t+1}^2 &= \alpha_0 + \sum_{i=0}^{q} \alpha_{i+1} x_{t-i}^2 + \sum_{j=0}^{p} \beta_{j+1} \sigma_{t-j}^2 \end{aligned}$$

with the conditional density of the innovation term given by

$$\begin{aligned} f\left(z_{t+1} | \Omega_t, \eta_t, \lambda_t\right) &= b_t c_t \left(1 + \frac{1}{\eta_t - 2} \left(\frac{b_t z_t + a_t}{1 - \lambda_t}\right)^2\right)^{-\frac{(\eta_t+1)}{2}} & z_t < -\frac{a_t}{b_t} \\ &= b_t c_t \left(1 + \frac{1}{\eta_t - 2} \left(\frac{b_t z_t + a_t}{1 + \lambda_t}\right)^2\right)^{-\frac{(\eta_t+1)}{2}} & z_t \geq -\frac{a_t}{b_t} \end{aligned}$$

where

[6]Bond (1999) discusses how the model may be adapted to allow for a non-zero conditional mean.

$$
\begin{aligned}
a_t &= 4\lambda_t c_t \left(\frac{\eta_t - 2}{\eta_t - 1}\right) && (7)\\
b_t^2 &= 1 + 3\lambda_t^2 - a_t^2 && (8)\\
c_t &= \frac{\Gamma\left(\frac{\eta_t+1}{2}\right)}{\sqrt{\pi\left(\eta_t - 2\right)}\Gamma\left(\frac{\eta_t}{2}\right)} && (9)
\end{aligned}
$$

$-1 < \lambda_t < 1$, and $\lambda_t$ is a parameter to control for the skewness of the distribution. If $\lambda_t = 0$, the distribution will collapse to the standardised $t$ distribution with $\eta_t$ degrees of freedom. The semi-variance can be derived from the numerical integration of the estimated density function

$$
sv\left(x_{t+1}\right) = \sigma_{t+1}^2 \int_{-\infty}^{\tau} z_{t+1}^2 f\left(z_{t+1} | \Omega_t, \eta_t, \lambda_t\right) dz_{t+1}. \tag{10}
$$

# 3 Portfolio Weights in a Two Asset Mean–Semi-variance Framework

The mean–semi-variance utility function, which was first suggested by Markowitz (1959), takes the form

$$
V\left(R_{t+1}\right) \equiv R_{t+1} - b \min\left[R_{t+1} - \tau, 0\right]^2 \tag{11}
$$

where $R_t$ is the portfolio return at time $t$, $\tau$ is the pre-defined target rate of return, which may or may not be time varying, $b$ is a parameter with $b \geq 0$. The above utility function is presented in returns form to maintain consistency with Markowitz's original work, it also a convenient form of presentation for the asset allocation example provided in the next section. However, it is more appealing to consider utility in terms of total wealth or consumption rather than purely as a function of returns[7]. To maintain consistency with the essence of Markowitz's loss aversion utility function, yet recognise that wealth is an important determinant of utility, definition (1) re-expresses (11) and introduces wealth into the analysis.

**Definition 1** *Let the Bernoulli utility function of an individual investor be represented by a function* $U$*:*$\Re^1 \longrightarrow \Re^1$*, which takes the form*

$$U\left(W_{t+1}\right) \equiv W_{t+1} - \frac{b}{2}\min\left[W_{t+1} - W_t\left(1+\tau\right), 0\right]^2 \tag{12}$$

*where* $W_t$ *is the level of wealth in period* $t$*,* $b$ *is a parameter such that* $b > 0$ *and* $\tau$ *is a pre-defined target rate of return.*

## 3.1 Consistency with Expected Utility Theory

It is not known if the utility function (12) is consistent with the expected utility theory. However, Fishburn (1977, theorem 2) provides a theorem for a general class of two-piece utility functions. In the mean–semi-variance model, with the utility function described by

[7] It is noted that some researchers claim individuals evaluate their decisions based on returns rather than wealth (see Kahneman and Tversky 1979).

Fishburn, the measure of risk is given by the deviation below the risk free wealth level, hence

$$\varphi(t-x) = [W_{t+1} - W_t(1+\tau)]^2 \qquad \text{for} \quad W_{t+1} \leq W_t(1+\tau),$$

where $\varphi(y)$ for $y \geq 0$ is a nonnegative nondecreasing function in $y$ with $\varphi(0) = 0$ that expresses the 'riskinesss' of getting a return that is $y$ units below the target (Fishburn 1977, p 118). The next section will show how this risk measure is related to the semi-variance of returns. The expected utility hypothesis represents the mainstream of the decision making under uncertainty literature developed by von Neumann and Morgenstern (1953). While it is noted that the mean–semi-variance approach to downside risk appears to share much in common with the 'Prospect Theory' of Kahneman and Tversky (1979), in particular an emphasis on myopic loss aversion, the approach adopted here is consistent with expected utility theory, while that of Kahneman and Tversky is not[8].

## 3.2 Implementing the Mean–Semi-variance Model

Returning to the implementation of the mean–semi-variance model, as the data available for evaluation are in returns form, the utility function may be re-expressed as follows. Define

$$W_{t+1} \equiv W_t(1+R_{t+1}) \tag{13}$$

[8]Further, the mean–semi-variance utility function displays risk aversion below the target rate of wealth and is risk neutral above target. This is in contrast to the loss function used in prospect theory, which emphasises convex preferences below the target rate of return (risk loving behaviour) and risk averse behaviour above.

where $R_{t+1}$ is the return on a portfolio consisting of a risky asset and a risk free asset. Substituting into equation (12) gives

$$U\left(W_{t+1}\right) \equiv W_t\left(1+R_{t+1}\right)-\frac{b}{2} W_t^2 \min \left[R_{t+1}-\tau, 0\right]^2, \tag{14}$$

with the similarities between (11) and (14) clearly evident. Further extension of the utility function (14) is undertaken by allowing for returns to be decomposed into excess returns $(XR)$ and a risk free return $(RF)$. Also let the target rate equal the risk free rate of return, then the expected utility function is given by

$$E_t\left[U\left(W_{t+1}\right)\right]=W_t\left(1+RF_{t+1}\right)+W_t \omega E_t\left[XR_{t+1}\right]-\frac{b}{2} W_t^2 \omega^2 sv_t\left[XR_{t+1}\right]. \tag{15}$$

Note that the conditional expectation of the term $\min\left[XR_{t+1}, 0\right]^2$ is the semi-variance and for simplicity is denoted by $sv_t\left[XR_{t+1}\right]$. Expected utility of wealth is maximised at the point where $\omega$, the portfolio weight assigned to the risky asset $XR_{t+1}$, is equal to

$$\omega=\frac{E_t\left[XR_{t+1}\right]}{bW_t sv_t\left[XR_{t+1}\right]} \tag{16}$$

The following proposition is made regarding the general solution to this problem.

**Proposition 1** *For the one-period maximisation of the expected utility function* $E_t\left[U\left(W_{t+1}\right)\right]$, *given by equation* (15), *the value of* $\omega$, *the portfolio weight assigned to the risky asset, is*

1. $\omega=0$ if $E_t\left[XR_{t+1}\right] \leq 0$
2. $\omega=1$ if $E_t\left[XR_{t+1}\right] \geq bW_t sv_t\left[XR_{t+1}\right]$

3. $\omega = \frac{E_t(XR_{t+1})}{bW_t sv_t[XR_{t+1}]}$ if $0 \leq E_t[XR_{t+1}] \leq bW_t sv_t[XR_{t+1}]$.

**Proof.** This results follows from standard constrained optimisation of equation (15).

This result is quite intuitive as it implies no allocation for the risky asset if the excess return on the asset is expected to be below zero and full allocation if the expected return ($E_t[XR_{t+1}]$) is above the penalty weighted risk term ($bW_t sv_t[XR_{t+1}]$). Note that in the case of the GARCH-SV model, that conditions on the regime of the model, the appropriate conditions are listed below in corollary 1.

**Corollary 1** *For $E_t[XR_{t+1}] > 0$ and assuming a GARCH-SV model of the form*

$$\begin{aligned} sv_t(XR_{t+1}) &= \alpha_0^- + \sum \alpha_t^- e_t^2 + \sum \beta_1^- sv^-(XR_t) + \sum \gamma_1^- sv^+(XR_t) \quad & XR_t < 0 \\ sv_t(XR_{t+1}) &= 0 & XR_t \geq 0 \end{aligned}$$

*the value of $\omega$ which maximises the utility function $(15)$, above, is:*

1. *if $XR_t \geq 0$ the maximisation of expected utility will occur at the boundary solution $\omega = 1$.*

2. *if $XR_t < 0$ the results of Proposition 1 will apply.*

**Proof.** From proposition 1, $sv_t[XR_{t+1}] = 0$ when $XR_t \geq 0$, hence $\omega = 1$ when $E_t[XR_{t+1}] \geq 2bW_t sv_t[XR_{t+1}]$. As $sv_t[XR_{t+1}] = 0, E_t[XR_{t+1}] \geq 0$ and $\omega = 1$.

### 3.3 Setting Parameter Values for Evaluation

In performing the utility based comparison of the dynamic semi-variance models it is necessary to select a value of the parameter of the risk term in the utility function above. While any specification of a parameter for a utility function used in numerical evaluation will be subject to claims of arbitrariness, the parameter is chosen by specifying the level of relative risk aversion (RRA) and then calculating the corresponding $b$ parameter. This method of parameter specification is consistent with the approach taken by West, Edison and Cho (1993). The values of RRA selected by West, Edison and Cho were one and 10, and those levels of RRA will also be used in this study along with the very extreme level of RRA equal to 30[9].

For each country studied the initial wealth endowment is taken to be the equivalent in local currency of GBP 50,000. This may not necessarily be the most realistic endowment level and low values of the portfolio may result in excessively high transactions costs[10], as a proportion of the size of the portfolio. However, these complications are placed to one side as the size of initial endowment is unlikely to be influential to the final results.

[9]Such high levels of risk aversion are observed in the literature on habit persistence and the equity premium puzzle (e.g. Campbell and Cochrane 1999, Mehra and Prescott 1985).

[10]Note that transaction costs are not considered in this study.

## 3.4 Using Utility as a Model Selection Criteria

Utility functions have been used in financial econometrics for two distinct but closely related purposes. In the first instance, they have been used to evaluate the cost of a sub-optimal strategy. This is done in the following way: given an optimal investment strategy, $\hat{x}_0$, and a sub-optimal strategy $\hat{x}_s$, compute $V_0 = E\left[U\left(W_0\left(1+\hat{x}_0'\tilde{r}\right)\right)\right]$ and $V_s = E\left[U\left(W_0\left(1+\hat{x}_s'\tilde{r}\right)\right)\right]$ where $W_0$ is initial wealth, $U$ is the Bernoulli utility function, $V_0$ and $V_s$ are the values of the von Neumann Morgenstern utility function and $\tilde{r}$ is the vector of actual rates of return. It is now possible to ask questions such as: what level of wealth $W_s$ will just compensate for the sub-optimal choice $\hat{x}_s$? That is, choose $W_s$ such that

$$V_0 = E\left[U\left(W_s\left(1+\hat{x}_s'\tilde{r}\right)\right)\right]. \tag{17}$$

Alternatively, what is the certainty-equivalent cost '$\pi$' of $x_s$? That is,

$$V_0 = E\left[U\left(W_0\left(1+\hat{x}_s'\tilde{r}-\pi\right)\right)\right]. \tag{18}$$

In (17), the cost is measured in wealth terms ($W_0$), whilst in (18), it is measured as a rate of return.

Ang and Bekaert (1999) assume Markov-switching returns to compute their optimal portfolio weights versus ignoring this information and solving a myopic (one-period) expected utility calculation. They assume power utility and a T period horizon with all proceeds re-invested. Campbell and Viceira (1999) evaluate sub-optimal strategies, in the context of multi-period Epstein-Zin utility functions, by computing the percentage loss of the optimal value function, i.e. $(V_0 - V_s)/V_0 \times 100$. McCulloch and Rossi (1990) (hereafter MR) evaluate exponential utility with normal returns to arrive at optimal weights $\hat{x}_0$ which lie on the mean–variance frontier, which in turn leads to a mean excess return $\mu^*$ and variance ${\sigma^*}^2$. MR then computed the impact of positive $a's$ in a linear factor model by calculating $ce(\alpha)$ i.e. $ce$ with an alpha vs $ce(0)$, $ce$ without an alpha. They employed a Bayesian analysis with an exact expression for the posterior of $a$. From this an exact expression for the certainty equivalence distribution as alpha varies can be computed. Whilst their calculations employed two different pdf's one is nested in the other in that $\alpha = 0$ is, in a classical sense a nested relation of a more general model. Kandel and Stambaugh (1996) carried out a similar calculations but they used the same density in both cases.

West, Edison and Cho (1993) propose an alternative approach to selecting between models. Their paper is the closest to this one in terms of the motivation, as they were comparing the performance of different models of conditional volatility. The means of comparison is based on an average utility measure for each model. Once the average utility for each model is calculated the cost of using a sub-optimal model (model $m$) in contrast to using the optimal model (model 1) is presented as a per period fee.

### 3.5 Implementation of the Utility Comparison

In this paper, the evaluation of the semi-variance models is undertaken using an expected utility approach following the work of West, Edison and Cho (1993) described above. While there are many alternative ways to present such an analysis, the work of West, Edison and Cho has a neat congruence with the construction of sample moments in statistics. West (1996) has also provided a stronger theoretical basis to such utility comparison of models. The proposition below outlines how the evaluation is undertaken in the current example.

**Proposition 2** *For a von Neumann Morgenstern lower partial moment utility function of the form of equation (12), computation of the average utility arising from use of a model m of the first moment and second lower partial moment is*

$$\begin{aligned}\overline{U}_m &= (T-N+1)^{-1}\sum_{t=N}^{T} W\left(1+RF_t\right)\left(1+\frac{\widehat{\mu}_{m,t+1}}{RRA\,\widehat{sv}_{m,t}\left[XR_{t+1}\right]}\right. \\ &\quad \left.\left(\mu_{t+1}-\frac{1}{2}\frac{\widehat{\mu}_{m,t+1}}{\widehat{sv}_{m,t}\left[XR_{t+1}\right]}sv_t\left[XR_{t+1}\right]\right)\right) \qquad (19)\end{aligned}$$

| | | |
|---|---|---|
| *where* | $\widehat{\mu}_{m,t+1}$ | *is an estimate of the conditional mean from model m* |
| | $\mu_{t+1}$ | *is the population conditional mean of excess returns, which is replaced by the ex post value of* $XR_{t+1}$ |
| | $\widehat{sv}_{m,t}\left[XR_{t+1}\right]$ | *is an estimate of the conditional semi-variance from model m* |
| | $sv_t\left[XR_{t+1}\right]$ | *is the population estimation of conditional semi-variance, replaced for evaluation by* $\min^2\left[XR_{t+1},0\right]$ |
| | $W$ | *is a constant level of wealth* |
| | $T$ | *is the last observation in the sample* |
| | $N$ | *is the number of ex post observations in the forecast evaluation.* |

**Proof.** See Appendix.

The above proposition applies when there are no restrictions on the portfolio weight assigned to the risky asset. When short-sale and borrowing constraints are applied, proposition 3 in the appendix is used.

# 4 Utility Based Comparison and Results

## 4.1 Data and Format of Study

Data from three emerging market countries are used in this evaluation. The three countries, Singapore, Malaysia and Taiwan, were chosen primarily because of the availability of a suitably long data series (for both equities and a risk free rate). Data from emerging market countries provide a particularly interesting illustration as the use of semi-variance as a measure of risk is most efficient when the conditional distribution of returns is skewed (Bond and Satchell 2002). It is strongly expected that such conditions will be found in emerging market equity data (see for instance Bekaert et al. 1998 or Hwang and Satchell 1999). The summary statistics for monthly excess returns are shown in Table A1.

The Kuala Lumpur Composite Index is used to represent a risky asset class in Malaysia. Monthly returns are available over the period February 1986 to July 1999. Returns are calculated from the value of the index at the end of each month (this is the same for all series). The one month deposit rate is chosen as the cash alternative. For Singapore, the Singapore All Share Index is used along with the interbank one month rate to represent the return on cash. The data on these series covers the period from May 1986 to July 1999. Finally the Taiwan Stock Exchange Price Index is chosen for Taiwan and a 30 day money market rate is used as well. The data extends from February 1986 to July 1999. All data are taken from the Datastream data service.

Certain compromises are immediately apparent in the selection of the dataset. In each case it was difficult to obtain a pure risk free one month cash rate. The nearest cash rate in terms of maturity was chosen although each of these series may contain a credit risk premium which would not be evident in say a Government cash rate. Further difficulties existed in finding a measure of total return in the equity series (that is including dividends in the return calculation). To this end the equity series will only capture returns associated with changing prices.

### 4.2 Results

The results for each country are presented in Tables A2 and A3 in Appendix A. Table A2 displays the expected utility value for each model based on the calculation in proposition 3. The difference in the magnitude of the expected utility values between countries is explained by the different levels of initial wealth when measured in local currency units.

The specification of the conditional mean equation was determined on the basis of model selection criteria (Akaike Information Criteria, Schwarz Bayesian Criteria and the Hannan-Quinn Criteria) over the pre-evaluation horizon data frame (the data for out of sample evaluation was not included when the specification of the model was chosen). When there was disagreement between the selection criteria, the specification which was preferred by two rather than all three of the criteria was chosen. This lead to a parsimonious specification for the conditional mean equations for Malaysia and Singapore with a constant term and one and two dummy variables for extreme outlying values respectively. An ARMA(2,2) model was selected for the conditional mean of the Taiwanese data set. A GARCH(1,1) specification was chosen for those models which explicitly capture time variation in the conditional second moment. While information criteria were not applied to the selection of the form for the conditional second moment equation, the GARCH(1,1) specification has been commonly applied to financial data and has been found to perform well in *ex post* forecast evaluation tests (see for instance West, Edison and Cho 1993).

In estimating the models two issues arose. Firstly, the estimation of the double gamma model often proved sensitive to the initial starting values. To overcome this problem an extensive set of starting values were tried until a model which successfully converged in each period of the forecast horizon was obtained. A constraint was also placed on the parameters of the density function to ensure that the density was continuous (that is, $\alpha > 1$). A second convergence difficulty occurred with the GARCH-SV model. Despite extensive application of a range of initial values successful convergence could not be obtained over the final 12 monthly horizons for the Singapore data set. This period coincided with large movements during the financial markets crisis in Asian countries. Even though the other models did converge successfully, the large movements in the series over this time suggest that the simple linear mean equations used in each model may ultimately be an unsatisfactory explanation of returns during this time. In the case of Singapore, because the GARCH-SV model did not converge successfully over the entire forecast horizon, the results for this model are omitted from the main tables, A2 and A3 in the Appendix. However, Tables A4 and A5 provide the results of all models over the first 12 monthly forecast horizons for the Singapore data set.

The models with the highest utility measure are highlighted in bold text in Table A2. From the table the linear model is found to be preferred in five of the nine instances, with the GARCH-ST model preferred in the three Malaysian calculations and the GARCH-SV model preferred in one case of the Taiwanese data. An alternative way of presenting the information is in Table A3, where the calculations show how much additional wealth is required, if a sub-optimal model is used, to provide a level of average utility consistent with that of the optimal model. It is immediately apparent that the issue of model selection is much more important at lower levels of relative risk aversion than at high levels. The differences are often quite marked between the models when RRA is equal to one but the difference largely disappears for RRA equal to 30. This occurs because the allocation to the risky asset declines rapidly as RRA increases.

The probability distribution associated with the measures in Table A2 and A3 are unknown, therefore caution must be exercised when assessing which models are preferred to others. In this case the model with the highest average utility model is preferred but it is noted that the difference between models may not necessarily be significant in the sense of a Neyman-Pearson hypothesis test.

Tables A4 and A5 repeat the analysis for the Singapore dataset over a series of twelve monthly horizons as opposed to twenty four as in the previous two tables. This is done because a satisfactory model could not be obtained for the GARCH-SV model over the entire 24 month horizon. When the results are calculated for all of the models it is found that the GARCH-SV model is preferred. However, caution must be exercised when considering these figures as the calculations are over a much shorter time-frame and individual observations may have a disproportionately large influence on the results.

What can be concluded from these results? No one model was consistently the best in all data sets. The linear model was preferred in five instances (three for Singapore and two for Taiwan), the GARCH-ST was the preferred model for Malaysian data and the GARCH-SV was preferred in the case of high risk aversion in the Taiwanese dataset. The GARCH-SV model was also preferred when only the first 12 monthly forecast horizons were considered in the case of Singapore (when convergence problems prevented the GARCH-SV model being applied across the entire period).

There does not appear to be a relationship between the characteristics of the data (as described by the summary statistics) and the preferred model. Originally it was anticipated that for strongly skewed series, models which explicitly capture skewness (the GARCH-ST and GARCH-DG models), would be found to perform best. From Table A1, the Malaysian and Singapore data sets are found to exhibit a high level of skewness. While the GARCH-ST model is preferred in the case of Malaysia, it is one of the worst performing models for the Singapore data set. Likewise the GARCH-DG model, while not the preferred model, is found to be one of the best performing models for the Singapore dataset and the GARCH-ST is one of the worst.

In spite of the seeming lack of consistency of the results, two conclusions become apparent. There appears to be less return from modelling the entire conditional density of the risky asset to indirectly calculate the risk measure (as in the case of the GARCH-DG or GARCH-ST), than in allowing the risk term to be modelled in a direct fashion (through a linear or regime process). Perhaps this result is unsurprising, given the technical complexity required to estimate density functions that capture departures from symmetry or time variation in higher moments and the errors likely to be induced in the estimation process (through, for example, sensitivity to starting values). It suggests that less complex but robust techniques may ultimately be of most assistance in risk management applications.

A second observation, and one that is also important for risk management, is that in no instance was the GARCH-$t$ model preferred. If an investor has a known aversion to returns falling below a benchmark, it is always in the investor's interest to use a model which explicitly models downside risk rather than trying to approximate downside risk using a fixed proportion of a symmetric risk measure such as variance (or standard deviation). In each data set for each level of risk aversion, the simple linear model of downside risk is (almost) always preferred to the results based on the GARCH-$t$ model.

# 5 Conclusion

This paper has attempted to draw together the models introduced to assess the relative merits of (a) attempting to model a conditional measure of semi-variance and assess its use in a financial management application, and (b) determine which model or approach to modelling semi-variance performs best. Three dynamic semi-variance models were examined in this analysis, where the GARCH-SV, GARCH-DG and GARCH-ST models represent a regime based approach and two distributional approaches, respectively. The GARCH-SV model is a direct attempt to model the lower partial moment, whereas the distributional approaches are indirect in so far as the emphasis of the models is on estimating the parameters of the conditional distribution and then calculating the semi-variance from the distributional information.

In determining the performance of the semi-variance models, a utility based criteria was employed instead of the traditional statistical approach to model selection. The criteria is developed using a recursive portfolio example based on a mean–semi-variance utility function. While it is acknowledged that this approach assesses the complete model (expected returns as well as semi-variance) and cannot be used to purely analyse the semi-variance forecasting performance, it is not necessarily the limitation that it would first appear. As expected returns and volatility of asset returns are intimately linked in financial theory, an assessment which considers all aspects of the model provides information about the applicability of this approach to financial applications.

The recursive portfolio example was carried out using monthly return data on three East Asian stockmarkets. For simplicity, the analysis was limited to two assets, a risky asset represented by the index returns on the national stockmarkets and a cash rate. Over 24 one month horizons each model was used to derive an estimate of expected returns and semi-variance.

In terms of the first goal of this paper, which was to model and assess the usefulness of a conditional measure of semi-variance, it can be concluded that the semi-variance models do indeed appear useful in financial management applications. In all cases at least one measure of conditional semi-variance was preferred to a simple interpolated estimate derived from the variance under the assumption of symmetry of returns. The second goal of this paper was to discriminate between the dynamic semi-variance models discussed. The results presented in Section 4 found that direct measures of downside risk, obtained through either the linear model or regime model (GARCH-SV), were generally preferred to measures derived from the estimated conditional density function (GARCH-ST, GARCH-DG). This is an interesting insight and seems to go against the research outlined in Diebold, Gunther and Tay (1998) which highlighted the increasing importance of density forecasts in financial risk management. Of course in more complex and realistic circumstances their approach could well lead to better results. Further research is required to delineate the precise assumptions that make one modelling technique superior to the other.

# 6 Appendix A – Tables

Table A1
Summary Statistics – Monthly Excess Returns
February 1986 to July 1999

| Statistic | Malaysia | Singapore* | Taiwan |
|---|---|---|---|
| Mean (%) | 0.358 | 0.319 | 0.799 |
| Variance | 98.536 | 51.819 | 172.636 |
| Semi-variance | 56.681 | 29.712 | 86.586 |
| Skewness | -1.242 | -1.262 | -0.481 |
| [p-value] | 0.000 | 0.000 | 0.154 |
| Excess Kurtosis | 6.786 | 6.287 | 2.247 |
| [p-value] | 0.000 | 0.000 | 0.117 |
| Minimum (%) | -57.184 | -39.309 | -46.281 |
| Maximum (%) | 27.570 | 21.300 | 43.580 |

* Data covers period May 1986 to July 1999.

Table A2
Utility Calculation for Recursive Portfolio Exercise ($\overline{\mathrm{U}}_m$)

| | | Semi-variance Model | | | | |
|---|---|---|---|---|---|---|
| Country | RRA | GARCH-t | GARCH-DG | GARCH-ST | GARCH-SV | Linear |
| Malaysia | 1 | 272,321 | 259,363 | 297,482 | 225,769 | 289,265 |
| | 10 | 312,360 | 311,064 | 314,876 | 309,067 | 314,054 |
| | 30 | 315,326 | 314,894 | 316,164 | 315,237 | 315,890 |
| Singapore | 1 | 119,536 | 124,902 | 115,308 | na | 126,437 |
| | 10 | 131,045 | 131,582 | 130,622 | na | 131,736 |
| | 30 | 131,898 | 132,076 | 131,757 | na | 132,128 |
| Taiwan | 1 | 2,301,840 | 2,314,647 | 2,228,550 | 2,073,595 | 2,317,874 |
| | 10 | 2,336,006 | 2,337,286 | 2,328,677 | 2,324,713 | 2,337,609 |
| | 30 | 2,338,536 | 2,338,963 | 2,336,093 | 2,343,315 | 2,339,071 |

Initial levels of wealth used in this analysis are set equal to a local currency value equivalent to GBP 50,000. Conversion rates are as follows: GBP 1 : MYR 6.30, GBP 1 : SGD 2.64, GBP 1 : TWD 46.54.

Table A3
Additional Wealth Required to Obtain Equivalent Expected Utility

| Country and Optimal Model | RRA | For Sub-optimal Semi-variance Model GARCH-t (%) | GARCH-DG (%) | GARCH-ST (%) | GARCH-SV (%) | Linear (%) |
|---|---|---|---|---|---|---|
| Malaysia | | | | | | |
| GARCH-ST | 1 | 9.24 | 14.70 | - | 31.76 | 2.84 |
| GARCH-ST | 10 | 0.81 | 1.23 | - | 1.88 | 0.26 |
| GARCH-ST | 30 | 0.27 | 0.40 | - | 0.29 | 0.09 |
| Singapore[1] | | | | | | |
| Linear | 1 | 5.77 | 1.23 | 9.65 | na | - |
| Linear | 10 | 0.53 | 0.12 | 0.85 | na | - |
| Linear | 30 | 0.17 | 0.04 | 0.28 | na | - |
| Taiwan | | | | | | |
| Linear | 1 | 0.70 | 0.14 | 4.01 | 11.78 | - |
| Linear | 10 | 0.07 | 0.01 | 0.38 | 0.55 | - |
| GARCH-SV | 30 | 0.02 | 0.19 | 0.31 | - | 0.18 |

[1] Convergence problems were encountered with the GARCH-$SV$ model for Singapore. As a satisfactory model could not be estimated, the GARCH-$SV$ model was excluded from the calculations in this instance.

Table A4
Utility Calculation for Recursive Portfolio Exercise ($\overline{\mathrm{U}}_m$)

| Country | RRA | Semi-variance Model GARCH-t | GARCH-DG | GARCH-ST | GARCH-SV | Linear |
|---|---|---|---|---|---|---|
| Singapore | 1 | 92012 | 104524 | 82024 | 128673 | 110551 |
| | 10 | 128476 | 129727 | 127477 | 132415 | 130329 |
| | 30 | 131177 | 131593 | 130844 | 132692 | 131795 |

Table A5

Additional Wealth Required to Obtain Equivalent Expected Utility

(Forecast Horizon Reduced to 12 One Month Periods)

| Country and Optimal Model | RRA | For Sub-optimal Semi-variance Model GARCH-t | GARCH-DG | GARCH-ST | GARCH-SV | Linear |
|---|---|---|---|---|---|---|
| | | (%) | (%) | (%) | (%) | (%) |
| Singapore[1] | | | | | | |
| GARCH-SV | 1 | 39.84 | 23.10 | 56.87 | - | 16.39 |
| GARCH-SV | 10 | 3.07 | 2.07 | 3.87 | - | 1.60 |
| GARCH-SV | 30 | 1.16 | 0.83 | 1.41 | - | 0.68 |

[1] Convergence problems were encountered with the GARCH-SV model for Singapore. The model is estimated over a reduced forecast horizon of 12 one month periods unlike in the tables above which are based on 24 one month forecast periods.

# 7 Appendix B

**Proposition 3** *For a von Neumann Morgenstern lower partial moment utility function of the form of (15), and given short-sales and borrowing are not permitted, computation of the average utility arising from use of a model m of the first moment and second lower partial moment is*

$$\overline{U}_m = (T - N + 1)^{-1} \sum_{t=N}^{T} W\left(1 + RF_t\right) \Psi_m\left(\omega_{t+1}\right) \tag{20}$$

$$\textit{where} \quad \Psi_m\left(\omega_{t+1}\right) \quad = \quad \begin{cases} \left(1 + \frac{\widehat{\mu}_{m,t+1}}{RRA\,\widehat{SV}_{m,t}[XR_{t+1}]}\left(\mu_{t+1} - \frac{1}{2}\frac{\widehat{\mu}_{m,t+1}}{\widehat{SV}_{m,t}[XR_{t+1}]} SV_t\left[XR_{t+1}\right]\right)\right) \\ \text{for } 0 \leq \widehat{\mu}_{m,t+1} \leq bW\,\widehat{SV}_{m,t}\left[XR_{t+1}\right] \\ 1 \\ \text{for } \widehat{\mu}_{m,t+1} \leq 0 \\ \left(1 + RF_{t+1} + \mu_{t+1} - \frac{RRA}{2(1+RF_{t+1})} SV_t\left[XR_{t+1}\right]\right) / \left(1 + RF_{t+1}\right) \\ \text{for } \widehat{\mu}_{m,t+1} \geq bW\,\widehat{SV}_{m,t}\left[XR_{t+1}\right] \end{cases}$$

*and the remaining variables are as described in proposition 2.*

**Proof. Proposition 2:**

From equation (15)

$$\begin{aligned} E_t\left[U\left(W_{t+1}\right)\right] &= E_t\left[W_t\left(1 + RF_{t+1}\right) + \omega_{t+1} W_t XR_{t+1} - \frac{b}{2} W_t^2 \omega_{t+1}^2 \min\left[XR_{t+1}, 0\right]^2\right] \\ &= W_t\left(1 + RF_{t+1}\right) + \omega_{t+1} W_t \mu_{t+1} - \frac{b}{2} W_t^2 \omega_{t+1}^2 sv_t\left[XR_{t+1}\right]. \end{aligned} \tag{21}$$

Replace $E_t\left[XR_{t+1}\right]$ by $\widehat{\mu}_{t+1}$ and $sv_t\left[XR_{t+1}\right]$ by $\widehat{sv}_t\left[XR_{t+1}\right]$ to indicate that the expectations are derived from an underlying model, then maximise expected utility wrt $\omega_{t+1}$

$$\omega_{t+1} = \frac{\widehat{\mu}_{t+1}}{bW_t \widehat{sv}_t\left[XR_{t+1}\right]}. \tag{22}$$

Substituting (22) into (21) and evaluating utility at this optimal point gives

$$U(W_{t+1}) = W_t(1 + RF_{t+1}) + \frac{\widehat{\mu}_{t+1}}{b\widehat{sv}_t[XR_{t+1}]}\left(\mu_{t+1} - \frac{1}{2}\frac{\widehat{\mu}_{t+1}}{\widehat{sv}_t[XR_{t+1}]}sv_t[XR_{t+1}]\right)$$

Recall that $b = RRA/W_t(1 + RF_{t+1})$, therefore

$$U(W_{t+1}) = W_t(1 + RF_{t+1})\left(1 + \frac{\widehat{\mu}_{t+1}}{RRA\,\widehat{sv}_t[XR_{t+1}]}\left(\mu_{t+1} - \frac{1}{2}\frac{\widehat{\mu}_{t+1}}{\widehat{sv}_t[XR_{t+1}]}sv_t[XR_{t+1}]\right)\right)$$

■

**Proof. Proposition 3:** When $\omega_{t+1} = 0$, there is no allocation to the risky asset and utility is simply equal to

$$U(W_{t+1}) = W_t(1 + RF_{t+1}). \tag{23}$$

Under the conditions outlined in propositions (1) and (1), when $\omega_{t+1} = 1$

$$U(W_{t+1}) = W_t\left(1 + RF_{t+1} + \mu_{t+1} - \frac{RRA}{2(1 + RF_{t+1})}sv_t[XR_{t+1}]\right).$$

By denoting a function $\Psi_m(\omega_{t+1})$, such that

$$\Psi_m(\omega_{t+1}) = \begin{cases} \left(1 + \frac{\widehat{\mu}_{m,t+1}}{RRA\,\widehat{sv}_{m,t}[XR_{t+1}]}\left(\mu_{t+1} - \frac{1}{2}\frac{\widehat{\mu}_{m,t+1}}{\widehat{sv}_{m,t}[XR_{t+1}]}sv_t[XR_{t+1}]\right)\right) \\ 1 \\ \left(1 + RF_{t+1} + \mu_{t+1} - \frac{RRA}{2(1+RF_{t+1})}sv_t[XR_{t+1}]\right)/(1 + RF_{t+1}) \end{cases} \tag{24}$$

depending on the conditions given in the proposition and

$$U(W_{t+1}) = W_t(1 + RF_{t+1})\Psi_m(\omega_{t+1})$$

the result in proposition 3 is obtained. ■

## References


[1] Andersen, T.G. and Bollerslev, T. (1998) 'Answering the Skeptics: Yes, Standard Volatility Models Do Provide Accurate Forecasts', *International Economic Review*, 39, pp. 885-905.

[2] Ang, A. and Bekaert, G. (1999) 'International Asset Allocation with Time Varying Correlations', *NBER Working Paper W7056.*

[3] Balduzzi, P. and Lynch, A.W. (1999) 'Transaction Costs and Predictability: Some Utility Calculations', *Journal of Financial Economics*, 52, pp. 47-78.

[4] Barberis, N., Huang, M, and Santos, T. (2001) 'Prospect Theory and Asset Prices', *Quarterly Journal of Economics*, 116, pp. 1-53.

[5] Bekaert, G., Erb, C.B., Harvey, C.R. and Viskanta, T.E. (1998) 'Distributional Characteristics of Emerging Market Returns and Asset Allocation', *Journal of Portfolio Management*, Winter, pp. 102-116.

[6] Bond, S.A. (1999) *Time Varying Skewness and Downside Risk Measurement: A Comparison of Models*, paper presented to the Royal Economic Society Annual Conference, March-April 1999, University of Nottingham, Nottingham.

[7] _____ (2000) *Asymmetry and Downside Risk in Foreign Exchange Markets*, paper presented to the European Financial Management Association Conference, Athens, June 2000.

[8] _____ and Patel, K. (2003) 'The Conditional Distribution of Real Estate Returns: Are Higher Moments Time Varying', *Journal of Real Estate Finance and Economics,* 26, pp 319-339.

[9] _____ and Satchell, S.E. (2002) 'Statistical Properties of the Sample Semi-variance', Applied Mathematical Finance, 9, pp 219-239.

[10] Campbell, J.Y. and Cochrane, J.H. (1999) 'By Force of Habit: A Consumption-Based Explanation of Aggregate Stock Market Behavior', *Journal of Political Economy*, 107, pp. 205-249.

[11] Campbell, J.Y. and Viceira, L.M. (1999) 'Consumption and Portfolio Decisions When Expected Returns are Time Varying', *Quarterly Journal of Economics*, 114, pp. 433-495.

[12] Diebold, F.X., Gunther, T.A., and Tay, A.S. (1998) 'Evaluating Density Forecasts with Applications to Financial Risk Management', *International Economic Review*, 39, pp. 863-883.

[13] Fishburn, P.C. (1977) 'Mean-Risk Analysis with Risk Associated with Below-Target Returns', *American Economic Review*, 67, pp. 116-126.

[14] _____(1984) 'Foundations of Risk Measurement I. Risk as a Probability or Loss', *Management Science*, 30, pp. 396-406.

[15] _____ and Kochenberger, G.A. (1979) 'Two-piece von Neumann - Morgenstern Utility Functions', *Decision Sciences*, 10, pp 503-518.

[16] Granger, C.W.J. and Pesaran, M.H. (2000a) 'A Decision Theoretic Approach to Forecast Evaluation', in W.S. Chan, W.K. Li and H. Tong (eds.) *Statistics and Finance: An Interface*, Imperial College Press, London.

[17] Hansen, B.E. (1994) 'Autoregressive Conditional Density Estimation', *International Economic Review*, 35, pp. 705-730.

[18] Hwang, S. and Satchell, S.E. (1999) 'Modelling Merging Market Risk Premia Using Higher Moments', *International Journal of Finance and Economics*, 4, pp. 271-296.

[19] Kahneman, D. and Tversky, A. (1979) 'Prospect Theory: An Analysis of Decision Under Risk', *Econometrica*, 47, pp. 263-291.

[20] Kandel, S. and Stambaugh, R.F. (1996) 'On the Predictability of Asset Returns: An Asset Allocation Perspective', *Journal of Finance*, 51, pp. 385-424.

[21] Knight, J.L., Satchell, S.E. and Tran, K.C. (1995) 'Statistical Modelling of Asymmetric Risk in Asset Returns', *Applied Mathematical Finance*, 2, pp. 155-172.

[22] Markowitz, H.M. (1959) *Portfolio Selection: Efficient Diversification of Investments,* John Wiley and Sons, New York.

[23] McCulloch, R. and Rossi, P. E. (1990) 'Posterior, Predictive, and Utility-Based Approaches to Testing the Arbitrage Pricing Theory', *Journal of Financial Economics*, 28, pp. 7-38.

[24] Mehra, R. and Prescott, E. (1985) The Euity Premium: A Puzzle', *Journal of Monetary Economics*, 34, pp. 145-161.

[25] von Neumann, J. and Morgenstern, O. (1953) *Theory of Games and Economic Behaviour*, $3^{rd}$ Edition, Princeton University Press, Princeton.

[26] Perez-Quiros, G. and Timmermann, A. (2000) 'Firm Size and Cyclical Variation in Stock Returns', *Journal of Finance*, 55, pp 1229-1263.

[27] Pesaran, M.H. and Timmermann, A. (1995) 'Forecasting Stock Returns. An Examination of Stock Market Trading in the Presence of Transaction Costs', *Journal of Finance*, 50, pp. 1201-1228.

[28] Rabin, M. and Thaler, R.H. (2001) 'Risk Aversion', *Journal of Economic Perspectives*, 15, pp. 219-232.

[29] Satchell, S.E. and Sortino, F. (2001) *Managing Downside Risk*, Butterworth-Heinemann, Oxford.

[30] _____ and Timmermann, A. (1995) 'An Assessment of the Economic Value of Non-linear Foreign Exchange Rate Forecasts', *Journal of Forecasting*, 14, pp. 477-498.

[31] Simon, H.A. (1947) *Administrative Behavior: A Study of Decision-making Process in Administrative Organisations*, Macmillan, New York.

[32] West, K.D. (1996) 'Asymptotic Inference About Predictive Ability', *Econometrica*, 64, pp. 1067-1084.

[33] West, K.D. Edison, H.J., and Cho, D. (1993) 'A Utility Based Comparison of Some Models of Exchange Rate Volatility', *Journal of International Economics*, 35, pp. 23-45.