Double-Holdups in a Matching Model with Search Frictions and Endogenous Heterogeneity*

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Abstract

This paper investigates the interaction between education decisions by workers and investment decisions by firms in a random matching model with endogenous heterogeneity. I analyze the efficiency properties of the equilibrium and find that in the presence of search frictions and investment costs in both human and physical capital, while workers under-invest in education firms may efficiently invest in high-skill jobs. I find that due to a technological externality, under-investments by workers alleviate the holdup problem faced by firms. Moreover, I show that there is a combination of human and physical capital costs that solves the firms' holdup problem.

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1 Introduction

This paper analyzes the interaction between human and physical capital investments in a search-matching model. I focus on the analysis of the effect of investment costs on equilibrium and social optimum investment decisions in a framework in which the surplus of the matches is endogenous.

I present a two-sector matching model in which the distribution of both workers and firms across sectors is endogenous. Heterogeneity arises because before finding a job workers may have to pay a higher education cost for becoming high-skilled and firms may need to make higher investments in order to open a vacancy in a high-skill sector. It is assumed that a high-skill vacancy can only be filled by a high-skill worker and accordingly that low-skill workers can only be productive in the low-skill sector[1]. The sectorial distribution of vacancies is determined by a free-entry condition while the corresponding allocation of workers is driven by an unemployment value indifferent condition.

This paper is related to Acemoglu (2001) in many ways. As in that study I assume that each sector produces and intermediate good that is combined with the intermediate good produced by the other sector in an aggregate utility or production function. The prices of the goods are determined in a competitive market and represent the gross surplus of the match in the corresponding sector. In other words, each type i worker matched with a type i firm produces one unit of good i valued at p_i. The model I will present is also closely related to the ex-post segmentation case developed in Albrecht and Vroman (2002) with the addition of endogenous workers heterogeneity and endogenous prices.

One of the most influential papers analyzing the effect of holdup problems in the presence of frictions is Acemoglu and Shimer (1999), who present a one sector matching model with homogeneous workers in which firms make a range of investments before finding their employees. They prove that if wages are determined by ex-post bargaining, firms will under-invest in capital in equilibrium. In a similar vein, Acemoglu (2001) analyzes the efficiency of job composition in a two-sector matching model with homogeneous workers. As in the model I will present here he finds that when investments are more expensive in the high-skill sectors and wages are determined by ex-post bargaining, job composition will be inefficiently biased toward low-wage jobs. However, the total number of jobs in the economy is constrained efficient.

Though there are a number of papers that have analyzed two-sided investments in matching models (Davis, 1995; Acemoglu, 1996, 1997; Masters, 1998), none of them has studied the interaction between investment and education decisions in an equilibrium unemployment framework. This

[1] I rule out the possibility of mismatch between for example high-skill workers and low-skill firms analyzed in other papers (Albrecht and Vroman, 2002; Dolado, Jansen and Jimeno, 2003; Uren 2003). This might lead to a coordination externality that will complicate the analyisis of the effect of two-sided investments in the model. See Blázquez and Jansen (2003) for the efficiency analysis of Albrecht and Vroman (2002).

paper contributes by introducing these investment interactions in a Pissarides (2000) two-sector environment. In addition, compared to previous studies the efficiency implications of the model are somewhat different. The literature has found that in the presence of search frictions and/or technological externalities in equilibrium both workers and firms under-invest in human and physical capital, respectively.

Assuming the distribution of property rights in ex-post bargaining proposed by Hosios (1990), I find that in the presence of frictions and rent sharing investment levels by workers are inefficiently low but firms investment levels may be efficient or not. Additionally the total number of jobs may be efficient or not, but it is not possible to obtain both efficient job creation and job composition simultaneously. The next paragraphs summarize the idea of these results.

If opening a vacancy in the high-skill sector is more costly than opening a low-skill vacancy and education costs are zero, rent-sharing implies a too high fraction of firms opening bad-job vacancies compared to the efficient level. By the workers side, I find that even when education costs are zero the fraction of high-skill workers is too low in equilibrium. Workers decisions to educate do not consider positive production externalities of education and the net reduction in the social cost of frictions implied by making contacts to the more costly vacancies easier.

On the contrary, if capital creation costs in both sectors is the same but education is costly, there is under-investment in education by workers that leads to over-investment by firms in high-skill jobs. The reason of this result is a combination of a hold-up problem affecting workers and a production externality. When the fraction of high-skill workers in the population is too low, the price of the goods produced by the high-skill sector is too high and this leads to an increase in high-skill job creation above the efficient level. Finally, I find that the total number of jobs is too low in equilibrium.

The interesting point of the paper is that starting from an economy with higher capital creation costs in the high-skill sector and costless education, it is possible to eliminate the inefficiencies in job-composition. Since the fraction of high-skill jobs is increasing in the education cost due to the described price effects, a tax on education can lead the economy to the efficient job composition level. This experiment may or not reduce the inefficiencies in the total number of jobs in the economy, but it has the cost of increasing the inefficiencies in the skill-composition of the workforce.

The paper is structured as follows: In the next section I describe the environment. Section 3 presents the equilibrium, Section 4 analyzes the efficiency properties of the equilibrium. I conlcude in Section 5.

2 Model

2.1 Environment

1. **Population**. Consider a continuous-time economy populated by a continuum of workers with mass equal to 1 who live forever, are risk neutral and discount time at rate r. Workers can either decide to have a low-skill at no cost or to educate at a fixed cost c and get a high-skill level. As a consequence, there is a fraction π of the workforce that consists of low-skill workers, and $1-\pi$ of high-skill workers.

2. **Unemployed**. Given that jobs break-up at an exogenous rate s, a fraction u of the population remains unemployed. The skill-composition of the unemployed is described by a fraction γ of low-skill, and $1-\gamma$ of high-skill unemployed.

3. **Firms**. There is a larger continuum of infinitely lived and risk neutral firms. A firm can open a job in either of two intermediate good sectors denoted by b (bad) and g (good). This decision is irreversible and involves a fixed capital creation cost k_b or k_g, respectively. The mass of vacancies in each sector will be determined by a free entry condition. The total mass of vacancies is denoted by v, and ϕ indicates the fraction of vacancies that are low-skill.

4. **Skill requirements**. A sector b vacancy can only be filled by a low-skill worker. Accordingly, a g vacancy can only be filled by a high-skill worker.

5. **Aggregate production technology**. The economy produces one final consumption good Y with two intermediate inputs Y_b and Y_g according to a standard Cobb-Douglas technology[2],

$$Y = Y_b^{\alpha} Y_g^{1-\alpha}. \tag{1}$$

   Assuming that the good Y is sold in a competitive market and normalizing the price of the final consumption good to 1, we obtain the price of the intermediate goods,

$$p_b = \alpha \frac{Y}{Y_b} = \alpha \left(\frac{Y_g}{Y_b} \right)^{1-\alpha} \tag{2}$$

$$p_g = (1-\alpha) \frac{Y}{Y_g} = (1-\alpha) \left(\frac{Y_b}{Y_g} \right)^{\alpha}. \tag{3}$$

6. **Intermediate goods production**. Since matched agents produce one unit of the respective good, the gross surplus of a match is p_b or p_g, correspondingly. It also implies that the

[2] Assuming a general CES production function would not restrict the main results of the paper. However, for convenience in the exposition I use a Cobb-Douglas.

total production of any intermediate good i is equal to the number of matched agents (i.e. employment) in sector i,[3]

$$\begin{aligned} Y_b &= \pi - \gamma u && (4) \\ Y_g &= 1 - \pi - (1-\gamma) u. && (5) \end{aligned}$$

7. **Matching function**. Unemployed workers and firms with unfilled vacancies meet randomly through a constant returns to scale matching function $M(u, v)$. As usual, it is assumed that the matching function is twice differentiable, increasing in both arguments, and satisfies $M(0, v) = M(u, 0) = 0$. Therefore we can write

$$\begin{aligned} q(\theta) &= \frac{M(u,v)}{v} \\ q(\theta)\theta &= \frac{M(u,v)}{u} = \theta \frac{M(u,v)}{v}. \end{aligned}$$

   where $\theta = \frac{v}{u}$ represents the tightness of the labor market and determines the number of matches in the economy. The function $q(\theta)$ represents the flow rate at which vacancies meet unemployed workers and is decreasing in θ. Accordingly, $q(\theta)\theta$ is the contact rate of vacancies by an unemployed worker and $\frac{d[q(\theta)\theta]}{d\theta} > 0$. Given the properties of the matching technology $0 < \eta(\theta) < 1$, where $\eta(\theta) = -\frac{q'(\theta)\theta}{q(\theta)}$ is the elasticity of the matching function.

8. **Effective matching rates**. By the random matching assumption, every worker faces the same probability of meeting vacancies. However, since a low-skill worker is only suitable for working in any of the ϕv jobs opened in sector b, he faces an effective arrival rate of $q(\theta)\theta\phi$. Following the same reasoning, the effective arrival rate to a low-skill vacancy, which can only match with a fraction γ of the unemployed, is $q(\theta)\gamma$. Obviously, the same reasoning applies for the effective matching rates in the high-skill sector.

2.2 Bellman equations and wage determination

Denote by J_i^U the value of unemployment, J_i^E the value of employment, J_i^V the value of an unfilled vacancy and J_i^F the value of a filled job in sector i. These values are linked by the following flow equations:

$$\begin{aligned} rJ_b^U &= \theta q(\theta)\phi \left[J_b^E - J_b^U\right] && (6) \\ rJ_g^U &= \theta q(\theta)(1-\phi) \left[J_g^E - J_g^U\right] \end{aligned}$$

[3]Note how the price of the intermediate goods are strongly dependent on the skill composition of the workforce. This has important implications for the equilibrium I will describe below.

Equation (6) states that the flow value of unemployment in any sector is equivalent to the expected capital gain from finding a job an realizing a flow value rJ_i^E. For simplicity, and without loss of generality, I assume that the flow income from unemployment is zero. Accordingly, the flow values of employment are expressed by

$$rJ_i^E = w_i + s(J_i^U - J_i^E). \tag{7}$$

These equations reflect the fact that while employed in any sector i a worker receives a salary w_i but faces a probability s of returning to unemployment and incurring a capital loss $J_i^U - J_i^E$.

A similar interpretation applies for the flow values of filled and unfilled vacancies,

$$rJ_i^F = p_i - w_i + s(J_i^V - J_i^F) \tag{8}$$

$$\begin{aligned} rJ_b^V &= q(\theta)\gamma(J_b^F - J_b^V) \\ rJ_g^V &= q(\theta)(1-\gamma)(J_g^F - J_g^V). \end{aligned} \tag{9}$$

Generalized Nash Bargaining. When a matched is formed, a p_i is realized and wages are determined through rent sharing over the surplus of the match. More precisely, wages are determined through a generalized Nash Bargaining solution in which β is a measure of workers bargaining power. Therefore, the first order conditions of the Nash Bargaining solution in any sector i are,

$$(1-\beta)(J_i^E - J_i^U) = \beta(J_i^F - J_i^V). \tag{10}$$

2.3 Equilibrium conditions

1. **Free entry condition**. The number of vacancies in each sector is determined by a free entry condition which drives expected net profits for opening vacancies to zero,

$$rk_i = rJ_i^V. \tag{11}$$

2. **Non-arbitrage condition**. Workers can get the low-skill at no cost or become high skilled through a costly human capital investment. Assuming a total education cost c, every high-skill worker has to make a flow payment rc for her education forever. The education decision is driven by the comparison of the unemployment values of being high or low-skill. Then in equilibrium the fraction π of low-skill workers must be such that makes the workers indifferent between being high or low-skill, controlling for the education cost c,

$$rc = rJ_g^U - rJ_b^U. \tag{12}$$

Any investment decision taken by an agent in this economy has two effects: Not only it affects the matching probabilities faced by other agents but also it affects the surplus of the matches through the effect on prices.

3. **Steady state unemployment**. As usual, in steady state the flow out of unemployment to any sector has to be equal to the flow into unemployment to that sector,

$$\phi q(\theta)\theta\gamma u = s(\pi-\gamma u) \tag{13}$$

$$q(\theta)\theta(1-\phi)(1-\gamma)u = s(1-\pi-(1-\gamma)u). \tag{14}$$

Equation (13) indicates that the flow out of low-skill unemployment is equal to the mass of low-skill unemployed (γu), times the effective arrival rate of vacancies $\phi q(\theta)\theta$. The corresponding flow into unemployment is determined by the fraction s of the jobs that are destroyed every period times the number of low-skill unemployed ($\pi - \gamma u$). The same argument explains equation (14) . As in the ex-post segmentation case of Albrecht and Vroman (2002), these two equations are used to solve for ϕ and u as a function of θ and γ[4],

$$\phi = \frac{\pi(1-\gamma)q(\theta)\theta + (\pi-\gamma)s}{q(\theta)\theta(\gamma+\pi-2\gamma\pi)} \tag{15}$$

$$u = \frac{s}{q(\theta)\theta + 2s}\left(\frac{\gamma+\pi-2\gamma\pi}{\gamma(1-\gamma)}\right). \tag{16}$$

Note that ϕ is decreasing in θ and γ and increasing in π. The unemployment rate is decreasing in θ, and $\frac{du}{d\gamma} \gtrless 0$ iff $\pi \lesseqgtr \gamma$. (see Appendix 1). With respect to π, we have that $\frac{\partial u}{\partial \pi} \gtrless 0$ iff $\gamma \lesseqgtr 1/2$. Note also that $\pi \gtrless \gamma$ iff $\phi \gtrless 1/2$.

[4] Alternatively, we could have writen γ and u as a function of θ and ϕ :

$$\gamma = \frac{\pi\left[q(\theta)\theta\left(1-\phi\right)+s\right]}{q(\theta)\theta\left[\pi+\phi-2\pi\phi\right]+s},$$

$$u = \frac{s}{q(\theta)\theta\phi+s}\frac{q(\theta)\theta\left[\pi+\phi-2\pi\phi\right]+s}{q(\theta)\theta\left(1-\phi\right)+s}.$$

3 Equilibrium

An equilibrium is defined as a proportion of low skilled workers π, bad jobs vacancies ϕ, low skilled unemployed workers γ, tightness of the labor market θ, unemployment rate u, value functions and prices of both goods such that equations (2)-(12) are satisfied. The steady state proportion of bad job vacancies and the unemployment rate are given by equations (15) and (16), respectively. Employment in each sector, equal to intermediate goods production, is given by $Y_b = \pi - u\gamma$ and $Y_g = 1 - \pi - u(1-\gamma)$.

Then equations (2) and (3) with the steady state equation for unemployment (16) substituted in gives equilibrium prices,

$$\begin{aligned} p_g &= (1-\alpha)\left(\frac{\pi - u\gamma}{1-\pi-u(1-\gamma)}\right)^{\alpha} = (1-\alpha)\left(\frac{\gamma}{1-\gamma}\frac{\pi(1-\gamma)q(\theta)\theta - s(\gamma-\pi)}{(1-\pi)\gamma q(\theta)\theta + s(\gamma-\pi)}\right)^{\alpha} \\ p_b &= \alpha\left(\frac{1-\pi-u(1-\gamma)}{\pi-u\gamma}\right)^{1-\alpha} = \alpha\left(\frac{1-\gamma}{\gamma}\frac{(1-\pi)\gamma q(\theta)\theta + s\,(\gamma-\pi)}{\pi(1-\gamma)q(\theta)\theta - s(\gamma-\pi)}\right)^{1-\alpha}. \end{aligned}$$

It can be shown that the price of the good (bad) job output is increasing (decreasing) in π and decreasing (increasing) in γ. The derivatives with respect to θ are $\frac{\partial p_b}{\partial \theta} \gtreqless 0$ and $\frac{\partial p_g}{\partial \theta} \lesseqgtr 0$ iff $\pi \gtreqless \gamma$ (see Appendix 1). To get an intuition of this last result, if for example $\pi > \gamma$ and increase in θ implies a relatively higher increase in Y_g than in Y_b which leads to a reduction (increase) in p_g (p_b).

Using equations (7) to (11) we obtain the standard expressions for wages in any sector i,

$$w_i = \beta(p_i - rk_i) + (1-\beta)rJ_i^U. \tag{17}$$

Therefore, the wage in any sector is a weighted average between the net surplus of the match and the value of unemployment.

Using (17) and equations (8) and (9), the free entry conditions (11) imply that the equilibrium flow value of vacancies can be expressed as

$$rJ_b^V = rk_b = \frac{q(\theta)\gamma(1-\beta)(p_b - rJ_b^U)}{r+s+(1-\beta)q(\theta)\gamma} \tag{18}$$

$$rJ_g^V = rk_g = \frac{q(\theta)(1-\gamma)(1-\beta)(p_g - rJ_g^U)}{r+s+(1-\beta)q(\theta)(1-\gamma)}. \tag{19}$$

In equilibrium the flow value of an unfilled vacancy in any sector is the weighted average of the expected net profits obtained from filled and an unfilled jobs (unfilled jobs generate a zero profit with probability $(r+s)$). Finally the equilibrium flow values of unemployment are given by (6) with (8), (9), and (17) substituted in,

$$rJ_b^U = \frac{\beta\theta q(\theta)\phi\,(p_b - rk_b)}{r+s+\beta\theta q(\theta)\phi} \tag{20}$$

$$rJ_g^U = \frac{\beta\theta q(\theta)(1-\phi)\,(p_g - rk_g)}{r+s+\beta\theta q(\theta)(1-\phi)}. \tag{21}$$

Again as usual, the equilibrium flow values of unemployment rJ_b^U and rJ_g^U are weighted averages of the flow values of employment and leisure.

3.1 Uniqueness

Combining equations (18) and (19) with the unemployment flow values (20) and (21), respectively, we can conveniently express the equilibrium flow values of vacancies as[5]

$$rk_b = \frac{q(\theta)\gamma(1-\beta)p_b}{r+s+(1-\beta)q(\theta)\gamma+\beta\theta q(\theta)\phi} \tag{22}$$

$$rk_g = \frac{q(\theta)(1-\gamma)(1-\beta)p_g}{r+s+(1-\beta)q(\theta)(1-\gamma)+\beta\theta q(\theta)(1-\phi)} \tag{23}$$

It turns out that for given π the equilibrium values (θ,γ) are determined by the intersection of a bad-job locus and a good-job locus given by equations (22) and (23) , respectively. In Appendix 2 I show that both curves intersect once where the bad-job locus is upward sloping and the good-job locus is downward sloping[6]. As θ increases, γ must increase to satisfy the bad-job locus equation (22) and decrease to satisfy the good-job locus equation (23). Therefore, for any given π, the intersection of both loci determines (θ,γ) and $rJ_g^U - rJ_b^U$ (the right-hand side of equation (12)). Given that in equilibrium a greater π implies an increase in γ[7], the difference $rJ_g^U - rJ_b^U$ is increasing in π too. Therefore, there is only one value of π (associated with θ and γ) that satisfies equation (12).

[5] Accordingly, the equilibrium unemployment values are

$$rJ_b^U = \frac{q(\theta)\theta\phi\beta p_b}{r+s+(1-\beta)q(\theta)\gamma+\beta\theta q(\theta)\phi}$$

$$rJ_g^U = \frac{q(\theta)\theta(1-\phi)\beta p_g}{r+s+(1-\beta)q(\theta)(1-\gamma)+\beta\theta q(\theta)(1-\phi)}.$$

[6] Though as shown in Appendix 2 the slope of both curves can be either positive or negative, given that in equilibrium the slope of the bad-job (good-job) locus is positive (negative), throughout the paper I will refer to an upward sloping bad-job locus and a downward sloping good-job locus.

[7] The intuition is that an increase in π leads to a greater p_g and a lower p_b. Then, there is an outward shift in the good-job locus and an inward shift in the bad-job locus that causes an increase in γ. The greater γ means a lower ϕ. Then rJ_b^U decreases and rJ_g^U increases.

Therefore we can conclude that $(22),(23)$ and the non-arbitrage condition (12) determine the unique vector (θ,γ,π) that solves the model. The solutions for ϕ and u are given by the steady state equations (15) and (16), respectively.

3.2 Comparative statics

In the next section I will analyze the efficiency implications of the equilibrium by focusing on their dependence on investments made in the two sides of the market. For that reason it is useful to analyze the comparative statics of the equilibrium with respect to changes in the capital creation cost in the good-job sector (k_g) and the education cost (c). For simplicity I will limit the analysis of the model assuming $\alpha = 1/2$ throughout the rest of the paper. This assumption is relevant for some results but not relevant for the main argument of the paper.

The general idea is summarized by,

Proposition 1 *In equilibrium,*

$$\frac{d\phi}{dk_g} > 0, \frac{d\pi}{dk_g} > 0, \text{ and } \frac{d\theta}{dk_g} < 0,$$
$$\frac{d\phi}{dc} < 0, \frac{d\pi}{dc} > 0, \text{ and } \frac{d\theta}{dc} \lesseqgtr 0 \text{ iff } k_g \lesseqgtr k_g^*,$$

where k_g^* is a threshold value defined by (22) *and* (23).

The intuition of these results will become clear after presenting a few related remarks that will matter for the efficiency analysis. Letting $j(k_g, c)$ denote the value of j for a given k_g and c we have that:

Remark 1 *If $k_g = k_b$, $c = 0$ the equilibrium is characterized by*

$$\pi(k_b, 0) = \gamma(k_b, 0) = \phi(k_b, 0) = 1/2.$$

Also,

$$p_g(k_b, 0) = p_b(k_b, 0).$$

This result comes from the fact that when the capital creation cost in both sectors is the same $(k_g = k_b)$ and education is costless $(c = 0)$, both firms and workers can decide at no cost on which sector to locate (for the case of the firms after having decided to open a vacancy) and the aggregate production technology leads the economy to the referred symmetric equilibrium. In other words if for example all the firms decide to open a vacancy in the good-job sector (just because it is good)

p_g would fall to zero and p_b would go to infinity. Then it would not be possible that all firms open good-jobs and $\phi = 0$ can not be a equilibrium.

Starting from the above remark, an increase in c leads to the following result:

Remark 2 *1. If $k_g = k_b$ and $c > 0$ the equilibrium is characterized by*

$$\begin{gathered}\pi(k_b, c) > \pi(k_b, 0) = 1/2, \\ \gamma(k_b, c) > \gamma(k_b, 0) = 1/2, \\ \phi(k_b, c) < \phi(k_b, 0) = 1/2, \\ \theta(k_b, c) < \theta(k_b, 0) \ \text{ and} \\ p_g(k_b, c) > p_g(k_b, 0) = p_b(k_b, 0) > p_b(k_b, c).\end{gathered}$$

2. If $k_g > k_b$ and $c > 0$ the equilibrium is characterized by

$$\begin{gathered}\pi(k_g, c) > \pi(k_b, c) > \pi(k_b, 0) = 1/2, \\ \gamma(k_g, c) < \gamma(k_b, c), \\ \phi(k_g, c) > \phi(k_b, c), \\ \theta(k_g, c) < \theta(k_b, c) < \theta(k_b, 0) \ \text{ and} \\ p_g(k_g, c) > p_g(k_b, c) > p_g(k_b, 0) = p_b(k_b, 0) > p_b(k_b, c) > p_g(k_g, c).\end{gathered}$$

An increase in c will obviously have a primary impact on workers through equation (12). Of course, the original equilibrium will not be sustainable because high skill workers would have a negative net value of unemployment and that would violate the non-arbitrage condition. Then π must increase to adjust to the increased education cost. The increased π affects prices (increase p_g, decrease p_b) in such a direction that leads firms to open more vacancies in sector g. Putting it differently, the greater price in the good-job sector implies a required increase in γ to satisfy the good-job locus equation (22), and the opposite is needed in the other sector. Then the increase in c leads to a shift-in the bad-job locus and shift-out in the good-job locus. As intuition would suggest, the effect on θ is shown to be negative (see Appendix 3)[8].

An increase in k_g will shift-in the good-job locus and so both θ and γ has to decrease to satisfy equations (22) and (23). The decrease in both θ and γ leads to an increase in ϕ that is reflected

[8] If $rk_g > rk_b$ and rk_g is high enough $\left(\geq rk_g^*\right)$ the price differences are so big that there is a positive net job creation at the aggregate. This occurs when p_g is too high and p_b is too low. Then the required increase in γ to satisfy the good-job locus is greater than the necessary to restore the equilibrium in the bad-job locus equation. This is a situation in which the economy gains a lot for good-job creation and lose a little for bad-job destruction, so the effect of c on θ would be positive.

in equation (15). By the workers side, the larger ϕ and the reduced θ implies by equation (12) a reduction in $J_g^U - J_b^U$. Then π also needs to increase. This last effect might imply a further reduction in θ and a partial compensation of the initial increase in ϕ. The increased cost of the input k_g must lead to an increase in p_g and a fall in p_b, as confirmed by the increase in π and the reduction in θ and γ.

Accordingly, the next remark describes the effect of an increase in k_g starting from the equilibrium with $k_g = k_b$ and $c = 0$.

Remark 3 *1. If $k_g > k_b$ and $c = 0$ the equilibrium is characterized by*

$$\begin{gathered}\pi(k_g, 0) > \pi(k_b, 0) = 1/2, \\ \gamma(k_g, 0) < \gamma(k_b, 0) = 1/2, \\ \phi(k_g, 0) > \phi(k_b, 0) = 1/2, \\ \theta(k_g, 0) < \theta(k_b, 0) \text{ and} \\ p_g(k_g, 0) > p_g(k_b, 0) = p_b(k_b, 0) > p_b(k_g, 0).\end{gathered}$$

2. If $k_g > k_b$ and $c > 0$ the equilibrium is characterized by

$$\begin{gathered}\pi(k_g, c) > \pi(k_g, 0) > \pi(k_b, 0) = 1/2, \\ \gamma(k_g, c) > \gamma(k_g, 0), \\ \phi(k_g, c) < \phi(k_g, 0), \\ \theta(k_g, c) \lesseqgtr \theta(k_b, 0) \text{ if } k_g \lesseqgtr k_g^* \text{ and} \\ p_g(k_g, c) > p_g(k_g, 0) > p_g(k_b, 0) = p_b(k_b, 0) > p_b(k_g, 0) > p_b(k_g, c).\end{gathered}$$

Though showing these last results might seem unnecessary since they have the same qualitative implications as Remark 2, the quantitative results of Remarks 2.1 and 3.1 regarding the relations between ϕ and γ will be useful for the analysis of the efficiency implications of the equilibrium.

4 Efficiency

The social planner sets time paths of the triplet (θ, ϕ, π) that maximize the present value of the net total surplus of the economy subject to the same restrictions that govern private decisions (Equations (13) and (14)). Before presenting the planner's problem, I will introduce some notation for the number of unemployed and vacancies in the high and low-skill sectors (u_b, u_g, v_b, v_g)

$$\begin{aligned} u_b &\equiv \gamma u \\ u_g &\equiv (1-\gamma)u \\ v_b &\equiv \phi\theta u \equiv \phi\theta\frac{u_b}{\gamma} \\ v_g &\equiv (1-\phi)\,\theta u \equiv (1-\phi)\,\theta\frac{u_g}{(1-\gamma)}. \end{aligned}$$

For convenience, I expressed v_i as depending on the group of unemployed type i vacancies match with (u_i). Then, for give initial conditions of u_b and u_g, the planner's problem[9] can be written as

$$\max\int_0^\infty \left\{ \begin{array}{c} (\pi(t)-u_b(t))(p_b(t)-rk_b)+(1-\pi(t)-u_g(t))(p_g(t)-rk_g)- \\ v_b(t)rk_b - v_g(t)\,rk_g-(1-\pi(t))\,rc \end{array} \right\} e^{-rt}dt$$

subject to

$$\begin{aligned} \dot{u_b} &= s(\pi(t)-u_b(t))-\phi(t)\,q(\theta(t))\theta(t)\,u_b(t) \\ \dot{u_g} &= s(1-\pi(t)-u_g(t))-q(\theta(t))\theta(t)\,(1-\phi(t))u_g(t)\,. \end{aligned}$$

The flow of net output of the economy at any point in time consists of the number of workers in good-jobs $(1-\pi(t)-u_g(t))$ times their net output $(p_g(t)-rk_g)$, plus the number of workers in bad-jobs $(\pi(t)-u_b(t))$ times their net output $(p_b(t)-rk_b)$, minus the flow cost of vacancy creation in the bad- job sector $(\phi(t)\,\theta(t)u(t)\,rk_b)$ and in the good-job sector $((1-\phi(t))\,\theta(t)u(t)\,rk_g)$, minus the flow cost of education for high-skill workers $(1-\pi(t))\,rc$.

The current-value Hamiltonian with multipliers λ_b and λ_g can be expressed as,

$$\begin{aligned} H_c \equiv\; & He^{rt}=(\pi-u_b)(p_b-rk_b)+(1-\pi-u_g)(p_g-rk_g)-\frac{u_b\phi\theta}{\gamma}rk_b-\frac{u_g(1-\phi)\,\theta}{(1-\gamma)}rk_g \\ & -(1-\pi)\,rc+\lambda_b\,(s(\pi-u_b)-\phi q(\theta)\theta u_b)+\lambda_g\,(s(1-\pi-u_g)-q(\theta)\theta(1-\phi)u_g)\,. \end{aligned}$$

Since we are interested in the steady state implications of the model I write down the current-value Hamiltonian without the time dependence of the variables. Also, $\dot{\lambda}_b$ and $\dot{\lambda}_g$ are both equal to zero.

In what follows I will present the necessary conditions that solve the optimum control planner's problem[10]. First I show the steady state costate equation conditions that the derivative of H^c with

[9] This is a typical total planner's problem as defined in the Chapter 8 of Pissarides (2000) for a one-sector model, and extended in Acemoglu (2001) for the two-sector homogeneous workers case. Other papers (Acemoglu and Shimer, 1999; Shimer and Smith, 2001 and Blázquez and Jansen, 2003) use a different total surplus function considering the expected net output of the economy from the workers persective. The use of one or another specification would not change results.

[10] It is too difficult to obtain analytical solutions for the sufficient conditions for a maximization in this problem, as in Acemoglu and Shimer (1999). Except the mentioned study and Blázquez and Jansen (2003), the literature has ignored the sufficient conditions for a maximization. Maybe this is because focusing on the necessary conditions is enough to compare the social allocation with the decentralized equilibrium.

respect to each of the states variables, u_b and u_g, has to be equal to the flow shadow value of the state variables, λ_b and λ_g, respectively,

$$\frac{\partial H_c}{\partial u_b} = r\lambda_b = -(p_b - rk_b) - \frac{\phi\theta}{\gamma} rk_b - \lambda_b (s + \phi q(\theta)\theta) \tag{24}$$

$$\frac{\partial H_c}{\partial u_g} = r\lambda_g = -(p_g - rk_g) - \frac{(1-\phi)\theta}{(1-\gamma)} rk_g - \lambda_g (s + (1-\phi) q(\theta)\theta). \tag{25}$$

According to equation (24) the value of a low-skill unemployed λ_b is the present value of the sum of the forgone surplus she could generate when matched minus her search cost[11]. Equation (25) conveys the same interpretation.

Now, I turn to the standard necessary conditions for θ, ϕ and π[12],

$$\frac{\partial H_c}{\partial \theta} = -[\phi rk_b + (1-\phi) rk_g] - [q(\theta)(1-\eta(\theta))][\lambda_b\gamma\phi + \lambda_g (1-\gamma)(1-\phi)] = 0 \tag{26}$$

$$\frac{\partial H_c}{\partial \phi} = (rk_g - rk_b) - q(\theta)(\lambda_b\gamma - \lambda_g(1-\gamma)) = 0 \tag{27}$$

$$\frac{\partial H_c}{\partial \pi} = (p_b - rk_b) - (p_g - rk_g) + rc + s(\lambda_b - \lambda_g) = 0. \tag{28}$$

where $\eta(\theta) = -\frac{q'(\theta)}{q(\theta)}\theta < 0$ is the elasticity of the matching function for vacancies. The optimality condition for labor market tightness states that the total number of jobs in the economy has to be determined by equating the average cost of creating vacancies $[\phi rk_b + (1-\phi) rk_g]$ to the average expected value of the matches in both sectors, taking into account congestion externalities (which reduce meeting rates in $q(\theta)\eta(\theta)$). The second term of equation (26) considers that bad-job vacancies, a fraction ϕ of the total number of vacancies, face an effective rate $\gamma q(\theta)(1+\eta(\theta))$ of matching a low-skill unemployed (with value $-\lambda_b$). The same applies for good-job vacancies.

Equation (27) implies that job composition is efficient when the economy gets the full expected marginal social value of its investment decisions. The differential cost of creating a good-job vacancy is $rk_g - rk_b$ and the marginal expected surplus from the investment is $q(\theta)(\lambda_b\gamma - \lambda_g(1-\gamma))$.

Finally equation (28) indicates that the planner sets skill-composition taking into account the education costs and the difference in the social value associated to the education decision. A higher fraction of educated workers means an increase in the net surplus of the match $((p_g - rk_g) - (p_b - rk_b))$ if employed but an increase in the cost of being unemployed if the match is destroyed $(-s(\lambda_b - \lambda_g))$.

[11] For instance, the expression for the search cost in the low-skill sector $\frac{\phi\theta}{\gamma} rk_b = \frac{\phi v}{\gamma u} rk_b$ is interpreted as vacancy creation cost per low-skill worker, as suggested by the term $\frac{\phi v}{\gamma u}$.

[12] Note that price effects cancel out in all the necessary conditions. In fact, it can be proved that

$$\frac{\partial p_b}{\partial j}(\pi - \gamma u) = -\frac{\partial p_g}{\partial j}(1 - \pi - (1-\gamma) u)$$

where $j = u, \theta, \phi, \pi$.

4.1 Efficient solution

Below I will give a complete characterization of the social optimum that will allow me to compare it with the decentralized equilibrium solution. I will focus on the effect of education and investment costs in sector g in the planned and decentralized economy, respectively. The starting point is an economy with no education costs and no differential costs of investments in sector g. Then I will depart from that environment in two ways: first analyzing the effect of $c > 0$, and second by focusing on the implications of $k_g > k_b$.Throughout the analysis I will relate the two allocations by imposing the standard condition for constrained efficiency introduced by Hosios (1990) that $\beta = \eta(\theta)$.

4.1.1 Case $k_g = k_b,\ c > 0$

In Appendix 4 I show that the efficient solution is determined by the unique triplet (θ, ϕ, π) that solves

$$
\begin{aligned}
rk_b &= q(\theta)\gamma(1-\beta)\frac{p_b}{D_b} && (29)\\
rk_g &= q(\theta)(1-\gamma)(1-\beta)\frac{p_g}{D_g} && (30)\\
rc &= (r + q(\theta)\theta(1-\phi)\beta)\frac{p_g}{D_g} - (r + q(\theta)\theta\phi\beta)\frac{p_b}{D_b}, && (31)
\end{aligned}
$$

where

$$
\begin{aligned}
D_b &= r + s + (1-\beta)q(\theta)\gamma + \beta\theta q(\theta)\phi\\
D_g &= r + s + (1-\beta)q(\theta)(1-\gamma) + \beta\theta q(\theta)(1-\phi).
\end{aligned}
$$

Then we see that if $k_b = k_g$ and $\eta(\theta) = \beta$ the equilibrium and social bad and good-job locus equations are identical, but the social and non-arbitrage conditions differ. Indeed, the unique social allocation is determined by the intersection of an upward sloping good-job locus defined by equation (29) and a downward sloping good-job locus, given by equation (30), both evaluated at the value of π that solves a social non-arbitrage condition defined by equation (31). Similarly to what occurs in equilibrium, the planner's allocation (denoted by S) consists of values $\pi^S(k_b, c)$, $\gamma^S(k_b, c) > 1/2$ and $\phi^S(k_b, c) < 1/2$ for every $c > 0$. It also happens that:

Proposition 2 *When $k_g = k_b$ and $c = 0$, the decentralized equilibrium is constrained efficient if*

$\beta = \eta(\theta)$. *Then,*

$$\pi^S(k_b, 0) = \pi(k_b, 0) = 1/2,$$
$$\gamma^S(k_b, 0) = \gamma(k_b, 0) = 1/2,$$
$$\phi^S(k_b, 0) = \phi(k_b, 0) = 1/2,$$
$$\theta^S(k_b, 0) = \theta(k_b, 0) \;\; \textit{and}$$
$$p_g(k_b, 0) = p_g^S(k_b, 0) = p_b(k_b, 0) = p_b^S(k_b, 0).$$

Note that (31) exceeds (12) in a term $r\left(\frac{p_g}{D_g} - \frac{p_b}{D_b}\right) > 0$. However, only when $c = 0$ it will be the case that $p_g = p_b$ and so that term becomes zero. Then the decentralized equilibrium coincides with the social optimum[13].

When $c > 0$ these standard efficiency implications no longer hold and we have:

Proposition 3 *If $k_g = k_b$, $c > 0$, and $\beta = \eta(\theta)$ the decentralized equilibrium generates too few jobs with a too low fraction of bad-jobs and a lower than optimal proportion of workers invests in education. Then,*

$$\pi(k_b, c) > \pi^S(k_b, c) > 1/2,$$
$$\gamma(k_b, c) > \gamma^S(k_b, c) > 1/2,$$
$$\phi(k_b, c) < \phi^S(k_b, c) < 1/2, \;\; \textit{and}$$
$$\theta(k_b, c) < \theta^S(k_b, c).$$

In fact, the optimality condition for π evaluated at the decentralized equilibrium parameters (always imposing Hosios' condition) is negative, meaning that π is too high. As mentioned above, when making education decisions, workers do not take into account how the economy as a whole benefits when more high-skill jobs are available, and this leads to an under-investment in education. Then π is inefficient for two reasons: because workers do not internalize a positive externality of future better matches and because they have to share with firms the proceeds of the higher surplus implied by good-job matches. I shown that the division of property rights that satisfies the Hosios' condition is not enough to give workers the correct incentives to education.

[13] Also, in a model with $rk_b = rk_g$ in which π and $p_g > p_b$ are exogenous (Albrecht and Vroman, 2002) the decentralized equilibrium is also efficient when $\eta(\theta) = -\beta$. The intuition of this result comes from the efficiency condition for ϕ (equation (27)). Given that there is not a differential cost of investment in both sectors that condition is always fulfilled in equilibrium. Indeed, when $rk_b = rk_g$ the efficiency condition becomes $\lambda_b\gamma = \lambda_g(1-\gamma)$ that means $(1-\gamma)\frac{p_g}{D_g} = \gamma\frac{p_b}{D_b}$ and this is exactly the expression we get by equating the two equilibrium value of vacancies equations.

The interesting point is that the inefficiency in π translates into changes in ϕ and θ. Regarding the former, note that a too high fraction π implies that the price of the goods produced in the good-job sector (p_g) is too high, and as well, p_b is too low. A higher price p_g in the equation for the equilibrium value of vacancies in the good-job sector implies that γ has to increase to satisfy (23). Similarly, considering the other equilibrium equation (22) when p_b decreases, γ has to increase. This means that compared to the social optimum, γ is too high and so ϕ too low. Then, attracted by the higher prices, a greater than optimal number of firms open good job vacancies when workers under-invest in education[14].

The analysis of the efficiency properties of θ is straightforward. Given that θ is decreasing in π (as shown in Appendix 3), we have that $\theta\left(k_b, c\right) < \theta^S\left(k_b, c\right).$

4.1.2 Case $k_g > k_b,\ c = 0$

Now I turn to the analysis of the efficiency implications of the equilibrium when capital creation costs are higher in the good-job sector, but for convenience in the exposition I will assume $c = 0$. After much algebra (see Appendix 4) we obtain that the social allocation satisfies,

$$
\begin{aligned}
rk_b &= q(\theta)\gamma\left(1-\beta\right)\frac{p_b}{D_b} - q\left(\theta\right)\beta\left(1-\phi\right)\left(rk_g - rk_b\right)\left(1-\left(1-\beta\right)A_b\right) && (32)\\
rk_g &= q(\theta)\left(1-\gamma\right)\left(1-\beta\right)\frac{p_g}{D_g} + q\left(\theta\right)\beta\phi\left(rk_g - rk_b\right)\left(1-\left(1-\beta\right)A_g\right) && (33)\\
rc &= \left(r + q(\theta)\theta\left(1-\phi\right)\beta\right)\frac{p_g}{D_g} - \left(r + q(\theta)\theta\phi\beta\right)\frac{p_b}{D_b} + && (34)\\
&\quad q\left(\theta\right)\beta\left(rk_g - rk_b\right)\left[\begin{array}{c}\phi\left[1 - A_g\left(\left(1-\beta\right) - \frac{s}{1-\gamma}\right)\right] + \\ \left(1-\phi\right)\left[1 - A_b\left(\left(1-\beta\right) - \frac{s}{\gamma}\right)\right]\end{array}\right],
\end{aligned}
$$

where

$$
\begin{aligned}
D_b &= r + s + (1-\beta)q(\theta)\gamma + \beta\theta q(\theta)\phi\\
D_g &= r + s + (1-\beta)q(\theta)\left(1-\gamma\right) + \beta\theta q(\theta)\left(1-\phi\right)\\
A_b &= \frac{\left(\gamma - \phi\theta\right)}{D_b}\\
A_g &= \frac{\left(1 - \gamma - \left(1-\phi\right)\theta\right)}{D_g}.
\end{aligned}
$$

Note that compared to the case $k_g = k_b$ equations (32) to (34) include new terms associated with the difference $k_g - k_b > 0$. One can show that the second term of equation (32) is increasing in

[14] As mentioned above the condition for efficient job composition is always satisfied in equilibrium. But the inefficiency in π leads to a ϕ that is lower than the social optimum.

γ while the corresponding of (33) is decreasing in γ, but it is not possible to show that both terms are always decreasing in θ. So in order to avoid complications with uniqueness I will restrict the analysis to cases in which $k_g - k_b$ is not too big. This maintains the fact that the social allocation is determined by the intersection of a downward sloping good-job social locus and an upward sloping bad-job social locus at the unique level of π that solves the planner's indifference condition (34)[15]. Then the efficient allocation is characterized by $\pi^S(k_g, 0)$, $\phi^S(k_g, 0) > 1/2$ and $\gamma^S(k_g, 0) < 1/2$.

Comparing the decentralized equilibrium to the social allocation, the main result is given by the following proposition.

Proposition 4 *If $k_g > k_b$, $c = 0$, and $\beta = \eta(\theta)$ the decentralized economy generates too many jobs with a too high fraction of bad-jobs, and a lower than optimal proportion of workers invests in education. Then,*

$$\pi(k_g, 0) > \pi^S(k_g, 0) > 1/2,$$
$$\gamma(k_g, 0) < \gamma^S(k_g, 0) < 1/2,$$
$$\phi(k_g, 0) > \phi^S(k_g, 0) > 1/2, \text{ and}$$
$$\theta(k_g, 0) > \theta^S(k_g, 0).$$

Concerning job composition, in contrast to what happens in the case $k_g = k_b$, it is no longer possible to make firms internalize congestion externalities by imposing the Hosios' value of β. Due to the random matching assumption, any firm posting a vacancy in any sector increases labor market tightness making it harder to all the firms in both sectors to meet an unemployed worker. However, when capital creation costs differ the amount of the surplus that is transferred to workers by Nash Bargaining in bad-job matches represents less than the value of the congestion externality implied by bad-job creation. On the contrary, the fraction β of the surplus of good-job matches that is shared with workers is too high in relation to the value of the congestion externality generated by good-job vacancies. Then, as I show in Appendix 5, this leads to a too high fraction of bad-job vacancies in the economy. Another way of understanding this result is by looking at the efficiency condition for ϕ presented in equation (27). Since firms only get a fraction $1 - \beta$ of the expected marginal benefit of their investment the efficiency condition is not satisfied in equilibrium and then $\phi(k_g, 0) > \phi^S(k_g, 0)$. Over-extraction of rents in Nash Bargaining (a hold-up problem) makes firms under-invest in good-jobs[16].

[15] Looking at the necessary condition for π (equation (28)) we can see that (34) is increasing in π. Then there is a unique π that solves the planner's problem.

[16] As mentioned, this result is caused by the random matching assumption. As suggested by Acemoglu and Shimer (1999) and Acemoglu (2001), in a model where workers can direct their search to the different sectors (but still

With respect to θ, we can see that the total number of jobs is inefficient even when Hosios conditions hold. In Appendix 5 I also show that the condition for θ (equation (26)) evaluated in the decentralized equilibrium is negative which means that the private economy creates too many jobs. In other words the negative externality of bad-job creation is greater than the positive externality of good-job creation leading to an overall over-creation of jobs[17]. Then $\theta(k_g, 0) > \theta^S(k_g, 0)$.

Finally, what remains to be done is to analyze the efficiency properties of the skill-composition π. In other words we need to check whether equation (34) is satisfied when evaluated in the decentralized equilibrium. Note that compared to the case $k_g = k_b$ the last term of equation (34) translates the net effect of congestion and thick market externalities into workers' education decisions. In Appendix 5 I show that unless k_g is too high that term is positive in equilibrium. Then comparing equations (12) to (34) we conlclude that the fraction of high-skill workers is too small in equilibrium. We can then suggest that the fraction of the surplus that workers get by rent sharing is too small to give them the appropriate incentives to education.

Note that these results apply even when education is costless. So firms under-invest because of a hold-up problem and a too low fraction of workers get the high-skill because they do not consider how the economy benefits from their decisions. This benefit comes from the higher expected surplus implied by high-skill matches and from the net reduction in the aggregate cost of frictions implied by the fact that good-job firms will find it easier to match high-skill workers.

4.2 Discussion

The results above indicate that both if either $c > 0$ or $k_g > k_b$ the decentralized economy will have a too high fraction of low-skill workers, then $\pi(k_g, c) > \pi^s(k_g, c)$. The efficiency implications for θ

with random search within sectors) these inefficiencies do not arise under Hosios' conditions. Applied to the model presented here, in that case the endogenous variables of the model would be θ_b and θ_g, denoting labor market tightness in each sector, instead of γ and θ. However, there would still be an inefficiency associated with the education decision. Then if $c > 0$ the economy would create too many high-skill jobs (θ_g too high) and too few low-skill jobs (θ_b too low), with a too high fraction of low-skill workers.

[17] In a model with $k_g > k_b$ and homogeneous workers (Acemoglu, 2001), under Hosios' conditions the positive externalities of good-job creation and the negative externalities of bad-job creation cancel out at the aggregate and lead to an efficient θ. So in our economy it is workers heterogeneity which causes an inefficient θ. Indeed, the fact that the price difference increases with π makes high-skill vacancies even more valuable than low-skill ones.

and ϕ when both c and $k_g - k_b$ are positive are more interesting. Recall the following results,

$$\begin{aligned}\phi(k_b, c) &< \phi^S(k_b, c) \\ \phi(k_g, 0) &> \phi^S(k_g, 0),\end{aligned}$$

$$\begin{aligned}\theta(k_b, c) &< \theta^S(k_b, c) \\ \theta(k_g, 0) &> \theta^S(k_g, 0).\end{aligned}$$

These expressions imply that regardless of $k_g \geq k_b$, an increase in c will cause a greater decrease in both ϕ and θ than in ϕ^S and in θ^S, respectively. Accordingly for any $c \geq 0$, an increase in k_g implies a greater change in ϕ compared to ϕ^s, and a lower decrease in θ than in θ^S. Then

$$\frac{d\phi(k_g, c)}{dc} < \frac{d\phi^S(k_g, c)}{dc} \tag{35}$$

$$\frac{d\theta(k_g, c)}{dc} < \frac{d\theta^S(k_g, c)}{dc} \tag{36}$$

$$\frac{d\phi(k_g, c)}{dk_g} > \frac{d\phi^S(k_g, c)}{dk_g} \tag{37}$$

$$\frac{d\theta(k_g, c)}{dk_g} > \frac{d\theta^S(k_g, c)}{dk_g}. \tag{38}$$

As a consequence the comparison of the efficient and decentralized economy solutions depends on the particular values assumed by c and rk_g. Additionally, the expressions (35) through (38) indicate that:

Proposition 5 *If $\beta = \eta(\theta)$ there is a pair (k_g, c) that can either achieve efficiency in ϕ or θ, but without removing inefficiencies in the other variables of the model.*

For example, starting from an economy with a hold-up problem by the firm side and zero education costs, we can introduce a lump sum tax on education in order to reduce ϕ and solve the hold-up problem. Labor market tightness may or may not approach to its efficiency level. Alternatively, we can adjust the education cost and make the economy create the total right number of jobs, with a reduced fraction of high-skill vacancies that may be above or below its efficient level. However, these experiments would always be at the cost of increasing the inefficiencies in π. Formally, this example suggests that there exists a $c^* > 0$ that implies

$$\begin{aligned}\phi(k_g, 0) &> \phi(k_g, c^*) = \phi^S(k_g, c^*) \\ \theta(k_g, 0) &< \theta(k_g, c^*) \gtreqless \phi^S(k_g, c^*) \\ \pi(k_g, 0) &< \pi(k_g, c^*) > \pi^S(k_g, c^*).\end{aligned}$$

I have then provided a way not explored before for solving a typical hold-up problem affecting firms in a two-sector matching model with frictions. In an important way these results rely on the production technology assumed in the model. Intuitively the fact that the surplus of the matches in the high-skill sector increases with investment costs makes reduce under-investments compared to an environment in which the surplus of the match is fixed.

5 Conclusions

This paper has studied the interaction between education and investment decisions in a two-sector matching model of the labor market. I have explored the normative implications of the model and shown that in the presence of investment costs the equilibrium skill-composition of the workforce is inefficiently biased toward low-skill workers. By the firms side, I shown that depending on the combination of human and physical capital costs job composition can be either efficient or not.

As mentioned in the paper under-investments by firms when capital creation costs are higher in the high-skill sector than in the low-skill sector arise because of the random matching assumption. Instead had we assumed that workers can direct their search to any particular sector and even with random matching within sectors we could solve the holdup inefficiency due to ex-post bargaining. However, if education costs are positive this would not allow us to eliminate the skill-composition inefficiency, and then we would expect to have too many good-jobs as in the case analyzed in the paper where capital creation costs across sectors is the same.

Among other limitations the model considers a fixed size of the worforce. It is possible to make the size of the workforce endogenous by introducing participation or immigration decisions. Regarding the latter, it may be interesting to analyze the effects of immigration on the efficiency properties of the equilibrium.

The model can also be extended to a more than two-sector environment without changing the main results. That would make it possible to test the macroeconomic implications of the model, since given a distribution of capital creation costs it might predict that countries with lower education costs might have a better skill-composition but a worse job-composition than countries with higher costs of education.

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6 Appendix

6.1 Appendix 1

6.1.1 Comparative statics for ϕ and u

Recall the expression for ϕ

$$\phi = \frac{\pi(1-\gamma)q(\theta)\theta + (\pi-\gamma)s}{q(\theta)\theta(\pi(1-\gamma)+\gamma(1-\pi))}.$$

The derivative $\frac{d\phi}{d\theta} < 0$ is straightforward. Note also that $\frac{d(1-\phi)}{d\theta} < 0$. The corresponding comparative statics results with respect to π and γ are

$$\begin{aligned} \frac{\partial\phi}{\partial\pi} &= \frac{(1-\gamma)\gamma\left(q(\theta)\theta+2s\right)}{q(\theta)\theta\left(\gamma+\pi-2\gamma\pi\right)^2} > 0 \\ \frac{\partial\phi}{\partial\gamma} &= \frac{\pi\left(\pi-1\right)}{\left(\gamma+\pi-2\gamma\pi\right)^2}\frac{\left(q(\theta)\theta+2s\right)}{q(\theta)\theta} < 0. \end{aligned}$$

Now consider the equation for steady state unemployment

$$u = \frac{s}{q(\theta)\theta+2s}\left(\frac{\gamma+\pi-2\gamma\pi}{\gamma(1-\gamma)}\right).$$

It is obvious that $\frac{\partial u}{\partial\theta} < 0$ and that $\frac{\partial u}{\partial\pi} \gtreqless 0$ iff $\gamma \lesseqgtr 1/2$. Regarding $\frac{\partial u}{\partial\gamma}$ we have

$$\frac{\partial u}{\partial\gamma} = \frac{s}{q(\theta)\theta+2s}\frac{\gamma^2\left(1-\pi\right)-\pi\left(1-\gamma\right)^2}{(1-\gamma)^2\gamma^2}. \tag{39}$$

Then if $\pi > \gamma \Longrightarrow (1-\pi) < (1-\gamma) \Longrightarrow \gamma^2\left(1-\pi\right) < \pi\left(1-\gamma\right)^2$. We conclude that

$$\frac{\partial u}{\partial\gamma} \lesseqgtr 0 \Longleftrightarrow \pi \gtreqless \gamma.$$

Finally the claim $\pi \lesseqgtr \gamma$ iff $\phi \lesseqgtr 1/2$ comes from the fact that $\phi = 1/2$ iff $\pi = \gamma$ and $\frac{\partial\phi}{\partial\pi} > 0$ and $\frac{\partial\phi}{\partial\gamma} < 0$.

6.1.2 Comparative statics for p_g and p_b

Recall the price equations

$$\begin{aligned} p_b &= \alpha\left[\frac{Y_g}{Y_b}\right]^{1-\alpha} \\ p_g &= (1-\alpha)\left[\frac{Y_b}{Y_g}\right]^{\alpha}. \end{aligned}$$

Then compute the derivatives $\frac{\partial p_i}{\partial u}$,

$$\begin{aligned}\frac{\partial p_b}{\partial u} &= -\alpha\left(1-\alpha\right)\left[\frac{Y_b}{Y_g}\right]^{\alpha}\frac{\left(1-\gamma\right)Y_b-\gamma Y_g}{Y_b^2} = -\left(1-\alpha\right)p_b\frac{\left(1-\gamma\right)Y_b-\gamma Y_g}{Y_gY_b} \lesseqgtr 0\\ \frac{\partial p_g}{\partial u} &= \alpha\left(1-\alpha\right)\left[\frac{Y_b}{Y_g}\right]^{\alpha-1}\frac{\left(1-\gamma\right)Y_b-\gamma Y_g}{Y_g^2} = \alpha p_g\frac{\left(1-\gamma\right)Y_b-\gamma Y_g}{Y_gY_b} \gtreqless 0.\end{aligned}$$

To obtain $\frac{\partial p_i}{\partial \theta}$ consider

$$\frac{\partial u}{\partial \theta} = -u\frac{q'\left(\theta\right)\theta+q\left(\theta\right)}{q\left(\theta\right)\theta+2s}.$$

Then,

$$\begin{aligned}\frac{\partial p_b}{\partial \theta} &= u\frac{q'\left(\theta\right)\theta+q\left(\theta\right)}{q\left(\theta\right)\theta+2s}\left(1-\alpha\right)p_b\frac{\left(1-\gamma\right)Y_b-\gamma Y_g}{Y_gY_b} \lesseqgtr 0\\ \frac{\partial p_g}{\partial \theta} &= -u\frac{q'\left(\theta\right)\theta+q\left(\theta\right)}{q\left(\theta\right)\theta+2s}\alpha p_g\frac{\left(1-\gamma\right)Y_b-\gamma Y_g}{Y_gY_b} \gtreqless 0.\end{aligned} \tag{40}$$

Note that

$$Y_b\left(1-\gamma\right)-Y_g\gamma = \pi\left(1-\gamma\right)-\left(1-\pi\right)\gamma = \pi-\gamma.$$

Then we conclude

$$\begin{aligned}\frac{\partial p_g}{\partial u} &\gtreqless 0,\ \frac{\partial p_b}{\partial u} \lesseqgtr 0 \Longleftrightarrow \pi \gtreqless \gamma \text{ or } \phi \gtreqless 1/2\\ \frac{\partial p_g}{\partial \theta} &\lesseqgtr 0,\ \frac{\partial p_b}{\partial \theta} \gtreqless 0 \Longleftrightarrow \pi \gtreqless \gamma \text{ or } \phi \gtreqless 1/2.\end{aligned} \tag{41}$$

Finally, results (39) and (41) imply that $\frac{\partial p_g}{\partial \gamma} < 0$ and $\frac{\partial p_b}{\partial \gamma} > 0$. Regarding the derivatives with respect to π we have that $\frac{\partial p_g}{\partial \pi} > 0$, $\frac{\partial p_b}{\partial \pi} < 0$. This is straightforward and come from the fact that $\frac{\partial Y_b}{\partial \pi} > 0$ and $\frac{\partial Y_g}{\partial \pi} < 0$,

$$\begin{aligned}\frac{\partial Y_b}{\partial \pi} &= 1-\gamma\frac{\partial u}{\partial \pi} = \frac{\left(1-\gamma\right)q(\theta)\theta+s}{\left(1-\gamma\right)\left[q(\theta)\theta+2s\right]}\\ \frac{\partial Y_g}{\partial \pi} &= -1-\left(1-\gamma\right)\frac{\partial u}{\partial \pi} = -\frac{\gamma q(\theta)\theta+s}{\gamma\left[q(\theta)\theta+2s\right]}.\end{aligned}$$

6.2 Appendix 2

The properties of the matching function and the production technology imply the existence of at least one interior equilibrium. In this section of the Appendix I show that the equilibrium is unique.

6.2.1 Uniqueness

The derivatives of the equilibrium value of vacancies equations (22) and (23) with respect to γ and θ are,

$$\begin{aligned}
\frac{\partial rk_b}{\partial\gamma} &= \frac{q(\theta)(1-\beta)}{D_b^2}\left\{p_b\left[r+s+\theta q(\theta)\beta\left(\phi-\gamma\frac{\partial\phi}{\partial\gamma}\right)\right]+\gamma D_b\frac{\partial p_b}{\partial\gamma}\right\}>0\\
\frac{\partial rk_g}{\partial\gamma} &= -\frac{q(\theta)(1-\beta)}{D_g^2}\left\{p_g\left[r+s+\theta q(\theta)\beta\left((1-\phi)-(1-\gamma)\frac{\partial\phi}{\partial\gamma}\right)\right]-(1-\gamma)D_g\frac{\partial p_g}{\partial\gamma}\right\}<0
\end{aligned}$$

where $D_b = r+s+(1-\beta)q(\theta)\gamma+\beta\theta q(\theta)\phi$ and $D_g = r+s+(1-\beta)q(\theta)(1-\gamma)+\beta\theta q(\theta)(1-\phi)$. As well,

$$\begin{aligned}
\frac{\partial rk_b}{\partial\theta} &= \frac{(1-\beta)q(\theta)\gamma\left\{p_b\left[(r+s+q(\theta)\theta\phi\beta)\frac{q'(\theta)}{q(\theta)}-\beta\frac{d[q(\theta)\theta\phi]}{d\theta}\right]+\frac{\partial p_b}{\partial\theta}D_b\right\}}{D_b^2}\gtrless 0\\
\frac{\partial rk_g}{\partial\theta} &= \frac{(1-\beta)q(\theta)(1-\gamma)\left\{p_g\left[(r+s+q(\theta)\theta(1-\phi)\beta)\frac{q'(\theta)}{q(\theta)}-\beta\frac{d[q(\theta)\theta(1-\phi)]}{d\theta}\right]+\frac{\partial p_g}{\partial\theta}D_g\right\}}{D_g^2}\gtrless 0.
\end{aligned}$$

Note that $\frac{\partial rk_i}{\partial\theta}$ can not be signed because the sign of the second term is ambiguous. For example, $\frac{\partial rk_b}{\partial\theta}$ is negative if $\phi\le 1/2$ because this implies $\frac{\partial p_b}{\partial\theta}\le 0$ too, but $\frac{\partial rk_b}{\partial\theta}\gtrless 0$ because if $\phi<1/2$ we have $\frac{\partial p_g}{\partial\theta}>0$. However, in the sequel I will show that in equilibrium it will always be the case that $\frac{\partial rk_i}{\partial\theta}<0$.

Then the expressions for the slope of the bad-job locus and good-job locus are

$$\begin{aligned}
\frac{d\gamma}{d\theta}_B &= -\frac{\frac{\partial rk_b}{\partial\theta}}{\frac{\partial rk_b}{\partial\gamma}} = -\frac{\gamma p_b\left[(r+s+q(\theta)\theta\phi\beta)\frac{q'(\theta)}{q(\theta)}-\beta\frac{d[q(\theta)\theta\phi]}{d\theta}\right]+\gamma D_b\frac{\partial p_b}{\partial\theta}}{p_b\left[r+s+\theta q(\theta)\beta\left(\phi-\gamma\frac{\partial\phi}{\partial\gamma}\right)\right]+\gamma D_b\frac{\partial p_b}{\partial\gamma}}\\
\frac{d\gamma}{d\theta}_G &= -\frac{\frac{\partial rk_g}{\partial\theta}}{\frac{\partial rk_g}{\partial\gamma}} = -\frac{p_g(1-\gamma)\left[(r+s+q(\theta)\theta(1-\phi)\beta)\frac{q'(\theta)}{q(\theta)}-\beta\frac{d[q(\theta)\theta(1-\phi)]}{d\theta}\right]+(1-\gamma)\frac{\partial p_g}{\partial\theta}D_g}{-p_g\left[r+s+\theta q(\theta)\beta\left((1-\phi)-(1-\gamma)\frac{\partial\phi}{\partial\gamma}\right)\right]+(1-\gamma)D_g\frac{\partial p_g}{\partial\gamma}}.
\end{aligned}$$

Clearly if $\phi=1/2$ the price derivatives are zero and we have that $\frac{d\gamma}{d\theta}_B>0$ and $\frac{d\gamma}{d\theta}_G<0$. Additionally, by equation (15) we know that $\phi=1/2$ iff $\pi=\gamma$. Then, for instance if $\pi=\gamma=1/2$ both curves intersect at $\gamma=1/2$ where the bad-job locus is upward sloping and the good-job locus is downward sloping. Accordingly at $\pi=\gamma>1/2$ we still have $\frac{d\gamma}{d\theta}_B>0$ and $\frac{d\gamma}{d\theta}_G<0$ and both loci intersect at a point below $\pi=\gamma>1/2$ where the slope of both curves remains unchanged. A similar rationale apply for $\pi=\gamma<1/2$. This rules out the possibility of having an intersection point in a region of the curves where the slope of both loci is either positive or negative.

We conclude that both curves intersect once and only once where the bad-job locus is upward sloping and the good-job locus is downward sloping.

6.3 Appendix 3

6.3.1 Proof that θ is decreasing in π if $rk_g \leq rk_g^*$

This can be proved by showing that the bad-job locus changes by more than the good-job locus after a change in π. Then, given the increase in p_g and the decrease in p_b the required increase in γ to satisfy the bad-job locus equation should be greater than the required increase in γ that satisfies the good-job locus equation.

It is crucial to show

$$\begin{aligned}\frac{\partial p_b}{\partial \pi} &= -(1-\alpha)\,\alpha\left(\frac{Y_g}{Y_b}\right)^{-\alpha}\frac{Y_g\frac{\partial Y_b}{\partial \pi}+Y_b\frac{\partial Y_g}{\partial \pi}}{Y_b^2}\\ \frac{\partial p_g}{\partial \pi} &= (1-\alpha)\,\alpha\left(\frac{Y_b}{Y_g}\right)^{\alpha-1}\frac{Y_g\frac{\partial Y_b}{\partial \pi}+Y_b\frac{\partial Y_g}{\partial \pi}}{Y_g^2}.\end{aligned}$$

Then,

$$-\frac{\partial p_b}{\partial \pi}Y_b = (1-\alpha)\,\alpha\left(\frac{Y_g}{Y_b}\right)^{-\alpha}\frac{Y_g\frac{\partial Y_b}{\partial \pi}+Y_b\frac{\partial Y_g}{\partial \pi}}{Y_b} = \frac{\partial p_g}{\partial \pi}Y_g.$$

Then $Y_b \geq Y_g$ as when $c \geq 0 \Longrightarrow \left|\frac{\partial p_g}{\partial \pi}\right| \geq \left|\frac{\partial p_b}{\partial \pi}\right|$.

Now, recall the equilibrium value of vacancies equations (22) and (23), and consider that for any θ the following weak inequality must hold

$$\frac{\gamma}{r+s+(1-\beta)q(\theta)\gamma+\beta\theta q(\theta)\phi}p_b \leq p_g\frac{(1-\gamma)}{r+s+(1-\beta)q(\theta)(1-\gamma)+\beta\theta q(\theta)(1-\phi)}.$$

Note that the expression in the $l.h.s$ ($r.h.s.$) is increasing (decreasing) in γ. Since $\left|\frac{\partial p_g}{\partial \pi}\right| \geq \left|\frac{\partial p_b}{\partial \pi}\right|$, for the case $k_g = k_b$ the required increase in γ in the $l.h.s$ has to be greater than the required increased in γ in the $r.h.s.$. We conclude that the bad-job locus shifts up by more than the good-job locus when π changes. If $d > 0$ this conclusion remains for a small enough $k_g \leq k_g^*$. From values of rk_g greater than this threshold the price difference would be so big that there would be required a higher change in γ in the good-job locus than in the bad-job locus, then θ would be increasing in c.

6.4 Appendix 4

6.4.1 Derivation of the efficient solution equations

Combining equations (26) and (27) we can get the next expressions,

$$\begin{aligned} rk_b &= -q(\theta)\lambda_b\gamma + q(\theta)\eta(\theta)\left[\lambda_b\gamma\phi + \lambda_g(1-\gamma)(1-\phi)\right] \\ &= -q(\theta)\lambda_b\gamma(1-\eta(\theta)) - q(\theta)\eta(\theta)(1-\phi)(rk_g - rk_b) \end{aligned} \tag{42}$$

$$\begin{aligned} rk_g &= -q(\theta)\lambda_g(1-\gamma) + q(\theta)\eta(\theta)\left[\lambda_b\gamma\phi + \lambda_g(1-\gamma)(1-\phi)\right] \\ &= -q(\theta)\lambda_g(1-\gamma)(1-\eta(\theta)) + q(\theta)\eta(\theta)\phi(rk_g - rk_b). \end{aligned} \tag{43}$$

Now we can combine equations (24) with (42), and (25) with (43) to get expressions for λ_b and λ_g,

$$-\lambda_b = \frac{p_b + \frac{1}{\gamma}\eta(\theta)(1-\phi)(rk_g - rk_b)(\gamma - \phi\theta)}{r + s + (1-\eta(\theta))q(\theta)\gamma + \eta(\theta)\theta q(\theta)\phi} \tag{44}$$

$$-\lambda_g = \frac{p_g - \frac{1}{(1-\gamma)}q(\theta)\eta(\theta)\phi(rk_g - rk_b)(1-\gamma-(1-\phi)\theta)}{r + s + (1-\eta(\theta))q(\theta)(1-\gamma) + \eta(\theta)\theta q(\theta)(1-\phi)}. \tag{45}$$

To interpret these expressions note that if θ is high enough $-\lambda_b < \frac{p_b}{r+s+(1-\eta(\theta))q(\theta)\gamma+\eta(\theta)\theta q(\theta)\phi}$ and $-\lambda_g > \frac{p_g}{r+s+(1-\eta(\theta))q(\theta)(1-\gamma)+\eta(\theta)\theta q(\theta)(1-\phi)}$ then the private value of good-jobs would be lower than its social value, and the opposite would occur for bad-jobs.

Now, we can substitute equations (44) and (45) in (42) and (43), respectively, to get the social value of vacancies equations:

$$\begin{aligned} rk_b = {} & \frac{q(\theta)\gamma(1-\eta(\theta))p_b}{r+s+(1-\eta(\theta))q(\theta)\gamma+\eta(\theta)\theta q(\theta)\phi} \\ & -q(\theta)\eta(\theta)(1-\phi)(rk_g - rk_b)\left(1 - \frac{(1-\eta(\theta))(\gamma-\phi\theta)}{[r+s+(1-\eta(\theta))q(\theta)\gamma+\eta(\theta)\theta q(\theta)\phi]}\right) \end{aligned}$$

$$\begin{aligned} rk_g = {} & \frac{q(\theta)(1-\gamma)(1-\eta(\theta))p_g}{r+s+(1-\eta(\theta))q(\theta)(1-\gamma)+\eta(\theta)\theta q(\theta)(1-\phi)} \\ & +q(\theta)\eta(\theta)\phi(rk_g - rk_b)\left(1 - \frac{(1-\eta(\theta))(1-\gamma-(1-\phi)\theta)}{r+s+(1-\eta(\theta))q(\theta)(1-\gamma)+\eta(\theta)\theta q(\theta)(1-\phi)}\right). \end{aligned}$$

Then the social value of vacancies and the shadow values evaluated at $\beta = \eta(\theta)$ are written in the text as

$$
\begin{aligned}
rk_b &= q(\theta)\gamma(1-\beta)\frac{p_b}{D_b} - q(\theta)\beta(1-\phi)(rk_g - rk_b)(1-(1-\beta)A_b) && (46)\\
rk_g &= q(\theta)(1-\gamma)(1-\beta)\frac{p_g}{D_g} + q(\theta)\beta\phi(rk_g - rk_b)(1-(1-\beta)A_g) && (47)
\end{aligned}
$$

$$
\begin{aligned}
-\lambda_b &= \frac{p_b}{D_b} + \frac{\beta(1-\phi)}{\gamma}(rk_g - rk_b)A_b && (48)\\
-\lambda_g &= \frac{p_g}{D_g} - \frac{\beta\phi}{1-\gamma}(rk_g - rk_b)A_g && (49)
\end{aligned}
$$

where $A_b = \frac{(\gamma-\phi\theta)}{D_b}$ and $A_g = \frac{(1-\gamma-(1-\phi)\theta)}{D_g}$.

Finally to get the social indifference condition for workers, substitute equations (46) through (49) in the necessary condition for π given by equation (28),

$$
\begin{aligned}
rc &= (p_g - rk_g) - (p_b - rk_b) - s(\lambda_b - \lambda_g)\\
&= (r + s + q(\theta)\theta(1-\phi)\beta)\frac{p_g}{D_g} + q(\theta)\beta\phi(rk_g - rk_b)(1-(1-\beta)A_g) -\\
&\quad (r + s + q(\theta)\theta\phi\beta)\frac{p_b}{D_b} + q(\theta)\beta(1-\phi)(rk_g - rk_b)(1-(1-\beta)A_b) -\\
&\quad s\left[\frac{p_g}{D_g} - \frac{\beta\phi}{1-\gamma}(rk_g - rk_b)A_g - \frac{p_b}{D_b} - \frac{\beta(1-\phi)}{\gamma}(rk_g - rk_b)A_b\right]
\end{aligned}
$$

$$
\begin{aligned}
rc &= (r + q(\theta)\theta(1-\phi)\beta)\frac{p_g}{D_g} - (r + q(\theta)\theta\phi\beta)\frac{p_b}{D_b} +\\
&\quad q(\theta)\beta(rk_g - rk_b)\left[\begin{array}{c}\phi\left[1 - A_g\left((1-\beta) - \frac{s}{1-\gamma}\right)\right] +\\ (1-\phi)\left[1 - A_b\left((1-\beta) - \frac{s}{\gamma}\right)\right]\end{array}\right].
\end{aligned}
$$

6.5 Appendix 5

6.5.1 Evaluating the necessary condition for θ in the decentralized equilibrium

Substitute equations (48) and (49) in (26), consider the equilibrium value of vacancies equations and assume Hosios' condition to get,

$$
\begin{aligned}
\frac{\partial H_c}{\partial \theta} &= -\left[\phi r k_b + (1-\phi) r k_g\right] - \lambda_b \gamma \phi \left[q(\theta)(1+\eta(\theta))\right] - \lambda_g (1-\gamma)(1-\phi)\left[q(\theta)(1+\eta(\theta))\right] \\
&= -\left[\phi r k_b + (1-\phi) r k_g\right] + \frac{p_b \gamma \phi q(\theta)(1-\beta)}{D_b} + \phi q(\theta)(1-\beta)\beta(1-\phi)(rk_g - rk_b) A_b \\
&\quad + \frac{(1-\gamma)(1-\phi) q(\theta)(1-\beta) p_g}{D_g} - \phi q(\theta)(1-\beta)\beta(1-\phi)(rk_g - rk_b) A_g \\
&= \phi q(\theta)(1-\beta)\beta(1-\phi)(rk_g - rk_b)(A_b - A_g)
\end{aligned}
$$

where $A_b = \frac{(\gamma - \phi\theta)}{D_b}$ and $A_g = \frac{(1-\gamma-(1-\phi)\theta)}{D_g}$.
Then the sign of this expression depends on the sign of $A_b - A_g$,

$$
\begin{aligned}
A_b - A_g &= \frac{(\gamma - \phi\theta) D_g - (1-\gamma-(1-\phi)\theta) D_b}{D_b D_g} \gtrless 0 \\
&= \frac{1}{D_b D_g}\left\{ \begin{array}{l} (\gamma - \phi\theta)\left[r + s + (1-\beta) q(\theta)(1-\gamma) + \beta\theta q(\theta)(1-\phi)\right] \\ -(1-\gamma-(1-\phi)\theta)\left[r + s + (1-\beta) q(\theta)\gamma + \beta\theta q(\theta)\phi\right] \end{array} \right\} \gtrless 0 \\
&= \frac{1}{D_b D_g}\left\{ (r+s)\left[2\gamma - 1 + \theta(1-2\phi)\right] + q(\theta)\theta(\gamma - \phi) \right\} \gtrless 0.
\end{aligned}
$$

Then given that $\phi(k_g, 0) > 1/2$ and $\gamma(k_g, 0) < 1/2$ this expression is negative, meaning that $\theta(k_g, 0)$ is too high compared to $\theta^S(k_g, 0)$. Note that we could assure $\theta(k_g, c) > \theta^S(k_g, c)$ if $\theta(k_g, c) \geq 1$ and c small enough such that $\gamma < \phi$. If c is high enough so that $\gamma > \phi$ and still if $\theta(k_g, c) \geq 1$ we would have $\theta(k_g, c) < \theta^S(k_g, c)$.

6.5.2 Evaluating the social value of vacancies equations in the decentralized equilibrium

Recall the social value of vacancies equations

$$
\begin{aligned}
rk_b &= q(\theta)\gamma(1-\beta)\frac{p_b}{D_b} - q(\theta)\beta(1-\phi)(rk_g - rk_b)(1-(1-\beta)A_b) \\
rk_g &= q(\theta)(1-\gamma)(1-\beta)\frac{p_g}{D_g} + q(\theta)\beta\phi(rk_g - rk_b)(1-(1-\beta)A_g).
\end{aligned}
$$

Note that

$$
\begin{aligned}
(1-(1-\beta)A_b) &= \frac{r + s + \beta\theta q(\theta)\phi + (1-\beta)\left[\phi\theta + \gamma(q(\theta)-1)\right]}{r + s + (1-\beta) q(\theta)\gamma + \beta\theta q(\theta)\phi} > 0 \\
(1-(1-\beta)A_g) &= \frac{r + s + \beta\theta q(\theta)(1-\phi) + (1-\beta)\left[(1-\phi)\theta + (1-\gamma)(q(\theta)-1)\right]}{r + s + (1-\beta) q(\theta)(1-\gamma) + \beta\theta q(\theta)(1-\phi)} > 0.
\end{aligned}
$$

Given $\phi > \gamma$ and the properties of the matching function, this result is true for any $\theta > 0$.

The second terms of both equations are not considered in the zero-profit equations of the decentralized equilibrium. This means that regardless of the values of θ and π, firms over-invest in bad-jobs.

Then $\phi(k_g, 0) > \phi^S(k_g, 0)$. Note that this might not be the case if k_g is small enough and c is high enough.

6.5.3 Evaluating the social indifference equation in the decentralized equilibrium

Recall the indifference equation

$$\begin{aligned} rc &= (r + q(\theta)\theta(1-\phi)\beta)\frac{p_g}{D_g} - (r + q(\theta)\theta\phi\beta)\frac{p_b}{D_b} + \\ & q(\theta)\beta(rk_g - rk_b)\begin{bmatrix} \phi\left[1 - A_g\left((1-\beta) - \frac{s}{1-\gamma}\right)\right] + \\ (1-\phi)\left[1 - A_b\left((1-\beta) - \frac{s}{\gamma}\right)\right] \end{bmatrix}. \end{aligned} \tag{50}$$

Note that given $(1 - (1-\beta)A_b) > 0$ and $(1 - (1-\beta)A_g) > 0$ if the expressions that accompany A_g and A_b are lower than $(1-\beta)$ then (50) is positive, which means that in equilibrium π is too high. In fact this would be the case if s is small enough. If $A_i \geq 0$ we do not need to impose any condition, while if $A_i < 0$ we know that the expressions $(1 - (1-\beta)A_i) > 1$ but this is not enough to know the sign of equation (50) evaluated in the decentralized equilibrium.

Note that if $k_g > k_b$ and $c = 0$ we know that $\gamma < 1/2$. Then it suffices to impose a condition $k_g \leq \overline{\overline{k}}_g$ where $\overline{\overline{k}}_g$ is the capital creation cost that makes $\gamma\left(\overline{\overline{k}}_g, 0\right) = \frac{s}{1-\beta}$. Then $k_g \leq \overline{\overline{k}}_g$ is sufficient to conclude that $\pi(k_g, 0) > \pi^S(k_g, 0)$. Though I could not prove this for any k_g, note that if $\gamma(k_g, 0) \leq \gamma\left(\overline{\overline{k}}_g, 0\right)$ we would expect the first term (related to the difference $J_g^U - J_b^U$) to be huge so we would still have that $\pi(k_g, 0)$ is too high.