Are Sunspots Inevitable?

Aditya Goenka[*]
(University of Essex)

Christophe Préchac[**]
(University of Paris I)

This version: November 21, 2002

Abstract: This paper examines the welfare of consumers in an incomplete markets economy with extrinsic uncertainty. It is shown that the utility of one consumer may be *minimized* at the Walrasian allocation relative to all other equilibrium allocations for a given security structure. Thus, this consumer will have no incentive to trade the new securities if they complete the insurance markets.

Keywords: Sunspot equilibrium, incomplete markets, financial innovation, welfare analysis.

JEL Classification Numbers: D52, E32, G10, D60.

∗ Department of Economics, University of Essex, Wivenhoe Park, Colchester, CO4 3SQ, U.K. *EMail*: goenka@essex.ac.uk

∗∗ CERMSEM, Maison des Sciences Economiques, Université Paris I Panthéon-Sorbonne, 106-112 Boulevard de l'Hôpital, 75647 Paris cedex 13, France. *EMail*: prechac@univ-paris1.fr

1 Introduction

One of the key ideas in the literature on sunspot equilibrium or extrinsic uncertainty is the "Philadelphia Pholk Theorem," [15] which says that if there is any distortion in the economy which renders equilibrium outcomes inefficient, then it is very likely that sunspots will have non-trivial effects. This has been confirmed in a wide variety of environments including restricted participation economies, the double infinity of overlapping generations economies, incomplete markets, externalities, public goods, rationing, bounded rationality, etc. A counterpart to the PPT is the sunspot ineffectivity result first due to David Cass and Karl Shell [6] and clarified in [12] (see also [2]) which says that in finite dimensional competitive economies, if consumers are risk averse, the set of feasible allocations convex, and markets complete, then sunspots will not matter. Thus, in finite economies the effect of sunspots hinges on these three conditions being satisfied. The first two conditions are given by the primitives of the model. The third is to an extent endogenous. If markets are incomplete, then a natural question is why do they remain so if introducing new securities can neutralize the sunspots so that the non-sunspot contingent Walrasian allocations are the only possible outcomes?

While the literature on financial innovations (which focusses primarily on the situation with intrinsic uncertainty) sheds some light on this issue (see [1] and [9] for surveys), in the sunspots literature a compelling viewpoint is that as soon as markets are complete, consumers can always introduce new sunspot variables to coordinate on and hence, the markets become incomplete once again ([4]). This however does not answer when will new securities be introduced. This paper also addresses the issue of market incompleteness. We take the viewpoint that if markets are incomplete, and financial innovation is to take place, we should ask whether the securities will in fact be traded in the market. We show there may be no incentives for the consumers to trade the new securities and hence, the economy will continue to be susceptible to sunspots in the face of financial innovations. This result is stark in that one consumer will prefer *any*

sunspot allocation to the Walrasian allocation and thus, have no incentive to trade the completing securities.

The environment we consider has only extrinsic uncertainty. We show in a one-good economy with only extrinsic uncertainty ([6]) and two consumers, the utility of one consumer is maximized while that of the other is minimized at the Walrasian allocation. This result is global under some preferences structures, and is local under general preferences (if some restrictions are satisfied). Thus, if the status quo has market incompletenss, any financial innovation that will complete the markets is unlikely to take place. We first give a parametric example and then generalize the result to an open class of preferences (consumers must have a sufficiently high precautionary savings motive ([14])) and general endowments. As we look at the case of one good in each state, effects relating to changes in relative prices within each state are not the key to the welfare effects (for multiple goods models see [13], [8], [5], and [10]).

We take the viewpoint that the consumers will only trade if the new allocation lies in the upper contour set of their status quo outcomes, i.e., trade is individually rational. This requires consumers to compare allocations across different security structures. As we are looking at rational expectations equilibria, it is already assumed that consumers understand the equilibrium map. We are, thus, requiring the consumers to use this information. To firmly ground this on strategic foundations, in the spirit of Cournot-Walras models of financial innovation (see for example [1]), we could model a two stage trading game where in the first stage consumers decide which securities to trade taking into account that they will act competitively *given* the set of securities. Alternatively, one could also think of the problem as one where a planner has to choose whether to introduce new securities, and the innovation takes place only if it makes no one worse off. Another interpretation would be that there is a two stage procedure where the consumers first vote whether to introduce a security, and then given the outcome, trade competitively. We are open about the interpretations and thus consider only the outcome of the second stage.

The result on the welfare properties of the equilibrium outcomes is consistent with what is commonly known about one-good incomplete market economies: the equilibrium outcomes are constrained Pareto efficient ([7], and [11]). However, as mentioned above we have a stronger statement on the Walrasian allocation vis-a-vis other equilibrium allocations that may emerge.

The plan of the paper is as follows. First we define the economy and then present a parametric example where the entire equilibrium set is characterised and the welfare effects identified. This is extended first to general preferences, and then to include non-corner endowments as well.

2   Economy with Financial Assets

Consider a pure-exchange economy with 2 periods. In period one there is one state $s = 0$, and $S < \infty$ states in the second period, $s = 1, \ldots, S$. These are indexed by the superscript s. In each state there is a single consumption good. There are 2 consumers indexed by the subscript $h = 1, 2$. The consumption plan for consumer h is $x_h = (x_h^0, x_h^1, x_h^2, \ldots, x_h^s, \ldots, x_h^S)$. The consumption set for the consumers, X_h, is the $S + 1$ dimensional positive orthant. Both the consumers have identical preferences represented by the utility function:

$$u_h(x_h) = v(x_h^0) + \frac{1}{S} \sum_{s=1}^{S} v(x_h^s). \tag{1}$$

The sub-utility functions $v_h(\cdot)$ are strictly increasing, strictly concave, and thrice-continuously differentiable. The endowments of the two consumers are $\omega_1 = (\alpha, 1 - \alpha, \ldots, 1 - \alpha)$ and $\omega_2 = (1 - \alpha, \alpha, \ldots, \alpha)$ respectively, with $\alpha \in (0, 1]$. The consumers can transfer wealth across the states using a nominal bond. The return matrix of the nominal bond is $R = (-1, 1, \ldots, 1)^T$. The purchase of the nominal bond for consumer h is denoted as θ_h, and the excess demand for commodities by z_h. The prices of the consumption goods are normalized so that $p = (1, p^1, \ldots, p^s, \ldots, p^S)$.

The budget constraints are given by:

$$z_h^0 + \theta_h = 0 \tag{2}$$

$$p^s z_h^s = \theta_h, \quad s = 1, \ldots, S \tag{3}$$

Definition 1: An GEI equilibrium in the economy is a vector (p, θ_1, θ_2) such that

(i) θ_h maximizes utility (1) for the consumers subject to the budget constraints (2-3).

(ii) The bond market clears, i.e., $\theta_1 + \theta_2 = 0$.

Definition 2: Sunspots do not matter if the allocations are independent of the states in period 2, i.e., if:

$$x_h^1 = x_h^2 = \cdots = x_h^S, \quad h = 1, 2.$$

3   The Leading Example

In this example, $S = 2$, $\alpha = 1$, and the preferences of the two consumers are restricted to be log-linear, i.e.,

$$u_h(x_h) = \log x_h^0 + \frac{1}{2}\left(\log x_h^1 + \log x_h^2\right). \tag{4}$$

To solve for the equilibria first solve for the demand of the two consumers.

For consumer 1, substitute the budget equations into the maximand to get

$$\text{Max} \quad \log(1 - \theta_1) + \frac{1}{2}\log\frac{\theta_1}{p^1} + \frac{1}{2}\log\frac{\theta_1}{p^2}$$

$$\Leftrightarrow \quad \text{Max} \quad \log(1 - \theta_1) + \log\theta_1 \tag{5}$$

Thus, $\theta_1^* = \frac{1}{2}$, which is independent of p^1, p^2.

For consumer 2, the maximization problem is:

$$\text{Max}\quad \log(-\theta_2) + \frac{1}{2}\log\left(1 + \frac{\theta_2}{p^1}\right) + \frac{1}{2}\log\left(1 + \frac{\theta_2}{p^2}\right)$$

The first order condition for consumer 2 is:

$$-\frac{1}{\theta_2} = \frac{1}{2p^1}\frac{1}{\left(1+\frac{\theta_2}{p^1}\right)} + \frac{1}{2p^2}\frac{1}{\left(1+\frac{\theta_2}{p^2}\right)}$$

$$\text{or}\quad 4p^1 p^2\left(1+\frac{\theta_2}{p^1}\right)\left(1+\frac{\theta_2}{p^2}\right) = -\theta_2\left(2p^1\left(1+\frac{\theta_2}{p^1}\right) + 2p^2\left(1+\frac{\theta_2}{p^2}\right)\right) \quad (6)$$

Market clearing in the bond market implies $\theta_1 = -\theta_2$, thus $\theta_1^* = \frac{1}{2}$ implies that in equilibrium it must be the case that $\theta_2^* = -\frac{1}{2}$.

Substitute this into the first order condition above to derive the equilibrium equation:

$$\left(2p^1 - 1\right)\left(2p^2 - 2\right) - \frac{1}{2}\left(2p^1 - 1\right) - \frac{1}{2}\left(2p^2 - 1\right) = 0$$

$$\text{or}\quad \left(2p^1 - 1 - \frac{1}{2}\right)\left(2p^2 - 1 - \frac{1}{2}\right) = \frac{1}{4}$$

$$\Rightarrow\quad \left(p^1 - \frac{3}{4}\right)\left(p^2 - \frac{3}{4}\right) = \frac{1}{16}. \quad (7)$$

There are, however, two solutions to the equilibrium conditions. The branch through $(\frac{1}{2}, \frac{1}{2})$ is not a solution as non-negativity conditions (for consumer 2) are violated. The only solution is the branch through $(1, 1)$. Thus, there is a unique (state) symmetric equilibrium which corresponds to the Walrasian equilibrium. In this economy, the symmetric equilibrium prices are $p_1 = p_2 = 1$.

Thus, the incompleteness of the markets is giving rise to the well known indeterminacy of equilibria ([4]). While the indeterminacy shown is price indeterminacy, there will be real indeterminacy as well.

From [3] it is known that the equilibrium allocations lie on the intersection of the offer curves of the consumers. The supporting prices (p and p' for the two consumers

respectively) however, in general need not be the same. If they are, then the economy is at a Walrasian equilibrium.

The offer curves of the two consumers in this economy are:

$$\Omega_1 = \left\{ \left(\frac{1}{2}, \frac{p^0}{4p^1}, \frac{p^0}{4p^2} \right) : \ p^0, p^1, p^2 > 0 \right\} \tag{8}$$

$$\Omega_2 = \left\{ \left(\frac{p^{1\prime} + p^{2\prime}}{2p^{0\prime}}, \frac{1}{4} + \frac{p^{2\prime}}{4p^{1\prime}}, \frac{1}{4} + \frac{p^{1\prime}}{4p^{2\prime}} \right) : \ p^{0\prime}, p^{1\prime}, p^{2\prime} > 0 \right\}. \tag{9}$$

Normalize $p^0 = p^{0\prime} = 1$. From the market clearing we know that $x_2^0 = \dfrac{1}{2}$ or from offer curve of consumer 2, $p^{1\prime} + p^{2\prime} = 1$. Substituting this into the offer curve of consumer 2, we have :

$$\Omega_2' = \left\{ \left(\frac{1}{2}, \frac{1}{4} + \frac{1 - p^{1\prime}}{4p^{1\prime}}, \frac{1}{4} + \frac{p^{1\prime}}{4(1 - p^{1\prime})} \right) : \ p^{0\prime}, p^{1\prime}, p^{2\prime} > 0 \right\}. \tag{10}$$

As we have taken the intersection of the two offer curves into account (the offer curve of consumer 1 being a plane at the x_1^0 coordinate of $\dfrac{1}{2}$), all the information of the equilibrium is contained in this equation.

Define the parameter $\lambda = \dfrac{1 - p^{1\prime}}{p^{1\prime}}$, with $\lambda \in (-1, \infty)$ (for $p^{1\prime} > 0$). The offer curve is now given by:

$$\Omega_2'' = \left\{ \left(\frac{1}{2}, \frac{1}{4} + \frac{\lambda}{4}, \frac{1}{4} + \frac{1}{4\lambda} \right) : \ \lambda \in (-1, \infty) \right\}. \tag{11}$$

From this we derive the equation for x_2^1, x_2^2 by setting the second coordinate of the above equal to x_2^1 and the third coordinate equal to x_2^2 and eliminating λ to obtain:

$$\left(x_2^1 - \frac{1}{4} \right) \left(x_2^2 - \frac{1}{4} \right) = \frac{1}{16}. \tag{12}$$

Thus, there are two solutions. The branch through $(\frac{1}{2}, \frac{1}{2})$ satisfies the equilibrium conditions. Thus, there is a 1-dimensional real indeterminacy as well.

To see examine the welfare effects, one can work with either equation (7) or equation (12). Working with the latter, it is easy to see that the Walrasian equilibrium minimizes $x_2^1 x_2^2$ subject to the equilibrium restriction $\left(x_2^1 - \frac{1}{4}\right)\left(x_2^2 - \frac{1}{4}\right) = \frac{1}{16}$, and $x_2^1, x_2^2 > \frac{1}{4}$.

Alternatively, for consumer 1, the indirect utility function

$$w_1(p^1, p^2) = \log\left(\frac{1}{2}\right) + \frac{1}{2}\log\left(\frac{1}{2p^1}\right) + \frac{1}{2}\log\left(\frac{1}{2p^2}\right)$$

can be reduced to $\zeta_1(w_1(p^1, p^2)) = -p^1 p^2$, where ζ_1 is a strictly increasing function.

The indirect utility of consumer 1 is maximized at the Walrasian prices on the equilibrium set. This can be seen, as $p^1 = p^2$ solves :

Min $\qquad\qquad p^1 p^2$

s.t. $\quad \left(p^1 - \frac{3}{4}\right)\left(p^2 - \frac{3}{4}\right) = \frac{1}{16}$

$\qquad\qquad p^1, \ p^2 \ > \ \frac{3}{4}.$

For consumer 2, the indirect utility function, up to a strictly increasing transformation is:

$$w_2(p^1, p^2) = \left(1 - \frac{1}{2p^1}\right)\left(1 - \frac{1}{2p^2}\right).$$

Using the equilibrium relation between the prices and eliminating p^2, this can be reduced to:

$$m_2(p^1) = \frac{1}{4}\frac{(2p^1 - 1)^2}{p^1(3p^1 - 2)}.$$

This function is minimized at the Walrasian prices. To see this compute

$$m_2'(p^1) = \frac{(2p^1 - 1)(2p^1 - 2)}{4(p^1(3p^1 - 2))^2}.$$

This function is decreasing in the interval $(\frac{3}{4}, 1)$ and increasing in $(1, \infty)$, with a critical point at 1. In fact, $m_2(1) = \frac{1}{4}$, and $\lim\limits_{p_1 \to \frac{3}{4}} m_2(p^1) = \lim\limits_{p^1 \to \infty} m_2(p^1) = \frac{1}{3}$.

The limiting allocations can also be computed:

$$x_1 = \left(\frac{1}{2}, 0, \frac{2}{3}\right) \quad \text{or} \quad \left(\frac{1}{2}, \frac{2}{3}, 0\right)$$

$$x_2 = \left(\frac{1}{2}, 1, \frac{1}{3}\right) \quad \text{or} \quad \left(\frac{1}{2}, \frac{1}{3}, 1\right).$$

Remarks:

1. The result on the maximum and minimum of utility at the Walrasian equilibrium is global for this economy.

2. Note that the equilibria will be constrained Pareto optimal as this is a 1-good economy.

3. The results of [11] for the one good economy do not apply as there: $H \geq \text{Max}\{2, J+1\}$ and $S \geq H + (J+1)$, where S is the number of states in the second period, J number of assets initially, and H number of consumers. In this economy $S = 2$, $H = 2$, $J = 1$.

4. The results of [10] and [5] do not apply. These results are for multiple good economies, and in [10] the restriction is : $S \geq H(J+2) + H(J+1)(J+2)/2$, while in [5] it is: $S - J \geq 2H - 1$.

5. This example can be generalized. This is the content of the next two proposi-tions.

4 Proposition 1

In this economy, for an open class of preferences the utility of consumer 2 is minimized (locally) at the Walrasian equilibrium. A sufficient condition is that consumer 2 has a sufficiently positive "precautionary savings motive":

$$\frac{v_2''}{4} + \frac{v'''}{8} > 0.$$

In other words, the Index of Absolute Prudence ([14], p. 61) should be strictly greater than two:

$$\frac{-v'''}{v''} > 2.$$

Proof:

Other than not imposing log-linearity on preferences, the structure of the economy is the same as in Example 1. As in equilibrium it must be the case that $\theta_1 = -\theta_2$, set $\theta_1 = \theta = -\theta_2$. After substitution of the budget constraints into the utility function of consumer 2 we have:

$$u_2(\theta, p^1, p^2) = v(\theta) + \frac{1}{2}v(1 - \frac{\theta}{p^1}) + \frac{1}{2}v(1 - \frac{\theta}{p^2}).$$

$$\nabla u_2(\theta, p^1, p^2) = \begin{pmatrix} v'(\theta) - \frac{1}{2p^1}v'(1 - \frac{\theta}{p^1}) - \frac{1}{2p^2}v'(1 - \frac{\theta}{p^2}) \\ \frac{\theta}{2(p^1)^2}v'(1 - \frac{\theta}{p^1}) \\ \frac{\theta}{2(p^2)^2}v'(1 - \frac{\theta}{p^2}) \end{pmatrix} \tag{13}$$

$$D^2 u_2 = \begin{pmatrix} v'' + \dfrac{1}{2(p^1)^2}v'' + \dfrac{1}{2(p^2)^2}v'' & \dfrac{1}{2(p^1)^2}v' - \dfrac{\theta}{2(p^1)^3}v'' & \dfrac{1}{2(p^2)^2}v' - \dfrac{\theta}{2(p^2)^3}v'' \\[3mm] \dfrac{1}{2(p^1)^2}v' - \dfrac{\theta}{2(p^1)^3}v'' & -\dfrac{\theta}{(p^1)^3}v' + \dfrac{\theta^2}{2(p^1)^4}v'' & 0 \\[3mm] \dfrac{1}{2(p^2)^2}v' - \dfrac{\theta}{2(p^2)^3}v'' & 0 & -\dfrac{\theta}{(p^2)^3}v' + \dfrac{\theta^2}{2(p^2)^4}v'' \end{pmatrix}$$

(14)

The equilibrium set consists of the 2 first order conditions (after substituting the market clearing conditions).

$$E = \begin{cases} \phi_1(\theta, p) & = & 0 & = & -v'(1-\theta) + \dfrac{1}{2p^1}v'(\dfrac{\theta}{p^1}) + \dfrac{1}{2p^2}v'(\dfrac{\theta}{p^2}) \\[4mm] \phi_2(\theta, p) & = & 0 & = & -v'(\theta) + \dfrac{1}{2p^1}v'(1-\dfrac{\theta}{p^1}) + \dfrac{1}{2p^2}v'(1-\dfrac{\theta}{p^2}) \end{cases}$$

(15)

Now $y = (\theta, p^2) = \psi(p^1)$, locally.

Thus, $\phi(p^1, \psi(p^1)) = 0 \Leftrightarrow \dfrac{\partial \phi}{\partial p^1} + \dfrac{\partial \phi}{\partial y}\psi'(p^1) = 0$ on E. In addition,

$$\frac{\partial \phi}{\partial p^1} = \begin{cases} -\dfrac{1}{2(p^1)^2}v'(\dfrac{\theta}{p^1}) - \dfrac{\theta}{2(p^1)^3}v''(\dfrac{\theta}{p^1}) \\[4mm] -\dfrac{1}{2(p^1)^2}v'(1-\dfrac{\theta}{p^1}) + \dfrac{\theta}{2(p^1)^3}v''(1-\dfrac{\theta}{p^1}) \end{cases}$$

(16)

$$\frac{\partial \phi}{\partial y} = \begin{pmatrix} v''(1-\theta) + \dfrac{1}{2(p^1)^2}v''(\dfrac{\theta}{p^1}) + \dfrac{1}{2(p^2)^2}v''(\dfrac{\theta}{p^2}) & -\dfrac{1}{2(p^2)^2}v'(\dfrac{\theta}{p^2}) - \dfrac{\theta}{2(p^2)^3}v'(\dfrac{\theta}{p^2}) \\[4mm] -v''(\theta) - \dfrac{1}{2(p^1)^2}v''(1-\dfrac{\theta}{p^1}) - \dfrac{1}{2(p^2)^2}v''(1-\dfrac{\theta}{p^2}) & -\dfrac{1}{2(p^2)^2}v'(1-\dfrac{\theta}{p^2}) + \dfrac{\theta}{2(p^2)^3}v'(1-\dfrac{\theta}{p^2}) \end{pmatrix}$$

(17)

At the Walrasian equilibrium $p^1 = p^2$, and $\theta = \dfrac{1}{2}$. Substituting into the above and using the expression for ψ', we have:

$$\psi' = -\begin{pmatrix} 2v'' & -\dfrac{1}{2}v' - \dfrac{1}{4}v'' \\[3mm] -2v'' & -\dfrac{1}{2}v' + \dfrac{1}{4}v'' \end{pmatrix}^{-1} \begin{pmatrix} -\dfrac{1}{2}v' - \dfrac{1}{4}v'' \\[3mm] -\dfrac{1}{2}v' + \dfrac{1}{4}v'' \end{pmatrix}$$

$$\psi' = -\dfrac{1}{-2v'v''}\begin{pmatrix} -\dfrac{1}{2}v' + \dfrac{1}{4}v'' & \dfrac{1}{2}v' + \dfrac{1}{4}v'' \\[3mm] 2v'' & 2v'' \end{pmatrix}\begin{pmatrix} -\dfrac{1}{2}v' - \dfrac{1}{4}v'' \\[3mm] -\dfrac{1}{2}v' + \dfrac{1}{4}v'' \end{pmatrix}$$

$$\psi' = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

The indirect utility of consumer 2 can be written as:

$$G(p^1) \quad = \quad w_2(p^1, \psi(p^1)) \tag{18}$$

$$\tag{19}$$

$$G'(p^1) \quad = \quad \dfrac{\partial w_2}{\partial p^1} + \dfrac{\partial w_2}{\partial y}\psi'(p^1) \tag{20}$$

$$G''(p^1) \quad = \quad \dfrac{\partial^2 w_2}{\partial (p^1)^2} + 2\dfrac{\partial^2 w_2}{\partial p^1 \partial y}\psi'(p^1) + \psi'(p^1)^t\dfrac{\partial^2 w_2}{\partial y^2}\psi'(p^1) + \dfrac{\partial w_2}{\partial y}\psi''(p^1). \tag{21}$$

Now, evaluate these expressions.

$$G'(p^1) = \dfrac{1}{4}v' + [0 \quad \tfrac{1}{4}v']\begin{bmatrix} 0 \\ -1 \end{bmatrix} = 0.$$

On the equilibrium set E :

$$\dfrac{\partial \phi}{\partial p^1} + \dfrac{\partial \phi}{\partial y}\psi'(p^1) = 0.$$

Thus, there are two equations (on E for $h = 1, 2.$):

$$\dfrac{\partial^2 \phi_h}{\partial (p^1)^2} + 2\dfrac{\partial^2 \phi_h}{\partial p^1 \partial y}\psi'(p^1) + \psi'(p^1)^t\dfrac{\partial^2 \phi_h}{\partial y^2}\psi'(p^1) + \dfrac{\partial \phi_h}{\partial y}\psi''(p^1) = 0.$$

$$\frac{\partial^2 \phi}{\partial (p^1)^2} = \begin{pmatrix} \frac{1}{(p^1)^3}v'(\frac{\theta}{p^1}) + \frac{\theta}{2(p^1)^4}v''(\frac{\theta}{p^1}) + \frac{3\theta}{2(p^1)^4}v''(\frac{\theta}{p^1}) + \frac{\theta^2}{2(p^1)^5}v'''(\frac{\theta}{p^1}) \\ \\ \frac{1}{(p^1)^3}v'(1 - \frac{\theta}{p^1}) - \frac{\theta}{2(p^1)^4}v''(1 - \frac{\theta}{p^1}) - \frac{3\theta}{2(p^1)^4}v''(1 - \frac{\theta}{p^1}) + \frac{\theta^2}{2(p^1)^5}v'''(1 - \frac{\theta}{p^1}) \end{pmatrix}$$

$$\frac{\partial^2 \phi}{\partial p^1 \partial p^2} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Hence,

$$\frac{\partial^2 \phi}{\partial p^1 \partial y}\psi' = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

By symmetry,

$$\frac{\partial^2 \phi}{\partial (p^2)^2} = \begin{pmatrix} \frac{1}{(p^2)^3}v'(\frac{\theta}{p^2}) + \frac{\theta}{2(p^2)^4}v''(\frac{\theta}{p^2}) + \frac{3\theta}{2(p^2)^4}v''(\frac{\theta}{p^2}) + \frac{\theta^2}{2(p^2)^5}v'''(\frac{\theta}{p^2}) \\ \\ \frac{1}{(p^2)^3}v'(1 - \frac{\theta}{p^2}) - \frac{\theta}{2(p^2)^4}v''(1 - \frac{\theta}{p^2}) - \frac{3\theta}{2(p^2)^4}v''(1 - \frac{\theta}{p^2}) + \frac{\theta^2}{2(p^2)^5}v'''(1 - \frac{\theta}{p^2}) \end{pmatrix}$$

Thus, the equation for E becomes:

$$\begin{pmatrix} v' + v'' + \frac{1}{8}v''' \\ \\ v' - v'' + \frac{1}{8}v''' \end{pmatrix} + \begin{pmatrix} v' + v'' + \frac{1}{8}v''' \\ \\ v' - v'' + \frac{1}{8}v''' \end{pmatrix} + \frac{\partial \phi}{\partial y}\psi'' = 0. \tag{22}$$

$$\psi'' = \frac{1}{2v'v''}\begin{pmatrix} -\frac{1}{2}v'' + \frac{1}{4}v'' & \frac{1}{2}v'' + \frac{1}{4}v'' \\ \\ 2v'' & 2v'' \end{pmatrix} \cdot 2 \begin{pmatrix} v' + v'' + \frac{v'''}{8} \\ \\ v' - v'' + \frac{v'''}{8} \end{pmatrix}. \tag{23}$$

Thus, solving:

$$\psi'' = \begin{pmatrix} \alpha \\ \\ 4 + \frac{v'''}{2v'} \end{pmatrix}.$$

This can be substituted into the expression for G'' to obtain:

$$G'' = \frac{v''}{4} + \frac{v'''}{8}.$$

If this is positive, then the utility of consumer 2 is minimized at the Walrasian allocation.   *Q.E.D.*

Remarks:

1. $G'' > 0$ for the following utility functions

    (a) The log-linear utility function: $\log x$.

    (b) The utility function: $\dfrac{x^{1-\epsilon}}{1-\epsilon}$, $\epsilon < 1$.

2. The intuition for the result is the following: Due to the precautionary motive, consumer 2 wants to consume less in period 1 as prices vary due to extrinsic uncertainty. Consumer 1 has fixed demand in period 1. Thus, consumer 1's endowment is 'relatively less valuable' and consumer 2's endowment is 'more valuable.' If the increase in value of endowment is large enough, then this will outweigh the loss in utility due to greater uncertainty. [1]

5 Proposition 2

This result extends to the case of $S < \infty$ states, and general endowment structures: $\omega_1 = (\alpha, 1 - \alpha, \ldots, 1 - \alpha)$ and $\omega_2 = (1 - \alpha, \alpha, \ldots, \alpha)$ with $\alpha \in (0, 1]$. A sufficient condition for the utility of consumer 2 to be minimized at the Walrasian allocation is:

$$\left(v'' + \left(\alpha - \frac{1}{2} \right) v''' \right) > 0 \tag{24}$$

Proof:

The structure of the economy is similar to that in the previous proposition. There

---

[1]We thank Todd Keister for suggesting this interpretation.

are, however, now S states in period 2. Thus, $s = 0, 1, \ldots, S$. The endowments of the two consumers are as given above. The preferences are given by:

$$u_h(x_h) = v(x_h^0) + \frac{1}{S} \sum_{s=1}^{S} v(x_h^s).$$

In the Walrasian equilibrium of the economy, we have:

$$x_h(0) = x_h^0 = \ldots = x_h^s = \ldots = x_h^S = \overline{x_h} = \frac{1}{2}.$$

The budget constraints are:

$$x_h^0 + \theta_h = \omega_h^0 \tag{25}$$

$$p^s x_h^s = p^s \omega_h^s + \theta_h, \ 1 \le s \le S. \tag{26}$$

Substituting the budget equations into the utility functions and imposing the market clearing equation $\theta_1 = -\theta_2 = \theta$, we get the following maximization problems for the two consumers.

$$\text{Consumer 1: } \text{Max} \quad v(\alpha - \theta) + \frac{1}{S} \sum_{s=1}^{S} v(1 - \alpha + \frac{\theta}{p^s}) \tag{27}$$

$$\text{Consumer 2: } \text{Max} \quad v(1 - \alpha + \theta) + \frac{1}{S} \sum_{s=1}^{S} v(\alpha - \frac{\theta}{p^s}) \tag{28}$$

The set of first order equations define the equilibrium set, E.

$$\phi_1(\theta, p) = -v'(\alpha - \theta) + \frac{1}{S} \sum \frac{1}{p^s} v'(1 - \alpha + \frac{\theta}{p^s}) \tag{29}$$

$$\phi_2(\theta, p) = -v'(1 - \alpha + \theta) + \frac{1}{S} \sum \frac{1}{p^s} v'(\alpha - \frac{\theta}{p^s}) \tag{30}$$

Let $(1 - \alpha) = \beta$. Then, the gradients of these two equations can be written.

$$\nabla\phi_1(\theta, p) = \begin{pmatrix} v''(\alpha - \theta) + \dfrac{1}{S}\sum \dfrac{1}{(p^s)^2}v''(\beta + \dfrac{\theta}{p^s}) \\[2ex] -\dfrac{1}{S(p^1)^2}v'(\beta + \dfrac{\theta}{p^1}) - \dfrac{\theta}{S(p^1)^3}v''(\beta + \dfrac{\theta}{p^1}) \\[2ex] \ldots \\[2ex] -\dfrac{1}{S(p^S)^2}v'(\beta + \dfrac{\theta}{p^S}) - \dfrac{\theta}{S(p^S)^3}v''(\beta + \dfrac{\theta}{p^S}) \end{pmatrix} \tag{31}$$

$$\nabla\phi_2(\theta, p) = \begin{pmatrix} v''(\beta + \theta) - \dfrac{1}{S}\sum \dfrac{1}{(p^s)^2}v''(\beta - \dfrac{\theta}{p^s}) \\[2ex] -\dfrac{1}{S(p^1)^2}v'(\alpha - \dfrac{\theta}{p^1}) + \dfrac{\theta}{S(p^1)^3}v''(\alpha - \dfrac{\theta}{p^1}) \\[2ex] \ldots \\[2ex] -\dfrac{1}{S(p^S)^2}v'(\alpha - \dfrac{\theta}{p^S}) + \dfrac{\theta}{S(p^S)^3}v''(\alpha - \dfrac{\theta}{p^S}) \end{pmatrix} \tag{32}$$

At the Walrasian point, we have

$$\nabla\phi_1(\theta, p) = \begin{pmatrix} 2v'' \\[2ex] -\dfrac{1}{S}(v' + (\alpha - \dfrac{1}{2})v'') \\[2ex] \ldots \\[2ex] -\dfrac{1}{S}(v' + (\alpha - \dfrac{1}{2})v'') \end{pmatrix} \tag{33}$$

$$\nabla\phi_2(\theta, p) = \begin{pmatrix} -2v'' \\[2ex] -\dfrac{1}{S}(v' - (\alpha - \dfrac{1}{2})v'') \\[2ex] \ldots \\[2ex] -\dfrac{1}{S}(v' - (\alpha - \dfrac{1}{2})v'') \end{pmatrix} \tag{34}$$

Thus, $\nabla\phi_1$ is not collinear to $\nabla\phi_2$, and E is locally a $(S-1)$ manifold.

The issue is now to see that the utility of consumer 2 is minimized at the Walrasian

equilibrium.

$$\text{Min} \quad u_2(\theta, p) = v(\beta + \theta) + \frac{1}{S}\sum_{s=1}^{S} v(\alpha - \frac{\theta}{p^s})$$

$$\text{s.t.} \qquad \phi_1(\theta, p) = 0$$

$$\phi_1(\theta, p) = 0.$$

The tangent space at the Walrasian point, w^*, is:

$$(\tilde{\theta}, \tilde{p}) \cdot \nabla\phi_1(w^*) = 0 \qquad (\tilde{\theta}, \tilde{p}) \cdot (\nabla\phi_1(w^*) + \nabla\phi_2(w^*)) = 0$$

$$\Leftrightarrow$$

$$(\tilde{\theta}, \tilde{p}) \cdot \nabla\phi_2(w^*) = 0 \qquad (\tilde{\theta}, \tilde{p}) \cdot (\nabla\phi_1(w^*) - \nabla\phi_2(w^*)) = 0$$

This implies:

$$\sum \tilde{p}^s = 0$$

$$\tilde{\theta} = 0.$$

The Lagrangian is:

$$L(\theta, p, \lambda_1, \lambda_2) = u_2(\theta, p) + \lambda_1 \phi_1(\theta, p) + \lambda_2 \phi_2(\theta, p).$$

The first order conditions are:

$$\frac{\partial L}{\partial \theta}(u^*, \lambda_1, \lambda_2) = 0 \quad \Leftrightarrow \quad 0 + \lambda_1 2v'' + \lambda_2(-2v'') = 0 \tag{35}$$

$$\Leftrightarrow \quad \lambda_1 = \lambda_2 = \lambda^* \tag{36}$$

$$\frac{\partial L}{\partial p^s}(u^*, \lambda_1, \lambda_2) = 0 \quad \Leftrightarrow \quad \frac{\theta^*}{S}v' + \lambda^*(-\frac{1}{S}2v') = 0 \tag{37}$$

$$\Leftrightarrow \quad \lambda^* = \frac{\theta^*}{2}. \tag{38}$$

Now,

$$\nabla u_2(\theta, p) = \begin{cases} v'(\beta + \theta) - \dfrac{1}{S}\sum \dfrac{1}{p^s}v'(\alpha - \dfrac{\theta}{p^s}) \\[3mm] \dfrac{\theta}{S(p^s)^2}v'(\alpha - \dfrac{\theta}{p^s}), \quad s \geq 1 \end{cases} \tag{39}$$

$$\frac{\partial^2 u_2}{\partial(p^s)^2}(u^*) \;=\; -\frac{2\theta^*}{S}v' + \frac{\theta^{*2}}{S}v'' \tag{40}$$

$$\frac{\partial^2 \phi_1}{\partial(p^s)^2}(W^*) \;=\; \frac{2}{S}v' + \frac{\theta}{S}v'' + \frac{3\theta}{S}v'' + \frac{\theta^2}{S}v''' \tag{41}$$

$$\frac{\partial^2 \phi_2}{\partial(p^s)^2}(u^*) \;=\; \frac{2}{S}v' - \frac{\theta}{S}v'' - \frac{3\theta}{S}v'' + \frac{\theta^2}{S}v'''. \tag{42}$$

This implies:

$$\frac{\partial^2 L}{\partial(p^s)^2}(u^*, \lambda^*) \;=\; -\frac{2\theta}{S}v' + \frac{\theta^2}{S}v'' + \theta(\frac{2}{S}v' + \frac{\theta^2}{S}v''')$$

$$=\; \frac{\theta^2}{S}(v'' + \theta v''')$$

$$=\; \frac{\theta^2}{S}(v'' + (\alpha - \frac{1}{2})v'''). \tag{43}$$

For a minimum we want this to be positive, which gives the desired result as $\dfrac{\theta^2}{S} > 0$.

Q.E.D.

Remarks :

1. If $v(x) = \dfrac{x^{1-\epsilon}}{1 - \epsilon}$, $\epsilon > 0$, we have $v'(x) = x^{-\epsilon}$, $v''x = -\epsilon x^{-1-\epsilon}$, $v'''(x) = \epsilon(\epsilon + 1)x^{-\epsilon-2}$. Then,

$$v''\left(\frac{1}{2}\right) + \left(\alpha - \frac{1}{2}\right)v'''\left(\frac{1}{2}\right) \quad = \quad -\epsilon 2^{\epsilon+1} + \left(\alpha - \frac{1}{2}\right)(2\epsilon(\epsilon+1)2^{\epsilon+1})$$

$$= \quad \epsilon 2^{\epsilon+1}(-1 + (2\alpha - 1)(\epsilon + 1))$$

$$> 0 \quad \text{iff} \quad \alpha > \frac{1}{2} + \frac{1}{2(\epsilon+1)}.$$

(a) If $\epsilon = 2$, for u_2 to be minimized at the Walrasian point, we need $\alpha > \dfrac{2}{3}$.

(b) If $\epsilon = 1$, then we need, $\alpha > \dfrac{3}{4}$.

References

[1] F. Allen and D. Gale, "Financial Innovation and Risk Sharing," Cambridge: MIT Press, 1994.

[2] Y. Balasko, Extrinsic uncertainty revisited, *J. Econ. Theory* **31** (1983), 203-201.

[3] Y. Balasko and D. Cass, The structure of financial equilibrium with exogenous yields: The case of incomplete markets, *Econometrica* **57** (1989), 135-162.

[4] D. Cass, Sunspots and incomplete financial markets: The leading example, *in* "The economics of imperfect competition and employment: Joan Robinson and beyond" (G. Feiwel, Ed.), Macmillan, London (1989).

[5] D. Cass and A. Citanna, Pareto improving financial innovation in incomplete markets, *Econ. Theory* **11** (1998), 467-494.

[6] D. Cass and K. Shell, Do sunspots matter? *J. Pol. Econ.* **91** (1983), 193-227.

[7] P. Diamond, The role of the stock market in a general equilibrium model with technological uncertainty, *Amer. Econ. Rev.* **57** (1967), 759-776.

[8] J. Dow, Arbitrage, Hedging, and Financial Innovation, *Rev. Finan. Stud.* **11** (1998), 739-755.

[9] D. Duffie and R. Rahi, Financial market innovation and security design: An introduction, *J. Econ. Theory* **65** 1995), 1-42.

[10] R. Elul, Welfare effects of financial innovation in incomplete markets economies with several consumption goods, *J. Econ. Theory* **65** (1995), 43-78.

[11] R. Elul, Welfare-improving financial innovation with a single good, *Econ. Theory* **13** (1999), 25-40.

[12] A. Goenka and K. Shell, When don't sunspots matter? *Econ. Theory* **9** (1997), 169-178.

[13] O.D. Hart, On the optimality of equilibrium when the market structure is incomplete, *J. Econ. Theory* **11** (1975), 418-443.

[14] M.S. Kimball, Precautionary savings in the small and in the large, *Econometrica* **58** (1990), 53-73.

[15] K. Shell, Sunspot equilibrium, *in* "The New Palgrave: A Dictionary of Economics" (J. Eatwell, M. Milgate, and P. Newman, Eds.), Macmillan, London, 1987.