Learning the CAPM through Bubbles

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Bubbles are generally considered the outcome of investor irrationality or informational asymmetry, both objectionable in efficient markets with rational investors. We introduce an Intertemporal-CAPM with market clearing between high- and low-risk-averse rational investors who learn the CAPM under incomplete, yet symmetric information. Periodic equilibrium prices make a lognormal price process that nests the classic CAPM with a potential for endogenous bubbles through learning. The absence of comparables through the introductory phase of new technologies results in unstable return dynamics that might burst to bubbles or decline to near-zero, "pink-sheet" valuations. When the technology shifts phase to generate real profits the return dynamics is convergent, revealing the classic CAPM. Once the real technology return is observable, over- and under-pricing can be assessed, resulting in prompt positive or negative price adjustments toward the CAPM valuation. Correspondence with the Abreu and Brunnermeier (2003) model of bubbles with rational arbitrageurs is presented as well.

Key Words: ICAPM; Bubbles; New Technologies; Rational Expectations.

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"I can calculate the motions of the heavenly bodies, but not the madness of people."

Sir Isaac Newton, following a substantial loss due to the South Sea bubble, 1720. (Carswell, 1960)

1 INTRODUCTION

Is it really madness? Must we relax rationality assumptions in order to explain bubbles and crashes? Most models of bubbles either assume that some investors are irrational and/or information is asymmetrically available to all investors. Both assumptions are objectionable. If irrational investors buy (sell) an overpriced (underpriced) asset, economists since Friedman (1953) showed that they would either loose all wealth to rational investors or abstain from the market, thus prices will be determined by rational investors. The asymmetric information approach is problematic as well since a few no-trade theorems (e.g., Milgrom and Stokey, 1982) preclude trade, and hence bubbles, save specific conditions in Allen, Morris and Postlewaite (1993) that allow bubbles in a game-theoretic set-up. We present a model where different degrees of risk aversion with rational, CRRA[1] preferences and incomplete, yet symmetric information yield an Intertemporal Capital Asset Pricing Model (ICAPM) that is consistent with bubbles and crashes.

Our model differs from existing rational asset pricing models in two major aspects. First, we present a specific heterogeneity structure of risk preferences that facilitates endogenous price revelation through intertemporal trade, and the second is the explicit account of incomplete information through dynamic learning of the CAPM Rational Expectations Equilibrium (REE). For the first aspect, we prove a Theorem whereby optimal dynamic asset allocation strategies result in an intertemporal trade in *units of shares* that can be positive or negative, depending on the value of investor's Relative Risk Aversion (RRA) coefficient with respect to the ex-ante market price for variance risk, $\lambda_{t+\Delta t}^{f}$. Given a positive random news signal, trade will be positive, i.e,

[1] Constant Relative Risk Aversion.

demand for shares, if investor's coefficient of RRA is less than $\lambda^f_{t+\Delta t}$, but negative, (supply of shares) if greater than $\lambda^f_{t+\Delta t}$. The detailed economic rationale is presented in section 3.1. The principle idea is that by segregating rational investors to two mutually exclusive groups of high and low risk-averse investors we obtain periodic demand and supply that alternate with the random arrival of negative and positive news signals. This notion allows us to clear the market and solve for a unique periodic equilibrium price *endogenously*, given the dynamically changing weighted average risk aversion. The resulting price process is lognormal, facilitating the derivation of a rational, stationary ICAPM as in Merton (1971), which nests the single-period CAPM.

The second differentiating aspect of this model, given our incomplete information setup, is the notion of learning the CAPM solution through adaptive expectations. This approach, recently applied in macroeconomics, relaxes the assumption that investors' information-set is instantaneously complete, as typically structured in asset pricing models, and replaces it with an explicit learning algorithm that converges to the stationary REE solution in time.[2] The advantage of this approach is two fold: first, because it is quantitative, rather than conceptual, with respect to the formation of expectations it disentangles the simultaneity between the arrival of "news" signals and return realizations given the signals, i.e., realized returns become a function of operationally defined expectations. Second, this approach is capable of revealing multiple equilibria in the economic model, if exist, and facilitates the analysis of dynamic convergence to, and stability of each solution. We find that while the CAPM-REE solution (i.e., the Srarpe-

[2] Other approaches for the equilibrium analysis of asset pricing under incomplete information started with Dothan and Feldman (1986), Detemple (1986), Gennotte (1986), Feldman (1989, 1992), Kurz (1994, 1997), Coles, Loewenstein, and Suay, (1995), and others. In general, the objective is to solve explicitly for the unobservable moments endogenously and the solution shows that the realized process is a function of the unobservable (conditional) moments of the production factors, production function and agents' first and second utility function derivatives. This result holds in our economy as well and extended to a multiple asset CAPM in Kedar-Levy (2004).

Lintner-Mossin CAPM) is the only long-term stable solution, a few transitional equilibria exist, obtainable through more than one path. The complex dynamics arise from a nonlinear transitional return generating function, which does not alter investors' rationality.

We distinguish between two information sets, which heuristically and without loss of generality are associated with the introduction of new technologies. In the first information-set, we assume that new technologies are return distributions for which neither historical parameter estimates nor comparables exist thus only *market returns* are observable, giving rise to a specific learning model throughout the "Introductory Phase." Eventually though, all technologies enter the "Real Phase" where they yield observable *real profits*, hence define a new information-set and a different learning model applies. We show that since the price dynamics throughout the Introductory Phase is unstable, it does not converge to the CAPM solution. Only in this phase new technology share prices might diverge to bubbles obtaining "blue chip" valuations, decline to near-zero "Pink-Sheet" valuations or cycle between temporary episodes of price increases and declines. In time though, all viable technologies must generate real monetary profits that establish the linkage between rates of return and the nominal value of their stock prices. This is where "crashes" arrive; crashes in our model are significant price adjustments that might occur when technologies shift from the Introductory- to the Real-Phase; the price of past good performers adjusts downward and that of the past poor performers upwards. In the Real Phase prices converge to the only long-term stable rational expectations equilibrium: the classic CAPM solution that equates the returns in the equity market with the real profitability rate.

Recent contributions to bubbles literature address the issue in the context of irrational investors who trade with rational (informed) arbitrageurs.[3] The general notion refers to an economy where rational arbitrageurs buy an overpriced security from less informed, or irrational investors as the arbitrageurs believe they can eventually sell the asset to the irrational investors at an even higher price, thereby riding, or creating a bubble. An important recent model in this strand of literature is Abreu and Brunnermeier (2003) (henceforth AB (2003)) who show that bubbles can persist as long as arbitrageurs do not attack the bubble simultaneously. Since they have no incentive to synchronize their attack but have an incentive to ride the bubble, arbitrageurs can exacerbate the bubble until it bursts for exogenous or endogenous reasons. As in De-Long, Shleifer, Summers and Waldmann (1990) and Shleifer and Vishny (1997), the bubble in AB (2003) exists because of the presence of irrational investors. The unique and intriguing property in AB (2003) though is that their arbitrageurs *can* collectively correct the bubble, but facing the disincentive to synchronize their attack, they prefer to ride it. Apparently, the AB (2003) model and the model described below complement each other by providing a rational framework for bubbles where a bubble that emerges due to incomplete information with rationally learning investors persists in spite of the presence of rational arbitrageurs who lack an incentive to attack it.

We describe the economy in Section 2; segregate investors by intertemporal trade as a function of risk aversion, clear the market and solve for the CAPM in Section 3. In Section 4 we analyze dynamic stability in the Introductory and Real phases; describe the conditions for price cascades up to bubbles or down to Pink-Sheet valuations; analyze dynamic stability of the CAPM and relate our results to the AB (2003) model of rational arbitrage. Section 5 summarizes.

[3] Irrational models of bubbles include variants of noise trading (e.g., De-long et. al., (1990)) or overreaction to news (Jegadeesh and Titman, 1995, De-Bondt and Thaler, 1987, Daniel, Hirshleifer and Subrahmanyam, 1998). Asymmetric information or signaling models of bubbles include, among others, Allen and Gale (1992, 2000), and Allen and Gorton (1993).

2 THE ECONOMY

Consider an infinitely lived, discrete-time exchange economy with a riskless bond that yields a given rate of return *r* and a single risky asset, a stock, which is the equity claim on a real new technology asset owned by an unlevered company. The bond is available at unlimited supply whereas the stock is available at a constant supply of *N* shares, both traded in financial markets.[4] When the new technology is first sold in the financial market (IPO) neither its mean rate of return nor the return variance are known, or relevant comparables exist. Investors hold a correct belief that the return distribution is stationary with $\mu \sim N(\mu, \sigma^2)$ and $\mu_0^f > 0$ is the period zero belief. At the passage of time, investors have an increasing history of financial market return data from which they need to learn the return distribution properties. As long as the technology does not generate real profits (dividends) the technology is said to be in the "*Introductory Phase*" and "information set" refers to periodic return realizations in the stock market $\{\mu_1, \mu_2, ...\}$. When it does, the technology is at the "*Real Phase*" and "information set" is the set of market return realizations given investor's belief that the technology yields a stationary real return μ^R in the long run, a belief that is based on observations of real return. In either case, we assume incomplete information such that investors neither observe the moments of the stock price *process*, nor utilities and wealth of other investors. Investors hold a correct belief that the price process is a stationary Itô with drift $\mu_{t+\Delta t}^f = \mu(\mu_t, \mu_t^f)$ and a constant standard deviation σ^f.[5] $\mu_{t+\Delta t}^f$ is the period *t* homogenous forecast of next period return; μ_t^f is previous period forecast and μ_t is the

[4] Following Merton (1971, 1973) we assume that the stock is a claim on a real asset that represents a specific technology. Unlike Merton though, this model does not assume continuous quantity adjustment to the number of shares traded; rather, we allow clearing price to vary and assume that changes in the number of shares are exogenous.
[5] While accounting for a time-varying model of volatility is technically not too complex by augmenting a GARCH model, it turns the solution cumbersome and does not add to the fundamental reasoning of bubbles.

unconditional mean of a history of length *s* of return realizations such that

$$\mu_t = \frac{1}{s}\sum_{s=0}^{t-1}\mu_s \ . \tag{1}$$

Forecasts change over time through an adaptive expectations (learning) model of the form of (2a) throughout the Introductory Phase but through the mean-reverting model (2b) through the Real Phase.

$$\mu_{t+\Delta t}^f = \theta_t \mu_t + (1-\theta_t)\mu_t^f \ . \tag{2a}$$

$$\mu_{t+\Delta t}^f = \mu^R + \theta_t(\mu^R - \mu_t) \, . \tag{2b}$$

The learning coefficient θ_t is assumed to decrease in time, satisfying $0 < \theta_t \leq 1 \ \forall t$ and $\lim_{t\to\infty}\theta_t = 0$; μ^R, the assumed real return is not part of the information set at the Introductory Phase; it is homogenous when real returns are observable.[6, 7] We specify the Itô process as

$$\frac{\Delta P_{t+\Delta t}^f}{P_t} = \mu_t^f \Delta t + \sigma^f z_{t+\Delta t}\sqrt{\Delta t} \ , \tag{3}$$

where we emphasize that unlike conventional notation, this is not a process of *return realizations*, but a process of *forecasts*. This structure assumes that investors observe the normally distributed random *news signal* $z_{t+\Delta t} \sim N(0,1)$ as well as the prior realization P_t at the beginning of period *t*,

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[6] Evans and Honkapohja (2001) show that the assumption of a declining θ_t facilitates in many cases the convergence of a stochastic, non-linear dynamical system to its non-stochastic form, as we are about to encounter.

[7] Following Sargent (1993), the adaptive learning approach builds upon the observation that while the Rational Expectations (RE) approach assumes complete knowledge of the economy and solves for it, it is more realistic, and in some cases advantageous to assume that economic agents, acting as econometricians need to dynamically learn the parameters of the economy. Hence, they need to collect and analyze data of observable realizations in order to infer the "true" law of motion of the economy. In these models, agents establish a forecast for the next period price conditional on the expanding information set, which yields their "Perceived Law of Motion" (PLM) and apply it into their decision problem. Under Rational Expectations Equilibrium (REE) the PLM must comply with the Actual Law of Motion (ALM). Frydman and Phelps (1986) and Evans and Honkapohja (2001) provide a thorough discussion of REE conditions with macroeconomic applications. The underlying assumption in our model is that investors have a homogenous information set and beliefs, all share a single model of the economy hence we assume that in order to hold, the only remaining REE requirement is that the distribution of realized returns converges with expectations.

based on which, and based on their beginning-of-period t beliefs, μ_t^f and σ^f they forecast the next period price $P_{t+\Delta t}^f = P_t + \Delta P_{t+\Delta t}^f$. We further stress a distinction between the terms "forecast" price (equivalent to "anticipated" price) and "expected" price in order to avoid confusion with the latter's meaning as a mathematical operator. In the next section we clear the market and solve for the realized equilibrium stock price $P_{t+\Delta t}^*$ endogenously, given individual forecasts.

All investors in this economy are price-takers who have homogenous information and a time-additive utility function that satisfies Constant Relative Risk Aversion (CRRA.) Investors belong to one of two homogeneous groups that differ with respect to their degree of risk aversion. While the formal derivation of boundary conditions for the RRA coefficient of each group is detailed in the following section, we mention here that we have a group of high risk-averse investors (denoted as *C*) and a group of low risk-averse investors (group *D*.) Henceforth parameters of the price-taking investors have small letter subscripts *c* and *d*, while group parameters have a capital letter subscripts *C* and *D*.

Let $N_{k,t}$ be the number of shares held by investor *k* at *t* ($k \in \{c,d\}$), thus $S_{k,t} = N_{k,t}P_t, \forall t$ is the value of the risky asset held by investor *k* at *t*, and $\sum_k N_{k,t}P_t = NP_t = S_t$ is the market value of the stock at *t*. Let $D_{k,t} = Q_{k,t}B_t,\ \forall t$ be the bond value held by agent *k* at *t* where *Q* is quantity of bonds and *B* their price. Finally, let $\alpha_{k,t} = S_{k,t} / W_{k,t}$ and $1 - \alpha_{k,t} = D_{k,t} / W_{k,t}$ represent proportional allocation between risky and riskless assets by investor *k* at *t*. Denoting $W_{k,t+\Delta t}^f$ the total wealth investor *k* anticipates to have at the end of period $t + \Delta t$ given the estimate $P_{t+\Delta t}^f$ then

$$W_{k,t+\Delta t}^f = N_{k,t}P_{t+\Delta t}^f + Q_{k,t}B_{t+\Delta t}. \qquad (4)$$

Applying Itô's Lemma on (4) yields the optimal asset allocation rule $\alpha_{k,t}^* = \dfrac{1}{R_{W,k,t}} \dfrac{\mu_t^f - r}{\sigma^{f\,2}}$

where $R_{W,k,t} \equiv -U''(W_{k,t})W_{k,t} / U'(W_{k,t})$ is the Arrow-Pratt measure of relative risk aversion. The utility function is assumed identical for all investors, yet they differ by the RRA coefficient δ_k

$$U_k(W,t) = e^{-\rho_k t} \frac{\delta_k}{1-\delta_k}\left(\frac{W_k}{\delta_k}\right)^{1-\delta_k} . \qquad (\delta_k \neq 1) \tag{5}$$

Fixing the explicit solution of $R_{W,k,t}$ in the optimal rule yields

$$\alpha^*_{k,t} W_{k,t} = \frac{\lambda^f_t}{\delta_k} W_{k,t}, \tag{6}$$

where $\lambda^f_t \equiv (\mu^f_t - r)/\sigma^{f\,2}$, and by promoting the time index to $t+\Delta t$ in order to represent the decision-making point in time, at the beginning of the period, we obtain

$$\alpha^*_{k,t+\Delta t} W_{k,t+\Delta t} = N^*_{k,t+\Delta t} P^*_{t+\Delta t} = \frac{\lambda^f_{t+\Delta t}}{\delta_k}\left(N_{k,t} P^f_{t+\Delta t} + Q_{k,t} B_{t+\Delta t}\right). \tag{7}$$

By establishing forecasts as an adaptive expectations algorithm, which is a function of previous period state variables rather than contemporaneous ones, we obtain a closed form functional relationship between forecasts, on the right-hand-side and realized price on the left-hand-side of (7), subject to market clearing conditions as presented in the following section.

3 RISK PREFERENCES, MARKET CLEARING AND ICAPM

While the optimal rebalancing rule (7) is common to all investors, we show in Theorem 1 below that the number of shares agent k optimally trades between t and $t+\Delta t$, i.e., their intertemporal trade *in units of shares* depends on the value of their relative risk aversion parameter δ_k with respect to $\lambda^f_{t+\Delta t}$. We show that whenever one group optimally demands shares, the other group optimally supplies shares; the roles of buyers and sellers change with the sign of the random news signal $z_{t+\Delta t}$. The next step is clearing periodic excess demand and solving for the single equilibrium price at period $t + \Delta t$. In section 4 we apply rational expectations arguments and turn to analyze price dynamics.

3.1 Optimal Supply and Demand for Shares

The number of shares agent *k* holds at *t+Δt* is equal to the number of shares she held at *t*, plus a periodic trade over *Δt*: $\Delta N_{k,t+\Delta t} = N_{k,t+\Delta t} - N_{k,t}$. Replace the latter in (7), where we also replace the definition $\widetilde{D}_{k,t+\Delta t} \equiv Q_{k,t} B_{t+\Delta t}$ and solve for $\Delta N_{k,t+\Delta t}$

$$\Delta N_{k,t+\Delta t} = \frac{\lambda^f_{t+\Delta t}}{P^H_{t+\Delta t}\delta_k}\widetilde{D}_{k,t+\Delta t} + N_{k,t}\left(\frac{\lambda^f_{t+\Delta t}}{\delta_k} - 1\right), \tag{8}$$

where $P^H_{t+\Delta t}$ is some hypothetical (reservation) price. This yields a theorem of intertemporal trade:

Theorem – Intertemporal Trade

Given CRRA preferences as in (5), the functional relationship between hypothetical price $P^H_{t+\Delta t}$ *and periodic trade* $\Delta N_{k,t+\Delta t}$ *will be positive iff* $\delta_k < \lambda^f_{t+\Delta t}$ *or negative iff* $\delta_k > \lambda^f_{t+\Delta t}$*; it will be concave iff* $\delta_k < \lambda_t$ *and convex iff* $\delta_k > \lambda_t$ *and in both cases monotonic.*

Proof

At the end of each period optimal allocation must hold with prevailing prices based on the wealth conserving budget constraint, thus

$$S_{k,t} = \frac{\lambda_t}{\delta_k}\left(S_{k,t} + D_{k,t}\right),$$

where $S_{k,t} \equiv N_{k,t} P_t$ and $D_{k,t} \equiv Q_{k,t} B_t$; solve for the amount invested in the riskless asset in terms of the risky asset

$$D_{k,t} = S_{k,t}\left(\frac{\delta_k}{\lambda_t} - 1\right). \tag{9}$$

Replace (9) in $\widetilde{D}_{k,t+\Delta t} = D_{k,t}(1+r\Delta t)$ and then into (8), and reorganize to obtain

$$\Delta N_{k,t+\Delta t} = \frac{\lambda^f_{t+\Delta t}(1+r\Delta t)}{P^H_{t+\Delta t}\delta_k} S_{k,t}\left(\frac{\delta_k}{\lambda_t} - 1\right) + N_{k,t}\left(\frac{\lambda^f_{t+\Delta t}}{\delta_k} - 1\right). \tag{10}$$

A partial derivative of (10) with respect to the hypothetical price yields

$$\frac{\partial \Delta N_{k,t+\Delta t}}{\partial P^H_{t+\Delta t}} = -\frac{\lambda^f_{t+\Delta t}\delta_k(1+r\Delta t)S_{k,t}\left(\delta_k/\lambda_t - 1\right)}{(P^H_{t+\Delta t})^2\delta_k^2}, \tag{11}$$

which guaranties a positive slope (purchase of units of shares if $\Delta P^H_{t+\Delta t} > 0$) iff the numerator is negative, thus $\delta_k < \lambda^f_{t+\Delta t}$ must hold, and a negative slope (sell units of shares if $\Delta P^H_{t+\Delta t} > 0$) iff the numerator is positive, thus $\delta_k > \lambda^f_{t+\Delta t}$ holds. A second partial derivative of (11) yields

$$\frac{\partial^2 \Delta N_{k,t+\Delta t}}{\partial (P^H_{t+\Delta t})^2} = \frac{2\lambda^f_{t+\Delta t}(1+r\Delta t)S_{k,t}}{(P^H_{t+\Delta t})^3}\left(\frac{1}{\lambda_t} - \frac{1}{\delta_k}\right), \tag{12}$$

which is negative iff $\delta_k < \lambda_t$ or positive iff $\delta_k > \lambda_t$, thus (10) is concave for $\delta_k < \lambda_t$ and convex for $\delta_k > \lambda_t$, and in both cases monotonic. *Q.E.D.*

The Theorem formalizes intertemporal optimal trade in units of shares through heterogeneity of preferences. Intuitively it implies that investors whose RRA is lower than the anticipated market price for variance risk, denoted by subscript *D*, ($\delta_D < \lambda^f_{t+\Delta t}$) demand *units of shares* if the stock price is anticipated to increase, but will sell shares if the price is anticipated to decline. The economic motivation for this trade pattern stems from the result that given the anticipated increase in $P^H_{t+\Delta t}$, the low risk aversion of group *D* investors implies an optimal position in the risky asset at period $t+\Delta t$ ($N^*_{D,t+\Delta t}P^H_{t+\Delta t}$) that is greater than the anticipated value appreciation of their period *t* nominal number of shares ($N_{D,t}P_{t+\Delta t}^{H}$). As a result, they need to buy shares in order to obtain the optimal asset allocation and the opposite holds when the price is anticipated to decline. Hence, low risk-averse investors (group *D*) apply, de-facto, an *optimal Trend Chasing* strategy: they buy shares upon anticipation for price increase and sell shares when they anticipate a price decline. We therefore dub the low risk-averse investors "*the Trend group*," stressing that although Trend might be associated with a behavioral investment style, when formalized as in the intertemporal trade Theorem, it is in fact an optimal strategy.[8]

The intuition for the high risk-averse investors, which we refer to as the Contrarian group and denote with subscript *C* is a mirror-reflection of the low risk-averse investors, as discussed above. By the intertemporal trade Theorem, an investor whose RRA satisfy $\delta_C > \lambda_{t+\Delta t}^{f}$ applies an optimal asset allocation rule whereby a forecast for a positive change in price imply a *sell of some units of shares* in order to maintain an optimal asset allocation at $t+\Delta t$, though the *value* of their investment in the risky asset increases. This optimal intertemporal trade stems from the fact that for such agents the anticipated price increase makes the anticipated value of shares they will hold if the forecast materializes higher than their optimal asset allocation rule requires, thus they must sell some units of shares. Since the opposite holds for anticipated price declines, these investors apply an optimal *Contrarian* strategy, as they trade in an opposite direction to anticipated price moves. Finally, notice that each investor provides both supply and demand for shares, alternating with the anticipated price change and both strategies are mutually exclusive.

[8] Kedar-Levy (2003) solves for optimal intertemporal trade under HARA preferences for stocks and bonds with finite horizon, no learning and continuous time and Kedar-Levy (2002) solved for the stock position only under infinite horizon, HARA preferences and discrete time. Working papers are available upon request.

3.2 Market Clearing in ICAPM

If the anticipated price $P^f_{t+\Delta t}$ increases, it generates demand for shares by group *D* and supply of shares by group *C*, whereas an anticipated price decline results in a supply by *D* and a demand by *C*. Therefore, we can create a market for shares between these two investor groups and reveal a periodic clearing price $P^*_{t+\Delta t}$. Replace $N_{C,t} + N_{D,t} = N \ \ \forall t$, and $\Delta N_{K,t+\Delta t} = N_{K,t+\Delta t} - N_{K,t} \ \ \forall K \quad (K \in \{C, D\})$ and apply the clearing condition $\Delta N^*_{C,t+\Delta t} = -\Delta N^*_{D,t+\Delta t} \ \ \forall t$ in the optimal allocation rule (7). By aggregating demands we solve for the realized equilibrium share price $P^*_{t+\Delta t}$ as a function of forecasts $P^f_{t+\Delta t}$,

$$P^*_{t+\Delta t} = \frac{\lambda^f_{t+\Delta t}}{N}\left(\frac{N_{D,t}P^f_{t+\Delta t} + Q_{D,t}B_{t+\Delta t}}{\delta_D} + \frac{N_{C,t}P^f_{t+\Delta t} + Q_{C,t}B_{t+\Delta t}}{\delta_C}\right). \tag{13}$$

Transform (13) to equilibrium return realization at $t + \Delta t$ by dividing it by P_t, subtracting 1 from both sides and denoting bond holdings in terms of the equivalent stock holdings using the post-trade version of (7) at period *t*, as in the proof of the intertemporal trade Theorem. This yields

$$1 + \mu^*_{t+\Delta t}\Delta t = \frac{(\mu^f_{t+\Delta t} - r)^2 \Delta t}{\sigma^{f\,2}}\Psi_t + \frac{\mu^f_{t+\Delta t} - r}{\sigma^{f\,2}}\left(\frac{1 + r\Delta t}{\lambda_t} + \sigma^{f\,2} z_{t+\Delta t}\sqrt{\Delta t}\Psi_t\right). \tag{14}$$

where $\Psi_t \equiv \dfrac{\pi_{D,t}}{\delta_D} + \dfrac{\pi_{C,t}}{\delta_C}$, and $\pi_{C,t} \equiv N_{C,t}/N$ and $\pi_{D,t} \equiv N_{D,t}/N$ (i.e., $\pi_{C,t} + \pi_{D,t} = 1$) are proportional shareholdings of strategists *C* and *D* out of total outstanding shares, respectively. We refer to Ψ_t as the *Strategies-Weighted-Average of Risk-Tolerance*, henceforth denoted *SWART*. SWART is important in explaining equilibrium returns since the average shareholder risk-aversion normatively *determines* the required rate of return of the risky asset. The strict rank

$\delta_D < \lambda^f_{t+\Delta t} < \delta_C$ makes it possible to draw important conclusions about the dynamics of stock ownership throughout boom and crash cycles.

A few important implications stem from (14) with respect to the dynamics and equilibrium properties of realized rates of return. The first pertains to the distinction between long-term and transitional-equilibrium. Recall that the rational expectations paradigm requires (among other requirements, fn. 7), $E(\mu^*_{t+\Delta t}) = \mu^f_{t+\Delta t}$, which allows periodic realizations to differ from forecasts; this attribute is absent from many asset-pricing models that implicitly assume instantaneous equality of expectations and realizations, by construction. In our model, periodic equilibrium is granted by clearing the market and solving for realized return (14), while the adaptive expectations models (2a,b) drive the convergence between forecasts and realizations, asymptotically satisfying REE. We refer to convergence or divergence processes as Transitional-Phases.

Second, the expected value of realized return (14) throughout the Transitional-Phase is

$$1 + E(\mu^*_{t+\Delta t})\Delta t = \frac{(\mu^f_{t+\Delta t} - r)^2 \Delta t}{\sigma^{f\,2}} \Psi_t + \frac{\mu^f_{t+\Delta t} - r}{\sigma^{f\,2}} \left(\frac{1 + r\Delta t}{\lambda_t} \right), \tag{15}$$

and the variance of (14) is

$$Var(\mu^*_{t+\Delta t}) = \frac{(\mu^f_{t+\Delta t} - r)^2}{\sigma^{f\,2}} \Psi_t^2 . \tag{16}$$

Notice that the variance of return realizations throughout the Transitional-Phase increases with the proportion of Trend investors and declines with the proportion of Contrarian investors since (16) increases with SWART and SWART increases with $\pi_{D,t}$. Since Trend investors buy shares (from the Contrarians) as prices increase and sell them back as prices decline, their proportion among shareholders increases in a bull market and declines in a bear market. This prediction of the model fits the general observation of asset prices where return variability is high in bull and low in bear markets, when Contrarian investors dominate.

The third important implication from (14) refers to its REE conditions, stated as $\mu^E \equiv E(\mu^*_{t+\Delta t}) = \mu^f_{t+\Delta t} = \mu^f_t = \mu_t = \mu^R$ and $\sigma^E \equiv \sigma^f = \sigma_t$, where the "*E*" superscript represents REE values. These conditions imply consistency of market mean realizations over time, with forecasts and with the real, non-market return. Applying the REE conditions in (14) yields

$$E(\mu^E) = r + \frac{{\sigma^E}^2}{\Psi^E}, \tag{17}$$

$$Var(\mu^E) = {\sigma^E}^2, \tag{18}$$

which is *a linear, stationary CAPM* where the mean expected return is *endogenously* determined by risk preferences of the aggregate investor and the REE variance of the single risky technology. An extension of this CAPM to multiple assets and its equilibrium implications to the classic CML and SML are analyzed in Kedar-Levy (2004) in the context of financial market anomalies.

Equation (17) indicates that SWART, variance and the riskless return determine the equilibrium rate of return of an individual asset when the long term REE conditions are satisfied. Under REE, a greater proportion of high risk-averse investors (or higher degree of risk aversion by any investor group), imply a higher required rate of return as a compensation for bearing risk. It turns out that equation (14) facilitates an analysis of deviations from the REE solution (17) while (16) and (18) measure the variances without, and with the REE condition, respectively.

The last implication from (14) is that the realized equilibrium return $\mu^*_{t+\Delta t}$ is quadratic in the forecast risk premium $\mu^f_{t+\Delta t} - r$, and obtains more complex structures with no explicit solution when the learning models replace $\mu^f_{t+\Delta t}$. In the next section we show that this property results in unstable price dynamics when the adaptive expectations model is (2a) but under (2b) it converges to the stable, long-term Sharpe-Lintner-Mossin CAPM with no bubbles.

4 BUBBLES AND CRASHES

4.1 Separating Equilibria and Bubbles at the Introductory Phase

Throughout the introductory phase of a new technology, investors' information set is limited to returns generated in financial markets as the technology does not generate real profits. If investors apply learning model (2a), the CAPM relationship between risk and return can only be measured in percentage terms since no real profits link between market, percentage returns and real, monetary returns. This implies that while stock *price* can be calculated, it is uninformative as market values of the new technology are arbitrary. We therefore refer to the equilibrium functional relationships throughout the introductory phase as *temporary-percentage-CAPM*.

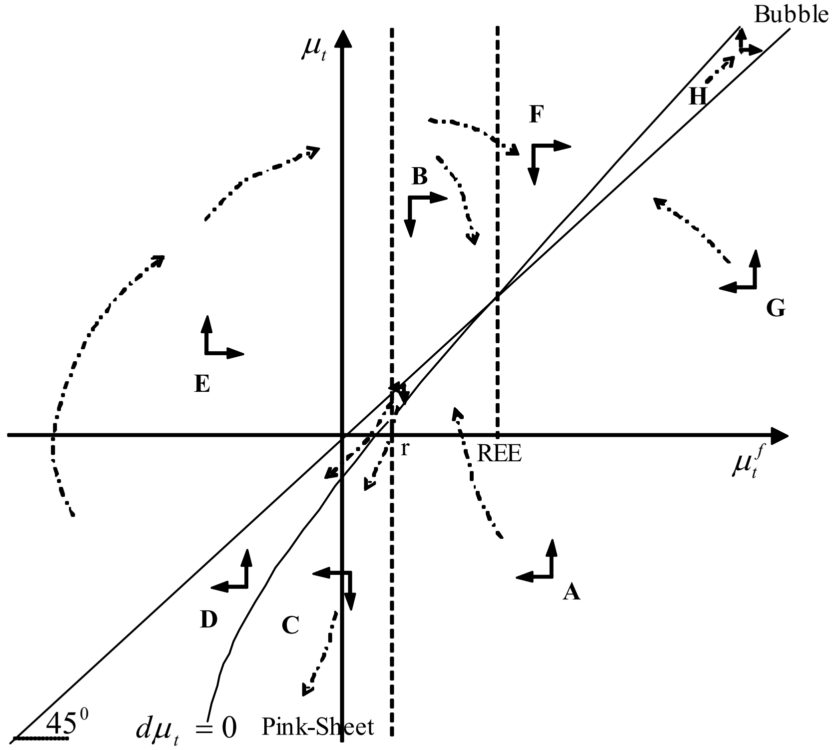
Once the learning algorithm (2a) has been introduced into (15), the time-dynamics of forecasts and realizations result in a four-dimensional non-linear system in μ_t, μ_t^f, Ψ_t and σ_t^2 (we assume $\Delta t = 1$).

$$1 + E(\mu_{t+\Delta t}^*) = \frac{(\theta_t \mu_t + (1-\theta_t)\mu_t^f - r)^2}{\sigma^{f\,2}} \Psi_t + \frac{(\theta_t \mu_t + (1-\theta_t)\mu_t^f - r)(1+r)}{\mu_t - r} \frac{\sigma_t^2}{\sigma^{f\,2}} \qquad (19)$$

This system cannot be solved explicitly for the most important state variables μ_t and μ_t^f, thus we turn to a numerical approximation. We ignore the crowding out and variance effects (Ψ_t, σ_t^2) since Ψ_t has minor effect on the main results and σ_t^2 can be learned fast. We also ignore the stochastic nature of the system given the property that under $\theta_t \to 0$, many stochastic systems converge to their non-stochastic form. Finally, we assume $\theta_t \to 0$ and constant since changes in θ_t are marginal and since we focus on the most influential state variables μ_t^f and μ_t. The return dynamics in the μ_t^f, μ_t plane is analyzed with the phase diagram in Figure 1.

Figure 1

Separated Equilibria throughout the Introductory Phase



The phase diagram shows that return dynamics will be unstable, resulting in one of the following: 1) Bubble path, with no price limit (stocks move from F or G to H); 2) decline to near zero "Pink-Sheet" valuation (from A or B to C) or 3) recurring cycles of price increases and declines (from A or B to D, to E then to B and again to D). Parameters assumed: $\delta_D = 2; \delta_C = 3; \pi_{C,D} = 0.5; r = 2\%$

Denoting by $d\mu_t^f \cong \mu_{t+\Delta t}^f - \mu_t^f$ the change over time of forecasts given (2a) and by $d\mu_t \cong E(\mu_{t+\Delta t}^*) - \mu_t$ the return realizations given the implementation of (2a) into (15), we find that $d\mu_t^f \cong \theta_t(\mu_t - \mu_t^f)$, which implies that forecasts will adjust downward (horizontally leftward) when $\mu_t^f > \mu_t$ (below the diagonal) and upward (rightward) when $\mu_t^f < \mu_t$ (above the diagonal). The condition for $d\mu_t = 0$ cannot be solved explicitly, but a numerical solution yields the approximate line "$d\mu_t = 0$" in Figure 1 for the parameters in its legend, which is discontinuous at the riskless return since $\mu_t = r$ is a singular point in the denominator of the right-most element in (19). Changes in μ_t are represented by the vertical arrows of the phase-diagram. The system is unstable about the (unstable) CAPM-REE, from which two separated divergent price paths can emerge, heuristically referred to as "Bubbles" and "Pink-Sheet." We illustrate the dynamical properties of the system below:

Section A: Shares for which investors' forecasts are above the riskless return, whether their realized return was positive (but under the lowest between the curve $d\mu_t = 0$ and the diagonal) or negative, will increase in value (left-up direction) until they meet either the curve or the diagonal. We start by analyzing the shares that meet the curve first, between *r* and REE. Once they meet the curve these stocks do not shift up anymore but still shift left due to learning, thus enter a declining price process toward the riskless return *r*.

Sections B: Shares that performed above expectations, and expectations were between *r* and REE, decrease in value either toward the diagonal (between *r* and REE) or toward the line $d\mu_t = 0$ (Section F). We analyze Section B first, where shares that meet the diagonal start moving left and down, toward *r* together with shares of Section A. As they meet *r* they will crash below it since the demand for a risky asset is zero if the anticipated equity premium is zero.

Section C: As they crash below *r*, shares might be in either Section C or D. If in section C, the shares will decline to near-zero value due to both selling pressure and learning. Since most investors will not hold these assets, the shift to the left due to learning might be slow, resulting in "Pink-Sheet" equilibrium of neglected shares that are not attractive, thus illiquid.

Sections D and E: Whether declined below *r* to section D or shift from section C to D through learning, shares in this section will shift upward and left toward section E, and once above the diagonal will shift rightward and up until they cross the riskless return as they generate positive returns. Once crossing *r*, they either start the Pink-Sheet cycle again through section B or shift to section F.

Sections F and G: Shares in section F shift rightward and down until they meet the curve $d\mu_t = 0$. While on the curve they do not shift downward, but the learning process shifts them to the right, into section H. Shares in section G shift upward and left until they too enter section H.

Section H: Shares in this section enter a Bubble since they shift rightward and up as long as the technology is in the introductory phase and the learning process is (2a). There is no limit to the booming price process, and we know from (17) that the return variability increases since the crowding-out process intensifies.

The above description of return dynamics predicts that after a new-technology IPO, as long as the true, real profitability rate is unknown, learning the REE based on historical realizations leads to one of three price patterns. 1) Stocks that perform above their unobservable, CAPM-REE (Section F) or stocks for which investors' forecasts are above the CAPM-REE (Section G) will diverge toward a bubble (Section H). 2) Stocks that perform better than the riskless return but less than the CAPM-REE, or stocks for which investors' forecasts are bounded in this range (Sections B and A, respectively) will drift either to near-zero, "Pink-Sheet" valuations (Section C) or to Section D, which represents a significant loss of value as well. Shares in Section C will eventually shift to Section D through learning, though the process might be lengthy. 3) Low valued stocks in Section D will shift in time to Section E and then to either B, starting the cycle again, or to F, which leads to a bubble.

These results seem to fit general attributes of new-technology IPOs in three major aspects. First, there is no upper limit to the valuation of new-technologies as long as they do not generate real profits, as exhibited in the internet bubble. Second, There is a large group of technology stocks that decline to near-zero valuations ("Penny stocks"), rarely traded and reported in the Pink-Sheets. Third, the model predicts that in order to increase the probability of generating high mid-term valuations, i.e., running into a bubble price process, entrepreneurs should undervalue their stocks at IPO. Entrepreneurs who adopt this strategy increase the probability that their stock either, perform above expectations in Section F, or that expectations will be above the (unobservable) REE, in Section G, and in both cases leading to a bubble, through Section H.

4.2 Return Dynamics at the Real Phase

Assume that the technology shifts to the real phase by generating real profits and investors agree on the expected value of the stationary real profitability rate μ^R, which is applied into learning model (2b). For the reasons mentioned in the preceding section we ignore the effects of stochastic noise, changing θ_t and crowding-out. Defining the approximate price dynamics as $d\mu_t \cong E(\mu^*_{t+\Delta t}) - \mu_t$ and solving (15) with (2b) we obtain:

$$\begin{aligned} d\mu_t \cong\; & \mu_t^2 \frac{\theta_t^2 \Psi_t}{\sigma^{f^2}} + \mu_t \left[\left(2r\theta_t - 2(1+\theta_t)\theta_t \mu^R\right) \frac{\Psi_t}{\sigma^{f^2}} - \frac{\theta_t \sigma_t^2 (1+r)}{\sigma^{f^2}(\mu_t - r)} - 1 \right] \\ & + \frac{((1+\theta_t)\mu^R - r)\sigma_t^2 (1+r)}{\sigma^{f^2}(\mu_t - r)} + \left((1+\theta_t)^2 \mu^{R^2} + r^2 - 2(1+\theta_t)\mu^R r\right) \frac{\Psi_t}{\sigma^{f^2}} - 1 \end{aligned}, \tag{20}$$

which is not available in a closed form, but a numerical solution yields

$$d\mu_t \cong \begin{cases} d\mu_t < 0 & \text{if } \mu_t > \mu^R \\ d\mu_t = 0 & \text{if } \mu_t = \mu^R \\ d\mu_t > 0 & \text{if } \mu_t < \mu^R \\ d\mu_t < 0 & \text{if } \mu_t < r \end{cases}. \tag{21}$$

The forecast dynamics can be approximated by $d\mu_t^f \cong \mu_{t+\Delta t}^f - \mu_t^f$, which yields zero movement when equated to zero, yielding the negatively steep linear line (22)

$$\mu_t = \mu^R + \frac{\mu^R - \mu_t^f}{\theta_t} \quad \text{for } d\mu_t^f = 0\,, \tag{22}$$

which passes through the point $\mu_t = \mu^R$ when $\mu_t^f = \mu^R$. This point is a Rational Expectations Equilibrium point since it implies that market returns, forecasts and real returns are equal. All stocks that lie on the right of the (almost vertical) line (22) will move leftward, and stocks on its left will move rightward due to the learning process (2b).

The return dynamics for profitable technologies as in Figure 2 can be described as follows:

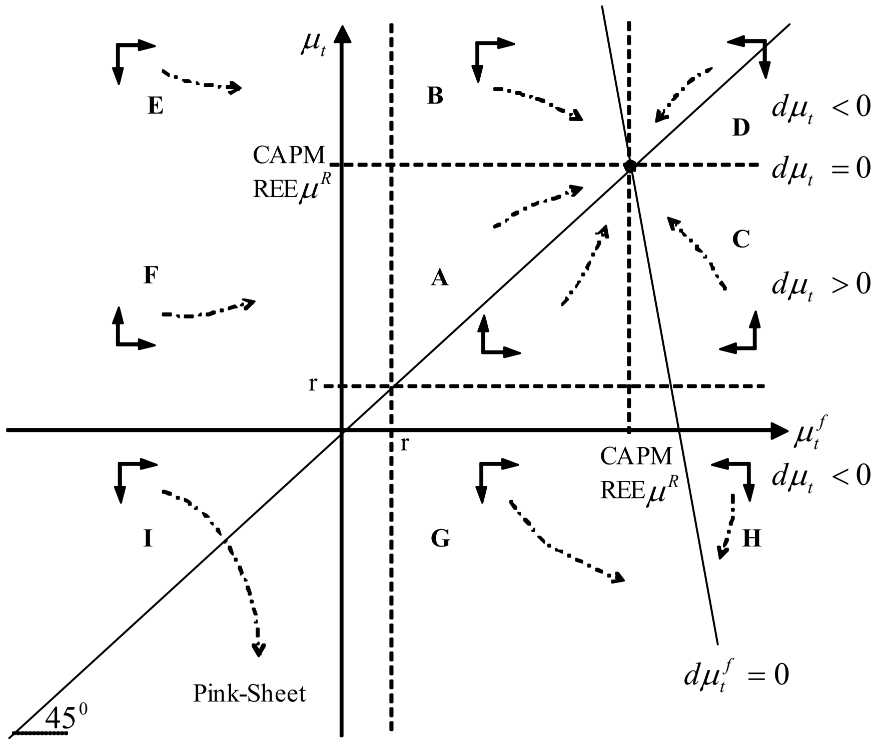
Sections A, B, C and D: Stocks for which the return forecast is greater than *r*, either left of the linear boundary line $d\mu_t^f = 0$ (Sections A, B) or on its right (Sections C, D) will move toward the CAPM REE point either through a process of price increase (Sections A, C) or price decline (Sections B, D). This implies that the long-term CAPM REE is dynamically stable (sink point).

Sections E and F: Learning process (2b) will make shares in Sections E and F move rightward since they generate positive equity premium while investors' forecasts are temporarily negative. Stocks that earn above REE returns will decline (Section E) while stocks that earn positive equity premium less than the CAPM REE will increase in value, converging to the stable solution through Sections A and B.

Sections G and H: Stocks in Sections G and H generated less than riskless returns in the market, though investors' forecasts are still above riskless return. These Sections are temporary, as market realizations must exceed the riskless return for all viable technologies. These shares will be sold and generate further negative returns as long as they are in these temporary Sections.

Figure 2

Convergence to CAPM throughout the Real Phase



Under the learning algorithm (2b) throughout the Real Phase all shares that generate real returns above the riskless return will converge to the long-term, stable CAPM REE. Stocks that temporarily generate less than riskless returns will obtain near-zero Pink-Sheet valuations until they generate positive returns. Non profitable technologies are excluded from the analysis as they trivially perish.

Section I: Stocks in Section I have not yet transferred their positive real profitability prospects to market returns and unlike stocks in Sections G and H, investors' forecasts are yet negative. As long as mean return realizations are below *r*, they will be neglected and remain in a "Pink-Sheet" category.

Our conclusions from the return dynamics in the Real Phase are straight-forward: the CAPM-REE is the only long-term stable equilibrium. Pink-Sheet valuations are transitional.

4.3 The Crash

We concluded above that when investors' information set is limited to market returns and investors apply learning model (2a), bubbles might evolve with no limit on stock price. The absence of real return data implies that market valuations are not necessarily "too high" or "too low," since only upon the revelation of the real rate of return the monetary value of the stock price will be comparable with its real value, and thus informative. We therefore infer that the "crash" of a bubble is merely the process of adjusting the stock-market price of a share to its real value once the real value is observable. This notion suggests that we can acknowledge the existence of a bubble ex-post, but not ex-ante, if in the term "bubble" we refer to an overpriced security since "overpricing" is defined when a benchmark exists. This argument also implies that investors are not-necessarily irrational when buying new-technology shares over a bubble. Throughout the Introductory Phase prices are not informative and thus are not "overpriced" or "underpriced".

4.4 *Crowding Out*

Whenever a price trend such as a prolonged bull or bear market persists for a stock, proportional holdings in the stock might change through trade in a systematic way that affects the required rate of return. This notion is related to the role of arbitrageurs, as presented in the next section. It follows from the Intertemporal Trade Theorem that trade takes place such that investors of group *D* buy shares from group *C* upon anticipation for price increase, and sell shares to group *C* upon anticipation for price decline. This is a bi-directional crowding-out effect that stems, and therefore is consistent with, the optimal asset allocation rules of both investor groups given their RRA.[9] This result is relevant to our analysis since it implies that low-risk-averse investors of group *D* will constitute the majority of shareholders in a booming stock, while the opposite holds for low-priced stocks. If upon the transition from the Introductory Phase to the Real Phase prices decline, the sellers will mainly be of group *D*. Conversely, a stock that was "underpriced" will

[9] This result is related to Wang (1996) who showed that less risk-averse agents (here Trend) crowd-out the more risk-averse agents (here Contrarian) completely out of the market in finite time. In our model however, asset pricing is endogenous and a function of weighted average risk aversion, resulting in an endogenous stable allocation at REE.

experience price appreciation where its patient, group C investors will be the sellers. Upon convergence to the CAPM-REE, the allocation between groups C and D will be identical if both shares generate identically distributed real returns.

4.5 The Role of Arbitrageurs

As described in the introduction, arbitrageurs create, or ride a bubble in models with irrational investors who are willing to trade improperly priced assets. The general notion is that arbitrageurs close gaps between "fundamental" and market prices, which implies that fundamental prices are observable and arbitrageurs can, and want to play this role. Abreu and Brunnermeier (2003) extended results from the literature of limits of arbitrage and showed that though arbitrageurs can collectively crash a bubble, they have no incentive to do so. In this subsection we are interested in a qualitative answer to the question: Does the AB (2003) result hold in an economy populated with only rational investors?

The major underlying assumptions in AB (2003) are first, that the bubble, defined as an exponential price increase at a rate higher than the fundamentally justified rate, is exogenously given and supported by irrational investors. Second, arbitrageurs weigh cost and benefit from riding the bubble, thus ride it as long as the benefit exceed the cost, and sell-out when the cost, in terms of loosing the bubble profits at the crash, are greater than the benefit. Third, arbitrageurs are heterogeneous with respect to the timing of being aware that the bubble exists, a characteristic that may proxy heterogeneity of beliefs, preferences, asymmetric information or other attributes. Fourth, individual arbitrageurs have limited price impact, but their aggregate impact can crash the bubble. In this setup, the important implication of the AB (2003) model is that arbitrageurs need to coordinate their attack on the bubble in order to burst it, but since they lack the incentive to do so, the bubble can grow until it bursts either due to endogenous or exogenous reasons, depending on the specific parameters.

We now map the AB (2003) assumptions with our model and ask whether in principle the two models can coexist; if they do, than there is a possibility that the model presented above can endure the presence of rational arbitrageurs, and the AB (2003) model can be cast with only rational investors. For the first assumption, our model shows that while the long-term CAPM-REE growth rate is the only sustainable, no-bubble solution, temporary episodes of higher than REE growth rates in asset prices evolve throughout the introductory phase of a new technology due to the absence of real return data. It appears than that the AB (2003) assumption that irrational investors cause and support the bubble can be replaced with our result. The second assumption whereby arbitrageurs ride a bubble as long as it is profitable is natural and need not be changed. The third assumption concerns heterogeneity of arbitrageurs and it can be mapped into our model as we recall that: 1) Trend investors, who have low RRA coefficient, increase their positions in the risky asset as its price appreciates, and sell out when they anticipate a price decline. This trend-chasing is optimal, and corresponds with the AB (2003) arbitrageurs. 2) Though we assumed that all our Trend (and Contrarian) investors are group wise homogeneous with respect to RRA, this assumption can be replaced with multiple investors who may be trivially aggregated. Thus, the AB (2003) arbitrageurs can be rational CRRA investors with RRA coefficient lower than the Trend group average, which makes them the lowest risk-averse investors among all market participants. As such, low risk-averse investors react less patiently to news – they will be first to buy and sell given positive and negative news, in accordance with stylistic facts. The fourth assumption, of the potential individual and aggregate price impact of arbitrageurs corresponds with our CAPM, which solves trade, liquidity and price impact in a general equilibrium setup.

Our conclusion from the above is that the AB (2003) model can in principle, be framed within a CAPM with rational investors, and our CAPM can, in principle, incorporate rational arbitrageurs, as both models are complementary. The technical solution is far more complex.

5 SUMMARY

We present a dynamic asset pricing model based on Merton (1971) budget dynamics under incomplete, yet symmetric information, and rational investors who differ in their risk aversion such that their optimal asset allocation strategies result in intertemporal trade in units of shares. We endogenously solve for an equilibrium price by clearing the market and find that the realized price process is lognormal, hence generalizing the Merton (1971) ICAPM. Unlike models that solve for the specific Rational Expectations Equilibrium (REE) price under the implicit assumption that investors possess all required information to reveal it, or that the price is given exogenously, we assume that agents dynamically learn the REE solution. Dynamic adaptive expectations reveal the process by which prices converge to the CAPM-REE solution, yet we find that convergence is not a guarantied result. We distinguish between learning in the Introductory Phase of a new technology, when only market returns are available, and in the Real Phase, when the technology generates real profits. We find that bubbles can only emerge throughout the Introductory Phase, if investors' forecasts and/or market realizations are above the CAPM solution. If not, technology shares after IPO might decline to near-zero, "Pink-Sheet" valuations. Both the bubble and Pink-Sheet valuations are temporary since all technologies eventually generate real profits. When they do, market prices promptly adjust being the "crash," if there was a bubble or a "come-back", if the asset was neglected at near-zero valuations. Once in the Real Phase, all technologies converge to the stable CAPM solution. Other than dynamic learning, we do not relax assumptions on investor rationality throughout.

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