The Dynamic Reform of Political Institutions

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Abstract

This paper formulates a model of dynamic, endogenous reform of political institutions. Specifically, a class of *dynamic political games (DPGs)* is introduced in which institutional choice is both *recursive* and *instrumental*. It is recursive because future political institutions are decided under current ones. The process is instrumental because institutional choices do not affect payoffs or technology directly.

DPGs provide a broad framework to address the question: which environments exhibit institutional reform? Which tend toward institutional stability? In any state, private (public) sector decisions are *essential* if, roughly, they cannot always be replaced by decisions in the public (private) sector. We prove that institutional reform occurs if public sector decisions are not essential. Conversely, private sector decisions are essential if institutional reform occurs. The results suggest that a relatively more effective public sector is conducive to institutional stability, while a more effective private sector is conducive to change. We also show that if the political rules satisfy a dynamic consistency property, then the game admits "political fixed points" of a recursive map from future (state-contingent) decisions rules to current ones. Since existence of political fixed points is a necessary condition of equilibrium, we apply the result to prove two equilibrium existence theorems, one of which implies that private and public sector decision rules that are smooth functions of the economic state.

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1 Introduction

Reforms of political institutions are common throughout history. They come in many varieties. In some cases, the reforms correspond to changes in the voting franchise. Periodic expansions of voting rights occurred in governments of ancient Athens (700- 338BC), the Roman Republic (509BC-25AD), and most of Western Europe in the 19th and early 20th centuries, to name just a few examples.[1]

In other cases, modifications were made to the voting procedure itself. Medieval Venice (1032-1300), for instance, gradually lowered the required voting threshold from unanimity to a simple majority in its Citizens' Council. Nineteenth century Prussia, where votes were initially weighted by one's wealth, eventually equalized the weights across all citizens. The U.S. changed its rules under the 17th Amendment to require direct election of senators. In still other instances, the scope of a government's authority changed. For example, England and France privatized common land during the 16th and 17th century enclosure movement thus reducing scope of rules governing the commons.[2] The U.S. government, on the other hand, increased its scope under the 16th Amendment by legalizing federal income tax in 1913.

In many instances, institutional change is gradual and incremental.[3] Consequently, this paper concerns the dynamics of endogenous institutional reform. Which environments tend toward institutional stability? Which environments admit institutional change? What are the relevant forces that drive these changes?

To address these questions, we introduce a class of *dynamic political games (DPG)* in which the rules for choosing public decisions are themselves part of the decision process. A dynamic political game (DPG) is an infinite horizon stochastic game in which at each date t, private and public sector decisions jointly determine the date $t+1$ distribution of states of the world. A state fully describes all the relevant "economic" (i.e., substantive) parameters and "political" (i.e., procedural) ones at a given point in time. The "political" parameters then describe the explicit process of political aggregation used to determine public decisions at that date. The aggregation process is referred to as *political rule*. If, for example, the current political rule is a simple majority rule, then the feasible set of outcomes of majority rule is the set of Condorcet Winners — the choices which cannot be defeated by any alternative choice

[1]See Fine (1983), Finer (1997), and Fleck and Hanssen (2003).

[2]See MacFarlane (1978) and Dahlman (1980).

[3]Consider the progess of reforms in the Roman Republic that gave increasing voice to the plebs (commoners). In 509BC, the Senate and Assembly were founded; in 494 BC the Patricians conceded the right of the plebs (the "commoners") to participate in the election of magistrates; in 336 BC one of the consulships became available for election by plebians; and in 287 BC Hortensian Law was introduced which gave resolutions in the plebian council the force of law. Gradual reform also characterized expantion of voting rights in 19th century England. In 1830, the voting franchise restricted to 2% of the population. In 1832, the First Reform Act extended the franchise to 3.5% of population. The Second Reform Act of 1867 extended it to some 7.7%. By 1884 it had been extended to 15% of population. Universal suffrage only passed in 1928 (see Finer (1997).

in a majority vote.

A Markovian equilibrium is a collection of state-contingent private and public sector decision rules such that (a) the private sector rule for each individual is optimal for him in each period and in each state, and (b) public sector decisions are consistent with the prevailing political rule in each period and in each state.

DPGs have two important features. First, the entire institutional design process is *recursive*; parameters of next period's political rule are a part of the explicit decision made under the current political rule. These parameters constitute the "political state" each period. Second, the institutional decisions are purely *instrumental.* That is, they do not affect payoffs or technology directly. Hence, society modifies its existing institutions not because the details of political procedures enter into the utility functions. Rather, institutions are modified because these changes modify *substantive* private and public sector decisions in the future.

There is a modest literature on dynamic, endogenous political institutions.[4] Informal discussions in North (1981) and Ostrom (1990) both hint at recursivity in the process of institutional change. In formal work, Messner and Polborn (2002) examine a model of endogenous changes to future voting rules under current ones in an OLG framework. Lagunoff (2001) examines a dynamic recursive model of voting over civil liberties.

A number of papers examine dynamic changes in the voting franchise. Seminal work by Acemoglu and Robinson (2000, 2001) examines a dynamic game in which a franchised elite can choose in any period whether to make a once-and-for-all extension of voting rights to the rest of the population. Since the choice of franchise is a one time decision, only the timing of the decision matters in their model. Models with gradual and incremental extensions of the voting franchise were examined by Justman and Gradstein (1999), Roberts (1998, 1999), Barbera, Maschler, and Shalev (2001), Jack and Lagunoff (2003), and Gradstein (2003).

In certain respects, the model in Jack and Lagunoff (2003) is a prototype for the present framework. The dynamics of institutional choices in that model are fully recursive and instrumental. The present paper extends the prototype to other institutional choices. At the same time, the generality of their economic environment is maintained here by allowing fairly arbitrary types of agent heterogeneity and by avoiding specific functional forms for payoffs and transition technologies.

Given these features, this paper proves two types of results. First, we characterize conditions under which an equilibrium exhibits institutional reform over time. By definition, a reform occurs whenever next period's political rule is chosen to be different than the present

[4]In focusing attention on dynamic models, I neglect a larger literature on static models of endogenous political institutions such as, for example, Lizzeri and Persico (2002), and Aghion, Alesina, and Trebbi (2002). I also neglect the work on infinite regress and self-selection of rules found in Lagunoff (1992), Barbera and Jackson (2000), Koray (2000).

one. If the class of political rules satisfies a dynamic consistency property, then institutional reform (alternatively, institutional stability) depends on whether either public or private sector decisions are *essential*. Roughly speaking, private (public) sector decisions are said to be *essential* if, on a set of states of positive measure, given a social payoff from a political rule and an alternative private (public) sector decision, one cannot equal or improve upon the social payoff by varying the public (private) sector decision alone. In other words, private (public) sector decisions are essential if they cannot be replaced by decisions in the public (private) sector.

It is proved that reform occurs if public sector decisions are *not* essential. Conversely, private sector decisions are essential if political reform occurs. Consider the particular case of endogenous voting rights. An elite will choose to extend the voting franchise if, under the existing franchise, policy concessions alone can never "buy off" the external threat of an uprising — the private sector decisions. This is, in fact, the crux of an argument by Acemoglu and Robinson (2000) as to why the voting franchise was extended in 19th century Europe.

Notice that the contrapositive restatement of this result is that the current institution is stable if private sector decisions are inessential, while public sector decisions are essential if the current institution is stable. The result therefore implies a crucial distinction between public and private sector activities. A relatively more effective public sector is conducive to institutional stability, while a more effective private sector is conducive to change.

A second set of results address the issue of equilibrium existence more generally. A necessary condition is that the implied map from future (state-contingent) decisions rules to current ones has a fixed point. We refer to this as the *political fixed point problem*. The problem is especially acute when political rules are voting rules since voting cycles may arise. Standard "fixes" such as single peakedness do not work in DPGs because public decisions are inherently multi-dimensional: both the current policy and the future political rule are chosen each period.

The findings thus far show that if the political rules satisfy a dynamic consistency property, then the DPG admits political fixed points. This is proved by showing that the associated "Bellman's map" has a fixed point in the space of continuation value functions. An important special case is the class of all voting rules. If stage game payoffs admit an affine representation then each voting rule is rationalized by the preferences of the median voter in each state. This affine preference representation is similar to (though more restrictive than) the class of Intermediate Preferences introduced by Grandmont (1978) to prove a Median Voter theorem when policies are multi-dimensional. Since these median voter preferences are dynamically consistent, political fixed points exist under voting rules and affine stage payoffs.

In Section 2, we present less formal version of the model in order to highlight some issues and problems of recursive institutional choice. The general model is described in Section 3. There, we introduce the class of *political rules*, and show how political rules are in process of recursive institutional choice. A dynamic political game combines both the public and private sectors of the environment. An "equilibrium" combines standard Markov Perfection in private decisions with an implementations requirement for public decisions. Section 4 examines the issue of institutional reform. Section 5 examines the political fixed point problem and describes the existence results more generally. Section 6 concludes with a discussion of dynamic consistency requirements of the political rules. Proofs of the main results are in the Appendix, Section 7.

1.1 A Model with Two Political Rules

A less formal, "stripped down" version of the model is presented here to highlight some basic issues in recursive institutional choice. Consider initially a simple model of period-by-period majority voting. This is a frequently studied model, and one in which the institutional environment is fixed.

At each date $t = 1, 2, \ldots$, a set $I = \{1, \ldots, n\}$ of individuals must vote to decide a policy p_t at date t. To fix ideas, one can think of p_t as an income tax schedule from a set, P, of feasible schedules. The current state is ω_t drawn from a set Ω. One can interpret ω_t as the distribution of incomes across individuals. Since current tax rates may affect savings behavior, the tax rate affects both one's current payoff and the future state. For now, omit (notationally) private sector behavior such as individuals' savings decisions.

A *policy rule* ψ is a function specifying the policy $p_t = \psi(\omega_t)$ as a function of state ω_t at date t. Given state ω_t, individual i's ($i = 1, \ldots, n$) dynamic payoff over policies is expressed in a recursive form by

$$U_i(\omega_t;\ \psi)(p_t) \ \equiv\ u_i(\omega_t, p_t) + \delta \int V_i(\omega_{t+1}; \psi) dq(\omega_{t+1}|\ \omega_t, p_t) \tag{1}$$

where δ is the discount factor, u_i is the stage game payoff received in each period, q denotes the stochastic transition function mapping current states and policies into probability distributions over future states, and V_i is i's continuation payoff given policy rule ψ.[5]

If p_t is chosen each period by a simple majority vote, then pairwise comparisons of policy are evaluated by each individual using his payoff function $U_i(\omega_t; \psi)(\cdot)$ in (1). The profile of payoff functions is $U(\omega_t; \psi) = (U_i(\omega_t; \psi))_{i=1}^n$. For each such profile, the outcome of majority voting is typically represented by the set of *Condorcet Winners* — outcomes that survive all pairwise comparisons in a majority vote. Denote this set by $C(U(\omega_t; \psi))$. For the policy rule ψ to be consistent with $C(U(\omega_t; \psi))$, it must satisfy the "political fixed point" problem

$$\psi(\omega_t) \in C(U(\omega_t; \psi)\ ),\ \forall \omega_t \tag{2}$$

[5]The transition q is assumed to satisfy the standard measurability assumptions.

If voting is cycle proof, then Condorcet Winners exist. But since voting takes place each period, the continuation payoff V_i already encodes future voting outcomes, and so the voting rule at date t is cycle-proof only if Condorcet Winners exist in all future dates. In certain cases, the problem is resolved by assuming that p_t is uni-dimensional and then showing that single peaked stage game preferences generate single peaked recursive preferences. However, single peakedness will not suffice if both the policy and the political institution itself are part of the decision problem.

Consequently, let θ_t denote a parameter that determines the political institution. θ_t is chosen from a finite index set Θ. For example, suppose $\Theta = \{\theta^1, \theta^2\}$ where $\theta_t = \theta^1$ means that the tax schedule is determined by a majority vote. By contrast, if $\theta_t = \theta^2$ then the tax schedule is imposed by a "dictator," whom we assume to be individual $i = 1$. The "political state" θ_t is then distinct from the "economic" state ω_t. The latter is directly payoff-relevant. The former summarizes the political process by which public decisions are made. In particular, the political state θ_t determines the political rule, in this case either majority rule or dictatorship, for choosing both the policy p_t and the subsequent political state, θ_{t+1}.

As before, ψ determines policy p_t. Again, omit the private sector. Now, the public sector decision includes the choice of institution for the following period. An *institutional decision rule* μ is a mapping that determines the future political state $\theta_{t+1} = \mu(\omega_t, \theta_t)$ given the current economic and political state. The institutional decision rule describes a recursive process of institutional change. The rule in period t produces the new rule for period $t + 1$. The *public sector decision* each period is a pair (p_t, θ_{t+1}).

To save on notation, let $s_t = (\omega_t, \theta_t)$ be the composite state. An individual's payoff now is

$$U_i(s_t;\ \psi, \mu)(p_t, \theta_{t+1}) \equiv u_i(\omega_t, p_t) + \delta \int V_i(s_{t+1};\ \psi, \mu) dq(\omega_{t+1}|\ \omega_t, p_t) \tag{3}$$

If $s_t = (\omega_t, \theta^1)$, then the set $C(U(s_t; \psi, \mu), s_t)$ describes the set of Condorcet Winning public decisions as before. However, if $s_t = (\omega_t, \theta^2)$, then $C(U(s_t; \psi, \mu), s_t)$ describes the public sector decisions that maximize dictator's payoff function $U_1(s_t; \psi, \mu)(\cdot)$. The political fixed point problem may be restated as

$$(\psi(s_t), \mu(s_t)) \in C(U(s_t;\ \psi, \mu), s_t),\ \forall s_t \tag{4}$$

The fixed point problems (2) and (4) are distinct in several respects. In (2), the map admits a fixed point in the space of policy rules. Since the institution — majority voting — was fixed, the recursive payoff profiles were required to admit Condorcet Winners for each economic state ω. This is a nontrivial problem by itself. However, the mapping in (4) varies by institutional state, θ_t, as well as by economic state ω_t, and so we further require that (4) admits solutions for all such institutions in Θ. Moreover, since the decision problem is multi-dimensional, the simplest Median Voter Theorems are not useful for solving (4). Finally, if private sector decisions are considered, then (4) must hold for individuals' private decision rules that best respond to public sector decisions and to each other in all states.

If, indeed, a satisfactory solution to the political fixed point problem is found, this model can determine when and if institutional change takes place. For instance, when (i.e., for what values of the economic state ω_t) is it true that dictators relinquish power: $\mu(\omega_t, \theta^2) = \theta^1$? When is it true that democracies turn over power to dictators: $\mu(\omega_t, \theta^1) = \theta^2$? The answer will depend crucially on the interaction between public and private sector decisions. In particular, without any private sector, we have:

Proposition *Let (ψ, μ) be a political fixed point (a pair that satisfies (4) in each state s_t) with the property that the Condorcet Winning choice in state θ^1 is the most preferred decision of one of the voters (e.g., a median voter). Then each political state $\theta \in \Theta$ is everywhere politically stable in the sense that for every ω, $\mu(\omega, \theta) = \theta$.*

The proof is a special case of a more general result proved later. The intuition is: by maintaining the current political state, the current pivotal decision maker (either the dictator in θ^2 or the pivotal voter in θ^1) holds on to power. By doing so, the decision problem reduces to a single agent dynamic programming problem. It is well known in such problems that the resulting sequence of decisions is optimal from the decision maker's point of view.

No other individuals make choices to offset the pivotal decision maker's choices. A private sector typically fills that role. Significantly, the absence of inalienable rights to make private decisions is the defining feature of a totalitarian government.[6] Consequently, both of these political rules are stable in a totalitarian state! Since most governments are not totalitarian to this extreme, we introduce a private sector in the general model that follows.

2 The General Model

2.1 Political Rules

In general, the set Θ can define a larger set of political institutions than merely "dictatorships" and "democracies". Following standard conventions, we will find it useful to drop time subscripts, and adopt instead the use of primes, e.g., θ' to denote subsequent period's variables, θ_{t+1}, and so on. Let $S = \Omega \times \Theta$ denote the composite state space. Let v_i denote an arbitary function expressing the payoff $v_i(p, \theta')$ over current policy p and next period's political state, θ'. When i's payoff was a dynamic recursive payoff, as in the previous Section,

[6] This is the traditional definition, according to which democracies can be totalitarian if *all* choices are filtered through the voting mechanism.

v_i is a notational shorthand, i.e., $v_i(p,\theta') = U_i(s;\ \psi,\mu)(p,\theta')$. Let $\mathcal{V}$ denote the set of all profiles, $v = (v_1,\ldots,v_n)$, of such payoff functions.

A *class of political rules* corresponds to a (possibly empty valued) correspondence $C : \mathcal{V} \times S \to\to P \times \Theta \cup \{\emptyset\}$ that associates to each profile v and to each state s, a set $C(v,s)$ of public decisions. If $(p,\theta') \in C(v,s)$, then (p,θ') is a feasible public decision under C. Each particular political rule in the class C is given by $C(\cdot,s)$. As a matter of definition, C may be empty valued, a possibility which arises quite naturally when C is a class of voting rules. The class of rules C is *single valued* if for all $v \in \mathcal{V}$ and for all s, there exists a (p,θ') pair such that $C(v,s) \subseteq \{(p,\theta')\}$.

Fix the state $s = (\omega,\theta)$. To get a better sense of how broadly the framework describes various political institutions, I sketch a few examples below.

I. Voting over the Voting Rule. The political state identifies the fraction, $\theta \geq 1/2$ of individuals required to pass a public decision. A supermajority voting rule therefore determines which supermajority rule is used in the future: formally $\Theta \subset (.5,1]$ and let $(p,\theta') \in C(v,s)$ if for all $(\hat{p},\hat{\theta}')$

$$|\{i \in I :\ v_i(\hat{p},\hat{\theta}') > v_i(p,\theta')\}|\ \leq \theta n$$

II. Voting over the Voting Franchise. The political state θ identifies the subset of individuals who currently possess the right to vote (the voting franchise). The chosen public decision is the one that is majority preferred within this restricted group. Each restricted voting franchise today uses a majority vote to determine what group of individuals have the right to vote tomorrow: formally, $\Theta \supseteq 2^I$, and let $(p,\theta') \in C(v,s)$ if for all $(\hat{p},\hat{\theta}')$,

$$|\{i \in \theta :\ v_i(\hat{p},\hat{\theta}') > v_i(p,\theta')\}|\ \leq\ \frac{1}{2}|\theta|$$

In this model, the current voting franchise decides on a new voting franchise in the following period. An interesting subclass of these rules is the class of **Delegated Dictatorship** rules. Under these rules, θ varies only over the singletons $\{i\}$, $i = 1,\ldots,n$. In each such state, the current dictator chooses his most preferred policy and then delegates the decision to possibly a new dictator in the future. If the dates t describes the length of a generation, then delegated dictator rule might be useful for describing a particular process of dynastic succession in which the king anoints his own successor.

III. Voting over the Scope of Government. The political state identifies the domain of public decisions. Let $P(\theta)$ denote the set of feasible policies in state θ. Then $P = \cup_\theta P(\theta)$. Let $(p,\theta') \in C(v,s)$ if $p \in P(\theta)$ and for all $(\hat{p},\hat{\theta}')$ satisfying $\hat{p} \in P(\theta)$,

$$|\{i \in I :\ v_i(\hat{p},\hat{\theta}') > v_i(p,\theta')\}|\ \leq \frac{1}{2}n$$

Numerous "hybrids" can be derived from these (e.g., delegated dictator from a limited oligarchy). In many (most?) cases, the political rules of interest are those that can rationalized by some social welfare criterion. Formally, a class of rules C is *(partially) rationalized by* a function $F : \mathbb{R}^n \times S \to \mathbb{R}$ if F is weakly increasing in each dimension of $\mathbb{R}^n$ and if

$$C(v, s) = (\supseteq)\arg\max_{p,\theta'} F(v(p, \theta'), s)$$

Clearly the Delegated Dictator Rule is rationalized by v_θ where θ identifies the dictator. When the political rule $C(\cdot, s)$ is some type of voting rule (e.g., Examples I and II above), then there are two well known conditions under which C is rationalized by the preferences of a Median Voter. The first is the standard restriction to single peaked preferences.[7] The second is the order restriction property of Rothstein (1990).[8]

In the present model, dynamic payoffs are of the time separable form, $v(p_t, \theta_{t+1}) = v_1(p_t) + \delta v_2(\theta_{t+1})$. In this case, most dynamic models of policy presume a government that is *dynamically consistent* in its decision making.[9] In the present context, a class of political rules, C, is said to be *dynamically consistent* if it is partially rationalized by a welfare function F that satisfies in every state $s_t = (\omega_t, \theta_t)$,

$$F\left(v_1(p_t) + \delta v_2(\theta_{t+1}),\ s_t\right) = F\left(v_1(p_t),\ \theta_t\right) + \delta F\left(v_2(\theta_{t+1}),\ \theta_t\right)$$

This definition is standard. However, its significance for questions of institutional reform/stability is not transparent. The subsequent results all assume dynamic consistency. In the concluding section, we discuss some potential implications of dynamically inconsistent aggregation.

2.2 Dynamic Political Games

Recall that $I = \{1, \ldots, n\}$ is the set of individuals in this society. P is the set of feasible policies in each period, and $S = \Omega \times \Theta$ is the composite state space. We now introduce private sector decisions. Let e_{it} denote i's private decision at date t, chosen from a feasible set E. A profile of private decisions is $e_t = (e_{1t}, \ldots, e_{nt})$. These decisions may capture any number of activities, including labor effort, savings, or investment activities. They may also include "non-economic" activities such as religious worship or one's participation in a social revolt. The private sector affects both the stage payoffs, as in $u_i(\omega_t, e_t, p_t)$, and the transition technology, as in $q(\cdot \mid \omega_t, e_t, p_t)$.

[7] See Arrow (1951) and Black (1958).

[8] Similar results can be found in application of single crossing properties by Roberts (1977) and by Gans and Smart (1996).

[9] A notable exception is Krusell, Kuruscu, and Smith (2002).

Formally, $q(B|\ \omega_t,\ e_t,\ p_t)$ is a probability that ω_t belongs to the (Borel measurable) subset B given the economic state ω_t, the private decision profile e_t, and the policy p_t. Given that economic states evolve according to q, each individual's dynamic objective is to maximize discounted payoff,

$$\sum_{t=0}^{\infty} \delta^t\ u_i(\omega_t,\ e_t,\ p_t\ ) \tag{5}$$

Denote the initial state by $s_0 = (\omega_0, \theta_0)$ and define a *Dynamic Political Game (DPG)* to be the collection

$$G \equiv \langle\ \overbrace{(u_i)_{i\in I}, q, E, P, \Omega}^{\text{economic structure}}\ ,\qquad \overbrace{\Theta, C}^{\text{political structure}}\qquad ,\qquad \overbrace{s_0}^{\text{initial state}}\qquad \rangle$$

The class of dynamic political games constitutes a broad set of problems in which institutional changes occur endogenously and incrementally. While this includes all the exogenous elements of the game, the relevant payoff inputs in C are endogenous recursive payoffs that depend on strategies to be defined in the next section. For tractability, we restrict attention to dynamic political games that satisfy one of the following two exclusive sets of assumptions.

(A1) Θ is a finite set. P and E are compact, convex subsets of Euclidian spaces, and Ω is a convex subset of a Euclidian space; the payoff function u_i for each i is continuous and uniformly bounded above by some $\bar{u}$; for each (ω, e, p), $q(\cdot|\ \omega, e, p)$ admits a norm continuous, conditional density $f(\cdot|\ \omega, e, p)$ with respect to a probability measure η.[10]

(A1') E, P and Ω are all finite sets.

Unless otherwise stated, all the subsequent results assume that either (A1) or (A1') holds.

2.3 Strategies and Equilibrium

To make the theory tractable, we restrict attention to Markov strategies. Such strategies only encode the payoff-relevent states of the game. Consequently, individuals are not required to coordinate on the history of past play.

Recall that $\psi : S \to P$ and $\mu : S \to \Theta$ describe the policy and institutional rules, respectively. A *private sector rule* for individual i is a function $\sigma_i : S \to E_i$ that prescribes

[10]These assumptions are fairly standard Theorems proving existence of Markovian equilibria in the dynamic games. They are not harmless. In particular, norm continuity of a density f precludes deterministic transitions. See Dutta (1994) for a cogent discussion of the role of these assumptions.

private action $e_{it} = \sigma_i(s_t)$ in state s_t. Let $\sigma = (\sigma_1, \ldots, \sigma_n)$. The strategy profile is therefore summarized by the triple

$$\pi \;=\; (\; \overbrace{\sigma,}^{\text{private sector rule}} \; \overbrace{\psi,}^{\text{policy rule}} \; \overbrace{\mu}^{\text{institutional rule}} \;)$$

Together, ψ and μ comprise the public sector rules. An individual deviation from π is denoted by, for example, $\pi\backslash\sigma_i$. For any $s_t = (\omega_t, m_t)$ the payoff to citizen i in profile π at date t is defined recursively by:

$$V_i(s_t;\ \pi) \;=\; u_i(\omega_t, \sigma(s_t), \psi(s_t)\ ) \;+\; \delta \int V_i(\omega_{t+1}, \mu(s_t); \pi) dq(\omega_{t+1}|\ \omega_t, \sigma(s_t), \psi(s_t)\ ) \qquad (6)$$

The function V depends on and varies with arbitrary Markov strategy profiles $\pi = (\sigma, \psi, \mu)$. Along an equilibrium path (defined below), the function V_i defines a Bellman equation for citizen i. Given any strategy π, and any state s_t at date t, an individual's public payoff function $U_i(s_t, \pi) : P \times \Theta \rightarrow \mathbb{R}$ is defined by

$$U_i(s_t, \pi)(p_t, \theta_{t+1}) \;\equiv\; u_i(\omega_t, \sigma(s_t), p_t) \;+\; \delta \int V_i(\omega_{t+1}, \theta_{t+1}; \pi) dq(\omega_{t+1}|\ \omega_t, \sigma(s_t), p_t\ ) \qquad (7)$$

Let $U(s_t, \pi) = (U_i(s_t, \pi)\ )_{i \in I}$ (recall that $\mathcal{V}$ is the set of payoff profiles defined on policies and future political states, $P \times \Theta$). We now drop time subscripts and define an equilibrium for any dynamic political game.

Definition 1 An *Equilibrium* of a dynamic political game, G, is a profile $\pi = (\sigma, \psi, \mu)$ of Markov strategies such that for all states $s = (\omega, \theta)$,

(a) *Private decision rationality*: For each citizen i, and each private decision rule, $\hat{\sigma}_i$,

$$V_i(s;\ \pi) \;\geq\; V_i(s;\ \pi\backslash\hat{\sigma}_i\ ) \qquad (8)$$

(b) *Public decision implementation*: The public decision pair $(\psi(s), \mu(s)\ )$ satisfies

$$(\psi(s), \mu(s)\ ) \;\in\; C\,(U(s, \pi),\ s\ ) \qquad (9)$$

According to (a), in each state, s, private sector actions are individually optimal. According to part (b), public sector decisions are consistent with political rules in the class C. In keeping with the standard definition of a stochastic game, both types of decisions are simultaneous. Therefore, an equilibrium of a DPG requires both Markov Perfection from individuals' private sector choices and recursive consistency of public sector choices with a political rule. The latter requirement must hold in each state s, and so the consistency condition also satisfies kind of "perfection constraint."

3 When Does Institutional Reform Occur?

By *institutional reform* is meant the simple idea that institutions are deliberately modified. Hence, institutional reform occurs in state s_t whenever $\mu(\omega_t, \theta_t) \neq \theta_{t+1}$. A fundamental question is when and whether institutional reform occurs. The answer depends, in part, on whether private and public sector decisions are *essential*.

Fix the political state θ. Formally, we will say that *private sector decisions are essential in* θ if, on a positive measure set[11] of economic states, ω, there exists a pair (e, p) of private and policy decisions, a uniformly bounded measurable function $x : \Omega \to \mathbb{R}^n$, and a private sector decision profile $\hat{e}$, such that for all policies $\hat{p}$,

$$F\left(u(\omega, e, p) + \delta \int x(\omega')dq(\omega'|\omega, e, p),\ \theta\right) \ > \ F\left(u(\omega, \hat{e}, \hat{p}) + \delta \int x(\omega')dq(\omega'|\omega, \hat{e}, \hat{p}),\ \theta\right) \quad (10)$$

Private sector decisions will be said to be *inessential in* θ if they are not essential in θ. In words, private decisions are essential if there is a feasible social payoff (the left-hand side of (10)), and an alternative private section profile $\hat{e}$ for which no policy can produce an alternative social payoff using the same continuation x such that the alternative social payoff matches or exceeds the original social payoff. Essentially, this means that the effect of at least one private sector decision cannot be replaced by any of those of the public sector.

If, for example, $\max_{e,p} F$ is always single valued, then the private sector is essential. An extreme case was illustrated earlier: when there is no private sector (the totalitarian state), then private decisions are, by definition, *in*essential. In general, anytime there is redundancy between public provision or private provision of a good — the private sector is inessential. This suggests that "essentiality" is a generic property. However, it is doubtful whether mathematical genericity is relevant here since redundancy, in this context, is partly an artifact of property rights arrangements. As such, it could be built into the political process itself. For example, if both the public and private sectors produce widgets using the same technology, then neither sector is essential if consumers have no inherent preference for where the good is produced.

An analogous definition exists for policy decisions. We will say that *policy decisions are essential in state* θ if, on a positive measure set of economic states, ω, there exists a (e, p), a uniformly bounded measurable function $x : \Omega \to \mathbb{R}^n$, and any alternative policy decision profile $\hat{p}$, such that for any Nash equilibrium private sector profile $\hat{e}$ of the game with payoffs, $u(\omega, \hat{e}, \hat{p}) + \delta \int x(\omega')dq(\omega'|\omega, \hat{e}, \hat{p})$, the Inequality (10) holds.

Theorem 1 *Consider a dynamic political game in which* C *is a single valued, dynamically consistent class of political rules. Fix a state* $s = (\omega, \theta)$. *Let* $\pi = (\sigma, \psi, \mu)$ *be an equilibrium*

[11]...using the probability measure η from (A1).

that is unique up to the given policy rule ψ.[12] *If institutional reform occurs in state* $s = (\omega, \theta)$ *(i.e., ,* $\mu(\omega, \theta) \neq \theta$*), then private sector decisions are essential in* θ*. Conversely, if public sector decisions are inessential in* θ*, then institutional reform occurs in state* s*.*

The result unifies a number of results in the literature on progressive expansion of voting rights. Under the external conflict explanation (e.g., Acemoglu and Robinson (2000)), the voting franchise was extended by an elite to head off social unrest. This conforms to a case in which public decisions are inessential under the restricted franchise — they can be undercut by the threat of revolt. Under the internal conflict explanation, rights are extended to gain support when there is ideological or class conflict among the elite. A special case appears in Jack and Lagunoff (2003).[13] They construct an example in which taxes sustain investment in public literacy. Conflict between the median voter within the elite, and the population median individual's private investment in literacy leads to an expansion of voting rights.[14]

If institutional reform does *not* occur, then the current institution could be said to be stable. Consequently, one could express the result, in shorthand, as either

public sector never essential ⇒ institutional reform ⇒ private sector sometimes essential

or, equivalently,

private sector never essential ⇒ institutional stability ⇒ public sector sometimes essential

Phrased in this way, the result suggests an effective public sector is conducive to institutional stability, while an effective private sector is conducive to change.

Unfortunately, the result does not apply to the most interesting cases, those that are intermediate. In these cases, both private and public decisions are essential some of the time. Reform then entails a balance between public and private sector effects. We offer no formal results on this, but the intuition above suggests that reform (or stability) may depend on the relative likelihood of states that sustain essential private sector decisions versus the states that sustain essential public sector decisions. In turn, this relative likelihood depends on the current state and current decisions. Hence, the process might exhibit natural hysteresis.[15]

[12] That is, there does not exist another equilibrium $\pi^* = (\sigma^*, \psi, \mu^*)$ with the same policy rule ψ.

[13] Internal conflict of a somewhat related type is found in Lizzeri and Persico (2002).

[14] Public investment is not essential when the franchise is too narrow, because the elite's chosen tax rate is too small to induce much private effort from the population.

[15] In a dynamic model of endogenous voting rights, Roberts (1999) generates endogenous hysteresis in the size of the voting franchise. The source of Roberts' hysteresis is quite different than what is contemplated here. In that model, there is no private sector, and payoffs vary directly with franchise decisions. Instead hysteresis comes from the single crossing structure between the economic state and the political state.

4 The Political Fixed Point Problem

Establishing existence of equilibria in dynamic political games is a nontrivial exercise. Roughly speaking, there are two main problems with establishing a fixed point of this map. First, there is the "standard" problem expressed by part (a) in the equilibrium definition. For even if there were no public decisions (or if C was constant in all states), known existence results generally employ restrictive conditions on feasible choice sets, preferences, and transition technology.

Second, there political fixed point problem expressed here in part (b) of the definition. Strictly speaking, the two problems cannot be separated. However, for conceptual reasons, it is useful to maintain the distinction. Given σ, a *political fixed point* is a pair (ψ, μ) that solves (9).

The following two results are proved in the Appendix. Both make use of either Assumption (A1) or (A1'). The first Lemma asserts that dynamic consistency of the political rule is sufficient for political fixed points to exist.

Lemma 1 *Suppose in a dynamic political game G, the political rule C is dynamically consistent. Then for any private decision rules, σ, the game admits political fixed points.*

The next result identifies a simple restriction on stage game payoffs — a restriction similar to Grandmont's (1978) Intermediate Preference assumption — that suffices for voting rules to be dynamic consistent. Under this restriction, any voting rule admits a political fixed point.

Lemma 2 *Suppose that in a dynamic political game G, the stage game payoffs admit the affine preference representation,*

$$u_i(\omega, e, p) = k(i)h(\omega, e, p) + g(\omega, e, p) \tag{11}$$

all i, where k is an increasing, real valued function. If C is a voting rule, then C is dynamically consistent.

Unfortunately, even with affine stage game preferences, a solution to the general existence problem is not guaranteed. To do that will require that institutional rules be extended to mixed strategies. This is taken up in the next two Subsections.

4.1 Extension to Mixed Strategies: A Simple Existence Theorem

The simplest way to resolve the existence issue is to assume that all sets, E, P, Ω and Θ are finite (Assumption (A1')) and extend the analysis to mixed strategies and or lotteries. The extension is fairly straightforward with one exception. Since the political rule C makes a joint determination of p and θ', it must be extended to the set of correlated distribution (lotteries), $\Delta(P \times \Theta)$.

Formally, we represent the public sector decision rule on p and θ' as a pair ψ^* and μ^* such that $\mu^* : S \to \Delta(\theta)$ where $\mu^*(\theta'|s)$ is the probability of θ' given s, and $\psi^* : S \times \Theta \to \Delta(P)$ where $\psi^*(p|s, \theta')$ denotes the conditional probability of p given s and the realized θ'. The associate mixed action is expressed as $\psi^* \times \mu^*$. Finally, the $\sigma_i^*(e_i|s)$ is the conditional probability of private decision e_i given s.

Let $\pi^* = (\sigma^*, \psi^* \times \mu^*)$ denote a profile of mixed Markov strategies. The payoff (6) is extended to these mixed actions in the usual way:

$$V_i(s;\ \pi^*) \ = \ E\left[u_i(\omega, e, p\ ) \ + \ \delta V_i(\omega', \theta'; \pi) q(\omega' |\ \omega, e, p\ ) \ \middle|\ (\psi^* \times \mu^*)(s), \sigma^*(s)\right] \tag{12}$$

Theorem 2 *For any dynamic political game, G, satisfying (A1'), suppose that the political rule C is dynamically consistent. Then there exists an equilibrium $\pi^* = (\sigma^*, \psi^* \times \mu^*)$ in possibly mixed Markov strategies.*

In the particular case of a voting rule, equilibria exist if stage game preferences admit affine representations.

Notice that since F is dynamically consistent, then for any political state θ, we can treat the social welfare function $F(\cdot, \theta)$ in state θ as a player in a standard dynamic game whose feasible pure actions from the set $P \times \Theta$ if θ is the current state, and are from the set $\emptyset$ if θ is not. This player then has dynamic preferences (in pure strategies) given by,

$$F\left(u(\omega, e, p),\ \theta\right) + \delta \int F\left(V(\omega', \theta';\ \pi),\ \theta\right) dq(\omega' | \omega, e, p)$$

Viewed in this way, the DPG can be transformed into a standard, finite stochastic game with $n + |\Theta|$ players. At this point, the Theorem 2 is just an application of a standard result, namely, that stochastic games with finite actions sets and finite states admit (possibly mixed strategy) Markov Perfect equilibria. We therefore ommit remainder of the proof.

4.2 A Smooth Existence Theorem

While the finite existence theorem is useful in many contexts, it is limited in a number of ways. First, it does not address existence in many economically relevant environments. In many models of economic interest, the economic states are unboundedly infinite, e.g., capital stocks.

Second, practical applications demand more structure. For example, Klein, Krusell, and Rios-Rull (2002)develop techniques for solving Euler equations politico-economic models of time consistent government policy. These are hybrid models in which private decisions aggregate through a price system. Their techniques allow one to estimate optimal policy functions that result from median voter and other political rules. These techniques can be partially adapted to complete, dynamic game environments, but only if Markovian equilibria exist and are differentiable in the economic states. Hence, a theory of dynamic institutional design has more immediate relevance, from an applied perspective, if it admits smooth Markovian equilibria.

An recent result of Horst (2003) asserts existence of Lipschitz-continuous, hence, almost everywhere smooth, Markovian equilibria in dynamic games. His results makes use of a "moderate social influence" assumption whereby, one's own actions have a relatively greater effect on one's own marginal dynamic payoff than those of all other individuals combined.[16] We adapt elements of his result, including the moderate social influence assumption, to show that dynamic political games with dynamically consistent rules admit equilibria that are differentiable in the economic state, ω.

To make sense of formal assumptions, we adopt the following definitions and notational conventions.

First, we endow the class of (smooth) C^∞ functions with the topology of C^∞-uniform convergence on compacta. Formally, $H^m \to H$ in this topology if, for any compact set K, H^m converges to H C^∞-uniformly on K (i.e., for each r and each rth partial derivative, $||D^r H^m - D^r H||_r \to 0$ on K).[17]

Next, define a real valued function, $g : \mathbb{R}^\ell \to \mathbb{R}$ to be α-*concave* with $\alpha > 0$ if $g(x)+\frac{1}{2}\alpha||x||^2$ is concave.[18]

Finally, given ϵ, we let E^ϵ and P^ϵ denote interior neighborhoods of E and P, respectively, such that any point $e \in E^\epsilon$ or $p \in P^\epsilon$ is ϵ in distance away from the respective boundaries in E and P.

[16]A similar condition is found in a local interaction model of Horst and Scheinkman (2002).

[17]The sup norm, $||\cdot||_r$ on the r^{th} derivative $D^r H : \mathbb{R}^\ell \to \mathbb{R}^{kr\ell}$ is defined by

$$||D^r H||_r = \sup_{x'} \sup_{j_1,\ldots,j_r} ||\frac{\partial^r H}{\partial x_{j_1} \cdots \partial x_{j_r}}(x')||.$$

In this notation, $||\cdot||_0$ is the standard sup norm on H.

[18]An equivalent definition is: g is α-concave if the matrix $D^2 g + \alpha I$, with I denoting the identity matrix, is negative semi-definite.

In addition to Assumption (A1), the following assumptions on the dynamic political game will be used.

(A2) For each i, the payoff function u_i is smooth, uniformly bounded above by $K > 0$ and is C^∞-uniformly bounded by $L > K$.[19] Furthermore, there is an $\alpha_i > 0$ such that u_i is α_i-concave in the policy and private decision pair, (e_i, p) pair, for each ω.

(A3) The conditional density $f(\cdot|\, \omega, e, p)$, with respect q, is norm continous in the variables (ω, e, p), and there is an $M > 0$ such that for each ω' and each ω, $f(\omega'|\, \omega, \cdot)$ is assumed to be C^∞-uniformly bounded by M.

(A4) There exists a $0 < \gamma < 1$ such that for all i, and all $s = (\omega, \theta)$,

$$\frac{(1-\delta)L_i + \delta K M}{\alpha_i} \leq \gamma(1-\delta).$$

(A5) There exists an $\epsilon > 0$ such that that for each each i, each e_{-i}, each p, and each pair of economic states ω and ω', both $u_i(\omega, e_{-i}, \cdot)$, and $f(\omega'|\omega, \cdot)$ achieve their upper bounds on $E^\epsilon \times P^\epsilon$.

Assumptions (A2) and (A3) are standard technical conditions, although the norm continuity of f is restrictive. It rules out, for instance, deterministic transitions. Assumption (A4) is the Moderate Social Influence (MSI) assumption adapted from (Horst, 2003) and Horst and Scheinkman (2002). Assumption (A5) ensures interior solutions. There can be no best responses at on boundaries of E or of $E \times P$.

Theorem 3 *Let G a dynamic political game satisfying (A1)-(A5). Suppose the political rule C is dynamically consistent. Then G admits an equilibrium, $\pi = (\sigma, \psi^* \times \mu^*)$, in which σ and $\psi^*(\cdot|\cdot, \theta')$ are pure strategies that are smooth in the state ω.*

5 Conclusion

This paper examines questions of institutional reform. It introduces a dynamic recursive framework in which the political institution is an instrumental object of choice each period. We show that reform depends on whether private or public sector decisions are essential.

The intuition from the "essentiality" result suggests that reform (or stability) may depend on the relative likelihood of states that sustain essential private sector decisions versus the

[19]A function $H : \mathbb{R}^\ell \to \mathbb{R}^k$, is C^∞-uniformly bounded if it is smooth and there is some some finite number $L > 0$ that uniformly bounds H and bounds all its higher order derivatives in sup norm.

states that sustain essential public sector decisions. Hence, the "intermediate" environments where both public and private sector decisions are sometimes essential is an obvious focal point for future work.

In general, dynamic, recursive models of political aggregation are not new. One of the first is the pioneering work of Krusell, Quadrini, and Ríos-Rull (1997). More recent examples include Klein, Krusell, and Ríos-Rull (2002) and Hassler, et. al. (2003).[20] In this literature the institution itself is fixed. Usually, some form of majority voting is assumed, and so the "political fixed point" problem outlined above can be resolved in certain cases when the policy space is single dimensional.

A few papers examine dynamic models of voting that specifically allow for multi-dimensional choice spaces (though keeping the voting mechanism fixed). These include Bernheim and Nataraj (2002), Kalandrakis (2002), and Banks and Duggan (2003).

The present framework in necessarily multi-dimensional. The political fixed point problem is compounded by fact that different institutions each have possibly distinct requirements achieving recursive consistency. Nevertheless, under certain conditions, the problem is resolved when political rules are dynamically consistent.

A strong case can be made that many if not most political rules observed in the world today are, in fact, dynamically *inconsistent*. Two sources of dynamic inconsistency are of particular interest. First, dynamically inconsistent choices arise because the political rules vary with economic states such as the income distribution. Arguably, modern financing of political campaigns has this property. A purer example of this is the wealth-is-power rule examined by Jordan (2002). In its simplest incarnation, policies are entirely determined by those with the most aggregate wealth. We define a slight variant as follows.

IV. Wealth-is-Power vs Dictatorship Let the economic state determine distribution of wealth, i.e, $\omega = (\omega_1, \ldots, \omega_n)$. Let $\Theta = \{\theta^a, \theta^b\}$ where θ^a is the dictatorship by individual $i = 1$. The political rule under θ^b is given by: $(p, \theta') \in C(v, \omega, \theta^b)$ if for all $(\hat{p}, \hat{\theta}')$,

$$\sum_{i \in M} \omega_i \; < \; \sum_{i \notin M} \omega_i$$

where $M = \{i \in I : \; v_i(\hat{p}, \hat{\theta}') > v_i(p, \theta')\}$. The state θ^b defines wealth-is-power rule under which policies are entirely determined by those with the most aggregate wealth.

Jordan (2002) shows that outcomes of the wealth-is-power rule correspond to the core of a certain cooperative game. He characterizes the set of wealth distributions that generate nonempty core, or in our context, generate political fixed points. The wealth-as-power rule

[20]See Persson and Tabellini (2001) for other references.

may possibly be unstable even if private decisions are inessential. A wealthy individual other than Player 1 may wish to switch to θ^a and, consequently, commit all decision authority to Player 1 as a "hedge" against more egalitarian wealth distributions which might arise in the future. It is worth noting that some have argued that unrestricted private funding of political campaigns would induce outcomes similar to those of the wealth-is-power rule.

A second form of dynamic inconsistency arises because the political rules are not time separable. An example is a weighted Rawlsian social choice rule under which society wishes to maximize the welfare of the person whose weighted payoff makes him least well off.

V. The Rawlsian Rule Let Θ be a finite subset of $\{\theta \in \mathbb{R}^n_+ : \sum_k \theta_k = 1\}$. For all states s,

$$C(v, \omega, \theta) = \arg\max_{(p,\theta')} \min_i \{\theta_1 v_1, \ldots, \theta_n v_n\}$$

It is not generally true that the least well off individual today also has the least well off continuation payoff next period. Hence, the political rule may choose a different weighting scheme in the following period, even if it involves a concession of decision authority in the future.

Under both types of inconsistencies, political reform could arise as a commitment against an institution's "future self." If the private sector is inessential, then the institution's own future self is possibly less trustworthy than that of another institution. A reform then occurs. However, if a private sector *is* essential, then the alternative institution's future self may be less trustworthy. That case could result in the stability of a dynamically inconsistent rule.

6 Appendix

Proof of Theorem 1 Fix a dynamic political game, a state $s = (\omega, \theta)$, and an equilibrium $\pi = (\sigma, \psi, \mu)$ all satisfying the hypothesis of the Theorem.

Since C is single valued and dynamically consistent, there is some social criterion F that rationalizes C and is invariant to the economic states ω. By definition,

$$(\psi(s), \mu(s)) = C(U(s;\pi), s) = \arg\max_{p,\theta'} F(U(s;\pi)(p,\theta'), \theta)$$

In other words,

$$F(V(s;\pi), \theta) \;=\; F(U(s;\pi)(\psi(s), \mu(s)), \theta) \;\geq\; F(U(s;\pi)(p,\theta'), \theta),\ \forall (p,\theta') \tag{13}$$

Suppose first that <u>private</u> decisions are <u>not</u> essential on a set of states $s' = (\omega', \theta)$ with full measure. This means that for all such states, for all pairs (e,p), all bounded measurable function $x : \Omega \to \mathbb{R}^n$, and all private sector profilea $\hat{e}$, there exists a policy $\hat{p}$, which violates (10). Specifically, there exists $\hat{p}$ such that for all $\hat{\theta}$,

$$\begin{aligned} & F\left(U(s';\pi)(\hat{p},\mu(\omega',\hat{\theta})),\ \theta\right) \\ = \ & F\left(u(\omega',\sigma(\omega',\theta),\hat{p}) + \delta\int V(\omega'',\mu(\omega',\hat{\theta});\pi)dq(\omega''|\omega',\sigma(\omega',\theta),\hat{p}),\ \theta\right) \\ \geq \ & F\left(u(\omega',\sigma(\omega',\hat{\theta}),\psi(\omega',\hat{\theta})) + \delta\int V(\omega'',\mu(\omega',\hat{\theta});\pi)dq(\omega''|\omega',\sigma(\omega',\hat{\theta}),\psi(\omega',\hat{\theta})),\ \theta\right) \\ = \ & F\left(V(\omega',\hat{\theta};\pi),\ \theta\right) \end{aligned} \tag{14}$$

Combining Equations (13) and (14), for almost every economic state ω',

$$\int F\left(V(\omega',\theta;\pi),\theta\right)\ dq(\omega'|\,\omega,e,p)\ \geq\ \int F\left(V(\omega',\hat{\theta}';\pi),\theta\right)\ dq(\omega'|\,\omega,e,p) \tag{15}$$

for all e and p and $\hat{\theta}$. Hence, by Equations (13), (15) and dynamic consistency, for all p and all $\tilde{\theta}$,

$$\begin{aligned} F\left(U(\omega,\theta;\pi)(p,\theta),\theta\right) &= F\left(u(\omega,\sigma(\omega,\theta),p),\theta\right) + \delta\int F\left(V(\omega',\theta;\pi),\theta\right)dq(\omega'|\,\omega,\sigma(\omega,\theta),p\ ) \\ &\geq F\left(u(\omega,\sigma(\omega,\theta),p),\theta\right) + \delta\int F\left(V(\omega',\tilde{\theta};\pi),\theta\right)dq(\omega'|\,\omega,\sigma(\omega,\theta),p\ )\ ) \\ &= F\left(U(\omega,\theta;\pi)(p,\tilde{\theta}),\theta\right) \end{aligned} \tag{16}$$

Hence, we have shown that a pair (ψ,μ) in which $\mu(\omega,\theta)=\theta$ is a solution to $\max_{p,\tilde{\theta}} F(U(\omega,\theta;\pi)(p,\tilde{\theta}),\theta)$. But because C is single valued, it is the only such solution. We therefore conclude that nonessentiality of private decisions implies no political reform.

Now suppose the equilibrium is such that $\mu(\omega,\theta)=\theta$, i.e, no political reforms occur in state s. Then (15) holds whenever $p=\psi(\omega,\theta)$ and $e=\sigma(\omega,\theta)$. Consequently, there is at least one political state $\hat{\theta}\neq\theta$, and there is a η-positive measure of ω' such that for each such ω',

$$F\left(V(\omega',\theta;\pi),\theta\right)\ >\ F\left(V(\omega',\hat{\theta};\pi),\theta\right)$$

By dynamic consistency of F, this means

$$
\begin{aligned}
&= F\left(V(\omega', \theta; \pi),\ \theta\right) \\
&= F\left(u(\omega', \sigma(\omega', \theta), \psi(\omega', \theta)),\ \theta\right) + \delta \int F\left(V(\omega'', \mu(\omega', \theta); \pi),\ \theta\right)\ dq(\omega'' |\ \omega', \sigma(\omega', \theta), \psi(\omega', \theta)\ ) \\
&\geq F\left(u(\omega', \sigma(\omega', \theta), \psi(\omega', \theta)),\ \theta\right) + \delta \int F\left(V(\omega'', \mu(\omega', \hat{\theta}); \pi),\ \theta\right)\ dq(\omega'' |\ \omega', \sigma(\omega', \theta), \psi(\omega', \theta)\ ) \\
&\geq F\left(u(\omega', \sigma(\omega', \hat{\theta}), \psi(\omega', \hat{\theta})),\ \theta\right) + \delta \int F\left(V(\omega'', \mu(\omega', \hat{\theta}); \pi),\ \theta\right)\ dq(\omega'' |\ \omega', \sigma(\omega', \hat{\theta}), \psi(\omega', \hat{\theta})\ )
\end{aligned}
\tag{17}
$$

with one of the inequalities in (17) strict.

Letting $x(\cdot) = V(\cdot, \mu(\omega', \hat{\theta}); \pi)$ notice that $\sigma(\omega', \hat{\theta})$ is a Nash equilibrium in private actions of the stage game with payoffs

$$
u(\omega', e, \psi(\omega', \hat{\theta})) + \delta \int x(\omega'')\ dq(\omega'' |\ \omega', e, \psi(\omega', \hat{\theta})\ )
$$

By assumption this Nash equilibrium is unique given the fixed policy rule ψ. Hence, we have shown that for some p — namely, $p = \psi(\omega', \hat{\theta})$, for any Nash equilibrium e^*

$$
\begin{aligned}
& F\left(u(\omega', e^*, \psi(\omega', \hat{\theta})),\ \theta\right) + \delta \int F\left(x(\omega''),\ \theta\right)\ dq(\omega'' |\ \omega', e^*, \psi(\omega', \hat{\theta})\ ) \\
< \ & F\left(u(\omega', \sigma(\omega', \theta), \psi(\omega', \theta)),\ \theta\right) + \delta \int F\left(x(\omega''),\ \theta\right)\ dq(\omega'' |\ \omega', \sigma(\omega', \theta), \psi(\omega', \theta)\ )
\end{aligned}
\tag{18}
$$

Since $F(x(\cdot), \theta)$ is also a bounded measurable function of the state, Inequality (18) proves that public policy decisions <u>are</u> essential. ∎

Proof of Lemma 1

Let $W : S \to \mathbb{R}^n$ be a profile of bounded, measurable continuation payoff functions. We call a continuation function W *feasible* if $W = V(\cdot, \pi)$ for some strategy profile, π. Let $\mathcal{W}$ denote the set of feasible continuations. By a slight abuse of our previous notation, we define the recursive public payoff with an arbitrary continuation W in (7) to

$$
U_i(s;\ \sigma, W)(p, \theta')\ =\ u_i(\omega, \sigma(s), p)\ +\ \delta \int W_i(\omega', \theta') dq(\omega' |\ \omega, \sigma(s), p\ ) \tag{19}
$$

We prove first that political fixed points are associated with the fixed points (in value function) of the associated Bellman's mapping. Specifically, let G be a dynamic political game in which rule C is partially rationalized by a function F. For each $\sigma \in \Sigma$, suppose that $\hat{W}$ is a fixed point of the "Bellman's" operator $B : \mathcal{W} \to \mathcal{W}$ defined by

$$(BW)(s), s) = U(s;\ \sigma, W)(\psi(s), \mu(s)) \tag{20}$$

where $(\psi(s), \mu(s))$ solves

$$\max_{p,\theta'} F\left(U(s;\ \sigma, W)(p, \theta'),\ s\right) \tag{21}$$

Then we show that the pair $(\hat{\psi}, \hat{\mu})$ that solves (21) when $W = \hat{W}$ is a political fixed point. To see this, fix σ and let $\hat{W}$ be a fixed point of the Bellman's operator defined in (20). Let $(\hat{\psi}, \hat{\mu})$ denote the solution to (21) under $\hat{W}$, i.e,

$$F\left(\hat{W}(s),\ s\right) = F\left(U(s;\ \sigma, \hat{W})(\hat{\psi}(s), \hat{\mu}(s)),\ s\right),\ \forall s$$

By our earlier (abuse of) notation, $U(s;\ \sigma, \hat{W}) = U(s;\ \sigma, \hat{\psi}, \hat{\mu})$. If C is either partially or fully rationalized by F, then $(\hat{\psi}, \hat{\mu}) \in C(U(s;\ \sigma, \hat{\psi}, \hat{\mu},\ s)$, and so $(\hat{\psi}, \hat{\mu})$ is a political fixed point.

Next, we prove that if C is fully rationalized by F, then the converse holds: namely, for every political fixed point $(\hat{\psi}, \hat{\mu})$, the corresponding value $\hat{W}$ of the Bellman's operator is a fixed point of (20). Suppose then that C is fully rationalized by F and let $(\hat{\psi}, \hat{\mu})$ be a political fixed point. Then

$$(\hat{\psi}, \hat{\mu}) \in C(U(s; \sigma, \hat{\psi}, \hat{\mu}),\ s) = \arg\max_{p,\theta'} F\left(U(s;\ \sigma, V(\cdot; \sigma, \hat{\psi}, \hat{\mu}))(p, \theta'),\ s\right)$$

By definition, $V(\cdot; \sigma, \hat{\psi}, \hat{\mu})$ is a fixed point of the Bellman operator.

We prove the remainder of the result as follows. Fix an arbitrary σ and let W be any bounded continuation profile, and by our abuse of notation, write $U(s, \sigma, W)$. Since C is dynamically consistent, the Bellman's operator defined in (20) is, in this case, given by

$$\begin{aligned} (BW)(s) &= \max_{p,\theta'} F\left(U(s, \sigma, W)(p, \theta'),\ \theta\right) \\ &= \max_{p,\theta'} \left\{ F\left(u(\omega, \sigma(\omega, \theta), p),\ \theta\right) + \delta \int F\left(W(\omega', \theta'),\ \theta\right) dq(\omega' | \omega', \sigma(\omega, \theta), p) \right\} \end{aligned} \tag{22}$$

Notice that under either (A1) or (A1'), (BW) in nonempty valued. It is easy to very that B satisfies two sufficient conditions, discounting and monotonicity, in a well known result of Blackwell (1965), implying that B is a contraction. Applying the Contraction Mapping Theorem, B has a fixed point, $\hat{W}$. By our previous argument relating fixed points in value space with those in strategy space, the conclusion follows. ∎

Proof of Lemma 2 Fix an arbitrary behavior profile π. Then, at each date t, for each state s_t,

$$\begin{aligned} V_i(s_t;\pi) &= E\left[\sum_{\tau=t}^{\infty}\delta^{\tau-t}[u_i(\omega_\tau,\sigma(s_\tau),\psi(s_\tau))\ ]\Big|s_t\right] \\ &= E\left[\sum_{\tau=t}^{\infty}\delta^{\tau-t}[k(i)h(\omega_\tau,\sigma(s_\tau),\psi(s_\tau))+g(\omega_\tau,\sigma(s_\tau),\psi(s_\tau))\ ]\Big|s_t\right] \\ &= k(i)E\left[\sum_{\tau=t}^{\infty}\delta^{\tau-t}[h(\omega_\tau,\sigma(s_\tau),\psi(s_\tau))\ ]\Big|s_t\right]+E\left[\sum_{\tau=t}^{\infty}\delta^{\tau-t}[g(\omega_\tau,\sigma(s_\tau),\psi(s_\tau))\ ]\Big|s_t\right] \\ &\equiv k(i)H(s_t)+G(s_t) \end{aligned}$$

Clearly, the continuation profile $V(\cdot,\pi)$ has an affine representation of the same form as stage payoffs in (11) for each π. Therefore,

$$\begin{aligned} U_i(s,\pi)(p,\theta') &= u_i(\omega,\sigma(s),p)+\delta\int V_i(\omega',\theta';\pi)dq(\omega'|\omega,\sigma(s),p) \\ &= (h(\omega,\sigma(s),p)k(i)+g(\omega,\sigma(s),p)\ ) \\ &\quad +\delta\int[k(i)H(\omega',\theta')+G(\omega',\theta')]dq(\omega'|\omega,\sigma(s),p) \end{aligned}$$

The profile $U(s,\pi)$ has also has an affine representation. Because k is strictly increasing, the profile $U(s,\pi)$ is order restricted with respect to linear order on i induced by k (i.e., $i \succ j$ iff $k(i) > k(j)$). By the Median Voter Theorem of Rothstein (1990), since C is a voting rule, C is partially rationalized by the function corresponding to the recursive preferences U_m, of the individual m for whom $k(m)$ is the median (individual m is the "median voter"). Formally,

$$\arg\max_{p,\theta'} U_m(s,\pi)(p,\theta') \subseteq C(U(s,\pi),s) \tag{23}$$

Given either Assumption (A1) or Assumption (A1'), the left hand side of (23) has a solution. ∎

Proof of Theorem 3 We first prove an existence Theorem without public decisions. That is, consider the standard dynamic game with only private decisions, e_i. In what follows, we exclude the public decision component from notation altogether. That is, we first assume that stage game payoffs are given by $u_i(\omega,e)$ while the density is given by $f(\omega'|\ \omega,e)$. Let $\bar{G} = \langle (u_i)_{i\in I},\Omega,q,E,\omega_0;\rangle$ denote the game with only private decisions. Restating the result, we first wish to prove

Theorem A Let $\bar{G}$ denote a dynamic game with only private decisions. Suppose $\bar{G}$ satisfies (A1)-(A5). Then the game has a Markov Perfect equilibrium σ that is smooth on Ω.

Let $\mathcal{X}$ denote the set of all uniformly bounded, Lipschitz continuous functions, $x : \Omega \to [0, c]^n$ with uniform Lipschitz bound given by $\max_i L_i$. Standard results show that $\mathcal{X}$ is compact in the topology of uniform convergence on compacta (see, for example, Mas Colell (1985, Theorem K.2.2). For each such function $x \in \mathcal{X}$, define a one shot game by the payoffs,

$$H_i(\omega, e,\ x) \ = \ (1-\delta)u_i(\omega, e) \ + \ \delta \int x_i(\omega')f(\omega'|\ \omega, e)d\eta \tag{24}$$

for each i. Then let $H = (H_i)_{i=1}^n$ be the vector valued function with components defined by (24).

Lemma 3 *For each state ω and each continuation value $x \in \mathcal{X}$, the one shot game defined by payoff profile, $H(\omega, \cdot,\ x)$, has a pure strategy Nash equilibrium profile,*

$$(\bar{\sigma}_1(\omega, x), \ldots, \bar{\sigma}_n(\omega, x)\ )$$

of private decisions. The profile $\bar{\sigma}$ is smooth with uniformly bounded first derivatives in ω, and is uniformly bounded and continuous in x.

Proof of Lemma 3

Observe, first, that by Assumptions (A2) and (A3), for each i, H_i is a smooth and C^∞-uniformly bounded function of (ω, e) (in the relative topology), with uniform bound given by

$$(1-\delta)L_i + \delta c M \tag{25}$$

Clearly, this bound is independent of x since x is itself uniformly bounded by c. Consequently, H_i is uniform bounded on its entire domain.

Next, we show that for each state ω, H_i is $\bar{\alpha}_i$-concave in e_i where $\bar{\alpha}_i = -(1-\delta)\alpha_i + \delta c M > 0$. To show this, we must show that for each ω, $D^2_{e_i} H_i(\omega, \bar{e}\ x) + \bar{\alpha}_i I$ is negative semi definite. To this end, fix ω. Observe that by α-concavity on stage utility functions, u_i, and uniform boundedness of the conditional densities, f, we have for every pair of profiles, $\bar{e}$ and e,

$$\begin{aligned}
& e_i^T \cdot D^2_{e_i} H_i(\omega, \bar{e}\ x) \cdot e_i, \\
\leq \ & e_i^T \cdot \left[(1-\delta) D^2_{e_i} u_i(\omega, \bar{e}) + \delta c \int D^2_{e_i} f(\omega' \big|\ \omega, \bar{e})\ d\eta \right] \cdot e_i \\
\leq \ & -(1-\delta)\alpha_i ||e_i||^2 \ + \ \delta c M ||e_i||^2 \\
= \ & -\bar{\alpha}_i ||e_i||^2
\end{aligned}$$

Since, by Assumption (A4), $\bar{\alpha}_i > 0$, it follows that H_i is $\bar{\alpha}$-concave. Consequently, by compactness of E, and by the smoothness and strict concavity of H_i in e_i, the best responses

$$g_i(\omega, e_{-i}, x) \equiv \arg\max_{e_i \in E} H_i(\omega, e, x)$$

for each i is nonempty and single valued.

Consider the best response function, g_i. By the Assumption (A5), $g_i(\omega, e_{-i}, x)$ defines a critical point, i.e.,

$$D_{e_i} H_i(\omega, g_i(\omega, e_{-i}, x), e_{-i}, x) = 0.$$

Then by strict concavity of H_i, the Implicit Function Theorem implies that g_i is a locally smooth function in a neighborhood of (ω, e_{-i}) (in the relative topology). In this neighborhood, the Implicit Function Theorem implies

$$Dg_i = -[D^2_{e_i} H_i]^{-1} \cdot [D_{\omega, e_{-i}} D_{e_i} H_i]$$

Given the C^∞-uniform bound on H_i given by (25), the $\bar{\alpha}_i$-concavity of H_i implies that there is a uniform bound on Dg_i given by $\frac{1}{\bar{\alpha}_i}(1-\delta)L_i + \delta c M$. Finally, since the choice of (ω, e_{-i}) was arbitrary, every such point is a regular point and so g_i is everywhere smooth with uniformly bounded first derivative.

We now show that there is a unique Nash equilibrium, $\bar{\sigma}(\omega, x)$ of the game with payoffs, $H(\omega, \cdot, x)$. Fixing, ω and x, consider the best response map

$$e \mapsto (g_1(\omega, e_{-1}, x), \ldots g_n(\omega, e_{-n}, x)\ )$$

by the arguments above, the conditions for Brouwer's Theorem are met and so this map has a fixed point. Since all best responses are interior — as shown above — the fixed point must be an interior point in E^n. To verify that this fixed point is unique, it suffices to show that the best response difference map

$$e \mapsto e - (g_1(\omega, e_{-1}, x), \ldots g_n(\omega, e_{-n}, x)\ ) \tag{26}$$

has no critical points. It suffices then to show that the Jacobian of this map at differentiable points is nonsingular. In turn, the Jacobian is nonsingular if it has a dominant diagonal. The Jacobian has a dominant diagonal if

$$||D_{e_{-i}} g_i||_1 < 1, \ \forall i, \tag{27}$$

at points (ω, e_{-i}, x) of the best response map, g_i. To verify (27), consider the best response map, g_i. Then we have:

$$
\begin{aligned}
||D_{e_{-i}} g_i||_1 &= ||-[D^2_{e_i} H_i]^{-1} \cdot [D_{e_{-i}} D_{e_i} H_i]||_2 \\
&\leq \frac{1}{\bar{\alpha}_i} ||(1-\delta) D_{e_{-i}} D_{e_i} u_i + \delta D_{e_{-i}} D_{e_i} \int V_i(\omega') f(\omega' \Big| \, \omega, e) d\eta||_2 \\
&\leq \frac{1}{\bar{\alpha}_i} (1-\delta) L_i + \delta c M \\
&< 1
\end{aligned}
\tag{28}
$$

The first equality is the Implicit Function Theorem,[21] the first *inequality* follows from the definition of $\bar{\alpha}$-concavity, the second follows from the bounds assumed by (A2) and (A3), and the last follows from the MSI condition (A4).

Next, we show that the profile $\bar{\sigma}$ is smooth with uniformly bounded first derivatives in ω with the bound uniform across all x as well. Observe that the nonsingularity of (26) implies that the unique Nash equilibrium $\bar{\sigma}(\omega, x)$ is implicitly defined by the Implicit Function Theorem. In turn, the IFT also implies that $D_\omega \bar{\sigma}$ is smooth and defined by

$$
D_\omega \bar{\sigma} \; = \; [D_e g]^{-1} [D_\omega g]
$$

Note that the inverse $[D_e g]^{-1}$ exists and is uniformly bounded over all ω and all x by the dominance diagonal condition, (28). Consequently, $D_\omega \bar{\sigma}$ exists everywhere and is uniformly bounded over all ω and x.

Finally, we now prove that σ is Lipschitz continuous in x with uniform Lipschitz constant. This follows from a result of Montrucchio (1987, Theorem 3.1) and later restated by Horst (2003, Theorem A1). In particular, their result implies that for each i, each ω, and any pair x, x',

$$
||g_i(\omega, \cdot, x) - g_i(\omega, \cdot, x')||_0 < \gamma ||x - x'||_0 \tag{29}
$$

where γ is the MSI bound in Assumption (A4). Using the difference map in (26) to define the fixed points, the Implicit Function Theorem again implies that (29) applies to the fixed point, $\bar{\sigma}$, as well. ∎

Using Lemma 3, let $\bar{\sigma}$ be the map that defines the unique Nash equilibrium $\bar{\sigma}(\omega, x)$ for the one shot game with payoffs, $H_i(\omega, e, x), \; i = 1, \ldots, n$.

[21] if e_i is one dimensional, then the sup norm picks out one such term,

$$
\frac{\partial g_i}{\partial e_j} \; = \; (\frac{\partial^2 H_i}{\partial e_i^2})^{-1} \frac{\partial^2 H_i}{\partial e_j \partial e_i}, \; \forall j \neq i
$$

Lemma 4 *The equilibrium payoff function $H_i(\cdot, \bar{\sigma}(\cdot), x)$ is smooth with a uniformly bounded first derivative in ω, and with the uniform bound applying across all x.*

Proof of Lemma 4 By definition,

$$H_i(\omega, \bar{\sigma}(\omega, x), x) \;=\; (1-\delta)u_i(\omega, \bar{\sigma}(\omega, x)\ ) \;+\; \delta \int V_i(\omega')f(\omega' |\, \omega, \bar{\sigma}(\omega, x)\ )d\eta$$

The smoothness therefore follows from the smooth of H in ω directly and from the smoothness of $\bar{\sigma}(\omega, x)$ in ω established in Lemma 3. The uniform boundedness of first derivatives in ω follows from the C^{∞}-uniform boundedness of H and the uniform boundness of first derivatives of $\bar{\sigma}$ established in Lemma 3. ∎

Lemma 5 *Let $\{x^{\ell}\}$ be a sequence such that $x^{\ell} \in \mathcal{X}$ for all ℓ and $x^{\ell} \to x \in \mathcal{V}$ with the convergence uniform on each compact set $K \subset \Omega$ as $\ell \to \infty$. Then for each ϵ there exists a $\bar{\ell}$ such that if $\ell > \bar{\ell}$,*

$$|| \int x_i(\omega')f(\omega'\Big|\cdot, \sigma(\cdot, x)\ )d\eta \;-\; \int x_i^{\ell}(\omega')f(\omega'\Big|\cdot, \sigma(\cdot, x^{\ell})\ )d\eta||_0 \;<\; \epsilon$$

Proof of Lemma 5 See Horst (2003, Lemma 5.2). ∎

Now define the operator, T defined on $\mathcal{X}^n$ by

$$(Tx)_i(\omega) = H_i(\omega, \bar{\sigma}(\omega, x), x)$$

for each $i = 1, \ldots, n$, or, in other words,

$$(Tx)(\omega) = (H_1(\omega, \bar{\sigma}(\omega, x), x), \ldots, H_n(\omega, \bar{\sigma}(\omega, x), x)\ ) \tag{30}$$

Clearly, from Lemma 4, the function $(Tx)(\cdot)$ is smooth in ω with uniformly bounded first derivative in ω over all ω and x. This implies, in particular, that Tx has uniform Lipschitz bound. Consequently, $Tx \in \mathcal{X}$ for all $V \in \mathcal{X}$. Hence, we have $T : \mathcal{X}^n \to \mathcal{X}^n$.

Lemma 6 *T is a continuous operator.*

Proof Lemma 6 Let $\{x^{\ell}\}$ be a sequence such that $x^{\ell} \in \mathcal{X}$ for all ℓ and $x^{\ell} \to x \in \mathcal{X}$ with the convergence uniform on each compact set $K \subset \Omega$ as $\ell \to \infty$. By Lemma 4, we also know that by Lipschitz continuity of $\bar{\sigma}$ in x, $||\bar{\sigma}(\cdot, x^{\ell}) - \bar{\sigma}(\cdot, x)|| \to 0$ uniformly in ω. Consequently, by the smoothness properties of u_i for each i and of f we can fix $\epsilon > 0$ and let $\bar{\ell}$ satisfy for all $\ell \geq \bar{\ell}$, all ω' and all i, $||u_i(\cdot, \bar{\sigma}(\cdot, x^\ell)\ ) - u_i(\cdot, \bar{\sigma}(\cdot, x)\ )||_0 < \epsilon$, and $|f(\omega' \Big| \omega, \sigma(\omega, x)\ ) - f(\omega' \Big| \omega, \sigma(\omega, x^\ell)\ )| < \epsilon$, and by Lemma 5,

$$|| \int x_i(\omega') f(\omega' \Big| \cdot, \sigma(\cdot, x)\ ) d\eta \ - \ \int x_i^\ell(\omega') f(\omega' \Big| \cdot, \sigma(\cdot, x^\ell)\ ) d\eta ||_0$$

With these results we see that:

$$\begin{aligned}
& ||(Tx^\ell)_i(\cdot) - (Tx)_i(\cdot)||_0 \\
= \ & ||H_i(\cdot, \sigma(\cdot, x^\ell), x^\ell) - H_i(\cdot, \sigma(\cdot, x), x)||_0 \\
\leq \ & (1-\delta)||u_i(\cdot, \sigma(\cdot, x^\ell)\ ) - u_i(\cdot, \sigma(\cdot, x)\ )||_0 \\
& \quad + \ \delta || \int x_i(\omega') f(\omega' \Big| \cdot, \sigma(\cdot, x)\ ) d\eta \ - \ \int x_i^\ell(\omega') f(\omega' \Big| \cdot, \sigma(\cdot, x^\ell)\ ) d\eta ||_0 \\
< \ & (1-\delta)\epsilon \ + \ \delta\epsilon \ = \ \epsilon
\end{aligned}$$

Hence T is continuous. ∎

The Rest of the Proof of Theorem A

Using Lemma 6, T maps continuously from the compact set $\mathcal{X}$ into $\mathcal{X}$. By Schauder's Fixed Point Theorem, T has a fixed point, x^*:

$$x^* = Tx^*.$$

Therefore, the profile, σ^* defined by $\sigma^* \equiv \sigma(\cdot, x^*)$ is a Markov Perfect equilibrium of the dynamic political game without public decisions.

The Extension to Public Decisions

Fix a dynamic political game G satisfying (A1)-(A5) and C dynamically consistent. We now define a simple transformation of the full game, G, with public decisions to one with private decisions game $\bar{G}$ such that $\bar{G} = \langle (\bar{u}_j)_{j \in J}, \bar{\Omega}, \bar{q}, \bar{E}, \omega_0; \rangle$ and the Markov Perfect equilibrium of transformed game with private decisions is an equilibrium of the original dynamic political game.

First, define the state space in the private decisions game to be $\bar{\Omega} = \Omega \times \Theta$, as expected. Next, observe that since C is dynamically consistent, the set of players in $\bar{G}$ is $J = I$. Use j to index this set. Now reinterpret public decisions in the original game G as private decisions in the private action game, $\bar{G}$, as follows. For each θ, let $\bar{E}_j(\theta)$ denote the feasible actions given

θ. Define $\bar{E}_j(\theta) = E$ if $j \in I$, and $\bar{E}_j(\theta) = P \times \Delta(\Theta)$ if $j = \theta$, and $\bar{E}_j(\theta) = \{p^\circ\}$ otherwise, where p° is a degenerate action.

For a player $j = \theta$, a part of his decision is a mixed action in $\Delta(\Theta)$, which we denote by β_θ. Hence, $\beta_\theta(\theta')$ is the probability assigned by player $j = theta$ to θ'.

Define the stage payoffs, $\bar{u}$ by:

$$\bar{u}_i(\omega, \theta, \bar{e}) = u_i(\omega, e, p) \text{ iff } \bar{e}_\theta = (p, \beta_\theta), \text{ and } \bar{e}_i = e_i, \forall i \in I$$

Similarly, let $\bar{f}$ denote the density admitted by $\bar{q}$ and defined by

$$\bar{f}(\omega', \theta' |\, \omega, \theta, \bar{e}) = f(\omega' |\, \omega, e, p)\beta(\theta') \text{ iff } \bar{e}_\theta = (p, \beta), \text{ and } \bar{e}_i = e_i, \forall i \in I$$

Now fix a realization θ' of next period's political state. Implicitly, this means that we ignore the mixed strategies $(\beta_\theta)_{\theta\in\Theta}$. Observe then that the restriction of $\bar{u}_i$ and $\bar{f}$ to the remaining variables of the game satisfies (A1)-(A5). Consequently, Theorem A implies that a Markov Perfect equilibrium, call it $\bar{\sigma}^*$ exists which depends on the realized θ'. We write $\bar{\sigma}^*(\omega, \theta, \theta')$ to denote the action profile condition on the state (ω, θ) and the realization θ' from the mixed strategies $(\beta_\theta)_{\theta\in\Theta}$. By Theorem A, $\bar{\sigma}^*$ is smooth in the economic state ω.

Now fix this $\bar{\sigma}^*$. Observe that since β_θ does not vary with the economic state, we can now consider the choice of (β_θ) as a mixed Markov strategy profile in a finite state, finite action dynamic game. Application of Theorem 2 implies the existence of an equilibrium profile (β_θ). Since (β_θ) and $\bar{\sigma}^*$ are chosen as if they are sequenced, we can define a corresponding correlated public decision rule $\psi^* \times \mu^*$ in the original game by:

$p = \psi^*(\omega, \theta, \theta') = \bar{\sigma}^*_\theta(\omega, \theta, \theta')$

$\mu^*(\theta'|\omega, \theta) = \beta_\theta(\theta'|\omega, \theta)$

Finally, the private decision rule is

$\sigma_i(\omega, \theta) = \bar{\sigma}^*_i(\omega, \theta)$. ∎

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