Non-Monotone Liquidity Under-Supply.

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Abstract

We define liquidity as the flexibility to move goods (money) from one project (investment) to another. We show that credit constraints on *demand* by themselves can cause an under-*supply* of liquidity, without the uncertainty, intermediation, asymmetric information or complicated international financial framework used in other models in the literature. In this respect liquidity is like a commodity: according to our offsetting distortions principle, a distortion in the demand for any good can often be understood as an inefficiency of supply.

We also show that the liquidity under-supply is a non-monotone function of the credit constraint. This result is also a particular case of a more general principle applying to any commodity with supply alternatives: second best supply inefficiency is non-monotone in the demand distortion. Defining liquidity as flexibility ensures that there will be alternatives, and thus non monotonicity.If we interpret the credit constraints as the degree of financial development in the economy, our second proposition suggests that when financial markets are very undeveloped, as in some emerging markets, financial innovation may paradoxically make government intervention (taxation) more necessary.

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Finally, we think about the magnitude of the under-supply in the context of a specific demand distortion. We model the credit constraint by assuming that borrowers will default unless their promises are covered by collateral. Further, we assume that only an exogenous proportion β of a durable good can serve as collateral. This parameter will represent the degree of financial development of the economy. We show that when the price of the collateral is endogenous, the magnitude of the under supply can be much larger. Any policy intervention that affects the interest rate in equilibrium will have two effects on the borrowing constraint: a direct effect, also present in the case when the credit constraint is exogenous, and an indirect effect through the price of the collateral. We explore our findings by solving and simulating a particular case in which utilities for the consumption good and collateral are quadratic.

Keywords: Liquidity Under-Supply, Credit Constraint, Non-Monotonicity, Multiplier, Collateral Equilibrium.

JEL Classification: D51, E44, F30, G15.

1 Introduction.

Liquidity has been defined in many different ways. We will adopt two definitions of liquidity, which we call Physical Liquidity and Financial Liquidity. They refer to the flexibility to move physical goods (money) across different projects (investments). The goal of this paper is to explain liquidity under-supply in equilibrium: when firms, in a decentralized way, optimally choose their own liquidity positions, the economy as a whole ends up with less liquidity than the second best efficient level.

There are already many explanations in the literature for the liquidity under supply phenomenon. For Holmstrom-Tirole(1998), liquidity is related to how complete the asset markets are, and in particular, with the ability of the private sector to buy assets in order to transfer wealth across time. With this definition, they get under-supply of liquidity in the presence of aggregate uncertainty. For Kiyotaki-Moore(2000), liquidity is money, and they obtain liquidity shortages in equilibrium in an infinite horizon economy under non-saleability assumptions. For Morris-Shin(2003) liquidity is the thickness of markets and they use global games techniques to show how liquidity under-provision can arise from asymmetric information. There is a big literature, including Diamond-Rajan(2001) and Chang-Velasco(1999), in which liquidity shortages and crise arises from bank intermediation and failures. For Geanakoplos(2002), liquidity declines when lenders endogenously raise margin requirementes. Finally, Caballero-Krishnamurthy (2001) argue that liquidity under provision is a characteristic of emerging market economies. They work in a model with two different liquidities (a domestic and an international), idiosyncratic and aggregate uncertainty, and credit constraints in both domestic and international markets.

In this paper, we use a simple model to show that a *sufficient* assumption to get liquidity under-*supply* is the presence of credit constraints on *demand*. In particular, we do not need any kind of uncertainty, asymmetric information, intermediation or a complicated international financial framework to prove the result. Our model also shows that the aggregate under-supply of liquidity is not necessarily a characteristic of emerging market economies, since the inefficiency can arise in mature markets as well, as long as there exist severe credit constraints.

The liquidity under-supply we describe is a particular example of a more general principle that we call the Offsetting Distortions principle. This principle states that a distortion in *demand* for any good can often be understood as an inefficiency of *supply*, even though the demanders are completely different from the suppliers. In our model, credit constraints play the role of demand distortions, which, as the principle suggests, can be understood as an inefficiency in the supply of liquidity.

Next, we explore the relationship between the liquidity under supply and the demand distortion that creates it. We find that the liquidity under-supply is a *non monotone* function of the credit constraint. If we interpret the credit constraints as the degree of financial development in the economy, our second proposition states that the liquidity under-supply is a non-monotone function of the degree of financial development. This suggests that when financial markets are very undeveloped, financial innovation may paradoxically make government intervention more necessary. It is difficult not to think about the financial innovation and simultaneous, dramatic, reduction of government participation in financial markets that took place in Latin American economies during the 90's. Numerous liquidity crises occurred in these economies during that period. The non monotonicity stems from the flexibility of liquid investments. This non monotonicity occurs for any commodity if we suppose that the central planner can affect only part of the supply curve: Second Best inefficiency is a non monotone function of the demand distortion.

In the last part of the paper we think about the *magnitude* of the liquidity under supply. We model the credit constraint by assuming that borrowers will default unless their promises are covered by collateral. Further, we assume that only an exogenous proportion β of one durable good can serve as collateral. This parameter β will represent the degree of financial development of the economy. We show that the magnitude of the under supply can be larger when the price of the collateral is endogenous, giving rise to a Liquidity Under Supply Multiplier. Any policy intervention that affects the interest rate in equilibrium will have two effects on the borrowing constraint: a direct effect, also present in the case when the credit constraint is exogenous, and an indirect effect through the price of the collateral.

Finally, we explore our findings in a particular example in which utilities for the consumption good and the collateral are quadratic. In this context, we can be more precise about the effects on liquidity of the degree of financial development, β, and the marginal utility λ, of collateral. First, in economies where the market value of collateral is low (because λ is low), there will always be liquidity under-supply, no matter how high β is. The government should intervene, for example, by taxing illiquid investments. Paradoxically, as β increases and the financial system becomes more efficient, the need for government intervention (taxation) increases. For higher values of λ, the liquidity under-supply is non-monotonic in β, increasing for low values and decreasing for high values of β. Similarly, the liquidity under-supply multiplier is increasing in β for low values of λ and becomes non-monotonic in β for high values of λ.

The paper is organized as follows. In section 2 we present diagrams explaining the general principles of Offsetting Distortions and Non Monotone Second Best Inefficiency. In section 3 we briefly discuss different definitions of liquidity used in the literature as well as the definitions we will consider. Section 4 presents the model and shows the under-supply and non monotonicity results for the case of physical liquidity. In Section 5 we argue why our results are valid for financial liquidity as well. In Section 6 we endogenize the credit constraint and prove the existence of the liquidity under supply multiplier. Also we solve and simulate the quadratic example. In section 7 we note that adding uncertainty does not cause any qualitative changes, but it does increase the magnitude of the under supply. Finally, section 8 concludes. All proofs are presented in the Appendix.

2 Offsetting Distortions and Non-Monotone Inefficiencies.

1. **Offsetting Distortions Principle.**

   Consider a market with its demand, D, supply, S, and initial equilibrium at A as shown in figure 1. There are no distortions of any kind in this market and hence the equilibrium is First Best efficient. Suppose now that we introduce some distortion to demand, say δ_2. The new demand is $D(\delta_2)$ instead of D and the new equilibrium price and quantities are smaller at point B. Given the demand distortion, the reduction in quantity can be interpreted as a supply inefficiency since a central planner could compensate for the demand dislocation by introducing a distortion to supply, shifting the curve to $S(\delta_2)$. At the new equilibrium C, the quantity is restored to its original First Best level, though the equilibrium price is lower than before.

   The Offsetting Distortions principle[1] states that a distortion in *demand* can often be understood as an inefficiency of *supply*, even though the demanders are completely different from the suppliers. If the central planner knows he cannot restore demand, then he must regard supply as in need of stimulation.

   In this paper we use the Offsetting Distortions principle to see how an aggregate under-supply of liquidity can arise from inefficiencies in the demand for credit. A collateral restriction that reduces the effective

[1] The Offseting Distortions principle is not novel. The idea, that a distortion on demand calls for a perturbation of supply, reminds us of what has been known as the Second Best Theory developed by Lancaster and Lipsey during the 1950's.

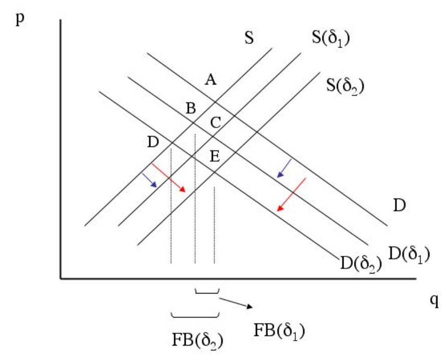

Fig. 1: Offsetting Distortions and First Best Monotonicity.

demand for loans can be interpreted as an inefficiency in the supply of loans. In effect, the collateral restriction on demand for loans manifests itself as an under-supply of liquidity, since loans can only be offered out of a pool of liquid capital created in the previous period.

2. **Non-Monotone Inefficiencies.**

   Define the First Best Inefficiency as the difference between the First Best quantity, and the quantity supplied in equilibrium under demand distortions but before intervention. For instance, in figure 1 the First Best Inefficiency associated with the demand distortion δ_2, what we denoted by $FB(\delta_2)$, is the difference between the quantities at C (or A) and B. Consider now a smaller demand distortion, say $\delta_1 < \delta_2$. The First Best inefficiency associated to δ_1, $FB(\delta_1)$, is given by the difference in quantities at E (or A) and D. Clearly, $FB(\delta_1) < FB(\delta_2)$. The First Best Inefficiency is a monotone function of the demand distortion.

   Suppose, however, that the central planner is constrained in how he can perturb supply. For example, suppose he cannot increase supply for prices below a lower limit of p^*. The equilibrium quantity attained after the planner intervenes optimally, but subject to his constraint, will be called the Second Best quantity. We may wonder at this point how our previous observations are modified by this constraint.

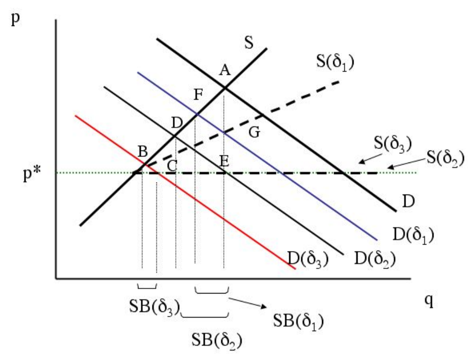

Fig. 2: Offsetting Distortions and Second Best Non-Monotonicity.

The first difference is obviously regarding the degree to which the central planner can reverse the quantity back to its First Best level. For instance, consider the demand distortion δ_3, as in figure 2. Given the constraint faced by the central planner, it is clear that the best he can do is to perturb supply to $S(\delta_3)$ resulting in a Second Best quantity at point C. Although this quantity is bigger than the one at B, it is still below the first best level at A, and not much bigger than at B . If the demand distortion were small enough, say $0 < \delta < \delta_2$, the central planner would be able to restore the quantity all the way back to its first best level.

Define the Second Best Inefficiency as the difference between the Second Best quantity and the equilibrium quantity attained without intervention. Surprisingly, the Second Best inefficiency is not monotone in the demand distortion. As we can see in figure 2, for $0 < \delta < \delta_2$, the Second Best inefficiency is equal to the First Best inefficiency, and grows with δ. But for $\delta > \delta_2$, the Second Best inefficiency declines as δ increases. A last observation is that the Second Best inefficiency becomes bigger when the demand curve becomes flatter. We will turn to this point at the end of the paper.

In figure 3 we can see the different behavior of the First Best and Second

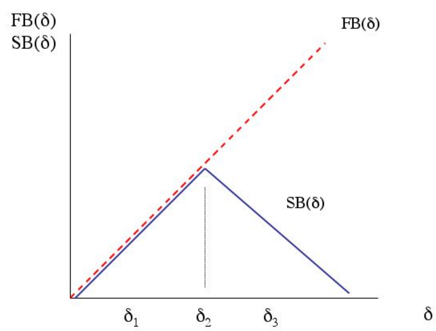

Fig. 3: First Best and Second Best Inefficiency.

Best Inefficiencies. The dotted curve represents the First Best Inefficiency as a function of the distortion δ, while the full one represents the Second Best inefficiency. While the First Best inefficiency is a monotone function of the demand distortion, the Second Best inefficiency is not.

3 Defining Liquidity.

Liquidity, and the closely related notion of flexibility, are intuitively understood by economists and others. However, when one tries to distil these notions into precise definitions, one finds that liquidity has been defined in many different ways.

According to some authors, like Shubik(1999) or Kiyotaki-Moore(2000), liquidity refers to a substance, like gold, which is accepted as a means of payment. An illiquid agent who is very rich in other goods may not be able to make purchases, at least for the moment, because he lacks gold or cash.

A second definition of liquidity refers to the thinness of the market for some good. An agent is in posession of an illiquid good if he cannot quickly sell it without a big discount. Clearly, this definition tries to capture the idea of flexibility mentioned before. It is about speed of reaction and about how costly it is to change financial position if needed. This is the most common notion used by market participants. It is also often used as a definition in academics as in Diamond(1986), Jones and Ostroy (1984), Morris-Shin(2003).

A third definition of liquidity stresses the completeness of markets. Holmstrom and Tirole (1998) define liquidity as the availability of instruments that can be used to transfer wealth across periods. An economy is more liquid than another if it has more markets. These authors particularly emphasize the ability of private agents to purchase a variety of assets that transfer wealth to the future. They suggest that the government can increase liquidity by creating and selling debt, which the private agents can then hold as a hedge against production emergencies.

A fourth definition is what we call financial liquidity. How liquid an agent is depends on his ability to borrow against the present value of his future income, that is, to sell contingent promises of future deliveries. Financial liquidity depends not only on the presence of various contingent promises, but also on the ability of agents to credibly commit to honor these promises, say by issuing collateral. This notion of liquidity is the one used in Geanakoplos(2001), Diamond-Rajan(2001), Caballero-Krishnamurthy(2001).

Finally, the last notion is what we call physical liquidity. As the name suggests, this notion refers to the flexibility to move goods across different projects. A project is liquid if an investor can move his physical imputs to another project as easily as he could if he has kept them in storage.

All these definitions try to capture in a different way the idea of flexibility: flexibility to make transactions, flexibility to exit a market and change portfolio composition quickly and without cost, flexibility to move wealth across time given that agents can buy or sell contingent promises and finally, flexibility to move goods across different activities or projects.

In this paper we will focus in the last two notions of liquidity: financial liquidity and physical liquidity. We shall find that when agents take liquidity decisions in a decentralized way, the economy as a whole will produce too little liquidity. In the next sections we will use the general principles of Offsetting Distortions and Non-Monotonicity of the Second Best Inefficiency to understand financial and physical liquidity under-supply as well as the non-monotone behavior of these inefficiencies.

4 Non Monotone Physical Liquidity Under-Supply.

1. **The Model.**

Consider an economy in which there are three periods, $t = 0, 1, 2$, and a single consumption good which is durable, and hence serves as a store of value as well as for productive investment. All consumption takes place in the last period. There is a continuum of firms of two types: firms of type L, lenders, and firms of type B, borrowers.

At $t = 0$ a type L firm is endowed with a single unit of the good. He has two investment options. The first is a long-term investment that has a constant gross return of H at $t = 2$, while the second is short-term and pays $h_1 < H$ at $t = 1$ per unit of investment. At stage 0 firm L decides on the percentage $\alpha \in [0, 1]$ of the good to invest in the short-term project.

An L firm arrives at period 1 with αh_1 wealth from his short term investment. At this point he has to decide how much to lend, y, to a type B firm, from which he gets a payoff of $y(1 + \rho)$ at $t = 2$, where ρ is the market interest rate. There exists another investment option which pays h_2 at $t = 2$. The decisions faced by an L firm at different periods can be seen in figure 4.

Type B firms only play a role at $t = 1$. Each B firm has an investment opportunity with a gross return of $R > 1$ at $t = 2$. We will assume that the return of this investment is extremely good, i.e. $Rh_1 > H > h_1h_2$. Each B firm has no endowment and therefore chooses to borrow x from type L firms. However, each B firm is credit constrained, i.e. the amount she can borrow has to satisfy,

$$(1 + \rho)x \leq \Pi$$

where Π is an exogenous limit to what the firm can promise. This constitutes the only market imperfection present in the economy.

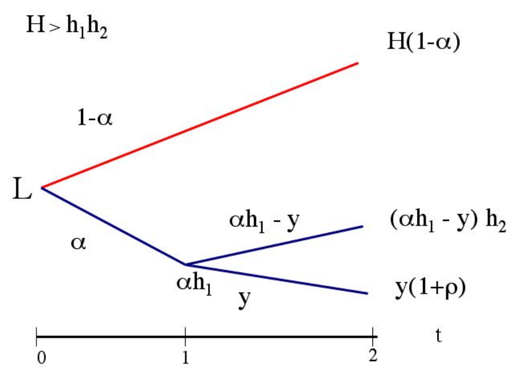

Fig. 4: L's decisions.

Finally, at period $t = 2$ debts are paid back and consumption is realized. Agents in this economy care only about output in $t = 2$. There is no uncertainty of any kind.

In this model, the measure of physical liquidity is given by α. The reason for this is quite simple. The short-term investment gives firms L the flexibility at $t = 1$ to reinvest the physical good into two different new projects: he can invest in a new project that returns h_2 at $t = 2$ or he can decide to enter in the credit market which would give him a return of $1+\rho$. On the other hand, the portion of the initial investment, $1 - \alpha$, devoted to the long-term investment gives him a return of H at $t = 2$ but no flexibility at all at $t = 1$ to move those physical goods to other projects.

At $t = 0$, L firms face the option of a liquid investment or an illiquid one. The illiquid investment has a higher return than the liquid investment, but the latter allows firms L to become lenders at $t = 1$. Every agent cares only about output in period 2. Any Pareto efficient allocation would maximize period 2 output. Hence, we take total output in period 2 as our measure of welfare.Since $Rh_1 > H$, the first best thing to do from a social point of view is to invest everything in the liquid option $\alpha = 1$, and then lend it all, $y = h_1$, to the B firms. As we will see, this will not be the outcome in equilibrium.

Let us be more precise about all this.

Definition 1:
An *equilibrium* in this economy consists of decisions $(\alpha_{EQ}, y_{EQ}, x_{EQ})$ and a price, ρ_{EQ} such that,

(a) L firms choose α_{EQ} in period 0 and y_{EQ} in period 1 such that (α_{EQ}, y_{EQ}) solves

$$\begin{cases} Max & (1-\alpha)H + (\alpha h_1 - y)h_2 + y(1+\rho_{EQ}) \\ \alpha, y & \\ s.t. & 0 \leq \alpha \leq 1 \\ & 0 \leq y \leq \alpha h_1 \end{cases}$$

(b) B firms choose x_{EQ} in period 1 such that (x_{EQ}) solves

$$\begin{cases} Max & Rx - (1+\rho_{EQ})x \\ x & \\ s.t. & 0 \leq (1+\rho_{EQ})x \leq \Pi \end{cases}$$

(c) $x_{EQ} = y_{EQ}$

Firms maximize consumption at period 2 taking the price as given and markets clear. The equilibrium in this model can be seen in Figure 5 in the case $\Pi < h_1 h_2$.

The decreasing curve represents the constrained demand of the B firms. The L firms will put all the good at $t = 0$ into the illiquid investment if $H > h_1(1+\rho)$, and all the good into the liquid investment if $H < h_1(1+\rho)$. Once they have invested α into the liquid investment, they will lend all αh_1 if $1+\rho > h_2$. Thus, the dotted and filled curves represent, respectively, the long and the short run supply of the L firms. When $\Pi < h_1 h_2$ the equilibrium is $\alpha_{EQ} = \Pi/H$, $y_{EQ} = x_{EQ} = \Pi h_1/H$ and $1 + \rho_{EQ} = H/h_1$. Clearly, equilibrium is not First Best efficient, since B firms borrow $\Pi h_1/H < h_1$, and output is $H(1-\Pi/H) + Rh_1\Pi/H < Rh_1$.

When $\Pi \geq R$, the equilibrium is $\alpha_{EQ} = 1$, $y_{EQ} = x_{EQ} = h_1$ and $1 + \rho_{EQ} = R$. For $h_1 h_2 \leq \Pi \leq R$, the equilibrium quantity is still First Best, although the interest rate is smaller, $1 + \rho_{EQ} < R$.

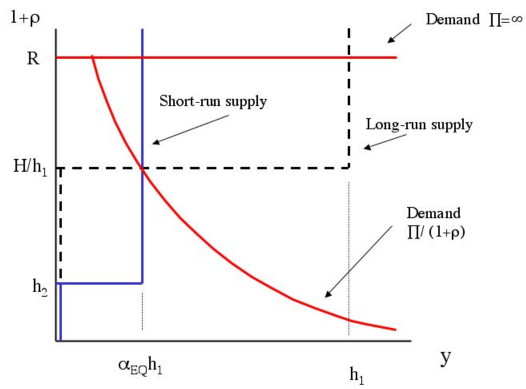

Fig. 5: Equilibrium in the case $\Pi < h_1 h_2$.

2. **Physical Liquidity Under-Supply.**

As we saw above, for low enough Π, B firms cannot borrow all they would like, and it is no surprise that the equilibrium is not First Best efficient. However, we now show that equilibrium is not even Second Best efficient. The constraint on borrowing at time 1 induces optimizing L firms to under-invest in the liquid option. Had they invested more in liquid capital, total output would have been higher, even if the borrowing constraint Π remained binding.

Let us be precise about what we mean by Second Best in this context.

Definition 2:
A pair of decisions $(\alpha_{CE}, y_{CE}, x_{CE})$ and prices ρ_{CE} is said to be *constrained efficient* if:

(a) A social planner chooses α_{CE} at $t = 0$ in order to maximize total output at t=2.
And at $t = 1$

(b) Given α_{CE} and ρ_{CE}, L chooses y_{CE} and B chooses x_{CE} to maximize their own output.

(c) $x_{EQ} = y_{EQ}$.

More precisely, we can think of a constrained efficient allocation $(\alpha_{CE}, y_{CE}, x_{CE}, \rho_{CE})$, as the one that solve the following maximization problem:

$$
\begin{cases}
\underset{\alpha, y, x, \rho}{Max} \quad (1-\alpha)H + (\alpha h_1 - y)h_2 + yR \\
s.t. \\
0 \leq \alpha \leq 1 \\
y \text{ solves} \quad \begin{cases} \underset{z}{Max} & (\alpha h_1 - z)h_2 + (1+\rho_{CE})z \\ s.t. & 0 \leq z \leq \alpha h_1 \end{cases} \\
x \text{ solves} \quad \begin{cases} \underset{w}{Max} & Rw - (1+\rho_{CE})w \\ s.t. & 0 \leq (1+\rho_{CE})w \leq \Pi \end{cases} \\
x = y
\end{cases}
$$

Now we are ready to state our first result.

Proposition 1: Physical Liquidity Under-Supply.

Suppose $\Pi < H$ *and* $h_1 h_2 < H < Rh_1$. *Then the equilibrium in this economy is not constrained efficient. In particular,* $\alpha_{EQ} < \alpha_{CE}$, *i.e. there is an under-supply of physical liquidity with respect to the constrained efficient allocation.*

Having in mind the offsetting distortions principle, this result should not come as a big surprise. In fact, it is a particular case of that principle. Credit constraints, Π, play the role of demand distortions, δ. As we saw, any distortion in *demand* can be understood as an inefficiency of *supply*, even though the demanders are completely different from the suppliers. Proposition 1 shows that this is also true for liquidity.

Consider first the case when $\Pi < h_1 h_2$ as shown in figure 6. The only new element added to figure 5 is the filled curve to the right, which represents the short run supply curve after the central planner's intervention. As we can see in the picture, $\alpha_{EQ} = \Pi/H < \alpha_{CE} = \Pi/h_1 h_2 < 1$. The equilibrium is not constrained efficient: there is a liquidity under-supply with respect to the central planner solution and this inefficiency is due to an externality.

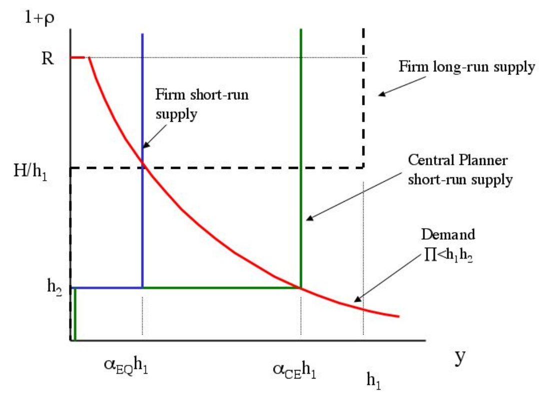

Fig. 6: Liquidity under-supply in the case $\Pi < h_1 h_2$.

Ex-ante, lenders L forecast a depressed equilibrium quantity at $t = 1$, the intersection of their long run supply curve and the constrained demand. Given this forecast, they optimally choose α which determines their short run supply. This curve intersects the demand at the same quantity and price H/h_1. Suppose now, that firms L had chosen a bigger α so that the short run supply curve had been the one to the right. At this equilibrium, the interest rate is lower and the quantity borrowed is larger. Exactly this constitutes the externality, for the increase in the supply not only has the usual effect of lowering the price, but also loosens the borrowing constraint of firms B. These firms can make strictly positive profits from each extra unit they borrow due to the wedge between the equilibrium interest rate and the return R. Ex-ante, lenders L do not internalize this effect. If the central planner knows he cannot restore demand for credit, then he must regard supply as in need of stimulation. This is exactly the central planner's action. He chooses α in order to increase supply, lowering interest rates and increasing the quantity up to the B' borrowing limit Π.

When $h_1 h_2 \leq \Pi < H$, a similar situation prevails, which we describe in figure 7. In this case $\alpha_{EQ} = \Pi/H < \alpha_{CE} = 1$ and the central planner intervention attains the first best quantity (even though not the first best interest rate.)

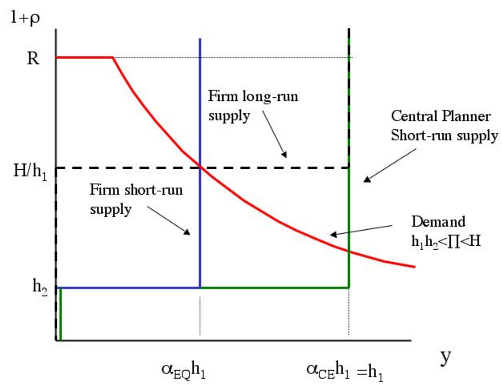

Fig. 7: Liquidity under-supply in the case $h_1 h_2 \leq \Pi < H$.

A last observation is that the result hinges only on the presence of credit constraints, in particular, it does not depend on the linear structure of the model.

3. **Implementing the Second Best.**

   We have seen that in equilibrium, optimizing agents will choose too little liquidity. A central planner could induce more liquid investments in several ways. Suppose however, that the only policy tool available to the planner is taxation. One method would be to tax the illiquid investment at time 0. (The lack of government intervention at time 1 presumably would be explained by supposing the government could not distinguish between liquid investments). [2]

4. **Non Monotonicity of the Liquidity Under-Supply.**

   The central planner in our model is constrained. Since in the second period there is an outside option with return h_2, the central planner cannot modify the supply of loans below $1+\rho = h_2$. This plays the role of p^* discussed before. Therefore, our model belongs to the Second Best

[2] One of the most common liquidity reducing moves agents make is borrowing money for a short term in order to invest in a long term illiquid deal. That behavior can be discouraged by taxing short term loans.

world. As a result, we should expect a non monotone relation between the liquidity under-supply, (in our previous terminology, the second best inefficiency), and the credit constraint (the demand distortion) that creates it. This is exactly what proposition 2 shows.

Definition 3:
For each Π, define the *Liquidity Under-Supply* as

$$LUS(\Pi) = \alpha_{CE}(\Pi) - \alpha_{EQ}(\Pi),$$

Proposition 2: Non monotonicity of the Liquidity Under Supply.

Suppose $\Pi < H$ *and* $h_1 h_2 < H < R h_1$. *Then the Liquidity Under-Supply is a non monotone function of the degree of financial development* Π. *In particular,*
$LUS'(\Pi) > 0$ *for all* $\Pi < h_1 h_2$
$LUS'(\Pi) < 0$ *for all* $\Pi \in [h_1 h_2, H]$.

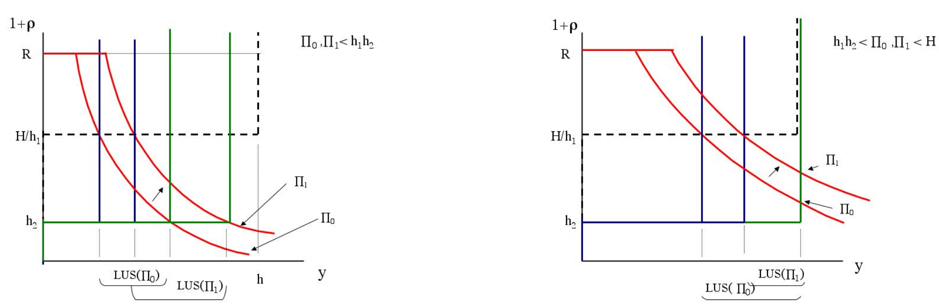

Fig. 8: Liquidity Under-Supply. Cases $\Pi < h_1 h_2$ and $\Pi \in [h_1 h_2, H]$

The non-monotonicity of the Liquidity Under-Supply can be seen in figure 8. It is increasing in the "low Π" zone while decreasing in the "high Π" zone. In this simple framework, we can think of Π as the degree of financial development of the economy. Any financial innovation (expressed by an increase in Π) at very low levels paradoxically makes government intervention (taxation) more necessary. As credit markets begin to get more sophisticated, the distortions if markets work in a decentralized way become bigger. On the other hand, once the credit markets are sophisticated, as in the "high Π" zone, credit market innovations lower the distortion and the costs of non intervention become smaller. Although the model is too simple to really think about policy implications, we think this non monotonicity property could have interesting implications for Emerging Markets economies.

The alternative investment opportunity h_2, is a critical ingredient of the non monotonicity. The essence of liquidity is flexibility, so the presence of such alternatives goes hand in hand with liquidity. The non monotonicity of the under-supply of liquidity is thus an inevitable consequence of its nature.

5 Non Monotone Financial Liquidity Under-Supply.

We can reinterpret our previous model in a slightly different way, so that propositions 1 and 2 apply also for Financial Liquidity.

How financially liquid an agent is depends , as we already defined at the beginning, on his ability to borrow against the present value of his future income. That is, to sell promises of future deliveries.

Suppose now that L firms at time 0 are endowed with a certain amount of cash. They can buy a long-term asset that pays H at $t = 2$. However, once in this position, they cannot sell any promise at $t = 1$ using as collateral the present value of their future income H. On the other hand, they can invest a proportion α of their initial cash holdings on a short-term investment with a return of h_1 at $t = 1$. An example could be deposits at a large foreign bank. In the second period they have two financial options, invest with a return of h_2 or enter into the credit market which will yield a return of $1 + \rho$. Clearly, with this new interpretation α now becomes a measure of financial liquidity.

Under the same assumptions on the parameters of the model we made before, proposition 1 and 2 hold. In equilibrium, there is an under-supply of financial liquidity with respect to the constrained efficient allocation. Moreover, the financial Liquidity Under-Supply is a non-monotone function of Π, the degree of financial development of the economy.

6 Endogenous Credit Constraints and the Liquidity Under-Supply Multiplier.

1. **The Model.**

   So far we have taken Π to be exogenous. Now we show that by introducing collateral explicitly into the model, the credit constraint can be taken to be endogenous. This is important, because once Π is endogenous, the demand for liquidity might become much more elastic, increasing the second best inefficiency and hence the liquidity under supply.

   Let us extend the model by introducing a perfectly durable and divisible good at time $t = 1$, owned in its entirety of one unit by B firms. We will assume that there is no market at time 1 for this good. It gives no utility to any agent at time $t = 1$, but it is desired by B firms for consumption at time $t = 2$. The good is useful because a B firm can use a proportion $\beta \in (0, 1]$ of it as collateral for his borrowing at time $t = 1$. We suppose that B firms have no incentive to repay any money on their loans, but that the collateral can be seized by the lender and sold to make them whole. Since the good is perfectly durable, if its price at time $t = 2$ is sure to be Π_2, then lenders will be willing to accept promises of $\beta\Pi_2$ due at time $t = 2$, and therefore the credit constraint faced by the borrowers is

   $$(1 + \rho)x \leq \beta\Pi_2$$

   In this new context, β can be interpreted as the degree of financial development of the economy. This is a natural interpretation since financial markets become more sophisticated as the proportion of the durable goods in the economy that can be used as collateral increases.

Before we move on, we need to extend the definition of competitive equilibrium for this model.

Definition 4:
A *collateral equilibrium* allocation consists of lenders and borrowers decisions $(\alpha^*, y^*, C_2^{L*}, c_2^{L*}, x^*, C_2^{B*}, c_2^{B*})$ and prices (ρ^*, Π_2^*) such that:

(a) L firms choose liquidity α^* in period 0, lending y^* in period 1 and collateral C_2^{L*} and consumption c_2^{L*} in period 2 such that they solve,

$$\begin{cases} Max & U_L(c_2^L) \\ \alpha, y, C_2^L, c_2^L & \\ s.t. & 0 \leq \alpha \leq 1 \\ & 0 \leq y \leq \alpha h_1 \\ & c_2^L + \Pi_2^* C_2^L \leq (1-\alpha)H + (\alpha h_1 - y)h_2 + y(1+\rho^*) \end{cases}$$

(b) B firms choose borrowing level x^* in period 1, collateral C_2^{B*} and consumption c_2^{L*} such that they solve

$$\begin{cases} Max & U_B(C_2^B, c_2^B) \\ x, C_2^B, c_2^B & \\ s.t. & 0 \leq (1+\rho^*)x \leq \beta \Pi_2^* \\ & c_2^B + \Pi_2^* C_2^B \leq Rx - (1+\rho^*)x + \Pi_2^* \end{cases}$$

(c) $x^* = y^*$

(d) $C_2^{L*} + C_2^{B*} = 1$

As before, a central planner in this model maximizes total output in period 2. Therefore, he chooses period 0 variables and lets firms maximize and markets clear from there on. Since the supply of the collateral good is fixed at 1, any Pareto efficient allocation maximizes the output of the consumption good c_2. Thus again, we take $c_2^L + c_2^B$ as our measure of welfare.

Suppose first that the utility of consumption of the output c_2^B and the collateral C_2^B by B firms at time $t = 2$ is

$$U_B(C_2^B, c_2^B) = c_2^B + \Pi_2 C_2^B$$

where Π_2 is constant. It is evident that an equilibrium in this case is precisely the same as the one we computed earlier in section 4. Moreover, from the credit constraint we get that

$$\tfrac{dx}{d\rho} = -\tfrac{\Pi_2}{(1+\rho)^2} d\rho$$

A change in the interest rate has only a *direct effect* on the demand for loans. The situation is different when Π_2 becomes an endogenous variable as we show next. We now turn to non-linear utility U_B.

Let us define and assume

$$\begin{aligned} \Pi_2(c_2^B) &\equiv \frac{\partial U_B(1, c_2^B)/\partial C_2^B}{\partial U(1, c_2^B)/\partial c_2^B} \\ \Pi_2'(c_2^B) &> 0 \end{aligned}$$

Proposition 3: Liquidity Under-Supply Multiplier.

Suppose $h_1 h_2 < H < R h_1$. Then

$$\tfrac{dx}{d\rho} = -\eta \tfrac{\Pi_2}{(1+\rho)^2} d\rho$$

where, $\eta = \frac{(1+\beta\Pi_2')}{1-\frac{\beta\Pi_2'}{1+\rho}[(R-(1+\rho)]} > 1$, *is the Liquidity Under-Supply Multiplier.*

Lower ρ, after intervention, enables the B firms to borrow more, even if Π_2 does not change. This is the direct effect. Lower ρ also increases the profit of B firms, even if they borrowed the same amount. Borrowing more further increases their profit, since on the margin $R > (1 + \rho)$. Increasing the profit of the B firms boosts their consumption of c_2^B (the consumption of C_2^B cannot increase, since supply is fixed at 1). By assumption, this increases the relative marginal utility of C_2^B, thus increasing Π_2 in the future. But lenders can forecast this future increment on Π_2, which in turn raises the ability of B firms to borrow, causing a big multiplier effect. This is the *indirect effect.* We can see this in figure 9.

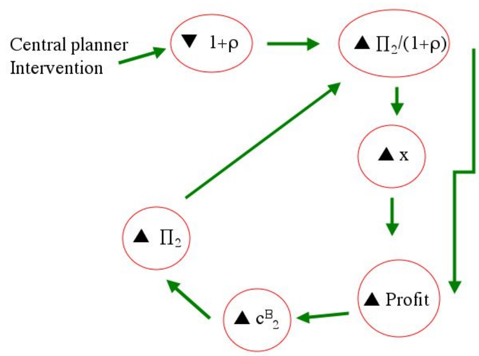

Fig. 9: The Liquidity Under Supply Multiplier.

2. An Example

In particular let us assume that $U_L(C_2^L, c_2^L) = c_2^L$ and that $U_B(C_2^B, c_2^B) = \mu c_2^B - (1/2)(c_2^B)^2 + \lambda C_2^B$

The following discussion is in terms of the parameters β and λ. As we said before, β is a measure of financial development of the economy, since the extent to which durable goods can be used as collateral depends on the presence of institutions like courts that guarantee that function. On the other hand, how efficient a good is in its function as collateral depends also on its market value, which depends on price times quantity. Since without loss of generality we choose units such that the total quantity of collateral is 1, this aspect is represented in our model by λ, the borrowers' marginal utility of collateral. The two variables are of vital importance. For instance, houses will not be a reasonable collateral if they have a very low value regardless of the presence of an efficient judicial system. Valuable goods will not be regarded as good collateral without a court system capable of enforcing confiscation. The following three propositions study the effect of β and λ on the liquidity under-supply, its behavior as a function of β and the liquidity under supply multiplier.

Proposition 4: Liquidity Under-Supply. (LUS)

Suppose $h_1 h_2 < H < Rh_1$, $\mu > Rh_1 + 1$ and $\lambda > 0$. Then $\exists\ \lambda_1 > 0$ such that:

(a) $\forall \lambda < \lambda_1$, $LUS(\beta) > 0, \forall \beta \in (0, 1]$.

(b) $\forall \lambda > \lambda_1$, $\exists\ \beta_1(\lambda) > 0$ *such that* $LUS(\beta) > 0, \forall \beta < \beta_1(\lambda)$.

In economies where the market value of collateral is low there will always be liquidity under-supply, no matter how high β is. If the government cannot change β or λ, it should, for example, tax illiquid investments. However, when market value of collateral is high enough, there exists liquidity under supply provided the level of financial development is low.

Proposition 5: Non-Monotonicity of LUS.

Suppose $h_1 h_2 < H < Rh_1$, $\mu > Rh_1 + 1$ and $\lambda > 0$. Then $\exists\ \lambda_0 < \lambda_1$ such that:

(a) $\forall \lambda > \lambda_0$, $LUS(\beta)$ *is a non-monotone function. This is,* $\exists\ \beta_0(\lambda) < \beta_1(\lambda)$ *such that* $LUS'(\beta) > 0, \forall \beta < \beta_0(\lambda)$ *and* $LUS'(\beta) \leq 0, \forall \beta > \beta_0(\lambda)$

(b) $\forall \lambda < \lambda_0$, $LUS'(\beta) > 0, \forall \beta \in (0, 1]$.

Proposition 6: Non-Monotonicity of the LUS Multiplier.

Suppose $h_1 h_2 < H < Rh_1$, $\mu > Rh_1 + 1$ and $\lambda > 0$. Then

(a) $\forall \lambda > \lambda_1$, $\eta(\beta)$ *is a non-monotone function, this is,* $\eta'(\beta) > 0, \forall \beta < \beta_1(\lambda)$ *and* $\eta'(\beta) \leq 0, \forall \beta > \beta_1(\lambda)$

(b) $\forall \lambda < \lambda_1, \eta'(\beta) > 0, \forall \beta \in (0, 1]$

Proposition 5 and 6 can be summarized as follows. When the market value of collateral is too low, any financial innovation expressed by an increase in β, regardless of how sophisticated the economy was to start with, will lead to a *bigger* liquidity under-supply. This inefficiency is even more dramatic, since the multiplier also gets bigger with the innovation. On the other hand, for higher levels of marginal utility of collateral, the behavior of the liquidity under-supply and the multiplier becomes non monotone. For very low levels of financial development, any increase in β will make the under-supply and the multiplier bigger. However, for developed markets, any innovation will reduce both.

The implications of the example for emerging markets are potentially interesting. These economies are often characterized by durable (often non-tradable) goods with low market values (at least during big crises) or weak court systems that reduce the range of goods that can serve as collateral. It is difficult not to think about the financial innovation and simultaneous, dramatic, reduction of government participation in financial markets that took place in Latin American economies during the 90's. Numerous liquidity crises occurred in these economies during that period.

Finally, to illustrate more clearly the example, we run a simulation of the model for parameters values of $R = 3, H = 2, h_1 = 1, h_2 = 1$ and $\mu = 7$. It turns out that the cut values for λ are $\lambda_0 = 6, \lambda_1 = 12$. The following graphs present the result of the simulation. The first one is for $\lambda = 5$, in which there is always liquidity under-supply, and both the liquidity under-supply and the multiplier are increasing.

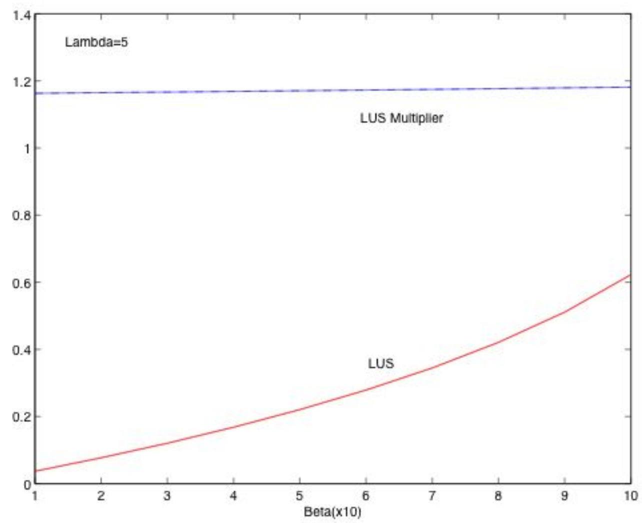

Fig. 10: Case $\lambda = 5$

The second corresponds to $\lambda = 10$, in which the liquidity-under supply is always positive but now becomes non monotone as a function of β, the multiplier is still increasing.

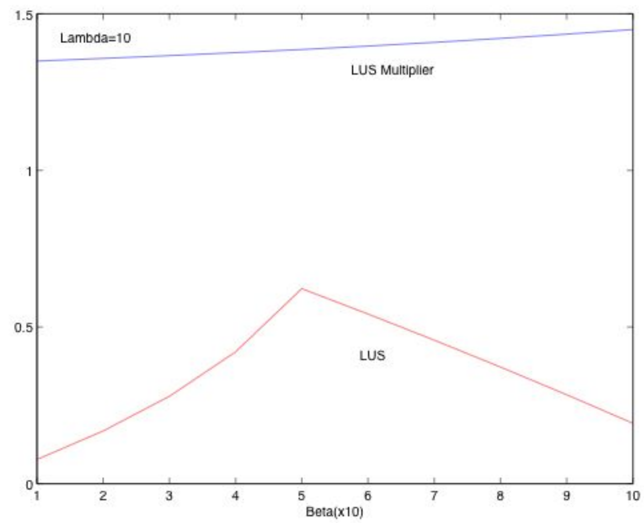

Fig. 11: Case $\lambda = 10$

Finally, for the case $\lambda = 20$, liquidity under-supply can be zero for high enough values of β, and both liquidity under-supply and multiplier become non monotone.

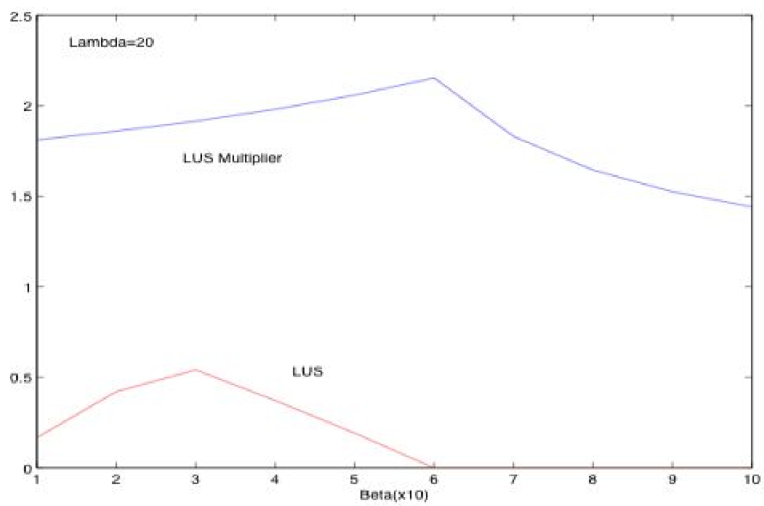

Fig. 12: Case $\lambda = 20$

7 Uncertainty.

Uncertainty has played no role in our model. Indeed, it is not needed to generate under-supply or non monotonicity. Introducing it does not change the qualitative features of the under-supply, but is can increase its magnitude.

Suppose now that the return R of the B firms is stochastic: with probability p it is R, as before, but with probability $(1-p)$ it is 0. The idea is that in normal states of nature, with probability $1-p$, the B firms have no opportunity to invest at time $t=1$. In extraordinary, perhaps crisis, situations they have a huge opportunity or need to invest with borrowed money. The question is, will liquidity providers, rationally anticipating these events, provide for the right amount of liquidity at $t=0$?

The answer is no, for the reasons given without uncertainty. Now, in equilibrium $H=(1-p)h_1h_2+ph_1(1+\rho)$, where ρ is the interest rate in the event $R>0$. It is easy to see that if p is small, and $H>h_1h_2$, then ρ can be enormous, choking off almost all borrowing by the B firms just when they need the money the most. The under-supply can therefore be much more severe.

8 Conclusion.

We show that very little is needed to create liquidity under-*supply* in equilibrium: only the presence of credit constraints on *demand*. We show that the under-supply is a non-monotone function of the demand distortion that causes it, a result that may have interesting implications for emerging markets economies. Finally, we show that the inefficiency can be large, due to the presence of a liquidity under supply multiplier arising from the endogeneity of the credit constraint. The inefficiency can also be large if there is uncertainty about the need for liquidity.

We use two very simple and general principles to show how liquidity under-supply can arise in equilibrium and be non monotone in the level of financial development. The Offsetting Distortions principle is simply the idea that any demand distortion can be interpreted as a supply inefficiency.

We define Second Best inefficiency as the difference between the equilibrium quantity under the demand distortion and the quantity after the central planner intervention. We show that the Second Best inefficiency is a non-monotone function of the demand distortion.

We briefly discussed the different definitions of liquidity. Our paper is about physical and financial liquidity. Physical liquidity refers to the flexibility to move physical goods across different projects. Financial liquidity refers to the ability an agent has to borrow against the present value of his future income, in particular the ability to sell contingent contracts.

We present a simple model in which we prove, as an application of the general principles mentioned before, the existence of under-supply of physical (financial) liquidity in equilibrium and that this under-supply is non monotone in the credit constraint that generates it.

Finally, we model the credit constraint by assuming that borrowers will default unless their promises are covered by collateral. Further, we assume that only an exogenous proportion β of one durable good can serve as collateral. This parameter β will represent the degree of financial development of the economy. We show that the magnitude of the under supply can be larger when the price of the collateral is endogenous, giving rise to a Liquidity Under Supply Multiplier. Any policy intervention that affects the interest rate in equilibrium will have two effects on the borrowing constraint: a direct effect, also present in the case when the credit constraint is exogenous, and an indirect effect through the price of the collateral. We explore further these results in a particular example in which utilities for consumption and collateral are quadratic and present the result of simulations.

9 Appendix

Proof of Proposition 1:

We prove the proposition calculating the unique equilibrium in this economy and comparing it with the social planner solution.
Solving for period $t = 1$
The demand for funds of a type B firm is obtained from their maximization problem. The borrowing constraint is going to be binding since $\Pi < H$, hence the demand for funds is given by

$$x = \begin{cases} 0, & 1+\rho > R \\ (0,\infty), & 1+\rho = R \\ \frac{\Pi}{1+\rho}, & 1+\rho < R \end{cases} \qquad (1)$$

On the other hand, the L firm's short run supply, is obtained solving the maximization problem given α. Since $Rh_1 > h_1h_2$, the short-run supply is given by

$$y = \begin{cases} 0, & 1+\rho < h_2 \\ (0,\alpha h_1), & 1+\rho = h_2 \\ \alpha h_1, & 1+\rho > h_2 \end{cases} \qquad (2)$$

Solving for equilibrium in the credit market, i.e. equating (1) and (2) we get that

$$1+\rho = \begin{cases} h_2, & \alpha \geq \Pi/h_1h_2 \\ \frac{\Pi}{\alpha h_1}, & \alpha \leq \Pi/h_1h_2 \end{cases} \qquad (3)$$

<u>Solving for period $t = 0$</u>

Competitive equilibrium. ($\Pi < \infty$):
To find the competitive solution, we need to solve the problem of the representative L firm, and we obtain that the long run supply is

$$y = \begin{cases} 0, & 1+\rho < H/h_1 \\ (0,h_1), & 1+\rho = H/h_1 \\ h_1, & 1+\rho > H/h_1 \end{cases} \qquad (4)$$

Condition (3) with (4) implies that in equilibrium $\alpha_{EQ} = \frac{\Pi}{H}$. Therefore the equilibrium is
$\{(\alpha_{EQ}, y_{EQ}, x_{EQ}), (1+\rho_{EQ})\} = \{(\frac{\Pi}{H}, \frac{\Pi h_1}{H}, \frac{\Pi h_1}{H}), \frac{H}{h_1}\}$

Planner solution:
In the case $\Pi < h_1h_2$,the planner chooses α in order to maximize total output at $t = 2$, it is clear that this is done for $\alpha = \frac{\Pi}{h_1h_2}$. Therefore the constrained efficient solution is $\{(\alpha_{CE}, y_{CE}, x_{CE}), (1+\rho_{CE})\} = \{(\frac{\Pi}{h_1h_2}, \Pi, \Pi), h_2\}$
When $\Pi \in [h_1h_2, H]$, by the same type of reasoning, we have that $\{(\alpha_{CE}, y_{CE}, x_{CE}), (1+\rho_{CE})\} = \{(1, h_1, h_1), \frac{\Pi}{h_1}\}$

Arrow-Debreu solution. ($\Pi = \infty$):
It is straightforward to see that when firms are not constrained the Arrow-Debreu equilibrium is
$\{(\alpha_{AD}, y_{AD}, x_{AD}), (1+\rho_{AD})\} = \{(1, h_1, h_1), R\}$.

Now, just notice that the equilibrium allocation is different from the constrained efficient allocation, and in particular, $\alpha_{EQ} = \frac{\Pi}{H} < \alpha_{CE} = \frac{\Pi}{h_1 h_2}$ since $h_1 h_2 < H$ in the case $\Pi < h_1 h_2$. In the case in which $\Pi \in [h_1 h_2, H]$ we have that $\alpha_{EQ} = \frac{\Pi}{H} < \alpha_{CE} = 1$.

Proof of Proposition 2:

It is a straightforward calculation to see that $LUS'(\Pi) = \frac{1}{h_1 h_2} - \frac{1}{H} > 0$ in the case $\Pi < h_1 h_2$, while $LUS'(\Pi) = -\frac{1}{H} < 0$ in the case $\Pi \in [h_1 h_2, H]$ using the allocations calculated in the proof of proposition 1.

Proof of Proposition 3:

In equilibrium when the borrowing constraint is binding we must have[3] $c_2^B = Rx - (1+\rho)x$ and $x = \beta\Pi_2(c_2^B)/(1+\rho)$. Therefore $x(\rho) = \beta\Pi_2(c_2^B)/(1+\rho) =$ $= \beta\Pi_2(Rx(\rho) - (1+\rho)x(\rho))/(1+\rho)$. Hence, $dx = -\frac{\beta\Pi_2}{(1+\rho)^2}d\rho + \frac{\beta\Pi_2'}{1+\rho}\{[R - (1+\rho)]dx - xd\rho\}$. Rearranging terms and using $x = \beta\Pi_2/(1+\rho)$ gives, $\frac{dx}{d\rho} = -\frac{\beta\Pi_2(1+\beta\Pi_2')}{1-\frac{\beta\Pi_2'}{1+\rho}[R-(1+\rho)]}\frac{d\rho}{(1+\rho)^2}$. Denote by $\eta = \frac{(1+\beta\Pi_2')}{1-\frac{\beta\Pi_2'}{1+\rho}[(R-(1+\rho)]}$ the Liquidity under supply Multiplier. Then, we have that $\frac{dx}{d\rho} = -\eta\frac{\beta\Pi_2}{(1+\rho)^2}d\rho$ Notice that as long as $\Pi_2' > 0$ and $R > (1+\rho)$, this is much more sensitive.

Proof of Proposition 4:

1. **Calculation of allocations.**

   <u>Solving for period $t = 2$</u>
   It is very clear that lenders and borrowers options are $C_2^L = 0$,$c_2^L = (1-\alpha)H + (\alpha h_1 - y)h_2 + (1+\rho)y$, $C_2^B = 1$, $c_2^B = Rx - (1+\rho)x$.

[3] It is clear that in equilibrium $C_2^B = 1$, hence the expression for consumption below follows from the budget constraint.

Solving for period $t = 1$

Lenders decisions are exactly the same in proposition 1, therefore, the short run supply is given by equation (2) above. In analyzing borrowers decisions there are two cases depending on wether the credit constraint is binding or not.

Credit constraint binding:

In this case we have that

$$x = \begin{cases} 0, & 1+\rho > R \\ (0,\infty), & 1+\rho = R \\ \frac{\Pi_2}{1+\rho}, & 1+\rho < R \end{cases} \qquad (5)$$

From the budget constraint $c_2^B = Rx - (1+\rho)x$. From the problem's first order conditions we have that $\Pi_2 = \frac{\lambda}{\mu - c_2^B}$. Combining these last three expressions we get that $\Pi_2 = \frac{-\mu + \sqrt{\mu^2 + 4\lambda\beta(1-R/(1+\rho))}}{2\beta(1-R/(1+\rho)}$. Plugging this expression for Π_2 into (5), we get the demand for loans. To solve for equilibrium we just solve equations (5) and (2) to get :

$$1+\rho = \begin{cases} h_2, & \alpha \geq \frac{\beta\Pi_2(h_2)}{h_1 h_2} \\ (1+\rho)^* & \alpha \leq \frac{\beta\Pi_2(h_2)}{h_1 h_2} \end{cases} \qquad (6)$$

where $(1+\rho)^*$ solves the equation $\alpha h_1 = \frac{\beta\Pi_2(1+\rho)}{1+\rho}$.

Credit constraint not binding:

In this case, the demand for loans is

$$x = \begin{cases} 0, & 1+\rho > R \\ (0,\infty), & 1+\rho \leq R \end{cases} \qquad (7)$$

Furthermore, $c_2^B = 0$ and from the first order conditions $\Pi_2 = \lambda/\mu$. Finally to solve for equilibrium solve equations (2) and (7) to get that $1+\rho = R$. (8)

Solving for period $t = 0$

Competitive equilibrium:

Credit Constraint binding:

The long run supply of lenders is given by (4) in the proof of Proposition 1. Taking into account (6) and (4) we have two cases:
In the first one $1+\rho = H/h_1$ and therefore $\alpha = \frac{\beta\Pi_2(H/h_1)}{H}$.
The equilibrium allocation is:
Lenders: $\alpha = \frac{\beta\Pi_2(H/h_1)}{H}$ $y = \alpha h_1$ $C_2^L = 0$, $\mathrm{c}_2^L = (1-\alpha)H + y(1+\rho)$
Borrowers:$x = \alpha h_1$, $\mathrm{C}_2^B = 1, c_2^B = Rx - (1+\rho)x$
Prices: $(1+\rho) = H/h_1, \Pi_2 = \frac{-\mu+\sqrt{\mu^2+4\lambda\beta(1-R/(1+\rho)}}{2\beta(1-R/(1+\rho)}$
In the second case, one of the two following conditions hold: I)$\frac{\beta\Pi_2(H/h_1)}{H} > 1$, II) $\mu^2 + 4\beta(1-R/(H/h_1) < 0$
In this case $\alpha = 1$ and $1+\rho$ is given implicitly by the equation $h_1 = \frac{\beta\Pi_2(1+\rho)}{1+\rho}$. The equilibrium allocation is:
Lenders: $\alpha = 1, y = \alpha h_1, C_2^L = 0, c_2^L = (1-\alpha)H + y(1+\rho)$
Borrowers: $x = \alpha h_1$, $\mathrm{C}_2^B = 1, c_2^B = Rx - (1+\rho)x$
Prices: $(1+\rho) =$ is given implicitly by the equation $h_1 = \frac{\beta\Pi_2(1+\rho)}{1+\rho}$ and $\Pi_2 = \frac{-\mu+\sqrt{\mu^2+4\lambda\beta(1-R/(1+\rho))}}{2\beta(1-R/(1+\rho)}$
Credit Constraint not binding:
Considering the long run supply (4), we have that since $(1+\rho) = R > H/h_1$ then $\alpha = 1$. The equilibrium allocation is:
Lenders: $\alpha = 1$, $y = h_1$, $C_2^L = 0$, $\mathrm{c}_2^L = Rh_1$
Borrowers: $x = h_1$, $\mathrm{C}_2^B = 1, c_2^B = 0$
Prices: $(1+\rho) = R, \Pi_2 = \frac{\lambda}{\mu}$

Central Planner:
Credit Constraint binding:
The central planner chooses α in order to maximize total output at $t = 2$, this is, $(1-\alpha)H + (\alpha h_1 - y)h_2 + Ry$. Since $Rh_1 > H$, it is clear that he will choose α so that the B firms can borrow all the way to the maximum, this time endogenously determined. Again we have two cases:
In the first case, $1+\rho = h_2$ and therefore $\alpha = \frac{\beta\Pi_2(h_2)}{h_1 h_2}$
The constrained efficient allocation is:
Lenders: $\alpha = \frac{\beta\Pi_2(h_2)}{h_1 h_2}$ $y = \alpha h_1$ $C_2^L = 0$, $c_2^L = (1-\alpha)H + y(1+\rho)$
Borrowers: $x = \alpha h_1, C_2^B = 1, c_2^B = Rx - (1+\rho)x$
Prices: $(1+\rho) = h_2, \Pi_2 = \frac{-\mu+\sqrt{\mu^2+4\lambda\beta(1-R/(1+\rho))}}{2\beta(1-R/(1+\rho))}$
In the second case, one of the two following conditions hold: I')$\frac{\beta\Pi_2(h_2)}{h_1 h_2} >$

1, II') $\mu^2 + 4\beta(1 - R/(h_2)) < 0$
In this case $\alpha = 1$ and $1 + \rho$ is given implicitly by the equation $h_1 = \frac{\beta\Pi_2(1+\rho)}{1+\rho}$.
The constrained efficient allocation is:
Lenders: $\alpha = 1, y = \alpha h_1, C_2^L = 0, c_2^L = (1 - \alpha)H + y(1 + \rho)$
Borrowers: $x = \alpha h_1$, $C_2^B = 1, c_2^B = Rx - (1 + \rho)x$
Prices: $(1 + \rho) =$ is given implicitly by the equation $h_1 = \frac{\beta\Pi_2(1+\rho)}{1+\rho}$ and $\Pi_2 = \frac{-\mu+\sqrt{\mu^2+4\lambda\beta(1-R/(1+\rho))}}{2\beta(1-R/(1+\rho))}$
Credit Constraint not binding.
This case is exactly as analyzed in the case of descentralized equilibrium above.

2. **Proof of (a) and (b)**

Consider $\triangle(\lambda, \beta, \rho) = \mu^2 + 4\lambda\beta(1 - R/(1 + \rho))$ and $\alpha(\lambda, \beta, \rho) = \frac{-\mu+\sqrt{\mu^2+4\lambda\beta(1-R/(1+\rho))}}{2(1-R/(1+\rho))H}$. Let λ' such that solve $\triangle(\lambda, 1, H/h_1) = 0$ and let λ'' such that solve $\alpha(\lambda, 1, H/h_1) = 1$. Define $\lambda_1 = min\{\lambda', \lambda''\}$.

a) Given the definition of λ_1, the fact that α is an increasing function of both β and λ and $\triangle$ is a decreasing function of both arguments, implies that $\alpha_{EQ}(\beta) < 1 \ \forall\beta \in (0, 1]$. In this case $LUS(\beta) = \frac{-\mu+\sqrt{\mu^2+4\lambda\beta(1-R/h_2)}}{2(1-R/h_2)h_1h_2} - \frac{-\mu+\sqrt{\mu^2+4\lambda\beta(1-Rh_1/H)}}{2(1-Rh_1/H)H}$. Since $H > h_1h_2$, LUS$(\beta) > 0 \ \forall\beta \in (0, 1]$. Obviously this holds even in the case $\alpha_{CE} = 1$.

b) Given the definition of λ_1 and the fact that α is a continuous increasing function of both β and λ and $\triangle$ is a continuous decreasing function of both arguments, implies for any $\lambda > \lambda_1$, $\exists\beta_1(\lambda) > 0$ such that $\alpha_{EQ} = 1$ and therefore $LUS(\beta) > 0, \forall\beta < \beta_1(\lambda)$

Proof of Proposition 5:

Let Let λ^+ such that solve $\triangle(\lambda, 1, h_2) = 0$ and let λ^{++} such that solve $\alpha(\lambda, 1, h_2) = 1$. Define $\lambda_0 = min\{\lambda^+, \lambda^{++}\}$. Clearly $\lambda_0 < \lambda_1$ since $H > h_1h_2$.

a) Given the definition of λ_0 by an analogous argument of b) in the proof of proposition 4, $\forall \lambda > \lambda_0$, $\exists \beta_0(\lambda) < \beta_1(\lambda)$ such that $\alpha_{CE} = 1$. Therefore $\forall \beta < \beta_0(\lambda)$, we have that
$LUS(\beta) = \frac{-\mu+\sqrt{\mu^2+4\lambda\beta(1-R/h_2)}}{2(1-R/h_2)h_1h_2} - \frac{-\mu+\sqrt{\mu^2+4\lambda\beta(1-Rh_1/H)}}{2(1-Rh_1/H)H}$.
And $LUS'(\beta) = \frac{\lambda(1-R/h_2)}{(1-R/h_2)h_1h_2\sqrt{\mu^2+4\lambda 1(1-R/h_2)}} - \frac{\lambda(1-Rh_1/H)}{(1-Rh_1/H)H\sqrt{\mu^2+4\lambda 1(1-Rh_1/H)}}$. It is easy to see that since $H > h_1h_2$, LUS'(β) > 0.
On the other hand, $\forall \beta_1(\lambda) > \beta \geq \beta_0(\lambda)$, we have that
$LUS(\beta) = 1 - \frac{-\mu+\sqrt{\mu^2+4\lambda\beta(1-Rh_1/H)}}{2(1-Rh_1/H)H}$.
and therefore, $LUS'(\beta) = -\frac{\lambda(1-Rh_1/H)}{(1-Rh_1/H)H\sqrt{\mu^2+4\lambda 1(1-Rh_1/H)}} < 0$.
Finally, it is clear that for $\beta \geq \beta_1(\lambda), LUS(\beta) = 0, and therefore LUS'(\beta) = o$.

(b) It follows from definition of λ_0 and the argument in the previous point for the case of $\beta < \beta_0(\lambda)$

Proof of Proposition 6:

From the equilibrium and the formula of Proposition 3,
$\eta = \frac{1+\frac{\lambda}{(\mu-\beta(\frac{R}{1+\rho}-1)\Pi_2)^2}}{1-\frac{\lambda}{(\mu-\beta(\frac{R}{1+\rho}-1)\Pi_2)^2}\frac{1}{1+\rho}(R-(1+\rho))} > 1$

Proof of a):

Case $\beta < \beta_1$.
First we prove that $\Pi_2(\beta)$ is an increasing function. In this case the interest rate is constant and equal to H/h_1, and
$\Pi_2 = \frac{-\mu+\sqrt{\mu^2+4\lambda\beta a}}{2\beta a}$, where $a = (1 - Rh_1/H)$. After some algebra , $\Pi'(\beta) = \frac{\mu-(\mu^2+2\lambda\beta a)/(\sqrt{\mu^2+4\lambda\beta a})}{2a\beta^2}$. Since the denominator is negative, we only need to check that the denominator is negative as well. Define $z(\beta) = \mu - (\mu^2 + 2\lambda\beta a)/(\sqrt{\mu^2 + 4\lambda\beta a})$. Then, $z(0) = 0$ and $z'(\beta) = -\frac{2\lambda a}{\sqrt{\mu^2+4\lambda\beta a}\,(\mu^2+4\lambda\beta a)}(2\lambda a\beta) < 0\ \forall \beta \in (0, 1]$. Therefore the numerator is negative as well and $\Pi'(\beta) > 0$.
Now we show that $\eta'(\beta) > 0\ \forall \beta < \beta_1$. From definition $\eta(\beta) = \frac{1+f(\beta)}{1-f(\beta)k}$, where $f(\beta) = \frac{\lambda}{(\mu-\beta(Rh_1/H-1)\Pi_2)^2}$ and $k = h_1/H(R - H/h_1) > 0$. Since $\Pi'(\beta) > 0$ it is clear that $f'(\beta) > 0$ as well. Now, $\eta'(\beta) = \frac{f'(\beta)+k}{(1-f(\beta)k)^2}$. Since $f'(\beta) > 0$ and $k > 0$, we have that $\eta'(\beta) > 0$ as we wanted to show.

Case $\beta > \beta_1$.

In this case the interest rate is a function of β as well. The first thing we show is that $\rho'(\beta) > 0$. For this, notice first that Π_2 is now a function of two variables,

$\Pi_2(\beta, \rho) = \frac{-\mu+\sqrt{\mu^2+4\lambda\beta(1-R/(1+\rho))}}{2\beta(1-R/(1+\rho)}$. Its partial derivatives are $\partial\Pi_2/\partial\rho = \frac{R/(1+rho)^2}{2\beta(1-R/(1+\rho))^2}w(\beta, b)$ and $\partial\Pi_2/\partial\beta = \frac{1}{2\beta^2(1-R/(1+\rho))}w(\beta, b)$, where $w(\beta, b) = \mu - (\mu^2 + 2\lambda\beta b)/(\sqrt{\mu^2 + 4\lambda\beta b})$ and $b(\rho) = (1 - R/(1 + \rho))$. Hence, we have that $b(H/h_1 - 1) = a$ and $w(\beta, a) = z(\beta)$. The same argument that shows that $z(\beta) < 0$ extends to show $w(\beta, b) < 0$ since in equilibrium $b < 0$ and hence $\partial\Pi_2/\partial\rho < 0$, $\partial\Pi_2/\partial\beta > 0$.

Define $F(\beta, \rho) = \beta\Pi_2 - h_1(1 + \rho)$. In equilibrium this is zero. Moreover, $F_\rho = \beta\partial\Pi_2/\partial\rho - h_1 < 0$, and hence different from zero. Therefore, by the Implicit Function Theorem, $\rho'(\beta) = -F_\beta/F_\rho$. Since $F_\beta = \Pi_2+\beta\partial\Pi_2/\partial\beta > 0$, it is immediate that $\rho'(\beta) > 0$.

Now we show that $\eta'(\beta) < 0$. Using the fact that $h_1 = \beta\Pi_2(\beta, \rho)/(1 + \rho)$ in equilibrium, it is true that $\eta(\beta) = \frac{1+h(\beta)}{1-h(\beta)k(\beta)}$, where $h(\beta) = \frac{\lambda}{(\mu-(R-(1+\rho))h_1)^2}$ and $k(\beta) = (R/(1 + \rho) - 1)$. Since $\rho'(\beta) > 0$, $h'(\beta) < 0$ and $k'(\beta) < 0$. Let $\beta_1 < \beta_2$. Then $\eta(\beta_1) = \frac{1+h(\beta_1)}{1-h(\beta_1)k(\beta_1)} > \frac{1+h(\beta_2)}{1-h(\beta_1)k(\beta_1)} > \frac{1+h(\beta_2)}{1-h(\beta_2)k(\beta_2)} = \eta(\beta_2)$.

Proof of b):

This case follows immediately from the definition of λ_1 and the proof above in the case $\beta < \beta_1$.

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