The Nature of Equilibria under Noncollusive Product Design and Collusive Pricing

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Abstract

It is well known that in a two stage duopoly model of product choice with quadratic transportation cost, the firms locate at the extreme endpoints of the market. This paper examines this model in an infinite horizon setting where in the initial period the firms choose locations and in subsequent periods choose prices. The firms collude in prices and share the profits on the profit possibility frontier. It is shown that under very general conditions, both the firms locating at the center is an equilibrium. It is not necessarily unique and multiple symmetric equilibria can exist. So, the products are not minimally differentiated and the degree of differentiation can vary. Sufficient conditions for three types of equilibria are given: a unique equilibrium at the center of the market, multiple symmetric equilibria and multiple asymmetric agglomerated equilibria. The first two cases obtain if the firms share profits equally when they are located at the same point and the last case otherwise.

1 Introduction

The literature on product differentiation originating from Hotelling (1929) is by now vast. d'Aspremont et al. (1979) and Neven (1985) have shown that, in a two stage model with quadratic transportation cost, where duopolists first choose locations and then compete in prices, the equilibrium locations are at the extreme endpoints of the market segment. In recent years, this model has been examined in a supergame setting. The firms choose locations in the initial period and prices in subsequent infinite periods. A justification for this is that redesigning the product is often more difficult than a change of price.

Even though locations are chosen independently, the firms can tacitly collude in the price setting stage. Friedman and Thisse (1993) (FT, henceforth) has termed this "partial collusion." In general, the optimal locations chosen in the beginning by the firms will depend on the collusive prices charged subsequently.

In Jehiel (1992) the prices are determined by the Nash (1950) bargaining solution at each pair of locations. In FT, the firms share profits on the profit possibility frontier (PPF) in proportion to the Nash equilibrium profits of the one–shot game. In both the cases, central agglomeration is the unique subgame perfect equilibrium (SPE) outcome.

The reason for this minimum differentiation is well known. In the one–shot game the firms locate at the market extremes to minimize competition and earn higher profits through higher prices. In a repeated setting with price collusion, competition is softened and each firm finds it advantageous to move towards the other. So, both firms locate at the market center.

Rath and Zhao (2003) (RZ, henceforth) shows that if the prices are determined either by the egalitarian [Kalai (1977)] or the Kalai–Smorodinski (1975) bargaining solutions then there are multiple symmetric equilibria. An inward move by a firm results in a lower price that does not compensate for the increase in market share. Profit goes down and the firms locate off the center. So, in equilibrium, the products are not necessarily minimally differentiated.

This paper focuses on these minimal and nonminimal product differentiation results. FT provides a set of sufficient conditions on profit sharing which ensure central agglomeration as the unique equilibrium outcome. Unfortunately, the conditions in FT are too strong, in the presence of some mild continuity requirements, there is exactly one profit sharing rule that satisfies those conditions. Under this, the firms charge identical prices at each pair of locations. Furthermore, this sharing rule cannot be supported as a SPE outcome at every pair of locations. This motivates a weakening of the conditions to capture a wider class of profit sharing rules. However, slightly weakened conditions may be insufficient for uniqueness. An example in section 5 demonstrates this. The conditions in FT are ordinal. A condition of a cardinal nature is given in section 6 for a unique equilibrium at the market center.

If the firms are colluding on the PPF then under very general conditions both the firms locating at the center is an equilibrium. The required conditions are: (a) the profits of the two firms be identical if they are symmetrically located and (b) the profit ratio be bounded above or below 1 at distinct asymmetric locations, depending on how asymmetrically the firms are located. In some cases, this equilibrium at the center of the market is the unique one as well. However, in some other cases, in addition, there may be multiple symmetric equilibria or, the firms can agglomerate at the same point off the center.

As is well known, if the firms locate together then the profit of each firm is indeterminate. This is a primary motivation behind Simon and Zame (1990) and it shows that alternative criteria for breaking ties have a critical bearing on the existence and nature of equilibria.

Profit sharing rules if the firms are at the same location have an important influence on the nature of equilibrium locations. Two such conditions which arise naturally are explored below. In one, the firms share the profits equally if they are identically located. Typically, this introduces a discontinuity in the profit functions at locations off the the center. As a result, in the presence of symmetry, any such location does not survive as an equilibrium. Therefore, all equilibria are symmetric. In some cases, the equilibrium at the center emerges as the unique one and in others there are multiple symmetric equilibria. In the other variant, the profit of each firm is determined by the limit of the profit as one firm approaches the other. Continuity of profits is retained in this case. If one of the profits is decreasing, then the firms will not agglomerate off the center. However, if the profits are increasing as the firms approach each other, the firms can agglomerate off the center and asymmetric equilibria can exist.

The paper is organized as follows. The one–shot model is given in the next section. Conditions on profit sharing are given in section 3 and possible equilibria are derived. Section 4 is devoted to characterization of prices, i.e., which consumer pays the reservation price when. In section 5, the conditions provided by FT for a unique equilibrium at the center of the market are discussed and an example is given to show that the conditions are not sufficient. Section 6 gives a sufficient condition for a unique equilibrium at the center of the market. This is in the form of a lower bound on the derivative of the profit ratio. Section 7 gives conditions for existence of multiple symmetric equilibria. Section 8 deals with agglomerated equilibria off the center. Some aspects of sustaining collusion are discussed in section 9. Section 10 concludes.

2 The Model

The consumers are uniformly distributed over the unit interval [0, 1]. The reservation price of the consumers is A. The quadratic transportation cost parameter is t. Each consumer buys a unit of the product per unit time (subject to the reservation price) from the producer with the lowest delivered price (price plus the transportation cost).

The producers are located at x_1 and x_2, $x_1 \leq x_2$. The production costs are zero. When $x_1 < x_2$, if the prices are p_1 and p_2, the consumer z who faces the same price from the two firms is

characterized by $p_1 + t(z - x_1)^2 = p_2 + t(z - x_2)^2$. The solution gives

$$z = \frac{p_2 - p_1}{2t(x_2 - x_1)} + \frac{x_1 + x_2}{2}$$

Each consumer in the market segment [0, z] buys a unit from Firm 1 and each consumer in [z, 1] buys a unit from Firm 2. The profits of the two firms are $\Gamma_1 = p_1 z$ and $\Gamma_2 = p_2(1 - z)$.

For each pair of locations a Nash equilibrium in prices exists. When $x_1 < x_2$ the Nash equilibrium profits are $\Gamma_{1N} = t(x_2 - x_1)(2 + x_1 + x_2)^2/18$ and $\Gamma_{2N} = t(x_2 - x_1)(4 - x_1 - x_2)^2/18$.

Let α denote the profit ratio Γ_1/Γ_2. The following preliminary claim will be useful later.

Claim 1 Suppose that $\min\{p_1, p_2\} \geq 2t(x_2 - x_1)$, the entire market is served at these prices and the profit of each firm is positive. Then each profit function is decreasing in own price and increasing in the other price. Consequently, α is decreasing in p_1 and increasing in p_2.

This can be proved by noting that the derivatives of z with respect to the two prices are $-1/[2t(x_2 - x_1)]$ and $1/[2t(x_2 - x_1)]$ respectively.

3 Some Conditions on Profit Sharing

Let the time periods be given by $\{0, 1, 2, \ldots\}$. Suppose that the firms choose locations in period 0 and in subsequent infinite periods compete in prices. Some general conditions on profit sharing by the two firms and possible equilibrium outcomes are examined below.

(C1) For any $x_1 \leq x_2$, (Γ_1, Γ_2) is on the PPF.

(C2) For any $x_1 \leq x_2$, $\Gamma_1 = \Gamma_2$ if $x_1 + x_2 = 1$.

(C3) For any $x_1 < x_2$, $\Gamma_1/\Gamma_2 < 1$ if $x_1 + x_2 < 1$ and $\Gamma_1/\Gamma_2 > 1$ if $x_1 + x_2 > 1$.

At any pair of locations the profit allocation is Pareto optimal by (C1). Lemma 3 in FT shows that if $A \geq 3t$ then at any PPF prices the entire market is served. This assumption $A \geq 3t$ will be maintained throughout. If $A \geq 3t$, then (C1) is equivalent to the conditions that at a pair of prices the entire market is served and some consumer pays the reservation price. (C2) is the symmetry condition. This requires that the profits be identical if the firms are symmetrically located. In both (C1) and (C2) the firms can agglomerate at the same point. In case of the latter it can only be at the center. The firms are located at distinct points in (C3). It provides an upper or lower bound on the profit ratio depending on the type of asymmetry, i.e., the sign of $1 - x_1 - x_2$.

The profits need to be specified if the firms are located at the same point $x_1 = x_2 \neq 1/2$. Two possible alternative conditions are examined. Let $x_1^* = x_2^*$.

(C4) $\Gamma_1(x_1^*, x_2^*) = \Gamma_2(x_1^*, x_2^*)$.

(C5) $\alpha(x_1^*, x_2^*) = \lim_{x_1 \to x_2^*} \alpha(x_1, x_2^*) = \lim_{x_2 \to x_1^*} \alpha(x_1^*, x_2)$.

(C4) stipulates that whenever the firms are located at the same point their profits are identical. (C5), on the other hand, requires that if the firms are located at the same point the profit ratio is determined by the limit of the profit ratios as one firm gradually moves towards the other. It is presupposed that these limits exist and are identical.

(C4) can always be exogenously imposed by the modeler. (C5), on the other hand, determines profit endogenously by the structure of the model. If $\lim_{x_1 \to x_2} \alpha \neq 1$ then (C4) introduces a discontinuity in the profit functions if the firms are located at the same point and if $\lim_{x_1 \to x_2} \alpha = 1$ then the choice between (C4) and (C5) is inconsequential. This is true for some specific solutions such as the Nash, Kalai–Smorodinski and egalitarian bragaining solutions. The implications of both these conditions are worth examining.

Lemmas 1 and 2 characterize possible equilibrium outcomes. These show that under (C1)–(C4), the equilibrium outcomes are the symmetric ones inside the market quartiles. A similar result is presented in Proposition 1 in RZ. These Lemmas can be proved along similar lines and the proofs are omitted. (C3) above is weaker than (P4) in RZ. So these results are more general than Proposition 1 in RZ.

If a particular pair of locations is claimed to be an equilibrium below, it is made under the caveat that it can be sustained in a repeated setting. This aspect is examined in section 9.

Lemma 1 Suppose that (C1)–(C3) hold.

(*i*) $x_1 = x_2 = 1/2$ is an equilibrium. Furthermore, any $x_1 < x_2 = 1/2$ or $1/2 = x_1 < x_2$ is not an equilibrium.

(*ii*) Let $x_1 \leq 1/2 \leq x_2$. If $x_1 + x_2 < 1$ then $\Gamma_1(x_1, x_2) < \Gamma_1(1 - x_2, x_2)$. If $x_1 + x_2 > 1$ then $\Gamma_2(x_1, x_2) < \Gamma_2(x_1, 1 - x_1)$. So, $1 - x_1 - x_2 \neq 0$, $x_1 \leq 1/2 \leq x_2$ is not an equilibrium.

(*iii*) Any symmetric pair of locations $(x_1, 1 - x_1)$ with $x_1 < 1/4$ is not an equilibrium.

If (C1)–(C3) hold then (1/2, 1/2) is an equilibrium. Part (*ii*) rules out asymmetric equilibria on opposite sides of the center of the market. The other candidates for equilibrium are the symmetric locations $(x_1, 1 - x_1)$ with $1/4 \leq x_1 < 1/2$ and locations on the same side of the center of the market, $x_1 < x_2 < 1/2$ or, $1/2 < x_1 < x_2$ or, $x_1 = x_2 \neq 1/2$. These asymmetric equilibria can be ruled out if (C4) holds.

Lemma 2 Let (C1) and (C4) hold. (*a*) If (C2) holds then $x_1 = x_2 \neq 1/2$ is not an equilibrium. (*b*) If (C3) holds then neither $x_1 < x_2 < 1/2$ nor $1/2 < x_1 < x_2$ can occur in equilibrium.

Thus, if (C1)–(C4) hold then (1/2, 1/2) is an equilibrium and the other candidates for equilibrium are the symmetric locations $(x_1, 1 - x_1)$ with $1/4 \leq x_1 < 1/2$.

In some cases, asymmetric equilibria can be ruled out without the aid of (C4). Suppose that (C1)–(C3) and (C5) hold. If Γ_1' is negative for all $1/2 < x_1 < x_2$ then Firm 1 will not locate at $x_1 \in (1/2,\, x_2]$.

If a similar condition also holds for the profit function of Firm 2 then under (C1)–(C3) and (C5), (1/2, 1/2) is an equilibrium and the other candidates for equilibria are the symmetric ones $(x_1, 1 - x_1)$ with $1/4 \leq x_1 < 1/2$.

A geometric intuition for multiple symmetric equilibria is given in RZ, section 5. As a firm moves inwards from symmetric locations (x_1, x_2), the symmetric profit pair $([A - tx_1^2]/2, [A - tx_1^2]/2)$ no longer belongs to the PPF, i.e., an inward move may result in a lower profit. So, firms may stay put at locations off the center and multiple equilibria can appear.

4 Characterization of Prices

If the firms collude in prices on the PPF then the prices fall into three possible cases, depending upon the consumer paying the reservation price. It is useful to know which of these cases obtains under which circumstances. (C3) above provides a bound on the profit ratio in one direction. A bound in the other direction helps in characterizing the collusive prices.

Claim 2 Let locations $x_1 < x_2$ be fixed and suppose that the hypothesis of Claim 1 holds. Then $p_1 \leq p_2$ iff $(x_1 + x_2)/(2 - x_1 - x_2) \leq \Gamma_1/\Gamma_2$.

From Claim 1, $\Gamma_1/\Gamma_2 = p_1 z/[p_2(1 - z)]$ is decreasing in p_1 and increasing in p_2. When $p_1 = p_2$, $z = (x_1 + x_2)/2$. So, $\Gamma_1/\Gamma_2 = z/(1 - z) = (x_1 + x_2)/(2 - x_1 - x_2)$.

(C6) Let $x_1 < x_2$. Then $(x_1+x_2)/(2-x_1-x_2) \leq \Gamma_1/\Gamma_2$ if $x_1+x_2 < 1$ and $(x_1+x_2)/(2-x_1-x_2)$ $\geq \Gamma_1/\Gamma_2$ if $x_1 + x_2 > 1$.

Lemma 3 Let $x_1 < x_2$ and suppose that (C1), (C3) and (C6) hold. (a) If $x_1 + x_2 < 1$ then $p_1 \leq p_2$. In this case, $p_1 = A - tx_1^2$ cannot hold. So, $p_2 = A - t(1 - x_2)^2$ or $p_2 = A - t(z - x_2)^2$ for some $z \in (0, 1)$. (b) If $x_1 + x_2 > 1$ then $p_2 \leq p_1$. In this case, $p_2 = A - t(1 - x_2)^2$ cannot hold. So, $p_1 = A - tx_1^2$ or $p_1 = A - t(z - x_1)^2$ for some $z \in (0, 1)$.

(C1), (C3) and (C6) ensure that the hypothesis of Claim 1 is fulfilled. If $x_1+x_2 < 1$ then $(x_1 + x_2)/(2 - x_1 - x_2) \leq \Gamma_1/\Gamma_2$ by (C6). From Claim 2, $p_1 \leq p_2$. Since $p_2 \leq A - t(1 - x_2)^2$, if $p_1 = A - tx_1^2$ then $p_1 - p_2 \geq t(1 - x_2)^2 - tx_1^2 > 0$, a contradiction.

All the solutions discussed below satisfy (C6). If the firms charge the prices determined by the Kalai–Smorodinski or the egalitarian bargaining solutions then (C6) is satisfied. It can be shown that if the profits are determined by the Nash bargaining solution then (C6) is violated. However, it is true that under the Nash bargaining solution some customer of Firm 1 (Firm 2) pays the reservation price if $x_1 + x_2 > 1$ $(x_1 + x_2 < 1)$. Therefore, (C6) is sufficient for the conclusions of Lemma 3, it is not necessary.

Some effects of an increase in market share as a firm changes its location on the profit ratio are captured in the following claim.

Claim 3 Suppose that (C1), (C3) and (C6) hold and z is an increasing function of x_1 and x_2. Let $x_1 < x_2$, $x_1 + x_2 - 1 \neq 0$.

If any of the following conditions holds then for small increases in the location of a firm Γ_1 increases and Γ_2 decreases, consequently Γ_1/Γ_2 increases.

(i) $p_1 + t(z - x_1)^2 = p_2 + t(z - x_2)^2 = A$ and either x_1 or x_2 increases.

(ii) $p_2 = A - t(1 - x_2)^2$ and x_1 increases.

(iii) $p_1 = A - tx_1^2$ and x_2 increases.

Consider two pairs of locations (x_1, x_2) and $(\bar{x}_1, x_2)$, $x_1 < \bar{x}_1 < x_2$. Denote the market share of Firm 1 at these locations by z and $\bar{z}$ respectively. Then $z < \bar{z}$.

Suppose that $p_2 = A - t(z - x_2)^2$. Then $\Gamma_2(x_1, x_2) = [A - t(z - x_2)^2](1 - z) > [A - t(\bar{z} - x_2)^2](1 - \bar{z}) = \Gamma_2(\bar{x}_1, x_2)$. $\Gamma_1(x_1, x_2) = [A - t(z - x_1)^2]z < [A - t(\bar{z} - x_1)^2]\bar{z} < [A - t(\bar{z} - \bar{x}_1)^2]\bar{z}$ $= \Gamma_1(\bar{x}_1, x_2)$. This proves *(i)*.

Suppose that $p_2 = A - t(1 - x_2)^2$. By (C6), $x_1 + x_2 - 1 < 0$ and $p_1 \leq p_2$. So, $2z - x_1 - x_2 \geq 0$. $\Gamma_2(x_1, x_2) = [A - t(1 - x_2)^2](1 - z) > [A - t(1 - x_2)^2](1 - \bar{z}) = \Gamma_2(\bar{x}_1, x_2)$. Since $2\bar{z} - x_1 - x_2 > 0$, $2\bar{z} - x_1 - x_2 > 2\bar{z} - \bar{x}_1 - x_2$ and $x_2 - x_1 > x_2 - \bar{x}_1 > 0$. $\Gamma_1(x_1, x_2) = [p_2 - t(x_2 - x_1)(2z - x_1 - x_2)]z$ $< [p_2 - t(x_2 - x_1)(2\bar{z} - x_1 - x_2)]\bar{z} < [p_2 - t(x_2 - \bar{x}_1)(2\bar{z} - \bar{x}_1 - x_2)]\bar{z} = \Gamma_1(\bar{x}_1, x_2)$. This proves *(ii)*. *(iii)* can be shown in analogous fashion.

5 Discussion on Some Conditions for a Unique Equilibrium

A combination of (a subset of) conditions (C1)–(C6) presented above guarantee the existence of an equilibrium at the center. To ensure the uniqueness of equilibrium some further conditions are needed. A set of such conditions are postulated in FT (p. 641–642). The implications of those conditions are examined in this section.

(C7) For any $x_1 \leq x_2$, $\Gamma_1 > \Gamma_{1N}$ and $\Gamma_2 > \Gamma_{2N}$.

(C8) Γ_1/Γ_2 is increasing in Γ_{1N}/Γ_{2N}.

(C9) $z/(1 - z)$ is increasing in Γ_{1N}/Γ_{2N}.

(C7) is a very intuitive condition. It requires that at any location pair the profit of each firm be higher than the corresponding Nash equilibrium profit. Its primary role is to ensure the existence of discount factors less than one to sustain collusion by reversion to Nash equilibrium profits. This is discussed in section 9, it is basically implied by (C1)–(C3) and (C6).

It is claimed in FT that if (C1), (C7) and (C8) hold then locations with $x_1 + x_2 = 1$ and $x_2 - x_1 \leq 1/2$ are the candidates for equilibrium. However, one of (C4) or (C5) is also needed.

FT mentions on p. 634 that if the firms are located at the same point then the profits are determined by their limits, i.e., by (C5). But in the proof of Lemma 6, (C4), equal division, is actually used when the firms are located at the same point.

Two further claims are made in FT. (*i*) (C9) implies (C8) [possibly in the presence of (C1), (C2) and (C7)]. (*ii*) If (C1), (C2), (C4), (C7), (C8) and (C9) hold then (1/2, 1/2) is the only equilibrium.

The first claim is false. The example given later shows that (C9) does not imply (C8). Under some mild continuity requirements, which are invariably met, the second claim is false as well. There is exactly one profit sharing rule that satisfies (C1), (C8), (C9) and continuity. Under this sharing rule the prices are identical at each pair of locations and this sharing rule cannot be supported in a repeated setting as a SPE outcome at every pair of locations.

Continuity: If $x_1 < x_2$ then (*a*) Γ_1/Γ_2 is continuous in x_1 for fixed x_2 and continuous in x_2 for fixed x_1; (*b*) $z/1-z$ is continuous in x_1 for fixed x_2 and continuous in x_2 for fixed x_1. These continuity requirements are extremely mild and plausible. If the location of one firm is held fixed then the ratio of profits and market shares are continuous functions in the location of the other firm as long as the firms are not agglomerated together. Essentially, continuity and (C8) and (C9) imply that the firms charge identical prices at each pair of locations.

$\Gamma_{1N}/\Gamma_{2N} = (2+x_1+x_2)^2/(4-x_1-x_2)^2$. Therefore, Γ_{1N}/Γ_{2N} increases iff $x_1 + x_2$ increases.

Suppose that (C1), (C8), (C9) and continuity (*a*) and (*b*) hold. Let $x_1 < \bar{x}_1 < \bar{x}_2 < x_2$ and $x_1 + x_2 = \bar{x}_1 + \bar{x}_2$. If the ratio of collusive profits Γ_1/Γ_2 at (x_1, x_2) is less than that at $(\bar{x}_1, \bar{x}_2)$ then by continuity, for some $\epsilon > 0$, the collusive profit ratio at $(x_1+\epsilon, x_2)$ is still less than that at $(\bar{x}_1, \bar{x}_2)$. This contradicts (C8) since $x_1+\epsilon+x_2 > \bar{x}_1+\bar{x}_2$. Similarly, by (C9) and continuity, the ratio of market shares are also the same at (x_1, x_2) and $(\bar{x}_1, \bar{x}_2)$. The ratio of collusive profits is the ratio of collusive prices times the ratio of market shares, so, the ratio of collusive prices p_1/p_2 and $\bar{p}_1/\bar{p}_2$ at the two pairs of locations (x_1, x_2) and $(\bar{x}_1, \bar{x}_2)$ are the same.

Since $p_2 = p_1 + t(x_2 - x_1)(2z - x_1 - x_2)$ and $\bar{p}_2 = \bar{p}_1 + t(\bar{x}_2 - \bar{x}_1)(2z - \bar{x}_1 - \bar{x}_2) = \bar{p}_1 + t(\bar{x}_2 - \bar{x}_1)(2z - x_1 - x_2)$, $p_1/p_2 = \bar{p}_1/\bar{p}_2$ implies that $[\bar{p}_1(x_2 - x_1) - p_1(\bar{x}_2 - \bar{x}_1)](2z - x_1 - x_2) = 0$. In particular, let $\bar{x}_1 = (2x_1 + x_2)/3$ and $\bar{x}_2 = (x_1 + 2x_2)/3$. Then $\bar{x}_1 < \bar{x}_2$, $x_1 + x_2 = \bar{x}_1 + \bar{x}_2$ and $x_2 - x_1 = 3(\bar{x}_2 - \bar{x}_1)$. Since $\bar{p}_1 \geq A - t$ and $p_1 \leq A$, $3\bar{p}_1 > p_1$, $2z - x_1 - x_2 = 0$, $z = (x_1 + x_2)/2$ and $p_1 = p_2$. So, the firms charge identical prices at each pair of locations. The profit ratio $\alpha = (p_1/p_2)[z/(1-z)] = z/(1-z) = (x_1+x_2)/(2-x_1-x_2)$ and it tends to zero and infinity as x_1+x_2 tends to zero and two respectively.

To show that this sharing rule cannot be obtained in a repeated setting as a SPE outcome at every pair of locations, consider $0 < \delta < 1$ and let $x_2 = 1$. Choose $1/2 < x_1 < x_2$ such that

$$\frac{1}{1-\delta}(A - tx_1^2)\frac{2-x_1-x_2}{2} \quad < \quad A - t$$

The inequality holds if $x_1 = x_2 = 1$ and hence for some $x_1 < x_2$. Since $1/2 < x_1$, $p_1 = A - tx_1^2 = p_2$ and $1 - z = (2 - x_1 - x_2)/2$. The LHS is the discounted profit of Firm 2 from the collusive prices. The RHS is the one–shot defection profit (from capturing the entire market). (One need not consider the profit from the continuation path following the defection.) This shows that this profit sharing rule cannot be supported in the repeated setting. The reason for this result is that under this sharing rule the market share of the firm at the edge of the market becomes arbitrarily small ($\alpha \to \infty$) and hence it has a strong incentive to undercut. [A similar situation also arises with linear cost, fixed locations and repeated pricing setting. All SPE equilibrium paths for some location are nonstationary, Rath (1998, section 5).]

The important ramification of this is that these conditions cannot be used to characterize equilibrium outcomes. Specifically, if the profits are shared in proportion to the Nash equilibrium market shares, $\Gamma_1/\Gamma_2 = (2+x_1+x_2)/(4-x_1-x_2)$, or, in proportion to the Nash equilibrium profits, $\Gamma_1/\Gamma_2 = (2+x_1+x_2)^2/(4-x_1-x_2)^2$ [as in Schmalensee (1987), FT], then (C8) is satisfied. In neither case $\Gamma_1/\Gamma_2 = (x_1+x_2)/(2-x_1-x_2)$. Therefore, (C9) cannot hold. There is a unique equilibrium at the market center under each of these sharing rules. But the result obtains not because these sharing rules satisfy the conditions discussed above.

The difficulty with (C8) and (C9) is that only the sum $x_1 + x_2$ matters in determining the ratio of profits and market shares. A natural question that emerges is: can one weaken these conditions and still obtain the uniqueness result. The example given below shows that this is not the case under a slight weakening of these conditions.

(C8*) If $x_1 < x_2$ then Γ_1/Γ_2 is increasing in x_1 for fixed x_2 and in x_2 for fixed x_1.

(C9*) If $x_1 < x_2$ then $z/(1-z)$ is increasing in x_1 for fixed x_2 and in x_2 for fixed x_1.

Consider the following example. Let $0 \leq \mu \leq 1$. Suppose that at any distinct pair of locations the firms remain on the PPF and charge prices such that

$$z \quad = \quad \frac{1}{2} + \frac{t\mu}{A}(x_1 + x_2 - 1)$$

Clearly, the sharing rule satisfies symmetry. Let $x_1 + x_2 - 1 > 0$. Then $z < (x_1 + x_2)/2$ is equivalent to $(t\mu/A)(x_1 + x_2 - 1) < (x_1 + x_2 - 1)/2$ which follows from $t\mu/A \leq 1/3$. This implies that $p_1 > p_2$. Since $z \geq 1/2$, $\Gamma_1/\Gamma_2 > 1$, i.e., (C3) holds. Since $p_1 > p_2$, (C6) holds.

Since $t\mu/A \leq 1/3$, $z \leq 5/6$ and $z/(1-z) \leq 5$. From $p_1 = p_2 - t(x_2 - x_1)(2z - x_1 - x_2)$, $p_1/p_2 \leq 2$. One can show that $1/10 \leq \Gamma_1/\Gamma_2 = p_1 z/[p_2(1-z)] \leq 10$. Arguments in section 9 can be used to show that (C7) holds.

As $x_1 \to x_2$, $p_1 \to p_2$ but z does not tend to 1/2 if $x_2 > 1/2$ and $\mu > 0$. So, $\lim_{x_1 \to x_2} \alpha(x_1, x_2) \neq 1$. So, whether (C4) or (C5) is assumed can make a difference in equilibrium outcomes. For the time being (C4) is assumed. The implications of (C5) are examined later.

By Lemmas 1 and 2, (1/2, 1/2) is an equilibrium and the other candidates for equilibria are the symmetric ones in [1/4, 3/4]. If $\mu = 0$, $x_1 + x_2 - 1 \geq 0$ and $1/4 \leq x_1 < x_2 \leq 3/4$ then $p_1 = A - tx_1^2$ and $z = 1/2$. For an inward move, price decreases and market share does not increase. So, all symmetric locations in [1/4, 3/4] are equilibria.

Assume that $\mu > 0$. The equilibrium outcomes are characterized in Appendix 1. The set of equilibria is the symmetric locations in an interval. If $\mu \leq 4A/(16A - t)$ then any pair of symmetric locations in [1/4, 3/4] is an equilibrium. If $\mu \geq 2A/(4A - t)$ then (1/2, 1/2) is the unique equilibrium. As μ increases, the set of equilibria becomes smaller. If $4A/(16A - t) < \mu < 2A/(4A - t)$ then the set of equilibria is a proper subset of [1/4, 3/4]. In particular, when $\mu \leq 1/2 < 2A/(4A - t)$ there are multiple equilibria.

If $\mu > 0$ then z is increasing in x_1. If $x_1 + x_2 - 1 > 0$ then $\alpha = \Gamma_1/\Gamma_2 > 1$. If $\mu = 0$ then $z = 1/2$ and $\lim_{x_1 \to x_2} \alpha(x_1, x_2) = 1$. So, for low μ, α is decreasing if x_1 is close to x_2. Therefore, an increasing z does not imply an increasing α. A different argument (also based on continuity with respect to μ) is given in Appendix 1. This means (C9*) does not imply (C8*). Moreover, in this example (C9) is satisfied. So, (C9) does not imply (C8) either.

Because of Claim 3, to determine whether α is increasing or not, one needs to consider $x_1 < x_2$, $x_1 + x_2 - 1 > 0$ and $p_1 = A - tx_1^2$. Since z increases with μ, Γ_1 is increasing and Γ_2 is decreasing in μ. So, for higher μ, α is increasing. It turns out that if $\mu \geq 2/5$ then α is increasing in x_1. However, if $\mu = 2/5$ then there are multiple equilibria. So, both z and α are increasing do not imply that there is a unique equilibrium, i.e., (C8*) and (C9*) together with the other conditions are not sufficient for uniqueness. The details are given in Appendix 1.

The preceding discussion shows that conditions like (C8), (C9), (C8*), (C9*) etc. do not help in characterizing equilibria (unique or not). In the subsequent sections conditions based on the derivative of the profit ratio are developed to characterize different types of equilibria.

6 A Sufficient Condition for a Unique Equilibrium

A sufficient condition for a unique equilibrium involving the derivative of α is given in the next theorem. It is worth noting that typically the profit functions are not differentiable at symmetric locations. Nevertheless, α might be differentiable. Derivatives with respect to x_1 are denoted by the prime symbol. Since only inward moves need to be examined because of Lemma 1 (*ii*), the derivatives are usually the right hand side derivatives at symmetric locations.

Theorem 1 Suppose that (C1)–(C4) hold. Let $x_1 \in [1/4,\ 1/2)$ and α be differentiable in x_1. If $\alpha' \geq t(1+2x_1)/(A-tx_1^2)$ at $(x_1, 1-x_1)$ then the pair is not an equilibrium. So, (1/2, 1/2) is the unique equilibrium if $\alpha' \geq t(1+2x_1)/(A-tx_1^2)$ at all symmetric locations $x_1 \in [1/4,\ 1/2)$.

By Lemmas 1 and 2, the possible equilibria are the symmetric ones in [1/4, 3/4]. The lower bound on α' ensures that the profit of Firm 1 is increasing at symmetric locations off the center. So, (1/2, 1/2) emerges as the unique equilibrium. The proof is given in Appendix 2.

$t(1+2x_1)/(A-tx_1^2)$ is an increasing function of x_1 and equals $8t/(4A-t)$ when $x_1 = 1/2$. Since $A \geq 3t$, $8t/(4A-t) \leq 8/11$. Therefore, if $\alpha' \geq 8/11$ for all $x_1 \in [1/4,\ 1/2)$ then central agglomeration is the unique equilibrium.

If the firms share profits on the PPF in proportion to the one–shot Nash equilibrium profits, the case examined in FT, then $\alpha = (2+x_1+x_2)^2/(4-x_1-x_2)^2$. If $x_2 \neq 1/2$ then α does not tend to 1 as x_1 tends to x_2. So, one needs to assume (C4). Clearly, (C2) and (C3) are satisfied. $\alpha' = 12(2+x_1+x_2)/(4-x_1-x_2)^3$ and equals 4/3 at symmetric locations. So, central agglomeration is the unique equilibrium. $\alpha = (2+x_1+x_2)/(4-x_1-x_2)$ if the profits are shared in proportion to the Nash equilibrium market shares [Schmalensee (1987), FT]. If (C4) is assumed then all the conditions are satisfied. At symmetric locations, $\alpha' = 2/3$. If $A \geq 4t$ then central agglomeration is the only equilibrium.

From (5), $p_1\alpha' = [4p_1 - 2t(1-2x_1)]z' + t(1-2x_1)$ at symmetric locations. So, $p_1\alpha' \geq t(1+2x_1) \Leftrightarrow [4p_1 - 2t(1-2x_1)]z' \geq 4tx_1$. In the example of the preceding section, this reduces to $\mu \geq 4Ax_1/[4p_1 - 2t(1-2x_1)]$. The RHS is increasing in x_1. Letting $x_1 = 1/2$ yields $\mu \geq 2A/(4A-t)$. This is exactly the bound obtained earlier for a unique equilibrium.

7 A Necessary and Sufficient Condition for Symmetric Equilibria Off the Center

Let $\sigma = [A - t(1-x_2)^2]/[2(A - tx_1^2)]$ and

$$Q \quad = \quad \frac{A - t(1-x_2)^2}{2[A - tx_1^2 + t(x_2 - x_1)(2\sigma - x_1 - x_2)](1-\sigma)}$$

Theorem 2 Let (C1)–(C4) hold. Suppose that $\alpha(x_1, x_2) = 1/\alpha(1 - x_2, 1 - x_1)$. A pair of symmetric locations $(1 - x_2,\, x_2)$, $1/2 < x_2 \leq 3/4$ is an equilibrium iff $\alpha(x_1, x_2) \leq Q(x_1, x_2)$ for all $1 - x_2 < x_1 < x_2$.

A pair of symmetric locations is an equilibrium if inward moves by neither firm is profitable. $\alpha(x_1, x_2) = 1/\alpha(1 - x_2, 1 - x_1)$ means that if the locations are flipped in a certain way then the profit ratio reverses, i.e., it is enough to check for inward moves of only one firm.

The intuition behind Theorem 2 is as follows. If $p_1 = A - tx_1^2$ then $p_1\sigma = A - t(1-x_2)^2/2$. So, if $p_1 = A - tx_1^2$ and the market share of Firm 1 is σ then Q is the ratio of the profits. If $x_1 + x_2 - 1 > 0$ then $\sigma < (x_1 + x_2)/2$. Therefore, $p_1 > p_2$ and $Q < (x_1 + x_2)/(2 - x_1 - x_2)$. If $p_2 = A - t(1-x_2)^2$ for some $x_1 + x_2 - 1 > 0$ then $p_1 \leq A - tx_1^2 < A - t(1-x_2)^2$ and the profit ratio is greater than $(x_1 + x_2)/(2 - x_1 - x_2)$. Therefore, $\alpha \leq Q$ implies that $p_1 = A - tx_1^2$.

If $(1 - x_2,\, x_2)$ is an equilibrium then $\Gamma_1(1 - x_2, x_2) \geq \Gamma_1(x_1, x_2)$ for all $x_1 \in (1 - x_2, x_2)$. If for some such x_1, $p_2 = A - t(1 - x_2^2)$ then $z > (x_1 + x_2)/2$ and $\Gamma_1(x_1, x_2) = [p_2 - t(x_2 - x_1)(2z - x_1 - x_2)]z > p_2/2 = \Gamma_1(1 - x_2, x_2)$. Therefore, $p_1 = A - tx_1^2$ for all $x_1 \in (1 - x_2, x_2)$.

$p_1\sigma = (A - tx_1^2)\sigma = [A - t(1-x_2)^2]/2$ is the profit of Firm 1 when the two firms are located symmetrically at $(1 - x_2, x_2)$. Thus, if the prices and market share (p_1, p_2, z) at (x_1, x_2) satisfies $z \leq \sigma$, then $\Gamma_1(x_1, x_2) = p_1 z \leq p_1\sigma = [A - t(1-x_2)^2]/2$ and vice versa.

Since $p_1 = A - tx_1^2$ and $p_2 = p_1 + t(x_2 - x_1)(2z - x_1 - x_2)$, $\alpha = p_1 z/[p_2(1-z)]$ is a strictly increasing function of z. Hence, $z \leq \sigma$ iff $\alpha \leq Q$. This proves the Theorem.

If $\alpha = (2 + x_1 + x_2)^2/(4 - x_1 - x_2)^2$ then $Q < \alpha$ iff $2(2 + x_1 + x_2)^2[A - tx_1^2 + t(x_2 - x_1)(2\sigma - x_1 - x_2)](1 - \sigma) - (4 - x_1 - x_2)^2[(A - t(1-x_2)^2] > 0$. At symmetric locations, the LHS is zero and its derivative is positive, i.e., the LHS is positive at near symmetric locations. So, $Q < \alpha$ for some $1 - x_2 < x_1 < x_2$ and there is a unique equilibrium at the center.

If $z = (1/2) + (t\mu/A)(x_1 + x_2 - 1)$ then $z \leq \sigma$ leads to $A(1 + x_1 - x_2) - 2\mu(A - tx_1^2) \geq 0$. The LHS is increasing in x_1. Solving as an equality at symmetric locations yields $t\mu x_1^2 + Ax_1 - A\mu$ $= 0$ which determines the set of equlibria in (1).

8 Existence of Asymmetric Equilibria

For some profit sharing rules, $\lim_{x_1 \to x_2} \alpha \neq 1$ for some $x_2 \neq 1/2$. To preserve the continuity of α, (C5) can be imposed instead of (C4). As Lemma 2 suggests, (C4) primarily rules out asymmetric equilibria. Without (C4) it may not be possible to rule those out. So, asymmetric equilibria can exist and the firms may agglomerate at a point off the center.

Theorem 3 Let (C1)–(C3) and (C5) hold. Suppose that $\alpha(x_1, x_2) = 1/\alpha(1-x_2, 1-x_1)$. If for some fixed x_2, $\alpha'/\alpha^2 \geq t(1+2x_1)/(A - tx_2^2)$ at all $x_1 < x_2$ then (x_2, x_2) is an equilibrium.

This is proved in Appendix 2. The proof shows that $\Gamma_1' > 0$, i.e., $\Gamma_1(x_1, x_2) < \Gamma_1(x_2, x_2)$ if $x_1 < x_2$. So, (x_2, x_2) is an equilibrium.

If $\alpha = (2 + x_1 + x_2)^2/(4 - x_1 - x_2)^2$ then $\alpha'/\alpha^2 = 12(4 - x_1 - x_2)/(2 + x_1 + x_2)^3$ and $(\alpha'/\alpha^2) - [t(1+2x_1)/(A - tx_2^2)]$ is decreasing in x_1 and x_2. To identify the equilibria one needs to solve $[3(2 - x_2)/(1 + x_2)^3] - [t(1 + 2x_2)/(A - tx_2^2)] \geq 0$. If $A \geq 3t$ then any point in the interval [0.4, 0.6] is an equilibrium. If $A \geq 9t$ then any point in the interval [0, 1] is an equilibrium. This also underscores the role of (C4), that the profits are identical if the firms are located together, to obtain central agglomeration as the unique equilibrium.

Let $\mu = 1$ in the example of section 5. Then $\Gamma_1' = (A - tx_1^2)(t/A) - 2tx_1 z > 0$ if $1/2 = x_1 < x_2$. So, there is an interval containing 1/2 such that any point in it is an equilibrium. Therefore, if μ is large then the firms may locate together off the center.

9 Sustaining Collusion

Let the firms choose locations in the initial period and prices in subsequent infinite periods. To obtain a specific pair of locations as a SPE outcome of the supergame, it needs to be shown that the collusive prices yield a SPE outcome of the repeated game for any given pair of locations. In this context, one can examine

optimal punishment paths as in Abreu (1988). However, for existence purposes reversion to Nash equilibrium prices of the one–shot game is sufficient.

Concentrate on Firm 1 with discount factor δ. Let Γ_1^D denote the defection profit from the collusive profits (Γ_1, Γ_2). The relevant inequality is $[1/(1-\delta)]\Gamma_1 \geq \Gamma_1^D + [\delta/(1-\delta)]\Gamma_{1N}$, or equivalently, $\delta \geq (\Gamma_1^D - \Gamma_1)/(\Gamma_1^D - \Gamma_{1N})$. So (C7) is a necessary condition for $0 < \delta < 1$.

Let $s = \inf\{\Gamma_1 - \Gamma_{1N}\}$ and $S = \sup\{\Gamma_1^D - \Gamma_{1N}\}$ for all $x_1 \leq x_2$. If $s > 0$, then collusion can be sustained by letting $\delta \geq (S - s)/S$.

To ensure this, let $1/r \leq \Gamma_1/\Gamma_2 \leq r$ for all $x_1 \leq x_2$, $1 < r < \infty$. Suppose that (C1)–(C3) and (C6) hold. Suppose further that either (C4) or (C5) holds. Then $s \geq t/(1 + r) > 0$.

If $x_1 + x_2 - 1 \geq 0$, then by (C1)–(C3) and (C4) or (C5), $\Gamma_1 \geq (1/2)(\Gamma_1 + \Gamma_2) \geq (1/2)(A - t) \geq t$ and $\Gamma_{1N} = t(x_2 - x_1)(2 + x_1 + x_2)^2/18 \leq t/2$. If $x_1 + x_2 - 1 < 0$ then $\Gamma_1 \geq (\Gamma_1 + \Gamma_2)/(1 + r) \geq (A - t)/(1 + r) \geq 2t/(1 + r)$. $\Gamma_{1N} \leq t(x_2 - x_1)/2 \leq tx_2/2$. If $x_2 \leq 2/(1 + r)$ then $\Gamma_1 - \Gamma_{1N} \geq t/(1 + r)$. Therefore, suppose that $x_2 > 2/(1 + r)$. By (C6), $\Gamma_1/\Gamma_2 \geq (x_1 + x_2)/(2 - x_1 - x_2)$, i.e., $\Gamma_1 \geq (\Gamma_1 + \Gamma_2)(x_1 + x_2)/2 \geq t(x_1 + x_2)$. So, $\Gamma_1 - \Gamma_{1N} \geq tx_2/2 \geq t/(1 + r)$.

Some type of bounds on the profit ratio are required to ensure that the collusive profits exceed the one–shot Nash equilibrium profits. It is important to verify this, especially because Theorems 1–3 do not require (C6) per se. All the examples considered above satisfy (C6).

10 Conclusion

This paper has explored the nature of equilibria when the firms noncollusively choose locations and set prices so as to be on the PPF. Under fairly general conditions both the firms locating at the center is an equilibrium. However, other symmetric locations or agglomeration off the center can emerge as equilibria as well.

One critical condition is the sharing of profits when the firms are located at the same point. If $\lim_{x_1 \to x_2} \alpha = 1$, then the choice of (C4) or (C5) is immaterial. With (C4), all equilibria are symmetric and Theorem 1 gives a sufficient condition for a unique equilibrium at the center. Without (C4), asymmetric equilibria can exist and firms may agglomerate off the center.

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Appendix 1

Derivatives with respect to x_1 are denoted by the prime symbol. Suppose that $p_1 = A - tx_1^2$. Then $\Gamma_1 = (A - tx_1^2)z$, $\Gamma_1' = -2tx_1z + (A - tx_1^2)z'$ and $\Gamma_1'' = -2tz - 4tx_1z' < 0$.

$p_2 = p_1 + t(x_2 - x_1)(2z - x_1 - x_2)$. So, $p_2' = -2tx_1 + t(x_1 + x_2 - 2z) + t(x_2 - x_1)(2z' - 1)$ $= -2tz + 2t(x_2 - x_1)z'$ and $p_2'' = -4tz'$. Since $\Gamma_2 = p_2(1 - z)$, $\Gamma_2' = p_2'(1 - z) - p_2z' = -[p_2 - 2t(x_2 - x_1)(1 - z)]z' - 2tz(1 - z)$ and $\Gamma_2'' = p_2''(1 - z) - 2p_2'z' = -2z'[p_2' + 2t(1 - z)] = -2z'[-2tz + 2t(x_2 - x_1)z' + 2t(1 - z)] = -4tz'[(x_2 - x_1)z' + 1 - 2z]$.

(1) To determine equilibria, one needs to examine inward moves, i.e., $x_1 + x_2 - 1 > 0$ and $x_2 \leq 3/4$. By (C6), $p_1 = A - tx_1^2$.

Since $\Gamma_1'' < 0$, to determine equilibria one needs to solve $\Gamma_1' \leq 0$ at symmetric locations, i.e., $(A - tx_1^2)(t\mu/A) - tx_1 \leq 0$. The LHS is decreasing in x_1. Therefore, it suffices to solve $t\mu x_1^2 + Ax_1 - A\mu = 0$. The quadratic formula gives

$$x_1 \quad = \quad \frac{-A + \sqrt{A^2 + 4At\mu^2}}{2t\mu} \tag{1}$$

The set of equilibria is given by the symmetric locations in the interval $[x_1, 1 - x_1]$. The RHS of (1) is increasing in μ. So, as μ increases, the set of equilibria becomes smaller.

If $x_1 = 1/4$ in (1) then $\mu = 4A/(16A - t) > 1/4$ and if $x_1 = 1/2$ in (1) then $\mu = 2A/(4A - t) > 1/2$. This determines the set of equilibria.

(2) Let $1/2 < x_1 < x_2$. Then $p_1 = A - tx_1^2$. Γ_1/Γ_2 is increasing or decreasing depends upon whether $\Gamma_2\Gamma_1' - \Gamma_1\Gamma_2'$ is positive or negative.

$$\begin{aligned} \Gamma_2\Gamma_1' - \Gamma_1\Gamma_2' &= \Gamma_2(-2tx_1z + p_1z') - \Gamma_1[p_2'(1 - z) - p_2z'] \\ &= \Gamma_2p_1z' + \Gamma_1p_2z' - 2t(x_2 - x_1)(1 - z)\Gamma_1z' + 2tz(1 - z)(p_1z - p_2x_1) \end{aligned}$$

When $\mu = 0$, $z = 1/2$ and $z' = 0$. In that case, $\Gamma_2\Gamma_1' - \Gamma_1\Gamma_2' = (t/2)[(p_1/2) - p_2x_1] = (t/4)(p_1 - 2p_2x_1) = (t/4)[p_1(1 - 2x_1) + 2t(x_2 - x_1)(x_1 + x_2 - 1)x_1]$

Let $x_1 = 3/4$ and $x_2 = 1$. Since $p_1 \geq 2t$, $\Gamma_2\Gamma_1' - \Gamma_1\Gamma_2' = (t/4)[-(p_1/2) + (2t/4)(3/4)(3/4)]$ $= (t/4)[-(p_1/2) + (9t/32)] = (t/128)(-16p_1 + 9t) \leq -(23t^2/128) \leq -(t^2/6)$.

So, $\Gamma_2\Gamma_1' - \Gamma_1\Gamma_2'$ is bounded away from zero when $\mu = 0$, $x_1 = 3/4$ and $x_2 = 1$. Since it is continuous in μ, it is also negative at those locations when μ is positive but small.

If μ is positive then z is increasing, i.e., (C9*) is satisfied. Since z is strictly increasing if μ is positive and Γ_1/Γ_2 can be decreasing, (C9*) need not imply (C8*).

(3) First it is shown that $\Gamma_2\Gamma_1' - \Gamma_1\Gamma_2'$ is decreasing in x_1, i.e., $\Gamma_2\Gamma_1'' - \Gamma_1\Gamma_2'' < 0$.

$\Gamma_2\Gamma_1'' - \Gamma_1\Gamma_2'' = -2t\Gamma_2(z + 2x_1z') + 4t\Gamma_1z'[(x_2 - x_1)z' + 1 - 2z]$. Since $z' = t\mu/A \leq 1/3$, $(x_2 - x_1)z' - z < 0$. So, it suffices to show that $-\Gamma_2z + 2\Gamma_1(1-z)z' = -p_2z(1-z) + 2p_1z(1-z)z'$ < 0.

This holds if $3p_2 - 2p_1 > 0$ and follows from $p_1 = p_2 - t(x_2 - x_1)(2z - x_1 - x_2) < p_2 + t(x_2^2 - x_1^2) \leq p_2 + t \leq 3p_2/2$.

It is enough to show that $\Gamma_2\Gamma_1' - \Gamma_1\Gamma_2' > 0$ when $x_1 = x_2$. In that case, $p_1 = p_2$. Ignoring p_1, $\Gamma_2\Gamma_1' - \Gamma_1\Gamma_2' = (1-z)(-2tx_2z + p_1z') + z[2tz(1-z) + p_2z'] = p_1z' + 2tz(1-z)(z - x_2)$. p_1z' $= t\mu(A - tx_2^2)/A > t/4$ when $\mu \geq 2/5$. On the other hand, $z(1-z) \leq 1/4$ and $|z - x_2| \leq 1/2$.

So, if $\mu \geq 2/5$ then α is increasing. So, both z and α are increasing do not imply that there is a unique equilibrium.

Appendix 2

Before the proof of Theorem 1, the expressions for Γ_1' for all possible prices are derived. From the definition of α, $p_1 z = p_2(1-z)\alpha$. Therefore,

$$p_1 z' + z p_1' \quad = \quad p_2(1-z)\alpha' + \alpha[p_2'(1-z) - p_2 z'] \tag{2}$$

If $p_2 = A - t(1-x_2)^2$ then $p_2' = 0$. $p_1 = p_2 - t(x_2 - x_1)(2z - x_1 - x_2)$ and $p_1' = 2t(z - x_1) - 2t(x_2 - x_1)z'$. Since $p_2(1-z) = p_1 z/\alpha$ and $\Gamma_1' = p_1 z' + z p_1'$, (2) yields

$$z' \quad = \quad \frac{p_1 z(\alpha'/\alpha) - 2t(z-x_1)z}{p_1 - 2t(x_2 - x_1)z + p_2\alpha}$$

$$\frac{\Gamma_1'}{\alpha z} \quad = \quad \frac{p_1(\alpha'/\alpha^2)[p_1 - 2t(x_2 - x_1)z] + 2t(z - x_1)p_2}{p_1 - 2t(x_2 - x_1)z + p_2\alpha} \tag{3}$$

If $p_1 + t(z - x_1)^2 = A = p_2 + t(z - x_2)^2$ for some $z \in (0,\, 1)$ then $p_1' + 2t(z - x_1)(z' - 1) = 0$ and $p_2' + 2t(z - x_2)z' = 0$. From (2),

$$z' \quad = \quad \frac{p_1 z(\alpha'/\alpha) - 2t(z - x_1)z}{p_1 - 2t(z - x_1)z + \alpha[p_2 + 2t(z - x_2)(1 - z)]}$$

$$\frac{\Gamma_1'}{\alpha z} \quad = \quad \frac{p_1(\alpha'/\alpha^2)[p_1 - 2t(z - x_1)z] + 2t(z - x_1)[p_2 + 2t(z - x_2)(1 - z)]}{p_1 - 2t(z - x_1)z + \alpha[p_2 + 2t(z - x_2)(1 - z)]} \tag{4}$$

If $p_1 = A - tx_1^2$ then $p_1' = -2tx_1$. $p_2 = p_1 + t(x_2 - x_1)(2z - x_1 - x_2)$ and $p_2' = 2t(x_2 - x_1)z' - 2tz$. From (2),

$$z' \quad = \quad \frac{p_1 z(\alpha'/\alpha) - 2tz(1 - z)\alpha + 2tzx_1}{p_1 + [p_2 - 2t(x_2 - x_1)(1 - z)]\alpha} \tag{5}$$

$$\frac{\Gamma_1'}{\alpha z} \quad = \quad \frac{p_1^2(\alpha'/\alpha^2) - 2tp_1(1 + x_1 - z) + 2t^2(x_2 - x_1)x_1(x_1 + x_2 + 2 - 4z)}{p_1 + [p_2 - 2t(x_2 - x_1)(1 - z)]\alpha} \tag{6}$$

Proof of Theorem 1 From Lemmas 1 and 2, (1/2, 1/2) is an equilibrium and the other candidates for equilibria are the symmetric ones with $1/4 \le x_1 < 1/2$. To eliminate these consider a pair of symmetric locations (x_1, x_2). Let $x_1^* \in (x_1, x_2)$. Then $x_1^* + x_2 - 1 > 0$.

Suppose that $p_2 = A - t(1 - x_2)^2$ at some x_1^*. Since $p_1 \le A - tx_1^{*2}$, $p_2 - p_1 > 0$ and $z > (x_1^* + x_2)/2 > x_1^*$. Since $\alpha' > 0$, (3) shows that $\Gamma_1' > 0$.

So, assume that $p_1 = A - tx_1^{*2}$ at all $x_1^* \in (x_1, x_2)$. At symmetric locations, $z = 1/2$ and $\alpha = 1$. From (6), $\Gamma_1' > 0$ if $p_1^2\alpha' - tp_1(1+2x_1) + 2t^2(1-2x_1)x_1 > 0$ which follows from $\alpha' \geq t(1+2x_1)/(A - tx_1^2)$ and $p_1 = A - tx_1^2$. Therefore, Γ_1 increases at symmetric locations and $(x_1, 1-x_1)$ is not an equilibrium. So, (1/2, 1/2) is the unique equilibrium. ∎

Proof of Theorem 3 In order to show that (x_2, x_2) is an equilibrium it needs to be shown that $\Gamma_1(x_1, x_2) < \Gamma_1(x_2, x_2)$ if $x_1 < x_2$.

First suppose that $x_2 > 1/2$. Lemma 1 shows that if $x_1 + x_2 - 1 < 0$ then $\Gamma_1(x_1, x_2) < \Gamma_1(1-x_2, x_2)$. Therefore, suppose that $x_1 + x_2 - 1 \geq 0$.

Consider the three cases for the prices.

If $p_2 = A - t(1-x_2)^2$ then as argued in the proof of Theorem 1, $z - x_1 > 0$. Since $\alpha' > 0$, $\Gamma_1' > 0$ from (3). If $p_1 = A - t(z-x_1)^2$ then $z - x_1 > 0$ and (4) shows that $\Gamma_1' > 0$. If $p_1 = A - tx_1^2$, by (6), $\Gamma_1' > 0$ if

$$\frac{\alpha'}{\alpha^2} > \frac{2t}{p_1^2}[p_1(1+x_1-z) - t(x_2-x_1)x_1(x_1+x_2+2-4z)]$$

The RHS is a decreasing function of z. $p_1 = A - tx_1^2$, $p_2 > A - tx_2^2$, so, $p_1 z + p_2(1-z) > A - tx_2^2$. (C3) ensures that $p_1 z > (A - tx_2^2)/2$ and $z > (A - tx_2^2)/[2(A - tx_1^2)] = z^*$, say. Then $z^* < 1/2$ and the RHS is less than $2t(1+x_1-z^*)/p_1$. It is easily verified that $2t(1+x_1-z^*)/(A-tx_1^2)$ $< t(1+2x_1)/(A-tx_2^2)$. Therefore, if $\alpha'/\alpha^2 \geq t(1+2x_1)/(A-tx_2^2)$ then $\Gamma_1' > 0$.

If $x_2 \leq 1/2$ then $p_2 = A - t(1-x_2)^2$. If $z \geq x_1$ then $\Gamma_1' > 0$ from (3). If $z < x_1$ then $p_1 > p_2$. $\Gamma_1' > 0$ if $(\alpha'/\alpha^2)[p_1 - 2t(x_2-x_1)z] > 2tx_1$. Since $1 + 2x_1 > 4x_1$, this is true if $2[p_1 - 2t(x_2-x_1)z] > A - tx_2^2$ which follows from $p_1 > p_2 = A - t(1-x_2)^2$ and $A \geq 3t$. ∎

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