Optimum Consumption and Portfolio Rules under Incomplete Information

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Abstract

We solve in closed form the optimal consumption / portfolio choice problem for an isoelastic utility agent under incomplete information about the mean return of the stock price. Upon observing the realizations of the stock and possibly outside market information, the investor can revise her beliefs about the true value of the mean return, which induces optimal allocations that can be significantly different from those of a myopic agent. The hedging demand for the risky security is positive (negative) and rises (falls) with more accurate information and the investor horizon, exactly when the intertemporal elasticity of substitution is above (below) unity.

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1. INTRODUCTION

Samuelson [17] and Merton [16] were among the pioneers to solve the optimal consumption / portfolio allocations for an agent maximizing her expected discounted utility with possibly a terminal date bequest in a multi-period horizon. Both authors assume that a small investor has complete information about the securities available on the market, i.e., can observe and know the dynamics of all the economic state variables relevant in her decision to allocate optimally her wealth between consumption and investment strategies. Unfortunately, in many real life cases, the investor does not know (or cannot observe) with perfect accuracy some characteristics of her investment opportunities. The main contribution of this paper is to solve in closed form Merton's problem [16] under incomplete information about the mean return of the stock price, which allows us to shed some light about the effects of learning on consumption and portfolio decisions.

1.1. Related Literature

Early attempts to introduce incomplete information about some non-observable fundamentals of the economy include Detemple [5], Dothan and Feldman [9] and Gennote [10]. These authors show that in a Markovian framework a separation principle holds: Agents first solve an inference problem to form their expectations, and second solve their dynamic optimization problem under the inferred information structure, incorporating learning as they update their beliefs. Detemple and Murthy [6], Feldman [9] and Zapatero [20] consider some equilibrium frameworks where investors have logarithmic utility preferences. Due to the specific feature of logarithmic preferences (myopia), the equilibrium interest rate is a weighted average (with weights equal to agents' relative wealth) that oscillates between the most pessimistic and the most optimistic agent's valuations. Regarding optimal consumption and investment decisions in a partial equilibrium framework, both a theoretical and empirical literature blossomed during the last decade. A rigorous mathematical treatment is presented in Lakner [14] who studies an investor who wishes to maximize the utility of her terminal wealth when investment opportunities are only partially observable. He provides some integral representation of portfolio allocations for several utility functions and multi-dimensional stochastic processes when agents have normally distributed beliefs about the mean return of the risky assets. Zohar [21] uses a similar framework to derive an extension of the Cameron-Martin formula which allows him to compute the optimal utility of the investor terminal wealth as well as the optimal portfolio in terms of Inverse Laplace Transform. Honda [11] studies a similar economy to the one presented in this paper. He uses dynamic programming techniques and relies on numerical methods to examine the optimal consumption and portfolio allocations. Our work departs from his as we apply martingale techniques to derive closed form solutions so we are able to perform a purely theoretical analysis that includes the effects of outside market information on portfolio choice, issue that is not addressed in Honda's article. Other papers aim at clarifying and estimating the role and importance of learning in portfolio decisions. Brennan [2] considers a CRRA investor who cannot observe the drift (known to be constant) of some risky asset and has

normally distributed beliefs about it. For a 1926-1994 period data base of annual returns on the S&P500 index, using numerical simulations, the author shows that incomplete information has a significant impact on investors' portfolio choice for a 20 year horizon, reducing the fraction of wealth invested in the risky security when agents are more risk averse than logarithmic preference investors. Barberis [1] calibrates a discrete time model using U.S. data of monthly real returns of NYSE stocks and Treasury bills for the time periods 1952-1995 and 1986-1995 in order to study the effects of the estimation risk and investment horizon for an investor maximizing her expected terminal wealth. The effects of the investor horizon on portfolio strategies can be quite different depending on whether the agent takes into account the new information to optimally rebalance her portfolio or chooses to ignore it. Xia [19] building on the continuous-time framework developed by Kim and Omberg [13] addresses similar issues when returns are predictable allowing for intermediate consumption and distinguishing two situations: (i) the investor knows the predictive relationship with certainty, (ii) the investor learns about the predictive relationship. Using U.S. stock market returns and dividend yields for the 1950-1997 period, she finds that the portfolio allocation is more sensitive to the predictive variable for a long horizon investor than for a short one, but learning mitigates this effect. Our model belongs to the category of learning without return predictability. Another issue raised in this paper is how the accuracy of information collected by an investor affects her portfolio selection. Veronesi [18] uses a general equilibrium model to investigate the impact of information quality on stock returns when the average growth of the dividend switches among several discrete states. He finds that the precision of the signal can enhance or dampen the volatility of the equilibrium asset price depending on whether the CRRA coefficient is above or below unity. The cut-off value of unit CRRA coefficient plays an important role in the results of this paper.

1.2. Results

We use a martingale approach to derive closed form solutions to Merton's problem [16] when CRRA preference investors have incomplete information about the mean return of a stock. Our information background is a continuous-time model of Bayesian learning as in Bolton and Harris [2] where the decision maker knows that the non-observable parameter is a constant but she hesitates over two possible values. After solving the filtering problem, we transform the investor's problem into an equivalent program that displays two important features:

 - the wealth dynamics are identical to those under complete information;

 - the utility function is altered by a multiplying stochastic factor that incorporates learning.

 We determine the consumption-wealth ratio and the demand for the risky asset disentangling the myopic demand and from the hedging demand. The consumption wealth-ratio is governed by the agent's desire to smooth her consumption across time. In particular, this ratio is increasing (decreasing) in optimism and a better quality information when the intertemporal elasticity of substitution (I.E.S.) is below (above). To achieve consumption smoothing the investor has two assets at her disposal; when she is willing to tolerate alter-

ations in her consumption plans, she relies more on the risky asset. As a consequence, the hedging demand for this latter is positive (negative) when the I.E.S. is greater (smaller) than one. It is increasing (decreasing) with optimism, the informativeness of the outside market signal and horizon time when the I.E.S. is above (below) unity. Better information means a higher variance of beliefs and faster updating which leads to more drastic changes in portfolio choices. Finally, we briefly discuss the case when investor has normally distributed beliefs about the mean return and find similar results.

The paper is organized as follows. Section 2 describes the economic setting and provides some insights on the structure of the optimal decision rules. Section 3 contains the derivation of the optimal consumption and portfolio allocations using martingale techniques and discusses the effects of changes in optimism, precision of the information received and investor horizon on the consumption-wealth ratio and the hedging demand. Section 4 concludes. Proofs of all results are collected in the appendix.

2. THE ECONOMIC SETTING

We consider an economy where an investor has to optimally allocate her wealth between a risk-free bond, a risky asset and consumption. We first examine the case of an infinite horizon.

Individual Preferences. There is a single perishable good available for consumption, the numéraire. Preferences are represented by a time additive utility function

$$U(c) = E\left[\int_0^\infty u(c(t))e^{-\theta t}dt\right],$$

where the instantaneous utility function u is twice continuously differentiable, increasing and strictly concave and θ denotes the subjective discount rate of future. In addition, u satisfies the following Inada conditions: $\lim_{c\to 0^+} u'(c) = \infty$ and $\lim_{c\to\infty} u'(c) = 0$.

The Financial Market and Information Structure. Uncertainty is modeled by a probability space $(\Omega, \mathcal{F}, P^w)$ on which is defined a *one* dimensional (standard) Brownian motion w. A state of nature ω is an element of Ω. $\mathcal{F}$ denotes the tribe of subsets of Ω that are events over which the probability measure P^w is assigned. For the sake of simplicity, there are only two securities available in the financial market:
  - a risk-free bond whose price B evolves according to

$$dB(t) = rB(t)dt,$$

where r is the constant interest rate, and
  - a risky non-paying dividend security (that can interpreted as a stock index) whose price S is given by a geometric Brownian motion

$$dS(t) = S(t)\left(\mu dt + \sigma dw(t)\right),$$

where $dw(t)$ is the increment of the standard Wiener process under P^w, μ is the mean return of the stock and σ is the instantaneous variance. The parameter μ is *unknown* to the investor. However, she knows that μ is a constant and it is either equal to h (high) or l (low). In the sequel, we assume that $-M < r < l < h$ with $M \geq 0$.

Even though the investor does not observe the true value for μ, she can observe the realizations of the value of the stock S and therefore infer the true value for the drift. Let $\mathcal{F}_t$ be the σ-algebra generated by the observations of the value of the stock up to time t, $\{S(s); 0 \leq s \leq t)\}$ and augmented. At time t, the investor's information set is $\mathcal{F}_t$. The filtration $\mathbb{F} = \{\mathcal{F}_t, \ t \in \mathbb{R}_+\}$ is the information structure and satisfies the usual conditions (increasing, right-continuous, augmented). At time t, let $p(t)$ be the probability or the investor's beliefs that μ is equal to h, i.e., $p(t) = \Pr(\mu = h \mid \mathcal{F}_t)$. Using Bayes' rules, the evolution across time of the posterior probability p is given by the following lemma.

Lemma 1. *The law of motion of the posterior beliefs P is*

$$dp(s) = \frac{h-l}{\sigma} p(s)(1 - p(s)) d\overline{w}(s),$$

where

$$
\begin{aligned}
d\overline{w}(s) &= \frac{1}{\sigma S(s)} \left(dS(s) - E^P\left[dS(s) \mid \mathcal{F}_s \right] \right) \\
&= dw(s) + \frac{1}{\sigma} \left(\mu - (p(s)h + (1 - p(s))l) \right) ds,
\end{aligned}
$$

is the increment of the standard Wiener process under P, relative to the filtration $\mathbb{F}$.

Proof. See Liptser and Shiryaev, [15] p 317 and for a more intuitive derivation see Bolton and Harris [2]. ∎

The innovation in beliefs is governed by the increment of a Brownian motion $\overline{w}$ which is adapted to the investor information structure whereas w is not. Changes in beliefs are increasing in the wedge $h - l$: when the two drifts differ significantly more information can be obtained and the investor can revise her beliefs more quickly. Similarly, when the quality of the signal is poor (high value of σ) or when the investor is almost certain of the value of μ (p close to 0 or 1), little information can be extracted and therefore beliefs do not change much. Finally, p is a martingale under P relative to $\mathbb{F}$ so on average, the investor's beliefs do not change[1].

Let P_μ be the probability measure under which the price process S is a geometric Brownian motion with constant mean return μ. Then, for $\mu \in \{l, h\}$, define the processes $\gamma_{p,\mu}$ and $\xi_{p,\mu}$ by

$$\gamma_{p,\mu}(t) = \frac{\mu - (p(t)h + (1 - p(t))l)}{\sigma},$$

---

[1]This is due to the fact that the non-observable process is a constant.

and

$$\xi_{p,\mu}(t) = \exp\left(-\int_0^t \gamma_{p,\mu}(s)dw(s) - \frac{1}{2}\int_0^t \gamma_{p,\mu}^2(s)ds\right).$$

$\xi_{p,\mu}$ is the density process of the Radon-Nikodym derivative of P with respect to P_μ, i.e.,

$$\xi_{p,\mu}(t) = \frac{dP(t)}{dP_\mu(t)}.$$

It can be shown that when $\mu = h$, then $\xi_{p,h}(t) = \frac{p_0}{p(t)}$ and when $\mu = l$, then $\xi_{p,l}(t) = \frac{1-p_0}{1-p(t)}$. To avoid degeneracy of the problem, we impose the following condition.

Assumption A1. The investor's beliefs are not trivial, i.e., $p(0) \in (0,1)$.

As mentioned in Veronesi [18], this implies that for all $t > 0$, $p(t) > 0$ and we choose to express conditional expectations under the probability measure P_l. In particular note that

$$\gamma_{p,l}(t) = -\frac{(h-l)(p(t))}{\sigma}.$$

Let us define $\phi(t) = \frac{p(t)}{1-p(t)}$. Under the probability measure P_l, the law of motion of the process ϕ is given by

$$\begin{aligned}
d\phi(t) &= \frac{dp(t)}{(1-p(t))^2} + \frac{(h-l)^2 p^2(t)(1-p(t))^2 dt}{(1-p(t))^3} \\
&= \frac{h-l}{\sigma}\phi(t)dw(t).
\end{aligned}$$

Hence ϕ is a geometric Brownian motion under P_l which would simplify greatly the analysis in the sequel. Notice that ϕ is strictly increasing in p or equivalently, the more optimistic the investor is that the average return of the risky security is equal to h, the higher the value of ϕ. Finally let E_t^P (respectively E_t^l) denote the conditional expectation with respect to probability P (P_l). We now describe the investor problem.

2.1. The Investor Problem

At time t, the investor's wealth is $W(t) = x(t) + z(t)$ where x is the amount invested in the risk-free bond and z the amount invested in the risky asset.
Feasibility. A consumption plan c is feasible if there is a trading strategy $(x, z) \in Q$ financing it such that

$$\begin{aligned}
dW(t) &= x(t)r(t)dt + z(t)\frac{dS(t)}{S(t)} - c(t)dt \\
W(t) &\geq -\underline{K}.
\end{aligned}$$

To rule out arbitrage opportunities, we impose that at all times t, the investor's wealth $W(t)$ must remain greater than a fixed amount $-\underline{K}$, where $\underline{K} > 0$ as exposed in Dybvig

and Huang [8]. Let $\mathcal{C}$ denote the set of feasible consumption plans and Q the set of admissible trading strategies. Under the investor's probability beliefs P, the price of the stock evolves according to

$$dS(s) = S(s)\left((p(s)h + (1 - p(s))l)ds + \sigma d\overline{w}(s)\right).$$

The agent must choose optimal portfolio rules (x, z) and consumption c in order to maximize her lifetime utility

$$J(W(t), p(t)) = \max_{(c \in \mathcal{C},\ (x,z) \in Q)} E_t^P \left[\int_t^{\infty} u(c(s))e^{-\theta(s-t)}ds \right]$$

$$
\begin{aligned}
\text{s.t. } & dW(s) = (rW(s) - c(s) + z(s)(p(s)h + (1 - p(s))l - r))\,ds + \sigma z(s)d\overline{w}(s) \\
& dp(s) = \tfrac{h-l}{\sigma}p(s)(1 - p(s))d\overline{w}(s) \\
& W(s) > -\underline{K},\ W(t) > 0,\ p(t) > 0 \text{ given.}
\end{aligned}
\qquad (P)
$$

Cuoco [4] provides technical restrictions on the stochastic processes r, c, x, z and σ to ensure existence of a solution. These conditions can be easily adapted to the infinite horizon case. In addition, due to the infinite time horizon, we need to impose a Non-Ponzi game or transversality condition.

Transversality Condition. The transversality condition for this problem can be written:

$$\lim_{T \to \infty} E_t^P \left[e^{-\theta(T+t)} J(W(t+T), p(t+T)) \right] = 0$$

This condition is satisfied when

$$\min\left(\theta + (b - 1)(r + \frac{(l - r)^2}{2b\sigma^2}), \theta + (b - 1)(r + \frac{(h - r)^2}{2b\sigma^2})\right) > 0.$$

To see this note that

$$J(W(t), p(t)) \leq p(t)J(W(t), 1) + (1 - p(t))J(W(t), 0),$$

and recall that

$$\theta + (b - 1)(r + \frac{(\mu - r)^2}{2b\sigma^2}) > 0$$

is the transversality condition for Merton's problem [16] when the mean return is known to be equal to μ.

We now use the Radon-Nikodym derivative of P with respect to P_l to rewrite program P into an equivalent program P' involving variable ϕ instead on p. Since

$$E_t^P \left[\int_t^{\infty} u(c(s))e^{-\theta(s-t)}ds \right] = \frac{1}{\xi_{p,l}(t)} E_t^l \left[\int_t^{\infty} \xi_{p,l}(s)u(c(s))e^{-\theta(s-t)}ds \right],$$

program P is equivalent to

$$\max_{(c \in \mathcal{C},\ (x,z) \in Q)} E_t^l \left[\int_t^{\infty} (1 + \phi(s))\, u(c(s))e^{-\theta(s-t)}ds \right]$$

$$\text{s.t. } dW(s) = (rW(s) - c(s) + z(s)(l - r))\,ds + \sigma z(s)dw(s)$$
$$d\phi(s) = \tfrac{h-l}{\sigma}\phi(s)dw(s) \tag{P'}$$
$$W(s) > -\underline{K},\ W(t) > 0,\ \phi(t) > 0 \text{ given.}$$

For program P' the evolution of wealth has been simplified since the drift of the price process S is now equal to lS. Also worth noticing is that program P' is identical to the usual Merton's problem [16] under complete information for an investor whose utility function is

$$v(\phi, c, t) = (1 + \phi)u(c, t).$$

This new utility function is state dependent and inherits all the smoothness and concavity properties of the usual utility function u with respect to the consumption argument c. Finally, note that this methodology can be applied to more general set ups.

3. OPTIMAL CONSUMPTION AND PORTFOLIO ALLOCATIONS

In this section, we derive the optimal consumption and portfolio allocations for Bernoulli beliefs and analyze how they respond to changes in optimism and quality of the information received. At the end of the section, we briefly investigate the case of normally distributed beliefs.

Program P' can be solved using standard martingale techniques as in the complete information case. Define the state price density

$$\pi_l(t) = \exp\left(-\int_0^t (r + \frac{1}{2}\kappa_l^2)ds + \int_0^t \kappa_l dw(s)\right),$$

where

$$\kappa_l = -\frac{l - r}{\sigma}.$$

It follows that

$$d\pi_l(t) = \pi_l(t)\left(-rdt + \kappa_l dw(t)\right). \tag{3.1}$$

As presented in Cuoco [4], the individual program is equivalent to

$$\max_{(c \in \mathcal{C},\ (x,z) \in Q)} E_t^l \left[\int_t^\infty (1 + \phi(s))u(c(s))e^{-\theta(s-t)}ds\right]$$

$$\text{s.t. } E_t^l \left[\int_t^\infty \pi_l(s)c(s)ds\right] = \pi_l(t)W(t)$$
$$d\phi(s) = \tfrac{h-l}{\sigma}\phi(s)dw(s) \tag{P}$$
$$W(s) > -\underline{K},\ W(t) > 0,\ \phi(t) > 0 \text{ given.}$$

The optimal condition is

$$(1 + \phi(s))\,u'(c(s))e^{-\theta s} = \lambda\pi_l(s), \tag{3.2}$$

with $\lambda > 0$ being the associated Lagrange multiplier. It follows that

$$c(s) = I\left(\frac{\lambda\pi_l(s)}{1 + \phi(s)}, s\right),$$

where I is the inverse of the marginal utility function and

$$W(t) = \frac{1}{\pi_l(t)} E_t^l \left[\int_t^\infty \pi_l(s) I(\frac{\lambda \pi_l(s)}{1 + \phi(s)}, s) \right].$$

The Lagrange multiplier λ is determined using the investor budget constraint, i.e. the previous relationship at the initial date $t = 0$. We now focus our analysis on the case of a CRRA preference agent

$$
\begin{aligned}
u(c) &= \frac{c^{1-b} - 1}{1 - b}, \ b \neq 1 \\
&= \ln c, \ b = 1.
\end{aligned}
$$

We start by recalling the main findings of Merton's model [16] under complete information for such an investor.

3.1. Benchmark Case: Merton's Problem

Within our financial market framework, the Merton's problem [16] for a CRRA investor is

$$\max_{(c \in \mathcal{C}, \ (x,z) \in Q)} E_t^\mu \left[\int_t^\infty \frac{c(s)^{1-b} - 1}{1 - b} e^{-\theta s} ds \right]$$

$$
\begin{aligned}
\text{s.t. } & dW(s) = (rW(s) - c(s) + z(s)(\mu - r))ds + \sigma z(s)dw(s) \\
& W(s) > -\underline{K}, \ W(t) > 0 \text{ given.}
\end{aligned}
\qquad (P)
$$

The optimal solution of this problem is given by

$$
\begin{aligned}
c(t) &= \lambda^{-\frac{1}{b}} \pi_\mu^{-\frac{1}{b}}(t) e^{-\frac{\theta}{b} t} \\
\frac{c(t)}{W(t)} &= \frac{\theta}{b} + \frac{b-1}{b} \left(r + \frac{(\mu - r)^2}{2b\sigma^2} \right) \\
\frac{z(t)}{W(t)} &= \frac{\mu - r}{b\sigma^2},
\end{aligned}
$$

where π_μ is the state price density defined as before and the Lagrange multiplier λ is given by

$$\lambda = \left(\frac{\theta}{b} + \frac{b-1}{b}(r + \frac{(\mu - r)^2}{2b\sigma^2}) \right)^{-b} W_0^{-b}.$$

Both the fraction of wealth invested into the risky asset and the consumption-wealth ratio are constant. As shown in the sequel, incomplete information alters this result.

3.2. Incomplete Information with Bernoulli beliefs

An explicit expression for the optimal condition (3.2) is

$$c(s) = \lambda^{-\frac{1}{b}}(1 + \phi(s))^{\frac{1}{b}}\pi_l^{-\frac{1}{b}}(s)e^{-\frac{\theta}{b}s}.$$

Hence, at time t, the investor's wealth is

$$W(t) = \frac{\lambda^{-\frac{1}{b}}}{\pi_l(t)}E_t^l\left[\int_t^\infty \pi_l^{\frac{b-1}{b}}(s)(1 + \phi(s))^{\frac{1}{b}}e^{-\frac{\theta}{b}s}ds\right].$$

Let denote $\rho = \frac{\theta}{b} + \frac{b-1}{b}(r + \frac{\kappa^2}{2b}) > 0$ and α and β be the positive and negative roots respectively of the quadratic

$$\frac{1}{2}\left(\frac{h-l}{\sigma}\right)^2 x^2 + \left(\frac{1-b}{b}\frac{(l-r)(h-l)}{\sigma^2} - \frac{1}{2}\left(\frac{h-l}{\sigma}\right)^2\right) x = \rho.$$

We have the following proposition.

Proposition 1. *The wealth process is given by*

$$W(t) = \frac{\lambda^{-\frac{1}{b}}\pi_l^{-\frac{1}{b}}(t)e^{-\frac{\theta}{b}t}}{\frac{1}{2}\left(\frac{h-l}{\sigma}\right)^2(\alpha-\beta)}\left[\int_0^\infty \left((1 + \phi(t)e^{-u})^{\frac{1}{b}}e^{\beta u} + (1 + \phi(t)e^u)^{\frac{1}{b}}e^{-\alpha u}\right) du\right],$$

and the fraction of wealth invested into the risky asset $\frac{z}{W}$ is given by

$$\frac{z(t)}{W(t)} = \frac{h-r}{b\sigma^2} - \frac{h-l}{b\sigma^2}\frac{\int_0^\infty \left((1 + \phi(t)e^{-u})^{\frac{1}{b}-1}e^{\beta u} + (1 + \phi(t)e^u)^{\frac{1}{b}-1}e^{-\alpha u}\right) du}{\int_0^\infty \left((1 + \phi(t)e^{-u})^{\frac{1}{b}}e^{\beta u} + (1 + \phi(t)e^u)^{\frac{1}{b}}e^{-\alpha u}\right) du}.$$

It is increasing with the degree of optimism ϕ and is always between the fraction of wealth invested into the risky asset when $\mu = l$ and $\mu = h$. Finally, the consumption-wealth ratio $\frac{c}{W}$ is given by

$$\frac{c(t)}{W(t)} = \frac{(1 + \phi(t))^{\frac{1}{b}}}{\frac{1}{\frac{1}{2}\left(\frac{h-l}{\sigma}\right)^2(\alpha-\beta)}\left[\int_0^\infty(1 + \phi(t)e^{-u})^{\frac{1}{b}}e^{\beta u} + (1 + \phi(t)e^u)^{\frac{1}{b}}e^{-\alpha u}du\right]}.$$

It is increasing (decreasing) in optimism ϕ exactly when $b > 1$ ($b < 1$).

Proof. See appendix A. ∎

As the agent becomes more optimistic above the value of the mean return, when $b < 1$ or equivalently when the elasticity of substitution $s = \frac{1}{b} > 1$, she increases her consumption-wealth ratio, reflecting consumption smoothing. We now examine in more detail the demand for the risky asset that can be rewritten (dropping the time index)

$$\frac{z}{W} = \frac{\frac{\phi h}{1+\phi} + \frac{l}{1+\phi} - r}{b\sigma^2} + \frac{h-l}{b\sigma^2}\left[\frac{1}{1+\phi} - \frac{\int_0^\infty\left((1 + \phi e^{-u})^{\frac{1}{b}-1}e^{\beta u} + (1 + \phi e^u)^{\frac{1}{b}-1}e^{-\alpha u}\right) du}{\int_0^\infty\left((1 + \phi e^{-u})^{\frac{1}{b}}e^{\beta u} + (1 + \phi e^u)^{\frac{1}{b}}e^{-\alpha u}\right) du}\right].$$

The first term is the myopic demand; it is increasing in optimism ϕ. The second term is the hedging demand. It is always equal to zero for a myopic investor ($b = 1$) and equal to zero when the investor knows the truth, i.e. when $p = 0$ or 1 or equivalently when ϕ equals 0 or ∞. Due to changes in investment opportunities, the hedging demand is no longer zero and it has the following property.

Proposition 2. *The hedging demand for the risky asset is positive (negative) exactly when the coefficient of risk aversion is below (above) unity.*

Proof. See appendix A. ■

Honda [12] conjectures this result and illustrates it using numerical simulations. However, he finds that for large enough values of the CRRA coefficient, the hedging demand can be positive as he assumes that the non-observable mean return can switch between l and h. We can reinterpret proposition 2 by recalling that the hedging demand aims at preparing and forearming the investor in the face of uncertainty. To achieve this, the investor has two assets at her disposal. When she does not mind too much altering her consumption plans, i.e. $s = \frac{1}{b} > 1$, she is more willing to hold the risky asset and the hedging demand is positive. The opposite occurs when $s < 1$.

For the sake of simplicity, we have considered an infinite horizon model. In the next section, we investigate the effect of the time horizon on the optimal consumption and portfolio strategies.

3.2.1. Finite Horizon

We assume that there is no terminal bequest. As shown in appendix A, within a finite horizon $T > 0$ the expressions of the consumption-wealth and the fraction of wealth invested into the risky asset are given by

$$
\frac{z(t)}{W(t)} = \frac{\frac{\phi(t)h}{1+\phi(t)} + \frac{l}{1+\phi(t)} - r}{b\sigma^2} + \frac{h-l}{b\sigma^2} \left[\frac{1}{1+\phi(t)} - \frac{E_t^l \left[\int_t^T (1 + X(s))^{\frac{1}{b}-1} e^{-\rho(s-t)} ds \right]}{E_t^l \left[\int_t^T (1 + X(s))^{\frac{1}{b}} e^{-\rho(s-t)} ds \right]} \right]
$$

$$
\frac{c(t)}{W(t)} = \frac{(1 + \phi(t))^{\frac{1}{b}}}{E_0^l \left[\int_t^T (1 + X(s))^{\frac{1}{b}} e^{-\rho(s-t)} ds \right]},
$$

where

$$
dX(s) = X(s) \left(\frac{1-b}{b} \frac{(l-r)(h-l)}{\sigma^2} ds + \frac{h-l}{\sigma} dw(s) \right)
$$
$$
X(t) = \phi(t).
$$

All the results of propositions 1 and 2 remain valid. To see this, one can refer to appendix A and realize that *all* the results were obtained using representations of the consumption-wealth ratio and demand for the risky asset involving *conditional expectations*. The same

proofs hold in the case of a finite horizon substituting $\int_t^\infty$ with $\int_t^T$. Next, we explore the impact of the investor horizon on the optimal allocations.

Proposition 3. *The demand for the risky asset increases (decreases) with the horizon time exactly when the coefficient of risk aversion is below (above) unity.*

Proof. See appendix A. ∎

In Merton's problem [16], the optimal portfolio strategies are independent of time. Portfolio managers often advise long run investors to allocate more aggressively in stocks. As pointed out and confirmed through numerical simulations in Barberis [1], when investors strive to learn about parameter uncertainty, it may be optimal to be more conservative and allocate less to equity at longer horizons. Brennan [3] obtains a similar result using numerical simulations. Proposition 3 clarifies the issue and shows the above argument of investment advisors is founded for agents willing to substitute consumption overtime or more precisely for the ones whose intertemporal elasticity of substitution is above unity. Finally, note that the hedging demand that has the same sign as $s - 1$ rises in *absolute terms* with the investor horizon or equivalently the deviation from the myopic demand is magnified by the investor time horizon.

3.3. Outside Market Information

In this section, the agent has a free access to an additional signal outside the market place, such as, for instance, some business and macroeconomic news, political news, release of corporations' earning reports, policymakers' statements. The aim is to study how optimal consumption and portfolio allocations respond to the quality or precision of the information received. Uncertainty is now modeled by a probability space $(\Omega, \mathcal{F}, P^w)$ on which is defined a *two* dimensional (standard) Brownian motion $w = (w_1, w_2)$ where w_1 and w_2 are independent. The investor observes the price of the risky security S and an additional signal A whose dynamics are given by

$$dS(t) = S(t) \left(\mu dt + \sigma dw_1(t) \right),$$

with $\mu \in \{l, h\}$ and

$$dA(t) = A(t) \left(\lambda dt + \Sigma_1 \sqrt{\frac{a}{1+a}} dw_1(t) + \Sigma_2 \sqrt{\frac{1}{1+a}} dw_2(t) \right),$$

where $a > 0$, $\Sigma_1 > 0$, $\Sigma_2 > 0$ and λ are known parameters and $(dw_1(t), dw_2(t))$ are the increments of two independent standard Wiener processes under P^w. The instantaneous covariance between the security price and the signal is

$$cov_t(dS(t), dA(t)) = \sigma \Sigma_1 \sqrt{\frac{a}{1+a}} S(t) A(t) dt.$$

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The higher a, the higher the correlation between the signal A and the price S, thus the more informative the signal is.

Let $\mathcal{F}_t$ be the σ-algebra generated by the observations of the values of the price S of the risky security and the signal A, $\{S(s), A(s); 0 \leq s \leq t)\}$ and augmented. At time t, the investor's information set is $\mathcal{F}_t$. The filtration $\mathbb{F} = \{\mathcal{F}_t, \ t \in \mathbb{R}_+\}$ satisfies the same conditions as before. We first solve the optimal filtering problem.

Proposition 4. *Given the observations of the stock price and the additional signal, the law of motion of the posterior beliefs P is given by*

$$dp(t) = \frac{h-l}{\sigma}(1-p(t))p(t)\left(d\overline{w}_1(t) - \sqrt{a}\frac{\Sigma_1}{\Sigma_2}d\overline{w}_2(t)\right),$$

where

$$
\begin{aligned}
d\overline{w}_1(t) &= dw_1(t) + \frac{1}{\sigma}\left(\mu - (p(t)h + (1-p(t))l)\right)dt \\
d\overline{w}_2(t) &= dw_2(t) - \sqrt{a}\frac{\Sigma_1}{\Sigma_2\sigma}\left(\mu - (p(t)h + (1-p(t))l)\right)dt,
\end{aligned}
$$

are the increments of two independent (standard) Wiener processes under P, relative to the filtration $\mathbb{F}$.

Proof. See appendix B. ∎

The evolution of beliefs is similar to the one previously obtained. It is worth noticing that the (instantaneous) variance of changes in beliefs is

$$\left(\frac{h-l}{\sigma}\right)^2 (1-p(t))^2 p^2(t)(1 + a\frac{\Sigma_1^2}{\Sigma_2^2}),$$

so the more informative signal A, the greater changes in beliefs and the higher the instantaneous variance Σ_2^2, the noisier signal A is, so the smaller changes in beliefs are. Then under the investor beliefs P, the security price, beliefs and the signal evolve according to the following laws of motion

$$
\begin{aligned}
dS(t) &= S(t)\left((p(t)h + (1-p(t))l)dt + \sigma d\overline{w}_1(t)\right) \\
dp(t) &= \frac{h-l}{\sigma}(1-p(t))p(t)\left(d\overline{w}_1(t) - \sqrt{a}\frac{\Sigma_1}{\Sigma_2}d\overline{w}_2(t)\right) \\
dA(t) &= A(t)\left(\lambda dt + \Sigma_1\sqrt{\frac{a}{1+a}}d\overline{w}_1(t) + \Sigma_2\sqrt{\frac{1}{1+a}}d\overline{w}_2(t)\right).
\end{aligned}
$$

As before, we choose to express conditional expectations under the probability measure P_l. We still have $\xi_{p,l}(t) = \frac{1-p_0}{1-p(t)}$. We define $\phi = \frac{p}{1-p}$ and using Ito lemma under probability measure P_l we obtain

$$d\phi(t) = \frac{h-l}{\sigma}\phi(t)\left(dw_1(t) - \sqrt{a}\frac{\Sigma_1}{\Sigma_2}dw_2(t)\right).$$

The state price density is defined as before

$$\pi_l(t) = \exp\left(-\int_0^t (r + \frac{1}{2}\kappa_l^2)ds + \int_0^t \kappa_l dw_1(s)\right),$$

where

$$\kappa_l = -\frac{l-r}{\sigma}.$$

The agent's program is exactly the same as before. In order to investigate how an additional source of information affects the optimal allocations, we look at instantaneous correlations[2] between the two state variables π_l and ϕ. We have

$$
\begin{aligned}
d\left[\pi_l, \pi_l\right](t) &= \pi_l^2(t)\kappa_l^2 dt \\
d\left[\phi, \phi\right](t) &= \left(\frac{h-l}{\sigma}\right)^2 \left(1 + \frac{a\Sigma_1^2}{\Sigma_2^2}\right)\phi^2(t)dt \\
d\left[\phi, \pi_l\right](t) &= \kappa_l\frac{h-l}{\sigma}\pi_l(t)\phi(t)dt.
\end{aligned}
$$

The only difference with the previous case is the instantaneous variance of the process ϕ. As explained in appendix B., we need to replace $\left(\frac{h-l}{\sigma}\right)^2$ by $\left(\frac{h-l}{\sigma}\right)^2\left(1 + \frac{a\Sigma_1^2}{\Sigma_2^2}\right)$ and thus define α and β respectively as the positive and negative roots of the new quadratic

$$\frac{1}{2}\left(\frac{h-l}{\sigma}\right)^2\left(1 + \frac{a\Sigma_1^2}{\Sigma_2^2}\right)x^2 + \left(\frac{1-b}{b}\frac{(l-r)(h-l)}{\sigma^2} - \frac{1}{2}\left(\frac{h-l}{\sigma}\right)^2\left(1 + \frac{a\Sigma_1^2}{\Sigma_2^2}\right)\right)x = \rho.$$

All the previous results derived in absence of an additional signal remain valid. However, the introduction of outside market information allows us to investigate how optimal consumption and portfolio allocations respond to a change in the informativeness of the signal. Such an analysis is impossible within the previous framework as parameters ($h - l$ and σ) determining the precision of the signal have multiple effects and one cannot properly disentangle their specific effect on the quality of the information received. We address this issue in the next proposition.

Proposition 5. *An increase in the informativeness of the outside market signal raises (lowers) the fraction of wealth invested into the risky asset $\frac{z}{W}$ when $b < 1$ ($b > 1$) and raises (lowers) the consumption-wealth ratio $\frac{c}{W}$ when $b > 1$ ($b < 1$).*

Proof. See appendix B. ∎

Proposition 5 can be reinterpreted as follows. When the intertemporal elasticity of substitution s is above unity, an investor with a more accurate information invests a higher

---

[2]If X and Y are Ito processes with diffusion vectors σ_X and σ_Y, their quadratic covariation is defined by
$[X, Y](t) = \int_0^t \sigma_X(s)^\intercal \sigma_Y(s)ds.$

fraction of her wealth into the risky asset and consumes a higher fraction of her wealth with respect to an identical but less informed investor. The reason is that the agent understands that when she receives a more informative signal, she can update her beliefs more quickly which in turn can lead to larger changes in her optimal consumption and portfolio allocations. She devotes more of her wealth into the risky asset only if she is willing to tolerate changes in her consumption pattern. Finally, notice that only the hedging demand depends on the quality of the information received by the investor. These results are consistent with those obtained numerically by Brennan [3] when the investor has normally distributed beliefs. Again, the same results apply to a finite horizon model.

In most of the existing literature - see for instance Brennan [3], Feldman [9], Lakner [14], authors assume that the investor has normally distributed beliefs. In the next section, we provide a closed form solution to the Merton's problem under this assumption.

3.4. Incomplete Information with Normally Distributed Beliefs

In this section, the horizon time T is finite and the investor has normally distributed beliefs about the non-observable mean return μ. As before, $\mathcal{F}_t$ is σ-algebra generated by the observations of the price S, $\{S(s); 0 \leq s \leq t)\}$ and augmented. Being at time t, a sufficient statistics vector for the beliefs is the conditional mean $m(t) = E^P[\mu \mid \mathcal{F}_t]$ and the conditional variance $\gamma(t) = E^P\left[(\mu - m(t))^2 \mid \mathcal{F}_t\right]$. Using Bayes' rules, the evolution across time of the posterior beliefs P is given by the following lemma.

Lemma 2. *The law of motion of the posterior beliefs P is*

$$
\begin{aligned}
dm(s) &= \frac{\gamma(s)}{\sigma} d\overline{w}(s) \\
\dot{\gamma}(s) &= -\frac{\gamma^2(s)}{\sigma^2},
\end{aligned}
$$

where

$$
\begin{aligned}
d\overline{w}(s) &= \frac{1}{\sigma S(s)}\left(dS(s) - E^P\left[dS(s) \mid \mathcal{F}_s\right]\right) \\
&= dw(s) + \frac{1}{\sigma}\left(\mu - m(s)\right) ds,
\end{aligned}
$$

is the increment of the (standard) Wiener process under P, relative to the filtration $\mathbb{F}$.

Proof. See Liptser and Shiryaev, [15] p 317. ∎

The variance γ is a measure of the precision of the knowledge about μ: it is deterministic, decreasing over time as knowledge about the true value of μ improves. Changes in γ are negatively related with the variance of the stock σ^2 which as mentioned before negatively affects the quality of information received. Changes in the mean m are increasing in γ (when γ is high, a lot remains to be learned so learning takes place at a faster speed) and

decreasing with σ. In addition, m is a martingale under P relative to $\mathbb{F}$ so on average, the investor's beliefs do not change as far as the mean is concerned but the accuracy of her beliefs improves across time. Finally, that under the probability measure P, the mean process m is given by

$$m(s) = m(t) + \int_t^s \frac{\gamma(u)}{\sigma} d\overline{w}(u).$$

So under P, given $\mathcal{F}_t$, $m(s)$ is normally distributed with mean $m(t)$ and variance $\int_t^s \frac{\gamma^2(u)}{\sigma^2} du = \gamma(t) - \gamma(s)$.

In order to have existence of the investor problem, we make the following technical assumption.

Assumption A2. $1 + \frac{b-1}{b} \frac{\gamma_0}{\sigma^2} T > 0$.

The condition is trivially satisfied for $b > 1$ and when $b \in [0,1]$, we must have

$$\gamma_0 < \frac{b\sigma^2}{(1-b)T},$$

which means that initial beliefs need to be precise enough. Note that this condition is similar to the one provided by Lakner [14] in proposition 4.6.

Contrarily to the Bernoulli belief case, it is here more convenient to work under the investor probability measure P. Then define the state price density

$$\pi_m(t) = \exp\left(-\int_0^t (r + \frac{1}{2}\kappa_m(s)^2) ds + \int_0^t \kappa_m dw(s) \right), \tag{3.3}$$

where

$$\kappa_m(t) = -\frac{m(t) - r}{\sigma}.$$

The investor's problem is

$$\max_{(c \in \mathcal{C}, \ (x,z) \in Q)} E_t^P \left[\int_t^T \frac{c(s)^{1-b} - 1}{1-b} e^{-\theta s} ds \right]$$

$$\text{s.t. } dW(s) = (rW(s) - c(s) + z(s)(m(s) - r)) ds + \sigma z(s) d\overline{w}(s)$$
$$dm(s) = \frac{\gamma(s)}{\sigma} d\overline{w}(s), \ \dot{\gamma}(s) = -\frac{\gamma^2(s)}{\sigma^2} \tag{3.4}$$
$$W(s) > -\underline{K}, \ W(t) > 0$$
$$m(t), \ \gamma(t) > 0 \text{ given.}$$

The optimal condition (3.2) becomes

$$c(s) = \lambda^{-\frac{1}{b}} \pi_m^{-\frac{1}{b}}(s) e^{-\frac{\theta}{b} s}.$$

Hence, at time t, the wealth process is given by

$$W(t) = \frac{\lambda^{-\frac{1}{b}}}{\pi_m(t)} E_t^P \left[\int_t^T \pi_m^{\frac{b-1}{b}}(s) e^{-\frac{\theta}{b} s} ds \right].$$

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The next step is to express the state price density as a function of state variables (m, γ, t) to exploit the fact that under P the mean return is normally distributed.

Lemma 3. *The state price density π_m can be written as a function of (m, γ, t) given by*

$$\pi_m(s) = \pi_m(t)\sqrt{\frac{\gamma(t)}{\gamma(s)}} \exp\left(-r(s-t) - \frac{1}{2\gamma(s)}(m(s)-r)^2 + \frac{1}{2\gamma(t)}(m(t)-r)^2\right).$$

Proof. See appendix C. ∎

In appendix C, using the previous expression of the state price density, we derive the optimal consumption and portfolio allocations. Results are gathered in the following proposition.

Proposition 6. *The wealth process is given by*

$$W(t) = \lambda^{-\frac{1}{b}}\pi_m^{-\frac{1}{b}}(t)e^{-\frac{\theta}{b}t}\int_0^{T-t} \frac{\left(1+\frac{\gamma(t)}{\sigma^2}u\right)^{\frac{b-1}{2b}}}{\sqrt{1+\frac{b-1}{b}\frac{\gamma(t)}{\sigma^2}u}}e^{-\frac{1}{b}(\theta+(b-1)(r+\frac{(m(t)-r)^2}{2b\sigma^2}\frac{1}{1+\frac{b-1}{b}\frac{\gamma(t)}{\sigma^2}u}))u}\,du,$$

the fraction of wealth invested into the risky asset $\frac{z}{W}$ is given by

$$\frac{z(t)}{W(t)} = \frac{m(t)-r}{b\sigma^2}\frac{\int_0^{T-t} \frac{\left(1+\frac{\gamma(t)}{\sigma^2}u\right)^{\frac{b-1}{2b}}}{\left(1+\frac{b-1}{b}\frac{\gamma(t)u}{\sigma^2}\right)^{\frac{3}{2}}}e^{-\frac{1}{b}(\theta+(b-1)(r+\frac{(m(t)-r)^2}{2b\sigma^2}\frac{1}{1+\frac{b-1}{b}\frac{\gamma(t)}{\sigma^2}u}))u}\,du}{\int_0^{T-t} \frac{\left(1+\frac{\gamma(t)}{\sigma^2}u\right)^{\frac{b-1}{2b}}}{\sqrt{1+\frac{b-1}{b}\frac{\gamma(t)}{\sigma^2}u}}e^{-\frac{1}{b}(\theta+(b-1)(r+\frac{(m(t)-r)^2}{2b\sigma^2}\frac{1}{1+\frac{b-1}{b}\frac{\gamma(t)}{\sigma^2}u}))u}\,du} \tag{3.5}$$

and the ratio consumption-wealth $\frac{c}{W}$ is given by

$$\frac{c(t)}{W(t)} = \frac{1}{\int_0^{T-t} \frac{\left(1+\frac{\gamma(t)}{\sigma^2}u\right)^{\frac{b-1}{2b}}}{\sqrt{1+\frac{b-1}{b}\frac{\gamma(t)}{\sigma^2}u}}e^{-\frac{1}{b}(\theta+(b-1)(r+\frac{(m(t)-r)^2}{2b\sigma^2}\frac{1}{1+\frac{b-1}{b}\frac{\gamma(t)}{\sigma^2}u}))u}\,du}. \tag{3.6}$$

Proof. See appendix C. ∎

Not surprisingly, we observe that the investor is willing to hold a long position in the risky asset as long as its expected return $m(t)$ exceeds the risk free rate r. Relationship (3.5) indicates that the demand for the risky asset is above (below) the myopic demand $\frac{m(t)-r}{b\sigma^2}$ exactly $b < 1$ $(b > 1)$ so as before, the hedging demand is positive[3] (negative) when the CRRA coefficient is below unity (above unity) provided that $m(t) > r$. As in the Bernoulli

---
[3] Brennan [3] also obtains this result using a dynamic programming approach and the non-satiation of the indirect utility function..

beliefs case, we show in appendix C that the demand for the risky asset is increasing in the mean return m and the more precise the beliefs are (the smaller the variance γ is), the higher (lower) the ratio $\left|\frac{z}{W}\right|$ is when $b > 1$ ($b < 1$). The effects of the investor horizon on the demand for the risky asset are the same as before, with the same cut off value of 1 for the I.E.S.. The consumption-wealth ratio is increasing (decreasing) in m exactly when $b > 1$ ($b < 1$). We also prove in appendix C that the ratio consumption over wealth is increasing in the precision of the beliefs, i.e. the lower the variance γ the higher $\frac{c}{W}$ when $b < 1$. When the investor's CRRA coefficient is above unity, results are more ambiguous. On the one hand, when $|m(t) - r|$ is small enough, then the lower γ, the lower the ratio $\frac{c}{W}$. On the other hand, when $|m(t) - r|$ is large enough, the opposite occurs.

Broadly speaking, the results obtained under normally distributed beliefs corroborate the findings in the Bernoulli framework.

4. CONCLUSION

We study the optimal consumption / portfolio allocations problem under incomplete information about the mean return of the risky asset. Our approach is to convert the investor's problem into an equivalent program for which standard martingale techniques can be easily implemented. Dynamic learning induces decisions that can be significantly different from the myopic behavior ones. The quality of the information received affects the speed at which the investor can revise her beliefs: the more accurate the information, the more drastic portfolio rebalances can be. The paper highlights the role of consumption smoothing in the optimal portfolio strategies with respect to the investor horizon, optimism and outside market information when the agent strives to learn about her investment opportunities. In particular, the conventional advice according to which long horizon investors should allocate aggressively their wealth to equity is founded only for agents whose intertemporal elasticity of substitution is above unity. The role of outside market information has been examined when its access is free and its amount is fixed. One possible extension would be to endogenize its acquisition and let the investor choose how much she wants to be informed. This is left for future research.

5. APPENDIX

APPENDIX A

Wealth process. Define $H(\pi_l, \phi) = E_t^l \left[\int_t^\infty \pi_l^{\frac{b-1}{b}}(s)(1+\phi(s))^{\frac{1}{b}} e^{-\frac{\theta}{b}(s-t)} ds \right]$. Then H satisfies the following PDE

$$\frac{\theta}{b} H = \pi_l^{\frac{b-1}{b}}(1+\phi)^{\frac{1}{b}} - r\pi_l H_1 + \frac{\kappa_l^2}{2}\pi_l^2 H_{11} + \kappa_l \frac{h-l}{\sigma}\pi_l \phi H_{12} \qquad (5.1)$$

$$+ \frac{1}{2}\left(\frac{h-l}{\sigma}\right)^2 \phi^2 H_{22}.$$

Due to the homogeneity of degree $\frac{b-1}{b}$ in π_l, we look for a solution of the type $H(\pi_l, \phi) = G(\phi)\pi_l^{\frac{b-1}{b}}$. It follows that G must satisfy the following ODE

$$\left(\frac{\theta}{b} - \frac{1-b}{b}(r + \frac{\kappa_l^2}{2b})\right) G = (1+\phi)^{\frac{1}{b}} + \frac{b-1}{b}\kappa_l \frac{h-l}{\sigma}\phi G'$$

$$+ \frac{1}{2}\left(\frac{h-l}{\sigma}\right)^2 \phi^2 G'',$$

with initial conditions

$$G(0) = \frac{1}{\frac{\theta}{b} - \frac{1-b}{b}(r + \frac{\kappa_l^2}{2b})}$$

$$G'(0) = \frac{1}{\theta - (1-b)(r - \kappa_l \frac{h-l}{\sigma} + \frac{\kappa_l^2}{2b})}.$$

The solution of this ODE is given by the Feyman-Kac representation (see Duffie [7], appendix E.)

$$G(\phi) = E_t^l \left[\int_t^\infty (1 + X(s))^{\frac{1}{b}} e^{-\rho(s-t)} ds \right],$$

where $\rho = \frac{\theta}{b} - \frac{1-b}{b}(r + \frac{\kappa_l^2}{2b}) > 0$, $X(t) = \phi$ and the law of motion of the process X is

$$dX(s) = X(s)\left(\eta ds + \Sigma dw(s)\right), \qquad (5.2)$$

with

$$\eta = \frac{b-1}{b}\kappa_l(h-l)$$

$$\Sigma = \frac{h-l}{\sigma}.$$

As presented in Harrison [11] p 45, let α and β be the positive and negative roots respectively of the quadratic

$$\frac{1}{2}\left(\frac{h-l}{\sigma}\right)^2 x^2 + \left(\frac{b-1}{b}\kappa_l \frac{h-l}{\sigma} - \frac{1}{2}\left(\frac{h-l}{\sigma}\right)^2\right) x = \rho.$$

Then

$$G(\phi) = \frac{1}{\frac{1}{2}\left(\frac{h-l}{\sigma}\right)^2(\alpha-\beta)}\left[\int_{-\infty}^0 (1+\phi e^u)^{\frac{1}{b}}e^{-\beta u}du + \int_0^\infty (1+\phi e^u)^{\frac{1}{b}}e^{-\alpha u}du\right]$$

$$= \frac{1}{\frac{1}{2}\left(\frac{h-l}{\sigma}\right)^2(\alpha-\beta)}\left[\int_0^\infty \left((1+\phi e^{-u})^{\frac{1}{b}}e^{\beta u} + (1+\phi e^u)^{\frac{1}{b}}e^{-\alpha u}\right)du\right].$$

Note that G is well defined as the condition $\theta - (1-b)(r - \kappa_l \frac{h-l}{\sigma} + \frac{\kappa_l^2}{2b}) > 0$ implies that $\alpha > 1$ and the transversality condition implies that $\frac{1}{b} < \alpha$. Finally,

$$W(t) = \frac{\lambda^{\frac{1}{b}}\pi_l^{-\frac{1}{b}}(t)e^{-\frac{\theta}{b}t}}{\frac{1}{2}\left(\frac{h-l}{\sigma}\right)^2(\alpha-\beta)}\left[\int_0^\infty (1+\phi(t)e^{-u})^{\frac{1}{b}}e^{\beta u} + (1+\phi(t)e^u)^{\frac{1}{b}}e^{-\alpha u}du\right].$$

Portfolio allocations. Applying Ito lemma, we obtain

$$dW(t) = \mu_W(t)dt - \frac{\kappa_l}{b}W(t)dw(t)$$

$$+ \frac{\lambda^{-\frac{1}{b}}\pi_l^{-\frac{1}{b}}(t)\frac{1}{b}\left(\frac{h-l}{\sigma}\right)\phi(t)}{\frac{1}{2}\left(\frac{h-l}{\sigma}\right)^2(\alpha-\beta)}\left[\int_0^\infty (1+\phi(t)e^{-u})^{\frac{1}{b}-1}e^{(\beta-1)u} + (1+\phi(t)e^u)^{\frac{1}{b}-1}e^{(1-\alpha)u}du\right]dw(t),$$

for some process μ_W. Identifying coefficients with the wealth dynamics in program P' yields

$$\frac{z}{W} = \frac{l-r}{b\sigma^2} + \frac{(h-l)\phi}{b\sigma^2}\frac{\int_0^\infty\left((1+\phi e^{-u})^{\frac{1}{b}-1}e^{(\beta-1)u} + (1+\phi e^u)^{\frac{1}{b}-1}e^{(1-\alpha)u}\right)du}{\int_0^\infty\left((1+\phi e^{-u})^{\frac{1}{b}}e^{\beta u} + (1+\phi e^u)^{\frac{1}{b}}e^{-\alpha u}\right)du}$$

$$= \frac{h-r}{b\sigma^2} - \frac{h-l}{b\sigma^2}\frac{\int_0^\infty\left((1+\phi e^{-u})^{\frac{1}{b}-1}e^{\beta u} + (1+\phi e^u)^{\frac{1}{b}-1}e^{-\alpha u}\right)du}{\int_0^\infty\left((1+\phi e^{-u})^{\frac{1}{b}}e^{\beta u} + (1+\phi e^u)^{\frac{1}{b}}e^{-\alpha u}\right)du}$$

$$= \frac{h-r}{b\sigma^2} - \frac{h-l}{b\sigma^2}\frac{E_t^l\left[\int_t^\infty(1+X(s))^{\frac{1}{b}-1}e^{-\rho(s-t)}ds\right]}{E_t^l\left[\int_t^\infty(1+X(s))^{\frac{1}{b}}e^{-\rho(s-t)}ds\right]}.$$

The desired result follows easily, and in particular, we have $\frac{z}{W} \in \left[\frac{l-r}{b\sigma^2}, \frac{h-r}{b\sigma^2}\right]$.

Risky asset demand and optimism. Writing $\phi_t = \phi$ and $\frac{z(t)}{W(t)} = \frac{z}{W}$

$$\frac{\partial\left(\frac{z}{W}\right)}{\partial\phi} = -\frac{h-l}{b\sigma^2}\frac{1-b}{b\phi D^2}\left(E_t^l\left[\int_t^\infty X(s)(1+X(s))^{\frac{1}{b}-2}e^{-\rho(s-t)}ds\right]E_t^l\left[\int_t^\infty(1+X(s))^{\frac{1}{b}}e^{-\rho(s-t)}ds\right]\right.$$

$$\left. -\frac{1}{b\phi}E_t^l\left[\int_t^\infty(1+X(s))^{\frac{1}{b}-1}e^{-\rho(s-t)}ds\right]E_t^l\left[\int_t^\infty X(s)(1+X(s))^{\frac{1}{b}-1}e^{-\rho(s-t)}ds\right]\right),$$

where

$$D = E_t^l\left[\int_t^\infty(1+X(s))^{\frac{1}{b}}e^{-\rho(s-t)}ds\right].$$

In order to show that $\frac{z}{W}$ is increasing in ϕ it is enough to show that Q is non-negative where

$$
\begin{aligned}
Q \;=\; & E_t^l\left[\int_t^\infty (1+X(s))^{\frac{1}{b}-1}e^{-\rho(s-t)}ds\right] E_t^l\left[\int_t^\infty X(s)(1+X(s))^{\frac{1}{b}-1}e^{-\rho(s-t)}ds\right] \\
& -E_t^l\left[\int_t^\infty X(s)(1+X(s))^{\frac{1}{b}-2}e^{-\rho(s-t)}ds\right] E_t^l\left[\int_t^\infty (1+X(s))^{\frac{1}{b}}e^{-\rho(s-t)}ds\right].
\end{aligned}
$$

Notice that if (x,y) is in $\mathbb{R}^2_{++}$ and $k > 0$ then

$$
\begin{aligned}
& x(1+x)^{k-1}(1+y)^{k-1} + y(1+y)^{k-1}(1+x)^{k-1} \\
& -x(1+x)^{k-2}(1+y)^{k} - y(1+y)^{k-2}(1+x)^{k} \\
=\;& (x-y)^2(1+y)^{k-2}(1+x)^{k-2} > 0.
\end{aligned}
\tag{5.3}
$$

Now consider two *independent* stochastic processes X and X' both starting at time t at ϕ and having the same law of motion under P_l given by relationship (5.2). Thus given identity (5.3), for times $s \geq t$ and $u \geq t$, we have

$$
\begin{aligned}
& E_t^l\left[X(s)(1+X(s))^{k-1}\right] E_t^l\left[(1+X'(u))^{k-1}\right] - E_t^l\left[X(s)(1+X(s))^{k-2}\right] E_t^l\left[(1+X'(u))^{k}\right] \\
=\;& \frac{1}{2}E_t^l E_t^{\prime l}\left[(X'(u)-X(s))^{k-2}(1+X'(u))^{k-2}(1+X(s))^{k-2}\right] > 0.
\end{aligned}
$$

Multiplying by $e^{-\rho(s-t+u-t)}$ and integrating with respect to s and u from t to infinity yields that Q is positive.

Consumption-wealth ratio and optimism. Given what precedes, we have

$$
\frac{c}{W} = \frac{(1+\phi)^{\frac{1}{b}}}{E_t^l\left[\int_t^\infty (1+X(s))^{\frac{1}{b}}e^{-\rho(s-t)}ds\right]}, \quad \text{so}
$$

$$
\begin{aligned}
\frac{\partial\left(\frac{W}{c}\right)}{\partial\phi} \;=\;& \frac{E_t^l\left[\int_t^\infty X(s)(1+X(s))^{\frac{1}{b}-1}e^{-\rho(s-t)}ds\right](1+\phi) - \phi E_t^l\left[\int_t^\infty X(s)(1+X(s))^{\frac{1}{b}}e^{-\rho(s-t)}ds\right]}{b\phi(1+\phi)^{\frac{1}{b}+1}} \\
\;=\;& \frac{-E_t^l\left[\int_t^\infty (1+X(s))^{\frac{1}{b}-1}e^{-\rho(s-t)}ds\right](1+\phi) + E_t^l\left[\int_t^\infty X(s)(1+X(s))^{\frac{1}{b}}e^{-\rho(s-t)}ds\right]}{b\phi(1+\phi)^{\frac{1}{b}+1}}
\end{aligned}
$$

In order to show that $\frac{\partial\left(\frac{W}{c}\right)}{\partial\phi} > 0$ (< 0) when $b < 1$ $(b > 1)$, it is equivalent to show that when $b < 1$, then

$$
\frac{1}{1+\phi} > \frac{E_t^l\left[\int_t^\infty (1+X(s))^{\frac{1}{b}-1}e^{-\rho(s-t)}ds\right]}{E_t^l\left[\int_t^\infty (1+X(s))^{\frac{1}{b}}e^{-\rho(s-t)}ds\right]},
$$

and the opposite inequality when $b > 1$. Actually, it is enough to show that for $u \geq t$, then

$$E_t^l(1 + X(u))^{\frac{1}{b}} \geq (1 + \phi)E_t^l(1 + X(u))^{\frac{1}{b}-1},$$

when $b < 1$ and the opposite inequality when $b > 1$. We write

$$
\begin{aligned}
E_t^l(1 + X(u))^{\frac{1}{b}} &= E_t^l\left[(1 + X(u))^{\frac{1}{b}-1}(1 + X(u))\right] \\
&= E_t^l(1 + X(u))^{\frac{1}{b}-1}E_t^l(1 + X(u)) + cov_t\left((1 + X(u))^{\frac{1}{b}-1}, (1 + X(u))\right).
\end{aligned}
$$

Note that when $b < 1$ ($b > 1$), then for $x \geq 0$, the function $x \mapsto (1 + x)^{\frac{1}{b}-1}$ is strictly increasing (decreasing) so $cov_t\left((1 + X(u))^{\frac{1}{b}-1}, (1 + X(u))\right) > 0$ (< 0). It follows that for $b < 1$,

$$
\begin{aligned}
E_t^l(1 + X(u))^{\frac{1}{b}} &\geq E_t^l(1 + X(u))^{\frac{1}{b}-1}E_t^l(1 + X(u)) \\
&\geq E_t^l(1 + X(u))^{\frac{1}{b}-1}(1 + \phi e^{\frac{1-b}{b}\frac{l-r}{\sigma^2}(h-l)(u-t)}) \\
&\geq E_t^l(1 + X(u))^{\frac{1}{b}-1}(1 + \phi) \text{ since } b < 1.
\end{aligned}
$$

The proof is similar for $b > 1$ using the fact that this time $E_t^l(1 + X(u)) \leq 1 + \phi$.

Hedging demand. Same proof as for the consumption-wealth ratio and optimism.

Risky asset demand and finite horizon time. Under a finite horizon T, the wealth process is given by

$$
\begin{aligned}
W(t) &= \lambda^{\frac{1}{b}}\pi_l^{-\frac{1}{b}}(t)e^{-\frac{\theta}{b}t}E_t^l\left[\int_t^T (1 + X(s))^{\frac{1}{b}}e^{-\rho s}ds\right] \\
&= \lambda^{\frac{1}{b}}\pi_l^{-\frac{1}{b}}(t)e^{-\frac{\theta}{b}t}\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}\int_t^T (1 + X(t)e^{(\eta-\frac{\Sigma^2}{2})(s-t)+\Sigma\sqrt{s-t}y})^{\frac{1}{b}}e^{-\frac{y^2}{2}}e^{-\rho s}dsdy.
\end{aligned}
$$

Hence using Ito lemma and Fubini theorem we obtain

$$
\begin{aligned}
dW(t) &= \mu_W(t)dt - \frac{\kappa_l}{b}W(t)dw(t) + \lambda^{-\frac{1}{b}}\pi_l^{-\frac{1}{b}}(t)\frac{1}{b}\left(\frac{h-l}{\sigma}\right) \times \\
&\quad \left[\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}\int_t^T X(t)e^{(\eta-\frac{\Sigma^2}{2})(s-t)+\Sigma\sqrt{s-t}y}(1 + X(t)e^{(\eta-\frac{\Sigma^2}{2})(s-t)+\Sigma\sqrt{s-t}y})^{\frac{1}{b}-1}e^{-\rho s}e^{-\frac{y^2}{2}}dsdy\right]dw(t) \\
&= \mu_W(t)dt - \frac{\kappa_l}{b}W(t)dw(t) \\
&\quad + \lambda^{-\frac{1}{b}}\pi_l^{-\frac{1}{b}}(t)\frac{1}{b}\left(\frac{h-l}{\sigma}\right)E_t^l\left[\int_t^T X(s)(1 + X(s))^{\frac{1}{b}-1}e^{-\rho s}ds\right]dw(t),
\end{aligned}
$$

for some process μ_W. Again, identifying coefficients with the wealth dynamics in program P' yields

$$\frac{z(t)}{W(t)} = \frac{h-r}{b\sigma^2} - \frac{h-l}{b\sigma^2}\frac{E_t^l\left[\int_t^T (1 + X(s))^{\frac{1}{b}-1}e^{-\rho(s-t)}ds\right]}{E_t^l\left[\int_t^T (1 + X(s))^{\frac{1}{b}}e^{-\rho(s-t)}ds\right]}.$$

It follows that

$$
\frac{\partial}{\partial T}\left(\frac{z(t)}{W(t)}\right) = -\frac{h-l}{b\sigma^2}\left(\frac{e^{-\rho T}\left[\int_t^T E_t^l(1+X_T)^{\frac{1}{b}-1}E_t^l(1+X(s))^{\frac{1}{b}}e^{-\rho(s-t)}ds\right]}{\left(E_t^l\left[\int_t^T(1+X(s))^{\frac{1}{b}}e^{-\rho(s-t)}ds\right]\right)^2}\right.
$$

$$
\left.-\frac{e^{-\rho T}\left[\int_t^T E_t^l(1+X(s))^{\frac{1}{b}-1}E_t^l(1+X_T)^{\frac{1}{b}}e^{-\rho(s-t)}ds\right]}{\left(E_t^l\left[\int_t^T(1+X(s))^{\frac{1}{b}}e^{-\rho(s-t)}ds\right]\right)^2}\right)
$$

$$
= \frac{h-l}{b\sigma^2}\frac{\int_t^T\left(E_t^l(1+X(s))^{\frac{1-b}{b}}E_t^l(1+X_T)^{\frac{1}{b}} - E_t^l(1+X_T)^{\frac{1-b}{b}}E_t^l(1+X(s))^{\frac{1}{b}}\right)e^{-\rho(s+T-t)}ds}{\left(E_t^l\left[\int_t^T(1+X(s))^{\frac{1}{b}}e^{-\rho(s-t)}ds\right]\right)^2}.
$$

We now show that for all $t \le s \le T$

$$
E_t^l(1+X(s))^{\frac{1}{b}-1}E_t^l(1+X_T)^{\frac{1}{b}} - E_t^l(1+X_T)^{\frac{1}{b}-1}E_t^l(1+X(s))^{\frac{1}{b}},
$$

is positive (negative) exactly when $b < 1$ ($b > 1$). Since X is a Markovian process, it is enough to show the property for $t = 0$. To do so, it is enough to prove that the function $F : s \mapsto \frac{E_0^l(1+X(s))^{\frac{1}{b}-1}}{E_0^l(1+X(s))^{\frac{1}{b}}}$ is decreasing (increasing) exactly when $b < 1$ ($b > 1$). For notational convenience, we set $k = \frac{1}{b}$ and we first compute

$$
\frac{\partial E_0^l\left[(1+X(s))^k\right]}{\partial s} = \frac{\partial}{\partial s}\left(\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}(1+\phi e^{(\eta-\frac{\Sigma^2}{2})s+\Sigma\sqrt{s}y})^k e^{-\frac{y^2}{2}}dy)\right)
$$

$$
= \frac{k}{\sqrt{2\pi}}\int_{\mathbb{R}}(\eta-\frac{\Sigma^2}{2}+\frac{\Sigma}{2\sqrt{s}}y)\phi e^{(\eta-\frac{\Sigma^2}{2})s+\Sigma\sqrt{s}y}(1+\phi e^{(\eta-\frac{\Sigma^2}{2})s+\Sigma\sqrt{s}y})^{k-1}e^{-\frac{y^2}{2}}dy
$$

$$
= k\eta E_0^l\left[X(s)(1+X(s))^{k-1}\right]
$$

$$
-\frac{k\Sigma\phi e^{(\eta-\frac{\Sigma^2}{2})s}}{\sqrt{2\pi}2\sqrt{s}}\left(\left[e^{\Sigma\sqrt{s}y-\frac{y^2}{2}}(1+\phi e^{(\eta-\frac{\Sigma^2}{2})s+\Sigma\sqrt{s}y})^{k-1}\right]_{-\infty}^{\infty}\right.
$$

$$
\left.-(k-1)\int_{\mathbb{R}}\Sigma\sqrt{s}\phi e^{(\eta-\frac{\Sigma^2}{2})s+\Sigma\sqrt{s}y}(1+\phi e^{(\eta-\frac{\Sigma^2}{2})s+\Sigma\sqrt{s}y})^{k-2}e^{-\frac{y^2}{2}}dy\right)
$$

$$
= k\eta E_0^l\left[X(s)(1+X(s))^{k-1}\right]
$$

$$
+k(k-1)\frac{\Sigma^2}{2}\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}\left(\phi e^{(\eta-\frac{\Sigma^2}{2})s+\Sigma\sqrt{s}y}\right)^2(1+\phi e^{(\eta-\frac{\Sigma^2}{2})s+\Sigma\sqrt{s}y})^{k-2}e^{-\frac{y^2}{2}}dy
$$

$$
= k\eta E_0^l\left[X(s)(1+X(s))^{k-1}\right] + k(k-1)\frac{\Sigma^2}{2}E_0^l\left[X^2(s)(1+X(s))^{k-2}\right].
$$

Hence

$$F'(s) = \frac{(k-1)\left(\eta E_0^l\left[X(s)(1+X(s))^{k-2}\right] + (k-2)\frac{\Sigma^2}{2}E_0^l\left[X^2(s)(1+X(s))^{k-3}\right]\right)E_0^l\left[(1+X(s))^k\right]}{\left(E_0^l\left[(1+X(s))^k\right]\right)^2}$$

$$- \frac{k\left(\eta E_0^l\left[X(s)(1+X(s))^{k-1}\right] + (k-1)\frac{\Sigma^2}{2}E_0^l\left[X^2(s)(1+X(s))^{k-2}\right]\right)E_0^l\left[(1+X(s))^{k-1}\right]}{\left(E_0^l\left[(1+X(s))^k\right]\right)^2}.$$

Now recall that

$$\eta = (k-1)\frac{(l-r)(h-l)}{\sigma}.$$

Thus F' has the same sign as $(k-1)(U+V)$, where

$$U = \frac{(l-r)(h-l)}{\sigma}\left((k-1)E_0^l\left[X(s)(1+X(s))^{k-2}\right]E_0^l\left[(1+X(s))^k\right]\right.$$
$$\left. - kE_0^l\left[X(s)(1+X(s))^{k-1}\right]E_0^l\left[(1+X(s))^{k-1}\right]\right)$$

$$V = \frac{\Sigma^2}{2}\left((k-2)E_0^l\left[X^2(s)(1+X(s))^{k-3}\right]E_0^l\left[(1+X(s))^k\right]\right.$$
$$\left. - kE_0^l\left[X^2(s)(1+X(s))^{k-2}\right]E_0^l\left[(1+X(s))^{k-1}\right]\right).$$

The next step is to show that U and V are negative. To do so, it is enough to show that

$$E_0^l\left[X(s)(1+X(s))^{k-2}\right]E_0^l\left[(1+X(s))^k\right] - E_0^l\left[X(s)(1+X(s))^{k-1}\right]E_0^l\left[(1+X(s))^{k-1}\right] < 0$$

$$E_0^l\left[X^2(s)(1+X(s))^{k-3}\right]E_0^l\left[(1+X(s))^k\right] - E_0^l\left[X^2(s)(1+X(s))^{k-2}\right]E_0^l\left[(1+X(s))^{k-1}\right] < 0.$$

To prove the first relationship, we note that if (x,y) is in $\mathbb{R}^2_{++}$, then

$$x(1+x)^{k-2}(1+y)^k + y(1+y)^{k-2}(1+x)^k$$
$$-x(1+x)^{k-1}(1+y)^{k-1} - y(1+y)^{k-1}(1+x)^{k-1}$$
$$= -(x-y)^2(1+x)^{k-2}(1+y)^{k-2} < 0. \tag{5.4}$$

Now consider two *independent* stochastic processes X and X' both starting at time 0 at ϕ and having the same law of motion under P_l given by relationship (5.2). Thus given identity (5.4), for times $s \geq 0$, we have

$$E_0^l\left[X(s)(1+X(s))^{k-2}\right]E_0^l\left[(1+X'(s))^k\right] - E_0^l\left[X(s)(1+X(s))^{k-1}\right]E_0^l\left[(1+X'(s))^{k-1}\right]$$
$$= -\frac{1}{2}E_0^lE_0^{\prime l}\left[(X'(s)-X(s))^2(1+X'(s))^{k-2}(1+X(s))^{k-2}\right] < 0.$$

To prove the second relationship, we proceed exactly as before using the following identity: if (x,y) is in $\mathbb{R}^2_{++}$, then

$$x^2(1+x)^{k-3}(1+y)^k + y^2(1+y)^{k-3}(1+x)^k$$
$$-x^2(1+x)^{k-2}(1+y)^{k-1} - y^2(1+y)^{k-2}(1+x)^{k-1}$$
$$= -(x-y)^2(x(1+y)+y(1+x))(1+x)^{k-3}(1+y)^{k-3} < 0.$$

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And again, we have to consider two identical but *independent* stochastic processes. Details of the rest of the proof for the second relationship are omitted. To sum up, we have proved that F' has the same sign as $1 - \frac{1}{b}$. We conclude that $\frac{\partial}{\partial T}\left(\frac{z(t)}{W(t)}\right)$ is positive (negative) exactly when $b < 1$ ($b > 1$).

APPENDIX B

Filtering problem. Observing S and A is equivalent to observing $x = \frac{\ln S}{\sigma}$ and $y = \frac{\sqrt{1+a}\sigma \ln A - \sqrt{a}\Sigma_1 \ln S}{\Sigma_2 \sigma}$. Using Ito's lemma, it is easy to check that

$$
dx(t) = \frac{1}{\sigma}\left(\mu - \frac{\sigma^2}{2}\right) dt + dw_1(t)
$$

$$
dy(t) = \frac{1}{\Sigma_2 \sigma}\left(\sqrt{1+a}\,m - \sqrt{a}\Sigma_1\left(\mu - \frac{\sigma^2}{2}\right)\right) dt + dw_2(t),
$$

with $m = \lambda\sigma - \frac{\alpha\Sigma_1^2}{2(1+a)} - \frac{\Sigma_2^2}{2(1+a)}$. Applying Bayes 'rule, we have

$$
p(t + dt) = \frac{p(t)H(h)}{p(t)H(h) + (1 - p(t))H(l)},
$$

where

$$
F(\mu) = \frac{1}{\sqrt{2\pi dt}}\exp\left(-\frac{\left(dx(t) - \frac{1}{\sigma}\left(\mu - \frac{\sigma^2}{2}\right) dt\right)^2}{2dt}\right)
$$

$$
G(\mu) = \frac{1}{\sqrt{2\pi dt}}\exp\left(-\frac{\left(dy(t) - \frac{1}{\Sigma_2\sigma}\left(\sqrt{1+a}\,m - \sqrt{a}\Sigma_1\left(\mu - \frac{\sigma^2}{2}\right)\right) dt\right)^2}{2dt}\right),
$$

and $H(\mu) = F(\mu)G(\mu)$ is the probability of observing $(dx(t), dy(t))$. Hence

$$
dp(t) = \frac{(1 - p(t))p(t)(\widetilde{H}(h) - \widetilde{H}(l))}{p(t)\widetilde{H}(h) + (1 - p(t))\widetilde{H}(l)},
$$

where

$$
\begin{aligned}
\widetilde{H}(\mu) = \ \ &\exp\left(\frac{1}{\sigma}\left(\mu - \frac{\sigma^2}{2}\right) dx(t) - \frac{1}{2\sigma^2}\left(\mu - \frac{\sigma^2}{2}\right)^2 dt\right. \\
&+ \frac{1}{\Sigma_2\sigma}\left(\sqrt{1+a}\,m - \sqrt{a}\Sigma_1(\mu - \frac{\sigma^2}{2})\right) dy(t) \\
&\left. - \frac{1}{2\Sigma_2^2\sigma^2}\left(\sqrt{1+a}\,m - \sqrt{a}\Sigma_1(\mu - \frac{\sigma^2}{2})\right)^2 dt\right).
\end{aligned}
$$

It follows that

$$\widetilde{H}(\mu) = 1 + \left(\frac{1}{\sigma}\left(\mu - \frac{\sigma^2}{2}\right)dx(t) + \frac{1}{\Sigma_2\sigma}\left(\sqrt{1+am} - \sqrt{a}\Sigma_1(\mu - \frac{\sigma^2}{2})\right)\right)dy(t)$$

$$- \left(\frac{1}{2\sigma^2}\left(\mu - \frac{\sigma^2}{2}\right)^2 + \frac{1}{2\Sigma_2^2\sigma^2}\left(\sqrt{1+am} - \sqrt{a}\Sigma_1(\mu - \frac{\sigma^2}{2})\right)^2\right)dt$$

$$+ \frac{1}{2}\left(\frac{1}{\sigma}\left(\mu - \frac{\sigma^2}{2}\right)dx(t) + \frac{1}{\Sigma_2\sigma}\left(\sqrt{1+am} - \sqrt{a}\Sigma_1(\mu - \frac{\sigma^2}{2})\right)dy(t)\right.$$

$$\left. - \left(\frac{1}{2\sigma^2}\left(\mu - \frac{\sigma^2}{2}\right)^2 + \frac{1}{2\Sigma_2^2\sigma^2}\left(\sqrt{1+am} - \sqrt{a}\Sigma_1(\mu - \frac{\sigma^2}{2})\right)^2\right)dt\right)^2$$

$$= 1 + \frac{1}{\sigma}\left(\mu - \frac{\sigma^2}{2}\right)dx(t) + \frac{1}{\Sigma_2\sigma}\left(\sqrt{1+am} - \sqrt{a}\Sigma_1(\mu - \frac{\sigma^2}{2})\right)dy(t),$$

where we have suppressed the terms of degree $dt^{\frac{3}{2}}$ and higher and use the fact that $(dx(t))^2 = (dy(t))^2 = 1$ and $dx(t)dy(t) = 0$. Therefore

$$dp(t) = \frac{\frac{h-l}{\sigma}(1-p(t))p(t)\left(dx(t) - \sqrt{a}\frac{\Sigma_1}{\Sigma_2}dy(t)\right)}{1 + \frac{1}{\sigma}(p(t)(h-l)+l) - \frac{\sigma^2}{2})dx(t) + \left(\sqrt{1+am} - \sqrt{a}\Sigma_1(p(t)(h-l)+l - \frac{\sigma^2}{2})\right)\frac{dy(t)}{\Sigma_2\sigma}}$$

$$= \frac{h-l}{\sigma}(1-p(t))p(t)\left(dx(t) - \sqrt{a}\frac{\Sigma_1}{\Sigma_2}dy(t) - \frac{1}{\sigma}(p(t)h + (1-p(t))l) - \frac{\sigma^2}{2})dx(t)\right.$$

$$\left. - \frac{1}{\Sigma_2\sigma}\left(\sqrt{1+am} - \sqrt{a}\Sigma_1(p(t)h + (1-p(t))l - \frac{\sigma^2}{2})\right)dy(t)\right)$$

$$= \frac{h-l}{\sigma}(1-p(t))p(t)\left[dw_1(t) + \frac{1}{\sigma}(\mu - (p(t)h + (1-p(t))l))dt\right.$$

$$\left. - \sqrt{a}\frac{\Sigma_1}{\Sigma_2}\left(dw_2(t) - \sqrt{a}\frac{\Sigma_1}{\Sigma_2\sigma}(\mu - (p(t)h + (1-p(t))l))dt\right)\right]$$

$$= \frac{h-l}{\sigma}(1-p(t))p(t)\left(d\overline{w}_1(t) - \sqrt{a}\frac{\Sigma_1}{\Sigma_2}d\overline{w}_2(t)\right),$$

with

$$d\overline{w}_1(t) = dw_1(t) + \frac{1}{\sigma}(\mu - (p(t)h + (1-p(t))l))dt$$

$$d\overline{w}_2(t) = dw_2(t) - \sqrt{a}\frac{\Sigma_1}{\Sigma_2\sigma}(\mu - (p(t)h + (1-p(t))l))dt.$$

Wealth process. As before, let us define

$H(\pi_l, \phi) = E_t^l \left[\int_t^\infty \pi_l^{\frac{b-1}{b}}(s)(1 + \phi(s))^{\frac{1}{b}} e^{-\frac{\theta}{b}(s-t)} ds \right]$ that satisfies the following PDE

$$\frac{\theta}{b} H = \pi_l^{\frac{b-1}{b}} (1 + \phi)^{\frac{1}{b}} - r\pi_l H_1 + \frac{\kappa_l^2}{2} \pi_l^2 H_{11} + \kappa_l \frac{h-l}{\sigma} \pi_l \phi H_{12}$$
$$+ \frac{1}{2} \left(\frac{h-l}{\sigma} \right)^2 \left(1 + \frac{a\Sigma_1^2}{\Sigma_2^2} \right) \phi^2 H_{22}.$$

This PDE is similar to PDE (5.1) obtained in appendix 1 but the coefficient of H_{22} is now $\frac{1}{2} \left(\frac{h-l}{\sigma} \right)^2 \left(1 + \frac{a\Sigma_1^2}{\Sigma_2^2} \right)$ instead of $\frac{1}{2} \left(\frac{h-l}{\sigma} \right)^2$. The analysis conducted in appendix 1 still holds but the roots α and β must now be defined as the solutions of the following quadratic

$$\frac{1}{2} \left(\frac{h-l}{\sigma} \right)^2 \left(1 + \frac{a\Sigma_1^2}{\Sigma_2^2} \right) x^2 + \left(\frac{b-1}{b} \kappa_l \frac{h-l}{\sigma} - \frac{1}{2} \left(\frac{h-l}{\sigma} \right)^2 \left(1 + \frac{a\Sigma_1^2}{\Sigma_2^2} \right) \right) x = \rho,$$

and the expression for the wealth obtained in appendix 1 remains valid.

Consumption-wealth ratio. The ratio $\frac{c}{W}$ is given by

$$\frac{c}{W} = \frac{(1 + \phi)^{\frac{1}{b}}}{E_t^l \left[\int_t^\infty (1 + X(s))^{\frac{1}{b}} e^{-\rho(s-t)} ds \right]},$$

where

$$dX(s) = X(s)\left(\eta ds + \Sigma dw(s) \right). \tag{5.5}$$

with this time

$$\Sigma = \frac{h-l}{\sigma} \sqrt{1 + a\frac{\Sigma_1^2}{\Sigma_2^2}}.$$

Set $k = \frac{1}{b} > 0$. We want to compute

$$\frac{\partial E_0^l \left[\int_0^\infty (1 + X(s))^k e^{-\rho s} ds \right]}{\partial \Sigma} = \int_0^\infty \left[\frac{\partial E_0^l \left[(1 + X(s))^k \right]}{\partial \Sigma} e^{-\rho s} ds \right],$$

where the introduction of the derivative under the integral sign can be justified using

Lebesgue's dominated theorem. Then

$$
\begin{aligned}
\frac{\partial E_0^l\left[(1+X(s))^k\right]}{\partial \Sigma} &= \frac{\partial}{\partial \Sigma}\left(\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}(1+\phi e^{(\eta-\frac{\Sigma^2}{2})s+\Sigma\sqrt{s}y})^k e^{-\frac{y^2}{2}}dy)\right) \\
&= \frac{k}{\sqrt{2\pi}}\int_{\mathbb{R}}(-\Sigma s+\sqrt{s}y)\phi e^{(\eta-\frac{\Sigma^2}{2})s+\Sigma\sqrt{s}y}(1+\phi e^{(\eta-\frac{\Sigma^2}{2})s+\Sigma\sqrt{s}y})^{k-1}e^{-\frac{y^2}{2}}dy \\
&= \frac{-k\phi\sqrt{s}e^{(\eta-\frac{\Sigma^2}{2})s}}{\sqrt{2\pi}}\left(\left[e^{\Sigma\sqrt{s}y-\frac{y^2}{2}}(1+\phi e^{(\eta-\frac{\Sigma^2}{2})s+\Sigma\sqrt{s}y})^{k-1}\right]_{-\infty}^{\infty}\right. \\
&\quad \left.-(k-1)\int_{\mathbb{R}}\Sigma\sqrt{s}\phi e^{(\eta-\frac{\Sigma^2}{2})s+\Sigma\sqrt{s}y}(1+\phi e^{(\eta-\frac{\Sigma^2}{2})s+\Sigma\sqrt{s}y})^{k-2}e^{-\frac{y^2}{2}}dy\right) \\
&= k(k-1)\Sigma s\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}\left(\phi e^{(\eta-\frac{\Sigma^2}{2})s+\Sigma\sqrt{s}y}\right)^2(1+\phi e^{(\eta-\frac{\Sigma^2}{2})s+\Sigma\sqrt{s}y})^{k-2}e^{-\frac{y^2}{2}}dy \\
&= k(k-1)s\Sigma E_0^l\left[X^2(s)(1+X(s))^{k-2}\right].
\end{aligned}
$$

It follows that this derivative is positive (negative) exactly when $k>1$ ($k<1$). Since

$$
\frac{c}{W} = \frac{(1+\phi)^{\frac{1}{b}}}{E_0^l\left[\int_0^\infty(1+X(s))^{\frac{1}{b}}e^{-\rho s}ds\right]},
$$

we obtain that this ratio is increasing (decreasing) in Σ exactly when $b>1$ ($b<1$).

Risky asset demand and precision of the signal. The ratio $\frac{z}{W}$ is given by

$$
\frac{z}{W} = \frac{h-r}{b\sigma^2} - \frac{h-l}{b\sigma^2}\frac{E_0^l\left[\int_0^\infty(1+X(s))^{\frac{1}{b}-1}e^{-\rho s}ds\right]}{E_0^l\left[\int_0^\infty(1+X(s))^{\frac{1}{b}}e^{-\rho s}ds\right]}.
$$

Hence

$$
\begin{aligned}
\frac{\partial\left(\frac{z}{W}\right)}{\partial\Sigma} &= -\frac{(h-l)\Sigma(k-1)}{b\sigma^2 D^2}\left((k-2)E_0^l\left[\int_0^\infty X^2(s)(1+X(s))^{k-3}se^{-\rho s}ds\right]E_0^l\left[\int_0^\infty(1+X(s))^k e^{-\rho s}ds\right]\right. \\
&\quad \left.-kE_0^l\left[\int_0^\infty X^2(s)(1+X(s))^{k-2}se^{-\rho s}ds\right]E_0^l\left[\int_0^\infty(1+X(s))^{k-1}e^{-\rho s}ds\right]\right),
\end{aligned}
$$

with

$$
D = E_0^l\left[\int_0^\infty(1+X(s))^{\frac{1}{b}}e^{-\rho s}ds\right].
$$

The derivative expression can be rewritten as

$$
\frac{\partial\left(\frac{z}{W}\right)}{\partial\Sigma} = \frac{(h-l)\Sigma(k-1)}{b\sigma^2 D^2}\left(2E_0^l\left[\int_0^\infty X^2(s)(1+X(s))^{k-3}se^{-\rho s}ds\right]E_0^l\left[\int_0^\infty(1+X(s))^k e^{-\rho s}ds\right]+M\right),
$$

where

$$M = E_0^l\left[\int_0^\infty X^2(s)(1+X(s))^{k-2}se^{-\rho s}ds\right]E_0^l\left[\int_0^\infty (1+X(s))^{k-1}e^{-\rho s}ds\right]$$
$$-E_0^l\left[\int_0^\infty X^2(s)(1+X(s))^{k-3}se^{-\rho s}ds\right]E_0^l\left[\int_0^\infty (1+X(s))^k e^{-\rho s}ds\right].$$

The next step is to show that $M > 0$ so we can conclude that $\frac{\partial\left(\frac{z}{W}\right)}{\partial\Sigma}$ is positive (negative) exactly when $b < 1$ ($b > 1$). Notice that if (x,y) is in $\mathbb{R}^2_{++}$, then

$$x^2(1+x)^{k-2}(1+y)^{k-1} + y^2(1+y)^{k-2}(1+x)^{k-1} \tag{5.6}$$
$$-x^2(1+x)^{k-3}(1+y)^k - y^2(1+y)^{k-3}(1+x)^k$$
$$= (x-y)^2(1+x)^{k-3}(1+y)^{k-3}(x+y+2xy) > 0.$$

Again, consider two *independent* stochastic processes X and X' both starting at time 0 at ϕ and having the same law of motion under P_l given by relationship (5.5). Thus given identity (5.6), for times $s \geq 0$ and $u \geq 0$, we have

$$E_0^l\left[X^2(s)(1+X(s))^{k-2}\right]E_0^l\left[(1+X'(u))^{k-1}\right] - E_0^l\left[(1+X(s))^k\right]E_0^l\left[X'^2(u)(1+X'(u))^{k-3}\right]$$
$$= \frac{1}{2}E_0^l E_0^{\prime l}\left[(X'(u)-X(s))^{k-2}(1+X'(u))^{k-3}(1+X(s))^{k-3}(X'(u)+X(s)+2X(s)X'(u))\right] > 0.$$

Multiplying by $sue^{-\rho(s+u)}$ and integrating with respect to s and u from 0 to infinity yields the desired result.

APPENDIX C

Under the probability measure P the process $U = \pi_m e^{rt}$ satisfies

$$dU(t) = -U(t)\left(\frac{m(t)-r}{\sigma}\right)d\overline{w}(t).$$

Writing $U(t) = U(m(t), \gamma(t))$ and using Ito lemma leads to

$$dU(t) = U_1 dm(t) + U_2\dot\gamma(t)dt + \frac{1}{2}U_{11}\frac{\gamma^2(t)}{\sigma^2}dt.$$

This implies

$$U_1(m,\gamma) = -U(m,\gamma)\left(\frac{m-r}{\gamma}\right) \tag{5.7}$$

$$\frac{1}{2}U_{11}(m,\gamma) - U_2(m,\gamma) = 0 \tag{5.8}$$

The general solution to (5.7) is

$$U(m,\gamma) = f(\gamma)\exp\left(-\frac{1}{\gamma}(\frac{m^2}{2} - rm)\right).$$

Plugging this expression into (5.8) and after simplification shows that f must satisfy

$$\frac{f'(\gamma)}{f(\gamma)} = -\frac{1}{2\gamma} + \frac{r^2}{2\gamma^2}.$$

Hence, $f(\gamma) = \frac{A}{\sqrt{\gamma}} \exp\left(-\frac{r^2}{2\gamma}\right)$ where A is a constant. Since $U(0) = 1$, we obtain

$$U(t) = \sqrt{\frac{\gamma_0}{\gamma(t)}} \exp\left(-\frac{1}{2\gamma(t)}\left(m(t) - r\right)^2 + \frac{1}{2\gamma_0}\left(m_0 - r\right)^2\right).$$

Wealth process. Using relationships (3.3), the result of lemma 3 and Fubini Theorem, the investor wealth at time t can be written

$$W(t) = \frac{\lambda^{-\frac{1}{b}} \pi_m^{\frac{b-1}{b}}(t) e^{\frac{b-1}{2b}\frac{(m(t)-r)^2}{\gamma(t)}} e^{-\frac{\theta}{b}t}}{\pi_m(t)} \int_t^T \left(\frac{\gamma(s)}{\gamma(t)}\right)^{\frac{1-b}{2b}} E_t^P\left[e^{\frac{1-b}{2b}\frac{(m(s)-r)^2}{\gamma(s)}}\right] e^{-\frac{\theta+(1-b)r}{b}(s-t)} ds.$$

Since under P, $m(s)$ is normally distributed with mean $m(t)$ and variance $\gamma(t) - \gamma(s)$, we have

$$\begin{aligned}
E_t^P\left[e^{\frac{1-b}{2b}\frac{(m(s)-r)^2}{\gamma(s)}}\right] &= \frac{1}{\sqrt{2\pi(\gamma(t)-\gamma(s))}} \int_{-\infty}^{\infty} e^{\frac{1-b}{2b}\frac{(u+m(t)-r)^2}{\gamma(s)}} e^{-\frac{u^2}{2(\gamma(t)-\gamma(s))}} du \\[2mm]
&= \frac{e^{\frac{1-b}{2b}\frac{(m(t)-r)^2}{\gamma(s)}} e^{\frac{(1-b)^2}{2b}\frac{(m(t)-r)^2}{\gamma(s)}\frac{1}{b-1+\frac{b\gamma(s)}{\gamma(t)-\gamma(s)}}}}{\sqrt{(\gamma(t)-\gamma(s))(\frac{b-1}{b\gamma(s)} + \frac{1}{\gamma(t)-\gamma(s)})}} \\[2mm]
&= \frac{e^{\frac{(1-b)(m(t)-r)^2}{2b\gamma(s)\left(1+\frac{b-1}{b}\frac{\gamma(t)}{\sigma^2}(s-t)\right)}}}{\sqrt{1 + \frac{b-1}{b}\frac{\gamma(t)}{\sigma^2}(s-t)}}.
\end{aligned}$$

Finally, the wealth process is given by

$$W(t) = \lambda^{-\frac{1}{b}} \pi_m^{-\frac{1}{b}}(t) e^{-\frac{\theta}{b}t} \int_0^{T-t} \frac{\left(1 + \frac{\gamma(t)u}{\sigma^2}\right)^{\frac{b-1}{2b}}}{\sqrt{1 + \frac{b-1}{b}\frac{\gamma(t)u}{\sigma^2}}} e^{-\frac{1}{b}(\theta + (b-1)(r + \frac{(m(t)-r)^2}{2b\sigma^2}\frac{1}{1+\frac{b-1}{b}\frac{\gamma(t)u}{\sigma^2}}))u} du.$$

Portfolio allocations. Using Ito lemma, we obtain

$$\begin{aligned}
dW(t) &= \mu_W(t)dt - \frac{\kappa_m(t)}{b} W(t) d\overline{w}(t) - \frac{(b-1)(m(t)-r)\gamma(t)}{b^2\sigma^3}\lambda^{-\frac{1}{b}}\pi_m^{-\frac{1}{b}}(t) \times \\[2mm]
&\quad \left[\int_0^{T-t} \frac{\left(1 + \frac{\gamma(t)u}{\sigma^2}\right)^{\frac{b-1}{2b}}}{\sqrt{1 + \frac{b-1}{b}\frac{\gamma(t)u}{\sigma^2}}} \frac{u}{1 + \frac{b-1}{b}\frac{\gamma(t)u}{\sigma^2}} e^{-\frac{1}{b}(\theta + (b-1)(r + \frac{(m(t)-r)^2}{2b\sigma^2}\frac{1}{1+\frac{b-1}{b}\frac{\gamma(t)u}{\sigma^2}}))u} du\right] d\overline{w}(t),
\end{aligned}$$

for some process μ_W. Identifying coefficients with relationship (3.4) leads to the desired result.

Demand for the risky asset and variance of beliefs. Let define
$$f(\gamma, t) = g(\gamma, t)h(\gamma, t),$$

with

$$g(\gamma, t) = \frac{\left(1 + \frac{\gamma t}{\sigma^2}\right)^{\frac{b-1}{2b}}}{\left(1 + \frac{b-1}{b}\frac{\gamma t}{\sigma^2}\right)^{\frac{3}{2}}}$$

$$h(\gamma, t) = e^{-\frac{1}{b}(\theta + (b-1)(r + \frac{(m-r)^2}{2b\sigma^2}\frac{1}{1 + \frac{b-1}{b}\frac{\gamma t}{\sigma^2}}))t}.$$

It follows that

$$\frac{g_1(\gamma, t)}{g(\gamma, t)} = \frac{3}{2\gamma}\frac{1}{1 + \frac{b-1}{b}\frac{\gamma t}{\sigma^2}} - \frac{b-1}{2b\gamma}\frac{1}{1 + \frac{\gamma t}{\sigma^2}} - \frac{3}{2\gamma} + \frac{b-1}{2b\gamma}$$

$$\frac{h_1(\gamma, t)}{h(\gamma, t)} = \frac{(1-b)^2}{2b\sigma^2}(m-r)^2\frac{\frac{\gamma}{\sigma^2}t^2}{(1 + \frac{b-1}{b}\frac{\gamma t}{\sigma^2})^2}.$$

Then

$$\frac{\partial}{\partial t}\log\left(\frac{h_1(\gamma, t)}{h(\gamma, t)}\right) = \frac{2}{(1 + \frac{b-1}{b}\frac{\gamma t}{\sigma^2})t} > 0.$$

So we can conclude that the function $t \mapsto \frac{h_1(\gamma, t)}{h(\gamma, t)}$ is increasing in t.

$$\frac{\partial}{\partial t}\log\left(\frac{g_1(\gamma, t)}{g(\gamma, t)}\right) = \frac{1-b}{2b\sigma^2}\left(\frac{3}{(1 + \frac{b-1}{b}\frac{\gamma t}{\sigma^2})^2} - \frac{1}{(1 + \frac{\gamma t}{\sigma^2})^2}\right).$$

Thus if $b < 1$, the function $t \mapsto \frac{g_1(\gamma, t)}{g(\gamma, t)}$ is also increasing in t and so is $t \mapsto \frac{f_1(\gamma, t)}{f(\gamma, t)}$. The demand for the risky asset can be written

$$\frac{z}{W} = \frac{\displaystyle\int_0^\tau f(\gamma, t)dt}{\displaystyle\int_0^\tau (1 + \frac{b-1}{b}\frac{\gamma t}{\sigma^2})f(\gamma, t)dt},$$

with $\tau = T - t$. Hence

$$\frac{\partial}{\partial\gamma}(\frac{z}{W}) = \frac{M(\tau)}{D^2},$$

with

$$M(\tau) = \int_0^\tau f_1(\gamma, t)dt\int_0^\tau (1 + \frac{b-1}{b}\frac{\gamma t}{\sigma^2})f(\gamma, t)dt$$
$$- \int_0^\tau f(\gamma, t)dt\int_0^\tau \left(\frac{b-1}{b}\frac{t}{\sigma^2}f(\gamma, t) + (1 + \frac{b-1}{b}\frac{\gamma t}{\sigma^2})f_1(\gamma, t)\right)dt$$
$$D = \int_0^\tau (1 + \frac{b-1}{b}\frac{\gamma t}{\sigma^2})f(\gamma, t)dt.$$

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$\frac{\partial}{\partial\gamma}(\frac{z}{W})$ and $M(\tau)$ have the same sign. Note that $M(0) = 0$ and after some simple algebra and re-arranging terms

$$M'(\tau) = \frac{(1-b)}{b} \int_0^\tau f(\gamma,t)f(\gamma,\tau) \left[\left(\frac{f_1(\gamma,\tau)}{f(\gamma,\tau)} - \frac{f_1(\gamma,t)}{f(\gamma,t)} \right) \frac{\gamma(\tau-t)}{\sigma^2} + \frac{t+\tau}{\sigma^2} \right] dt.$$

Now recall when $b < 1$, the function $t \mapsto \frac{f_1(\gamma,t)}{f(\gamma,t)}$ is also increasing in t. Hence M' is positive when $b < 1$. Since $M(0) = 0$, we conclude that M is positive when $b < 1$. To analyze the case when $b > 1$ we rewrite the demand for the risky asset as follows

$$\frac{z}{W} = \frac{\int_0^{\gamma\tau} G(y)e^{-\frac{\theta+(b-1)A(y)}{b\gamma}y}dy}{\int_0^{\gamma\tau} (1+\frac{b-1}{b\sigma^2}y)G(y)e^{-\frac{\theta+(b-1)A(y)}{b\gamma}y}dy},$$

with

$$G(y) = \frac{\left(1 + \frac{y}{\sigma^2}\right)^{\frac{b-1}{2b}}}{\left(1 + \frac{b-1}{b}\frac{y}{\sigma^2}\right)^{\frac{3}{2}}}$$

$$A(y) = r + \frac{(m-r)^2}{2b\sigma^2}\frac{1}{1+\frac{b-1}{b}\frac{y}{\sigma^2}}.$$

$$\frac{\partial}{\partial\gamma}(\frac{z}{W}) = \frac{1-b}{b\sigma^2}\frac{G(\gamma\tau)e^{-\frac{\theta+(b-1)A(\gamma\tau)}{b}\tau}}{D^2} \int_0^{\gamma\tau} (\gamma\tau-y)G(y)e^{-\frac{\theta+(b-1)A(y)}{b\gamma}y}dy + \frac{1-b}{b\sigma^2\gamma^2}\frac{N(\tau)}{D^2},$$

with

$$N(\tau) = \int_0^{\gamma\tau} (\theta+(b-1)A(y))y^2G(y)e^{-\frac{\theta+(b-1)A(y)}{b\gamma}y}dy \times \int_0^{\gamma\tau} G(y)e^{-\frac{\theta+(b-1)A(y)}{b\gamma}y}dy$$

$$- \int_0^{\gamma\tau} (\theta+(b-1)A(y))yG(y)e^{-\frac{\theta+(b-1)A(y)}{b\gamma}y}dy \times \int_0^{\gamma\tau} yG(y)e^{-\frac{\theta+(b-1)A(y)}{b\gamma}y}dy.$$

It is enough to show that N is positive to obtain that $\frac{\partial}{\partial\gamma}(\frac{z}{W})$ is positive when $b > 1$. After some simple algebra and re-arranging terms, we have

$$N'(\tau) = G(\tau)e^{-\frac{\theta+(b-1)A(\gamma\tau)}{b}\tau} \int_0^{\gamma\tau} R(\gamma\tau,y)G(y)e^{-\frac{\theta+(b-1)A(y)}{b\gamma}y}dy,$$

where

$$R(\gamma\tau,y) = \gamma^2\tau^2(\theta+(b-1)A(\gamma\tau)) + y^2(\theta+(b-1)A(y))$$
$$- \gamma\tau(\theta+(b-1)A(\gamma\tau))y - \gamma\tau(\theta+(b-1)A(y))y$$
$$= (\gamma\tau-y)^2 \left(\theta+(b-1)(r + \frac{(m-r)^2}{2b\sigma^2}\frac{1}{(1+\frac{b-1}{b}\frac{\gamma\tau}{\sigma^2})(1+\frac{b-1}{b}\frac{y}{\sigma^2})}) \right).$$

Hence, when $b > 1$, $R(\gamma\tau, y)$ is positive so is N'. It follows that N is strictly increasing and since $N(0) = 0$, we can conclude that N is positive and the proof is complete.

Consumption-wealth ratio and variance of beliefs. We look at the effect of beliefs on the wealth. We have

$$\frac{\partial W}{\partial \gamma} = \frac{1-b}{2b^2\sigma^2} \int_0^\tau \left(\frac{2b + (2b+1)\frac{\gamma t}{\sigma^2}}{(1 + \frac{b-1}{b}\frac{\gamma t}{\sigma^2})(1 + \frac{\gamma t}{\sigma^2})} + \frac{(1-b)(m-r)^2}{(1 + \frac{b-1}{b}\frac{\gamma t}{\sigma^2})^2} \right) tg(\gamma, t)h(\gamma, t)dt.$$

Hence, if $b < 1$, W is increasing in γ and consequently the consumption-wealth ratio $\frac{c}{W}$ is decreasing in γ. If $b > 1$ then when $|m - r|$ is small, given what precedes, the ratio $\frac{c}{W}$ can be increasing in γ. The opposite can occur if $|m - r|$ is large enough.

Demand for the risky asset and mean return. Let us rewrite

$$g(\gamma, t) = \frac{\left(1 + \frac{\gamma t}{\sigma^2}\right)^{\frac{b-1}{2b}}}{\left(1 + \frac{b-1}{b}\frac{\gamma t}{\sigma^2}\right)^{\frac{3}{2}}} e^{\frac{1-b}{2b^2\sigma^2}K(\gamma,t)(m-r)^2 - \frac{1}{b}(\theta + (b-1)r)t}$$

$$K(\gamma, t) = \frac{t}{1 + \frac{b-1}{b}\frac{\gamma t}{\sigma^2}}.$$

So

$$\frac{\partial}{\partial m}\left(\frac{z}{W}\right) = \frac{1}{b\sigma^2} \frac{\int_0^\tau f(\gamma, t)dt \int_0^\tau (1 + \frac{b-1}{b}\frac{\gamma t}{\sigma^2})(1 + \frac{b-1}{b}\frac{\gamma t}{\sigma^2})f(\gamma, t)dt + O(\tau)}{D^2},$$

with

$$O(\tau) = \frac{1-b}{b^2\sigma^2}(m-r)^2 \left(\int_0^\tau f(\gamma, t)K(\gamma, t)dt \int_0^\tau (1 + \frac{b-1}{b}\frac{\gamma t}{\sigma^2})f(\gamma, t)dt \right.$$
$$\left. - \int_0^\tau f(\gamma, t)dt \int_0^\tau tf(\gamma, t)dt \right).$$

In order to show that $\frac{\partial}{\partial m}(\frac{z}{W})$ is positive, it is enough to show that $O(\tau)$ is positive. Then after rearranging terms we have

$$O'(\tau) = \frac{(m-r)^2}{b\gamma} \int_0^\tau f(\gamma, t)f(\gamma, \tau) \left(\frac{b-1}{b}\frac{\gamma t}{\sigma^2} + \frac{b-1}{b}\frac{\gamma \tau}{\sigma^2} - \frac{\frac{b-1}{b}\frac{\gamma \tau}{\sigma^2}}{1 + \frac{b-1}{b}\frac{\gamma \tau}{\sigma^2}}(1 + \frac{b-1}{b}\frac{\gamma t}{\sigma^2}) \right.$$
$$\left. - \frac{\frac{b-1}{b}\frac{\gamma t}{\sigma^2}}{1 + \frac{b-1}{b}\frac{\gamma t}{\sigma^2}}(1 + \frac{b-1}{b}\frac{\gamma \tau}{\sigma^2}) \right) dt.$$

Set $X = \frac{b-1}{b}\frac{\gamma t}{\sigma^2}$ and $Y = \frac{b-1}{b}\frac{\gamma \tau}{\sigma^2}$. Note that $1 + X > 0$ and $1 + Y > 0$. In order to show that O' is positive, it is enough to show that

$$X + Y - \frac{X(1+Y)}{1+X} - \frac{Y(1+X)}{1+Y} > 0$$

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or equivalently

$$\frac{(Y-X)^2}{(1+X)(1+Y)} > 0,$$

which is always satisfied. So O is strictly increasing in τ and since $O(0) = 0$, we conclude that O is positive and the proof is complete.

Demand for the risky asset and finite horizon. Writing the demand for the risky asset as

$$\frac{z}{W} = \frac{m-r}{b\sigma^2} \frac{\displaystyle\int_0^\tau g(\gamma,t)dt}{\displaystyle\int_0^\tau (1 + \frac{b-1}{b}\frac{\gamma t}{\sigma^2})g(\gamma,t)dt},$$

it follows that

$$
\begin{aligned}
\frac{\partial}{\partial \tau}\left(\frac{z}{W}\right) &= \frac{m-r}{b\sigma^2} \frac{g(\gamma,\tau)\displaystyle\int_0^\tau (1 + \frac{b-1}{b}\frac{\gamma t}{\sigma^2})g(\gamma,t)dt - (1 + \frac{b-1}{b}\frac{\gamma \tau}{\sigma^2})g(\gamma,\tau)\displaystyle\int_0^\tau g(\gamma,t)dt}{D^2} \\
&= \frac{(1-b)\gamma(m-r)}{b^2\sigma^2} \frac{g(\gamma,\tau)\displaystyle\int_0^\tau (\tau-t)g(\gamma,t)dt}{D^2}.
\end{aligned}
$$

We conclude that $\left|\frac{z}{W}\right|$ is increasing (decreasing) in τ exactly when $b < 1$ ($b > 1$). $\blacksquare$

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