# An Optimal Auction with Identity-Dependent Externalities*

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**Abstract**

We analyze the problem of a seller who has multiple units of a good and faces a set of buyers with unit demands, private information, and identity-dependent externalities. We derive the seller's optimal mechanism and characterize its main properties. As an application of the model, we consider the problem of a shopping center's developer who wants to sell its stores to a set of potential firms whose willingness to pay depend on the flow of customers that will visit the mall, which is in turn affected by the composition of the firms that locate in the center. We show that a sequential selling procedure commonly used in practice is an optimal mechanism if externalities are sufficiently large.

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# 1 Introduction

Consider the problem faced by the developer of a new shopping center who wants to allocate retail space among a set of potential firms that are interested in locating in the mall. An important constraint in this allocation problem is that each firm is privately informed about some of its characteristics; e.g., some aspects of its cost of production or attributes of the demand for its product. Another feature of this problem that is of paramount importance is the existence of inter-store externalities: the mall's customer traffic (and hence the volume of sales) depends on the identities of the firms that locate in the center. Therefore, each firm's willingness to pay is also determined by the identities of the other firms that purchase some retail space. If the developer wants to maximize her profits, what is the optimal selling procedure?

This is an example of an auction problem with multiple units, private information, and identity-dependent externalities. For a contracting example that exhibits similar features, consider a firm trying to fill several positions in its R&D department; each applicant has private information about his disutility of effort, which depends also on the identities of the other people to be hired, with whom he can enjoy some complementarities. In this setting, the firm should take these features into account in the design of an optimal contract.

The present paper analyzes a simple model intended to capture the most salient aspects of the examples described above. We consider the problem of a seller who has two identical units of a good and who faces a set of potential buyers with unit demands. A buyer's valuation for the good is equal to the product of his privately known type and an externality parameter that depends on his identity and that of the buyer who obtains the other unit of the good. Using the mechanism design methodology, we characterize the optimal (profit maximizing) selling procedure in this context.

Our main results are the following. The seller should allocate the good to the pair of buyers that generates the largest sum of virtual surpluses, weighted by the external effects they enjoy. The allocation need not be ex-post efficient: first, as in the case without externalities, the seller sometimes keeps one or both units of the good; second, since the presence of external effects introduces an asymmetry in the model, the buyers who receive the good need not be the ones with the largest sum of valuations. Unlike the case without externalities, a buyer with a negative virtual surplus can receive a unit of the good so long as he creates a sufficiently large externality.

Regarding the pricing of the units, we characterize an optimal payment rule with the following features: a buyer only pays when he receives a unit, and the amount paid is equal to the value the good would have had to him had he submitted the lowest winning report, plus the sum of the increments in utility derived from the higher external benefits that the buyer would have enjoyed, had he submitted the lowest winning report that allows him to join a neighbor that provides a higher external benefit.

We also show that these results extend to more general quasi-linear utility functions that contain the multiplicative form as a special case.

As an application of the model, we consider the shopping center developer's problem. A common procedure used in practice is to sign the 'anchor stores' first (e.g., Sears, JC Penney), which are the main externality generators, and then approach the remaining firms interested in locating in the mall. The evidence also suggests that anchors receive large discounts that are increasing in the externalities they generate, and the firms that enjoy the external effects pay a premium that is also increasing in their magnitude. We characterize the equilibrium properties of the sequential procedure, and we show that it exhibits the main features documented by the empirical evidence. More importantly, we show that this is an optimal mechanism when the externality generated by the anchors is sufficiently large. For intermediate values of the external effects, however, the sequential procedure allocates a unit to an anchor store more often than the optimal mechanism.

To the best of our knowledge, this is the first paper that analyzes an optimal auction design problem with multiple units, private information, and identity-dependent externalities. As such, it is related to several strands of literature. To be sure, it is closely related to the papers by Jehiel et al. (1996, 1999), Jehiel and Moldovanu (2000), and Das Varma (2002), all of which analyze auctions with externalities.[1] Unlike our paper, these references deal with single unit auctions and focus mainly on the case in which the winner imposes a negative externality on the losers,

[1] Jehiel et al. use the mechanism design approach, and they allow the external effects to be private information as well. Jehiel and Moldovanu analyze second-price, sealed-bid auctions under the assumption that bidders interact after the auction; this interaction makes each agent's payoff to be a function of everybody's types. Das Varma studies the effects that (common knowledge) identity-dependent externalities have on bidding behavior in open ascending-bid auctions; among other results, he shows that when externalities are non-reciprocal the open auction yields a higher expected revenue to the seller than a sealed-bid auction.

making the individual rationality constraints endogenous.[2] Another related contribution is Segal (1999), who analyzes a very general model of contracting situations with externalities but under the assumption of complete information; he distinguishes between cases where the externalities are on nontraders and those in which they are present on efficient traders. Our model is an instance of a contracting situation with externalities on traders, but unlike Segal we also study the effects that private information has on the optimal contract. Regarding the literature on optimal multi-unit auctions, the paper relates to Maskin and Riley (1989), Bergin and Zhou (2001), and Levin (1997). Without externalities, the optimal mechanism we characterize reduces to the one obtained in Maskin and Riley in the unit demand case or in Bergin and Zhou with a cost function that is zero for the first two units and infinite for additional ones. Moreover, the optimal payment rule we derive is similar to the one derived by Levin in his analysis of an optimal selling mechanism for complements: in that paper, a buyer's payment internalizes the complementarities among the goods in an incremental way that resembles how our payment rule internalizes the external effects. Finally, the paper is related to the literature on pricing of space in shopping centers. Brueckner (1993) analyzes a model of retail space allocation and pricing under complete information, assuming that the developer has already chosen the firms that will locate in the mall. Among other things, he shows that the effects of the externalities on prices paid are ambiguous when the developer can perfectly discriminate; i.e., it is not necessarily true that firms pay less if they generate more externalities. Pashigian and Gould (1998) provide extensive empirical evidence on the subject: they estimate the effects of the externalities created by the anchor stores on rental prices, and they show that anchors receive rent subsidies that are increasing in the externalities created, while the rest of the firms pay rent premiums that are increasing in the external effects enjoyed. Unlike Brueckner, our model incorporates private information and allows the seller to choose the composition of the stores that will locate in the shopping center. Moreover, we show that when these features are present the characteristics of the optimal mechanism are consistent with the evidence presented by Pashigian and Gould.

The rest of the paper proceeds as follows. The next section presents the model and some

[2] Jehiel and Moldovanu (2001) analyze a general model that can accommodate multi-unit auctions with identity-dependent externalities. Their focus, however, is on efficiency rather than revenue maximization. We are grateful to Benny Moldovanu for pointing this reference out to us.

preliminary results. Section 3 characterizes the optimal mechanism, while Section 4 focuses on the application of the model to the shopping center developer's problem. Section 5 extends the results to more general quasi-linear payoffs, and Seccion 6 concludes.

## 2 The Model

There are $I+1$ risk-neutral agents: a seller, whom we call agent 0, and $I$ potential buyers, numbered $1, 2, ..., I$. The seller owns two identical units of an indivisible good, and buyers have unit demands (i.e., each one demands at most one unit of the good).[3]

The seller has zero valuation for the good.[4] Buyer $i$'s valuation for a unit depends on two factors. First, it depends on a parameter or type $\theta_i$ that is private information and it is distributed on $\Theta_i = [\underline{\theta}_i, \overline{\theta}_i]$, $0 \leq \underline{\theta}_i < \overline{\theta}_i$, with positive and atomless density $\phi_i(\cdot)$ and cumulative distribution function $\Phi_i(\cdot)$; moreover, buyers' types are independently distributed. Second, a buyer's valuation depends also on who gets the other unit. We model this by introducing a matrix of external effects $\{\alpha_{ij}\}_{1\leq i\leq I, 0\leq j\leq I}$, with $\alpha_{ij} \geq 1$, $\alpha_{ii} = \alpha_{i0} = 1$.[5] If buyer $i$ gets a unit of the good, $j$ gets the other unit, and $i$ pays the seller $-t_i$, then buyer $i$'s utility is $\alpha_{ij}\theta_i + t_i$. Notice that $i$ does not derive any extra utility if he gets the second unit, and he does not enjoy an external effect if the seller keeps the second unit. The matrix of external effects is common knowledge.

As the following examples illustrate, the model encompasses several interesting applications.

**Example 1:** Consider the case of a shopping center where each potential store owner is a branch of a large firm whose headquarters sets a uniform pricing policy for all of its stores in a particular region, a common practice of, for instance, some apparel stores. If we denote such a price by $P_i$, the privately known marginal cost of production by $c_i$, the flow of customers that would visit $i$ if $j$ were also located in the mall by $\alpha_{ij}$, and the quantity that $i$ would sell by $Q_i = k_i\alpha_{ij}$, $k_i > 0$, then $i$'s profits would be $\alpha_{ij}k_i(P_i - c_i) = \alpha_{ij}\theta_i$, where $\theta_i = k_i(P_i - c_i)$.

**Example 2:** Alternatively, suppose each firm $i$ faces a separate linear demand for the product it

[3]The extension to more than two units is immediate, albeit cumbersome in terms of notation.

[4]This is a standard simplifying assumption that can be relaxed without affecting the main results of the paper.

[5]This is just a normalization; it is only required that $\alpha_{ii} = \alpha_{i0} = \min_{i,j} \alpha_{ij}$. We only consider the case of positive externalities in this paper.

sells, and let the slope be private information and the intercept be affected by the identity of the other firm located in the shopping center; e.g. $P_i = \sqrt{\alpha_{ij}} - b_i Q_i$. Suppose the cost of production is zero. Then, firm $i$'s profit function (after choosing the optimal $Q_i$) is $\alpha_{ij} \frac{1}{4b_i} = \alpha_{ij}\theta_i$, where $\theta_i = \frac{1}{4b_i}$.[6]

**Example 3:** Finally, consider the case in which the seller is a firm that is trying to fill two positions in its R&D department. Each applicant $i$ is privately informed about his disutility of effort, which also depends on who will be the candidate $j$ that will fill the other position; e.g., $i$'s payoff would be $t_i - \alpha_{ij}\theta_i$. It is straightforward to see that a slight modification of the model described above subsumes this contracting problem with externalities as a special case.

The goal of the seller is to design a mechanism that maximizes her expected profits, taking into account that buyers have private information, that ownership entails external effects, and that participation is voluntary. If no trade occurs, then for simplicity the payoffs of all agents are normalized to zero.

By the Revelation Principle we can, without loss of generality, restrict the search for the optimal selling scheme to direct revelation mechanisms (DRM) that are incentive compatible and individually rational. In the present case, since the two units are identical and buyers have unit demands, we can describe a DRM as follows. Let $\Lambda = \{[i,j] \mid i,j = 0,1,...,I\}$ be the set of unordered pairs $[i,j]$ (that is, pairs $(i,j)$ and $(j,i)$ are regarded to be the same),[7] and let $y = (y_{[i,j]})_{0 \le i,j \le I}$ be a probability distribution over $\Lambda$: that is, $y_{[i,j]}$ is interpreted as the probability that $i$ gets one unit and $j$ gets the other unit. A DRM is a pair of functions $(y(\theta), t(\theta))$, where $y(\theta) = (y_{[i,j]}(\theta))_{0 \le i,j \le I}$, $t(\theta) = (t_0(\theta), ..., t_I(\theta))$, and $\theta = (\theta_1, ..., \theta_I)$; $y_{[i,j]}(\theta)$ is the probability that $i$ and $j$ get the units of the good when $\theta$ is the vector of announced types, and $t_i(\theta)$, $i = 1, ..., I$, is the amount of money transferred to buyer $i$ in that case.

Since $t_0(\cdot) = -\sum_{i=1}^{I} t_i(\cdot)$ and $1 - y_{[0,0]}(\cdot) = \sum_{[i,j] \neq [0,0]} y_{[i,j]}(\cdot)$, the seller's problem can be

[6] Examples 1 and 2 make the multiplicative case look more restrictive than it actually is. We could allow firms to make multiple decisions (investment, advertising, etc.) after the auction takes place, so long as the final profits are affected multiplicatively by the externality parameter.

[7] The notation is borrowed from Shiryaev (1996, p. 6). Notice that the number of unordered pairs is $\frac{(I+2)(I+1)}{2}$.

written as follows:

$$\max_{(y_{[i,j]}(\cdot))_{[i,j]\neq[0,0]},(t_i(\cdot))_{1\leq i\leq I}} E_\theta\left[-\sum_{i=1}^{I} t_i(\theta)\right] \tag{1}$$

$$\text{subject to} \quad U_i(\theta_i) \geq \theta_i \overline{v}_i(\hat{\theta}_i) + \overline{t}_i(\hat{\theta}_i) \quad \forall(i,\theta_i,\hat{\theta}_i) \tag{2}$$

$$U_i(\theta_i) \geq 0 \quad \forall(i,\theta_i) \tag{3}$$

$$y_{[i,j]}(\theta) \geq 0 \quad \forall([i,j],\theta) \tag{4}$$

$$1 - \sum_{[i,j]\neq[0,0]} y_{[i,j]}(\theta) \geq 0 \quad \forall\theta, \tag{5}$$

where $\overline{v}_i(\hat{\theta}_i) = E_{\theta_{-i}}[v_i(\hat{\theta}_i,\theta_{-i})] = E_{\theta_{-i}}[\sum_{j=0}^{I} \alpha_{ij} y_{[i,j]}(\hat{\theta}_i,\theta_{-i})]$ is the expected external effect buyer $i$ enjoys if he reports $\hat{\theta}_i$; $\overline{t}_i(\hat{\theta}_i) = E_{\theta_{-i}}[t_i(\hat{\theta}_i,\theta_{-i})]$ is $i$'s expected transfer if he reports $\hat{\theta}_i$; and $U_i(\theta_i) = \theta_i\overline{v}_i(\theta_i) + \overline{t}_i(\theta_i)$ is buyer $i$'s expected utility if his type is $\theta_i$ and he reports it truthfully.

Using a procedure that is standard in the literature on optimal auctions, we can simplify the seller's problem as follows. First, the incentive compatibility constraints (2) can be replaced by the following conditions:

**Lemma 1 (Myerson)** *A selling mechanism is incentive compatible if and only if for* $i = 1, 2, ..., I$

*(i)* $\overline{v}_i(\cdot)$ *is increasing;*[8] *and*

*(ii)* $U_i(\theta_i) = U_i(\underline{\theta}_i) + \int_{\underline{\theta}_i}^{\theta_i} \overline{v}_i(s)ds$, $\forall \theta_i \in \Theta_i$.

Second, the lemma reveals that the individual rationality constraints (3) are satisfied if and only if $U_i(\underline{\theta}_i) \geq 0$.

Third, since $-\overline{t}_i(\theta_i) = \theta_i\overline{v}_i(\theta_i) - U_i(\theta_i)$, we can use condition $(ii)$ in the lemma and rewrite the objective function as follows:

$$\begin{aligned} E_\theta\left[-\sum_{i=1}^{I} t_i(\theta)\right] &= \sum_{i=1}^{I} E_{\theta_i}\left[\theta_i\overline{v}_i(\theta_i) - \int_{\underline{\theta}_i}^{\theta_i} \overline{v}_i(s)ds\right] - \sum_{i=1}^{I} U_i(\underline{\theta}_i) \\ &= \sum_{i=1}^{I} E_{\theta_i}\left[\left(\theta_i - \frac{1-\Phi_i(\theta_i)}{\phi_i(\theta_i)}\right)\overline{v}_i(\theta_i)\right] - \sum_{i=1}^{I} U_i(\underline{\theta}_i) \\ &= \sum_{i=1}^{I} E_\theta\left[\left(\theta_i - \frac{1-\Phi_i(\theta_i)}{\phi_i(\theta_i)}\right)\left(\sum_{j=0}^{I} \alpha_{ij} y_{[i,j]}(\theta)\right)\right] - \sum_{i=1}^{I} U_i(\underline{\theta}_i), \end{aligned} \tag{6}$$

where the second line follows by integration by parts, and the last one uses the definition of $\overline{v}_i(\cdot)$.

[8] Throughout the paper, increasing is used in the weak sense.

Fourth, it is clear from the objective function that $U_i(\underline{\theta}_i) = 0$ at the optimum for $i = 1, 2, ..., I$.

Therefore, the seller's problem becomes:

$$\max_{(y_{[i,j]}(\cdot))_{[i,j]\neq[0,0]}} \sum_{i=1}^{I} E_\theta \left[ \left( \theta_i - \frac{1 - \Phi_i(\theta_i)}{\phi_i(\theta_i)} \right) \left( \sum_{j=0}^{I} \alpha_{ij} y_{[i,j]}(\theta) \right) \right] \tag{7}$$

subject to (4)-(5) and condition $(i)$ in Lemma 1.

# 3 Main Results

Consider the 'relaxed problem' of maximizing (7) subject to (4)-(5) only. Let

$$J_i(\theta_i) = \theta_i - \frac{1 - \Phi_i(\theta_i)}{\phi_i(\theta_i)},$$

and assume $J_i(\cdot)$ is a strictly increasing function. Rewrite the seller's objective function as follows:

$$E_\theta \left[ \sum_{i=1}^{I} \sum_{j=0}^{I} J_i(\theta_i) \alpha_{ij} y_{[i,j]}(\theta) \right]. \tag{8}$$

By inspection, the seller's relaxed problem is equivalent to solving, for each $\theta = (\theta_1, ..., \theta_I)$,

$$\max_{(y_{[i,j]}(\theta))_{[i,j]\neq[0,0]}} \sum_{i=1}^{I} \sum_{j=0}^{I} J_i(\theta_i) \alpha_{ij} y_{[i,j]}(\theta) \tag{9}$$

subject to (4)-(5).

Straightforward algebra reveals that the objective funtion in (9) is equal to (recall $\alpha_{i0} = \alpha_{ii} = 1$)

$$\sum_{i=1}^{I} J_i(\theta_i) y_{[i,0]}(\theta) + \sum_{i \geq j \geq 1} J_{[i,j]}(\theta_i, \theta_j) y_{[i,j]}(\theta), \tag{10}$$

where, with a slight abuse of notation, we have set

$$\begin{aligned} J_{[i,j]}(\theta_i, \theta_j) &= \alpha_{ij} J_i(\theta_i) + \alpha_{ji} J_j(\theta_j) \\ J_{[i,i]}(\theta_i, \theta_i) &= J_i(\theta_i). \end{aligned}$$

It is evident that the solution to this linear programming problem is given by the following *allocation rule*:

$$y^*_{[i,j]}(\theta) = \begin{cases} 1 & \text{if } J_{[i,j]}(\theta_i, \theta_j) \geq \max\{0, \max_i J_i(\theta_i), \max_{[l,k], l\neq k, l,k\geq 1} J_{[l,k]}(\theta_l, \theta_k)\} \\ 0 & \text{otherwise} \end{cases} \tag{11}$$

In words, the allocation is as follows: *"Given the buyers' reports, the seller chooses the pair that generates the largest combined virtual surplus weighted by the external effects. If this number is nonnegative and greater than the virtual surplus of every single buyer, then the seller allocates the goods to the aforementioned pair of buyers; otherwise, she gives the units (or only one of them) to the buyer who has the largest nonnegative virtual surplus. If there is no buyer or pair of buyers with nonnegative (combined) virtual surplus, the seller keeps the two units of the good."*

In order to prove that this is an optimal allocation rule for the seller, we still need to show that it satisfies condition (i) of Lemma 1. If $y^*_{[i,j]}(\cdot)$ were increasing in $\theta_i$ for any given $\theta_{-i}$, then condition (i) would easily follow. The next example reveals that $y^*_{[i,j]}(\cdot)$ need not be increasing in $\theta_i$ when externalities are present:

**Example 4:** There are three bidders with valuations distributed uniformly on $[0,1]$, so $J_i(\theta_i) = 2\theta_i - 1$. Let $\theta = (\theta_1, \theta_2, \theta_3)$, with $\theta_i > \frac{1}{2}$ for all $i$. Suppose the following conditions hold: (i) $y^*_{[1,2]}(\theta_1, \theta_{-1}) = 1$, (ii) $\alpha_{13} > \alpha_{12}$, (iii) $\alpha_{21}(2\theta_2 - 1) > \alpha_{31}(2\theta_3 - 1)$, (iv) $\frac{\alpha_{21}(2\theta_2-1)-\alpha_{31}(2\theta_3-1)}{\alpha_{13}-\alpha_{12}} < 1$. Under these assumptions, we will show that there exists a type $\theta_1' > \theta_1$, such that $y^*_{[1,2]}(\theta_1', \theta_{-1}) = 0$ and $y^*_{[1,3]}(\theta_1', \theta_{-1}) = 1$, proving that $y^*_{[1,2]}(\cdot, \theta_{-1})$ cannot be increasing. It suffices to show that there exist values $\theta_1 > \frac{1}{2}$ and $\theta_1' \in (\theta_1, 1]$ such that the following inequalities are satisfied:

$$\begin{aligned}
\alpha_{12}(2\theta_1 - 1) + \alpha_{21}(2\theta_2 - 1) &> \alpha_{13}(2\theta_1 - 1) + \alpha_{31}(2\theta_3 - 1) \\
\alpha_{13}(2\theta_1' - 1) + \alpha_{31}(2\theta_3 - 1) &> \alpha_{12}(2\theta_1' - 1) + \alpha_{21}(2\theta_2 - 1).
\end{aligned}$$

These inequalities and (ii) yield

$$\theta_1' > \frac{1}{2}\left(1 + \frac{\alpha_{21}(2\theta_2 - 1) - \alpha_{31}(2\theta_3 - 1)}{\alpha_{13} - \alpha_{12}}\right) > \theta_1.$$

But (iv) ensures that the value of the squeezed term belongs to the interval $(\frac{1}{2}, 1)$; therefore there exist numbers $\theta_1$ and $\theta_1'$ with the aforementioned properties. For a numerical illustration, set $\theta_2 = 0.8$, $\theta_3 = 0.6$, $\theta_1 = 0.7$, $\theta_1' = 0.95$, $\alpha_{21} = \alpha_{31} = 1$, $\alpha_{12} = 1.5$, $\alpha_{13} = 2$, and $\alpha_{23} = \alpha_{32} = 1.3$.

This example notwithstanding, it is easy to prove that the allocation rule $y^*_{[i,j]}(\cdot)$ is indeed optimal.

**Proposition 1** *The allocation rule given by (11) satisfies condition (i) in Lemma 1 and it is therefore optimal for the seller.*

**Proof:** To show that $\overline{v}_i(\cdot)$ is increasing, it is enough to prove that $v_i(\cdot, \theta_{-i}) = \sum_{j=0}^{I} \alpha_{ij} y^*_{[i,j]}(\cdot, \theta_{-i})$ is increasing. Take $\theta_i' > \theta_i''$; we will show that $v_i(\theta_i', \theta_{-i}) \geq v_i(\theta_i'', \theta_{-i})$. The result follows trivially when $v_i(\theta_i'', \theta_{-i}) = 0$, since $v_i(\cdot, \theta_{-i})$ is a nonnegative function. Thus, without loss of generality suppose $v_i(\theta_i'', \theta_{-i}) = \alpha_{il}$.

Notice that it must be the case that $v_i(\theta_i', \theta_{-i}) > 0$, for $\alpha_{il} J_i(\theta_i') + \alpha_{li} J_l(\theta_l) > \alpha_{il} J_i(\theta_i'') + \alpha_{li} J_l(\theta_l)$ implies that, if the seller finds it optimal to allocate a unit to $i$ when he reports $\theta_i''$, she must find it optimal to do so under $\theta_i'$ (recall $\theta_{-i}$ is kept fixed). Therefore, $v_i(\theta_i', \theta_{-i}) = \alpha_{ik}$ for some $k$.

To complete the proof, we show that $\alpha_{ik} \geq \alpha_{il}$. Since $y^*_{[i,l]}(\theta_i'', \theta_{-i}) = 1$ and $y^*_{[i,k]}(\theta_i', \theta_{-i}) = 1$, it follows that

$$
\begin{aligned}
\alpha_{il} J_i(\theta_i'') + \alpha_{li} J_l(\theta_l) &\geq \alpha_{ik} J_i(\theta_i'') + \alpha_{ki} J_k(\theta_k) \\
\alpha_{ik} J_i(\theta_i') + \alpha_{ki} J_k(\theta_k) &\geq \alpha_{il} J_i(\theta_i') + \alpha_{li} J_l(\theta_l). \qquad (12)
\end{aligned}
$$

These inequalities yield $(\alpha_{il} - \alpha_{ik})(J_i(\theta_i'') - J_i(\theta_i')) \geq 0$; but $J_i(\cdot)$ is strictly increasing and hence $\alpha_{ik} \geq \alpha_{il}$. ■

**Example 5:** Consider the special case with no external effects; i.e., $\alpha_{ij} = 1$ for all $i = 1, 2, ..., I$, $j = 0, 1, ..., I$. The allocation rule becomes

$$
y^*_{[i,j]}(\theta) = \begin{cases} 1 & \text{if } J_i(\theta_i) + J_j(\theta_j) \geq \max\{0, \max_i J_i(\theta_i), \max_{[l,k], l \neq k, l,k \geq 1} J_l(\theta_l) + J_k(\theta_k)\} \\ 0 & \text{otherwise} \end{cases}
$$

It follows that $y^*_{[i,j]}(\theta) = 1$ if and only if (i) $J_i(\theta_i) \geq 0$, (ii) $J_j(\theta_j) \geq 0$, (iii) $J_i(\theta_i) \geq J_k(\theta_k)$ for $k \neq j$, and (iv) $J_j(\theta_j) \geq J_k(\theta_k)$ for $k \neq i$. Thus, the seller simply allocates the units of the good to the buyers with the largest virtual surpluses, as in Maskin and Riley (1989). In particular, they show that if types are identically distributed then a standard auction with a reserve price is an optimal mechanism.

Notice that, unlike the single object and the multi-unit cases without externalities, it could happen under the allocation rule (11) that a buyer receives a unit despite having $J_i(\theta_i) < 0$, so long as he is paired with another buyer who enjoys a large external effect from being with $i$. In the special case in which $J_i(\underline{\theta}) > 0$ for every $i$, the allocation rule simply instructs the seller to allocate the goods to the pair of buyers with the largest combined virtual surplus.

To complete the characterization of an optimal mechanism, we need a *payment rule* that, along with the allocation rule (11), constitute an optimal selling procedure. From Lemma 1, it is enough to find one that guarantees incentive compatibility. Using condition (ii) in the lemma and the definition of $U_i(\theta_i)$ we obtain

$$-\bar{t}_i(\theta_i) = \theta_i \bar{v}_i(\theta_i) - \int_{\underline{\theta}_i}^{\theta_i} \bar{v}_i(s) ds, \tag{13}$$

which is equivalent to

$$E_{\theta_{-i}}\left[-t_i(\theta_i, \theta_{-i})\right] = E_{\theta_{-i}}\left[\theta_i v_i(\theta_i, \theta_{-i}) - \int_{\underline{\theta}_i}^{\theta_i} v_i(s, \theta_{-i}) ds\right]. \tag{14}$$

Therefore,

$$\begin{aligned} -t_i^*(\theta_i, \theta_{-i}) &= \theta_i v_i(\theta_i, \theta_{-i}) - \int_{\underline{\theta}_i}^{\theta_i} v_i(s, \theta_{-i}) ds \\ &= \theta_i \sum_{j=0}^{I} \alpha_{ij} y^*_{[i,j]}(\theta_i, \theta_{-i}) - \int_{\underline{\theta}_i}^{\theta_i} \left( \sum_{j=0}^{I} \alpha_{ij} y^*_{[i,j]}(s, \theta_{-i}) \right) ds \end{aligned} \tag{15}$$

for every $i$ and every $\theta = (\theta_i, \theta_{-i})$, is an optimal payment rule. We have thus shown the following result.

**Theorem 1** *The mechanism $(y^*, t^*)$ is an optimal selling procedure.*

Let's characterize the payment rule further. In the proof of Proposition 1, we show that $v_i(\cdot, \theta_{-i}) = \sum_{j=0}^{I} \alpha_{ij} y^*_{[i,j]}(\cdot, \theta_{-i})$ is an increasing function. Given the shape of $y^*_{[i,j]}(\cdot)$, it is actually a step function that, without loss of generality, can be assumed to be right-continuous. Let $\{\theta_i^1, \theta_i^2, ..., \theta_i^n\}$, with $\theta_i^1 < \theta_i^2 < ... < \theta_i^n$, be the set of points where the function jumps. Notice that, given $\theta_{-i}$, $\theta_i^1$ is the smallest type that $i$ could report and still obtain a unit of the good. It is evident from (15) that if $\theta_i < \theta_i^1$ then $-t_i^*(\theta_i, \theta_{-i}) = 0$. Suppose $\theta_i \geq \theta_i^1$, and let $j^1, j^2, ..., j^n$ be the identities of $i$'s neighbors at the jump points. Notice that $\alpha_{ij^1} < \alpha_{ij^2} < ... < \alpha_{ij^n}$ by the monotonicity of $v_i(\cdot, \theta_{-i})$. Then (setting $\theta_i^{n+1} = \theta_i$),

$$\int_{\underline{\theta}_i}^{\theta_i} v_i(s, \theta_{-i}) ds = \sum_{p=1}^{n} \alpha_{ij^p} (\theta_i^{p+1} - \theta_i^p),$$

and therefore,

$$
\begin{aligned}
-t_i^*(\theta_i, \theta_{-i}) &= \theta_i \alpha_{ij^n} - \sum_{p=1}^{n} \alpha_{ij^p} (\theta_i^{p+1} - \theta_i^p) \\
&= \theta_i^1 \alpha_{ij^1} + \sum_{p=1}^{n-1} (\alpha_{ij^{p+1}} - \alpha_{ij^p}) \theta_i^{p+1}. \qquad (16)
\end{aligned}
$$

Figure 1 provides an illustration of the payment rule in the case of three alternative neighbors with whom $i$ could be paired given $(\theta_i, \theta_{-i})$.

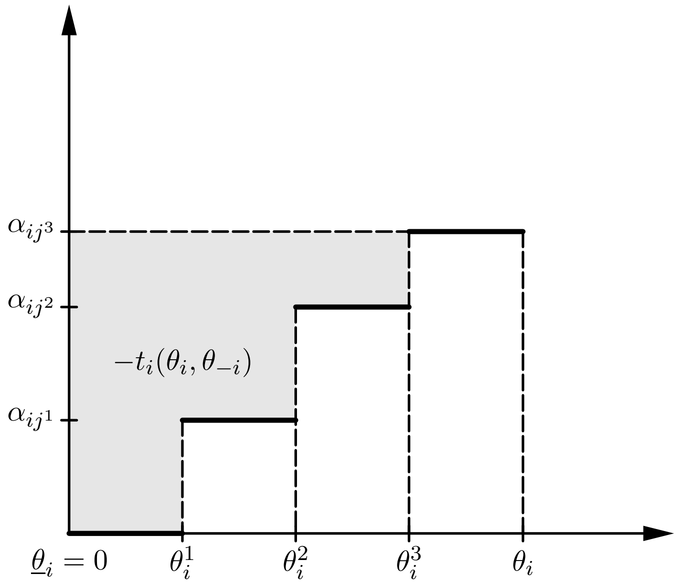


Figure 1

In summary, the optimal payment rule characterized above is given by

$$
-t_i^*(\theta_i, \theta_{-i}) = \begin{cases} \theta_i^1 \alpha_{ij^1} + \sum_{p=1}^{n-1} (\alpha_{ij^{p+1}} - \alpha_{ij^p}) \theta_i^{p+1} & \text{if } \theta_i \geq \theta_i^1 \\ 0 & \text{if } \theta_i < \theta_i^1. \end{cases} \qquad (17)
$$

The interpretation is the following. Suppose that, given $(\theta_i, \theta_{-i})$, $i$ obtains a unit of the good and $j^n$ receives the other unit. Then the amount that $i$ pays for the unit is the sum of two components: (i) he pays an amount $\alpha_{ij^1}\theta_i^1$, which is the value the good would have had to him had he submitted the lowest winning report given $\theta_{-i}$, namely $\theta_i^1$; (ii) he pays the sum of the increments in utility derived from the higher external benefits that $i$ would have enjoyed, had he submitted the lowest winning report that allows him to join a neighbor who provides a higher external benefit. For instance, the smallest winning report above $\theta_i^1$ that pairs $i$ with a neighbor that generates higher externalities than $j^1$ is $\theta_i^2$, and the incremental utility $i$ gets is $(\alpha_{ij^2} - \alpha_{ij^1})\theta_i^2$.

**Example 6:** Suppose there are three bidders and $J_i(\underline{\theta}_i) > 0$ for $i = 1, 2, 3$. Without loss of generality, let $\theta = (\theta_1, \theta_2, \theta_3)$ be a vector of reports such that $y^*_{[1,2]}(\theta) = 1$. That is, (i) $\alpha_{12}J_1(\theta_1) + \alpha_{21}J_2(\theta_2) \geq \alpha_{13}J_1(\theta_1) + \alpha_{31}J_3(\theta_3)$ and (ii) $\alpha_{12}J_1(\theta_1) + \alpha_{21}J_2(\theta_2) \geq \alpha_{23}J_2(\theta_2) + \alpha_{32}J_3(\theta_3)$. As an illustration, let us calculate $-t_1(\theta)$. There are two cases to consider:

**Case 1:** Suppose $\alpha_{12} < \alpha_{13}$. Then it must be the case that $\alpha_{21}J_2(\theta_2) > \alpha_{31}J_3(\theta_3)$, for otherwise the seller would prefer $[1,3]$ to $[1,2]$. Buyer 1's payment is

$$-t_1(\theta) = \theta_1^1 \alpha_{12}, \tag{18}$$

with $\theta_1^1 = \underline{\theta}_1$ if $\alpha_{12}J_1(\underline{\theta}_1) + \alpha_{21}J_2(\theta_2) \geq \alpha_{23}J_2(\theta_2) + \alpha_{32}J_3(\theta_3)$, and $\theta_1^1 = \hat{\theta}_1$ otherwise, where $\hat{\theta}_1 = J_1^{-1}(\frac{(\alpha_{23}-\alpha_{21})J_2(\theta_2)+\alpha_{32}J_3(\theta_3)}{\alpha_{12}})$ is the value of $\theta_1$ that satisfies (ii) with equality (see Figure 2). Notice that $\frac{\partial \hat{\theta}_1}{\partial \alpha_{21}} < 0$; that is, buyer 1 pays *less* the higher the external effect he imposes on buyer 2.

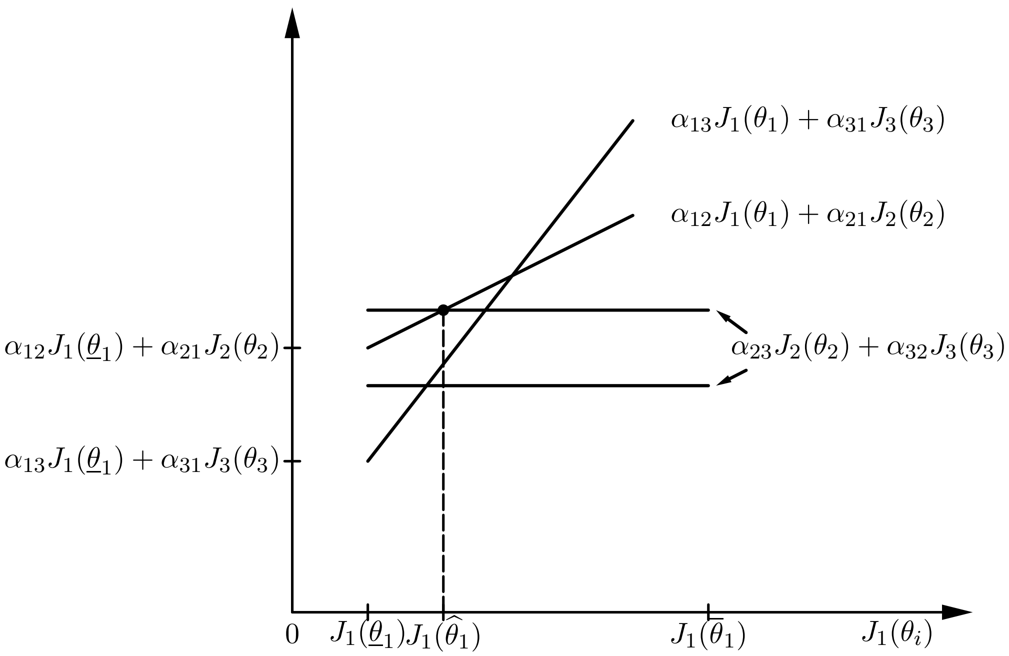


Figure 2

**Case 2:** Suppose $\alpha_{12} \geq \alpha_{13}$. In this case, either $\alpha_{21}J_2(\theta_2) \geq \alpha_{31}J_3(\theta_3)$ and we are back in Case 1, or $\alpha_{21}J_2(\theta_2) < \alpha_{31}J_3(\theta_3)$ and there are two subcases to consider, namely $\alpha_{13}J_1(\underline{\theta}) + \alpha_{31}J_3(\theta_3) \geq \alpha_{23}J_2(\theta_2) + \alpha_{32}J_3(\theta_3)$ and $\alpha_{13}J_1(\underline{\theta}) + \alpha_{31}J_3(\theta_3) < \alpha_{23}J_2(\theta_2) + \alpha_{32}J_3(\theta_3)$.

If $\alpha_{13}J_1(\underline{\theta}) + \alpha_{31}J_3(\theta_3) \geq \alpha_{23}J_2(\theta_2) + \alpha_{32}J_3(\theta_3)$, then

$$-t_1(\theta) = \theta_1^1 \alpha_{13} + (\alpha_{12} - \alpha_{13})\theta_1^2, \tag{19}$$

with $\theta_1^1 = \underline{\theta}_1$ and $\theta_1^2 = \tilde{\theta}_1$, where $\tilde{\theta}_1 = J_1^{-1}(\frac{\alpha_{31}J_3(\theta_3)-\alpha_{21}J_2(\theta_2)}{\alpha_{12}-\alpha_{13}})$ is the value of $\theta_1$ that satisfies (i) with equality (see Figure 3). As in the previous case, $\frac{\partial \tilde{\theta}_1}{\partial \alpha_{21}} < 0$, which makes buyer 1's payment a decreasing function of the external effect that buyer 2 enjoys.

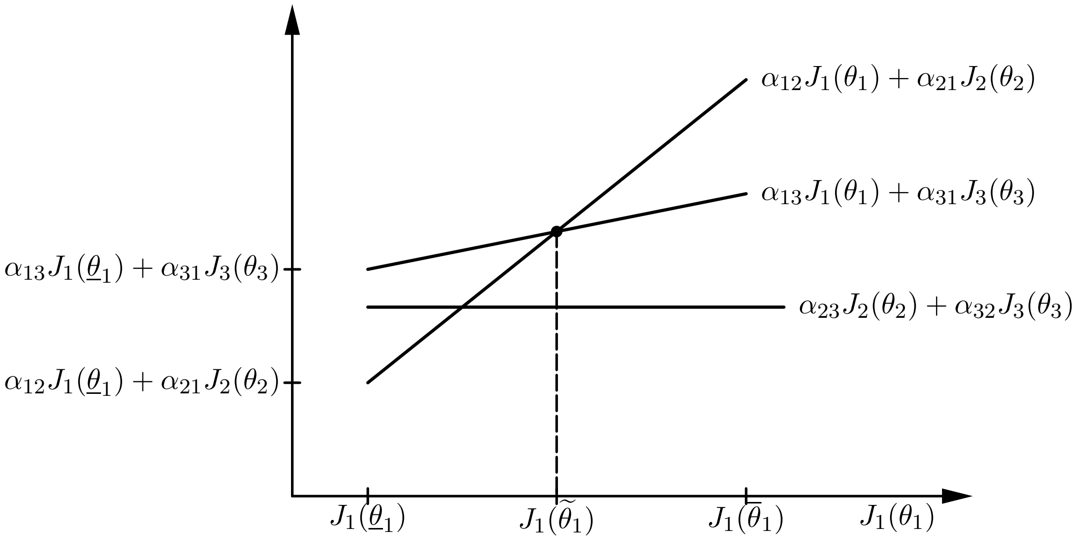

Figure 3

If $\alpha_{13}J_1(\underline{\theta}_1) + \alpha_{31}J_3(\theta_3) < \alpha_{23}J_2(\theta_2) + \alpha_{32}J_3(\theta_3)$, then either $\hat{\theta}_1 \geq \tilde{\theta}_1$ and $-t_1(\theta)$ is given by (18) with $\theta_1^1 = \hat{\theta}_1$, or $\hat{\theta}_1 < \tilde{\theta}_1$ and $-t_1(\theta)$ is given by (19) with $\theta_1^1 = \hat{\theta}_1$ and $\theta_1^2 = \tilde{\theta}_1$ (see Figure 4).

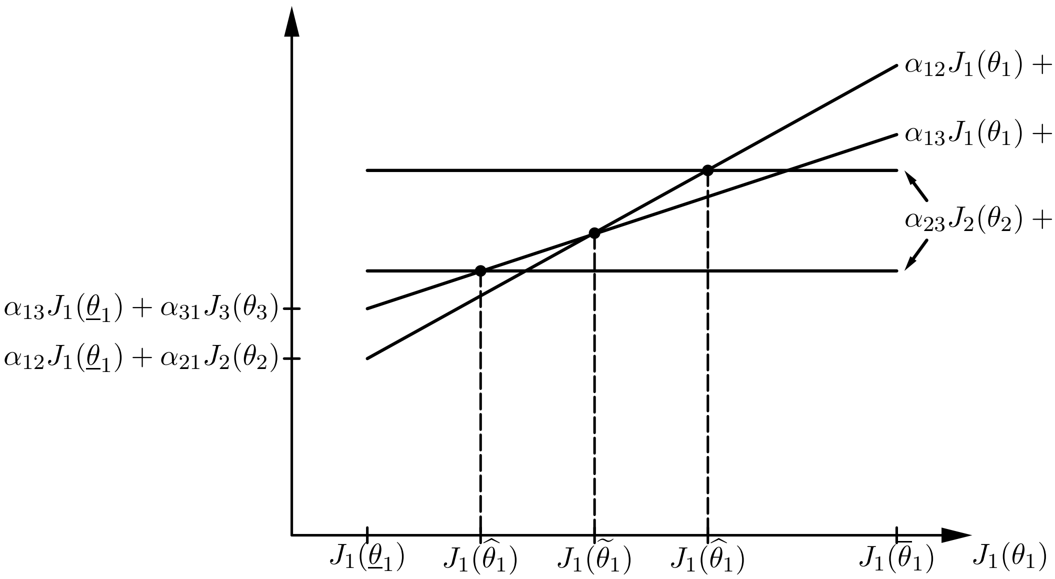

Figure 4

To conclude this section, notice that the allocation rule $y^*_{[i,j]}(\cdot)$ *need not* yield an (ex-post) efficient allocation of the goods. If types were common knowledge, the seller's optimal mechanism is to allocate the units of the goods to the pair $[i, j]$ with the largest $\alpha_{ij}\theta_i + \alpha_{ji}\theta_j$, and to charge $\alpha_{ij}\theta_i$ to buyer $i$ and $\alpha_{ji}\theta_j$ to $j$, thereby extracting all the surplus from the buyers who obtain the units of the good. Notice that the allocation that ensues is *efficient*. Moreover, since $\underline{\theta}_i \geq 0$ for all $i$ and $\alpha_{ij} \geq 1$, the seller should *always* sell the two units to *different* buyers. Under the optimal allocation rule $y^*_{[i,j]}(\cdot)$, however, the seller may keep one or both units, and even when she sells both units, she need not allocate them to the pair of buyers $[i, j]$ with the largest $\alpha_{ij}\theta_i + \alpha_{ji}\theta_j$, as the following example illustrates.

**Example 7:** There are three bidders with valuations distributed uniformly on $[\underline{\theta}, \overline{\theta}]$, $2\underline{\theta} > \overline{\theta}$, which implies that $J_i(\theta_i) = 2\theta_i - \overline{\theta} > 0$ for all $\theta_i$ and for all $i$; hence, the seller always sells both units. Suppose that the external effects are given by $\alpha_{21} = \alpha_{31} = \alpha > 1$, $\alpha_{12} = \alpha_{13} = \alpha_{23} = \alpha_{32} = 1$; that is, buyer 1 is the only one who generates externalities. Consider a vector of reports $\theta = (\theta_1, \theta_2, \theta_3)$, with $\theta_2 > \theta_3$. Notice that in this case it will never be optimal for the seller to allocate the goods to $[1, 3]$. It is easy to verify that the goods will be sold to $[1, 2]$ if and only if $(\theta_1 + \alpha\theta_2) - (\theta_2 + \theta_3) \geq \frac{\overline{\theta}}{2}(\alpha - 1) > 0$. Thus, it could happen that the seller allocates the two units to $[2, 3]$ even though $(\theta_1 + \alpha\theta_2) > (\theta_2 + \theta_3)$; i.e., the allocation is inefficient. For a numerical illustration, set $\underline{\theta} = 1.05$, $\overline{\theta} = 2$, $\alpha = 1.1$, $\theta_1 = 1.07$, $\theta_2 = 1.2$, and $\theta_3 = 1.1$.

Intuitively, the inefficiency illustrated in the example is a consequence of the *asymmetry* introduced by the presence of external effects in the allocation process. Another consequence of the asymmetric nature of the model is that it makes it extremely difficult to find an *indirect* mechanism that implements the optimal auction $(y^*, t^*)$, a daunting task even in the asymmetric single-unit case with no externalities. In the next section, we focus on a particular application of the model in order to shed some light on the optimality properties of a commonly used procedure.

## 4 Application: Selling a Shopping Center's Stores

As an application of the model, consider the problem of the owner (developer) of a shopping center who wants to sell its stores to a set of potential firms that are interested in locating there. The literature suggests that inter-store externalities are of paramount importance in the determination of the prices paid and of the composition of the firms that are awarded the stores.[9] Indeed, a common procedure used in practice by shopping mall developers is to first sign at lower per square foot prices the 'anchor stores' (e.g., Sears, JC Penney), which are the ones that typically generate the largest externalities, and only then the seller offers the remaining space to the rest of the potential neighboring firms (Pashigian and Gould (1993), pp. 119 and 130). The price paid by an anchor store is a *decreasing* function of the externalities it creates, and the prices paid by the rest of the firms are *increasing* functions of the size of the external effects they enjoy (Pashigian and Gould (1993), Sections IV and V, especially pp. 126-128 and 135).

To cast some light on the properties of this sequential mechanism, we will consider the case of three potential buyers in which only buyer 1 generates externalities; formally, $\alpha_{21} = \alpha_{31} = \alpha > 1$, $\alpha_{12} = \alpha_{13} = \alpha_{23} = \alpha_{32} = 1$. For simplicity, let $\theta_i$ be distributed on $[\underline{\theta}, \overline{\theta}]$ with density $\phi(\cdot)$ for all $i$, and let $\underline{\theta}\phi(\underline{\theta}) > 1$ (this ensures that $J(\underline{\theta}) > 0$ and hence $J(\theta_i) > 0$ for all $\theta_i$, $i = 1, 2, 3$).

The seller proceeds as follows: in the first stage, she makes a take-it-or-leave-it offer to buyer 1. If buyer 1 accepts, he obtains one store and the seller then uses a first-price auction to allocate the remaining unit between buyers 2 and 3 in the second stage. If buyer 1 rejects, then the seller sells one unit to buyer 2 and one unit to buyer 3 at a price $\underline{\theta}$ per unit in the second stage. Notice that, given the assumptions made, the seller uses an optimal mechanism in each possible case in which she deals with buyers 2 and 3.[10]

Consider the second stage. If there are two units left, buyers 2 and 3 accept the offer and the seller's revenue is equal to $2\underline{\theta}$. If there is only one unit left, then the buyer with the highest type between 2 and 3 receives the store; it is straightforward to see that the seller's expected revenue in this case is $2\alpha \int_{\underline{\theta}}^{\overline{\theta}} s(1 - \Phi(s))\phi(s)ds$, which is equal to the expected value of $\min\{\theta_2, \theta_3\}$ times the external effect.

In the first stage, the seller solves

$$\max_{\theta_1^* \geq \underline{\theta}} \; (1 - \Phi(\theta_1^*)) \left( \theta_1^* + 2\alpha \int_{\underline{\theta}}^{\overline{\theta}} s(1 - \Phi(s))\phi(s)ds \right) + \Phi(\theta_1^*)2\underline{\theta}, \tag{20}$$

where $\theta_1^*$ is the take-it-or-leave-it offer tendered to buyer 1.

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[9]See Pashigian and Gould (1993) for empirical evidence on the subject and for estimates of the size of the external effects.

[10]If $J(\underline{\theta}) < 0$, then the only modification needed is to introduce a 'reserve price' $\theta_r$ in the second stage determined by $J(\theta_r) = 0$.

The solution to this problem is the following: if $\alpha \geq \alpha^*$, then $\theta_1^* = \underline{\theta}$, while if $\alpha < \alpha^*$, then $\theta_1^*$ is the unique solution to

$$J(\theta_1^*) = 2\underline{\theta} - 2\alpha \int_{\underline{\theta}}^{\overline{\theta}} s(1 - \Phi(s))\phi(s)ds. \tag{21}$$

The threshold value of the external effect, $\alpha^*$, is given by

$$\alpha^* = \frac{\underline{\theta}\phi(\underline{\theta}) + 1}{2\phi(\underline{\theta}) \int_{\underline{\theta}}^{\overline{\theta}} s(1 - \Phi(s))\phi(s)ds}. \tag{22}$$

The properties of the sequential mechanism can be summarized as follows. Straightforward differentiation of (21) reveals that the seller makes buyer 1 an offer $\theta_1^*$ that is *decreasing* in the size of the external effect he generates; if he accepts the offer, then the price paid for the remaining unit is an *increasing* function of $\alpha$. These properties are consistent with the empirical evidence on the subject.

When the size of the externality is relatively small, there is a positive probability that buyer 1 will not obtain a store. However, if the externality that buyer 1 generates is sufficiently large, namely $\alpha \geq \alpha^*$, then the seller ensures that buyer 1 receives a unit with probability one, and her expected revenue is $\underline{\theta} + 2\alpha \int_{\underline{\theta}}^{\overline{\theta}} s(1 - \Phi(s))\phi(s)ds$.

It is worth pointing out that these features depend crucially on the existence of private information on the part of the buyers. For instance, if buyers' types were common knowledge and if the seller decided to sell a unit to buyer 1, then she would do it with probability one and would extract all of the surplus; the price paid by buyer 1 would be *independent* of the external effect he creates. That is, there would be no 'discounts' to anchors.

Let's compare the results of the sequential procedure with the optimal mechanism characterized in the previous section and analyzed in Example 6 for the case of three buyers with $J(\underline{\theta}) > 0$. The comparison relies on the following result.

**Lemma 2** *In the optimal mechanism buyer 1 obtains a unit of the good with probability one if and only if* $\alpha \geq \hat{\alpha} = \frac{\overline{\theta}\phi(\theta)}{\underline{\theta}\phi(\underline{\theta}) - 1}$.

**Proof:** We need to find the smallest value of $\alpha$ such that the following inequalities hold for every

$(\theta_1, \theta_2, \theta_3)$:

$$J(\theta_1) + \alpha J(\theta_2) \geq J(\theta_2) + J(\theta_3) \tag{23}$$

$$J(\theta_1) + \alpha J(\theta_3) \geq J(\theta_2) + J(\theta_3). \tag{24}$$

Consider (23); it holds if and only if, for every $(\theta_1, \theta_2, \theta_3)$,

$$\alpha \geq \frac{J(\theta_2) + J(\theta_3) - J(\theta_1)}{J(\theta_3)}.$$

In particular, this must hold for every $\theta = (\underline{\theta}, \overline{\theta}, \theta_3)$; that is,

$$\alpha \geq \frac{J(\overline{\theta}) - J(\underline{\theta}) + J(\theta_3)}{J(\theta_3)}.$$

The right side of this expression is of the form $f(x) = \frac{a+x}{x}$, with $a > 0$; notice that $\frac{\partial f(x)}{\partial x} < 0$. Hence, the right side achieves its maximum at $\theta_3 = \underline{\theta}$, and its value is $\frac{J(\overline{\theta})}{J(\underline{\theta})} = \frac{\overline{\theta}\phi(\underline{\theta})}{\underline{\theta}\phi(\underline{\theta})-1} = \hat{\alpha}$. Therefore, (23) holds if and only if $\alpha \geq \hat{\alpha}$, and a similar analysis reveals that this condition is necessary and sufficient for (24) as well. ■

It is straightforward to calculate the payments when $\alpha \geq \hat{\alpha}$ (see Example 6). Since buyer 1 always receives one store, his smallest winning report is $\underline{\theta}$, and this is the amount he pays. Buyer 2 obtains the other unit if and only if $\theta_2 \geq \theta_3$; the smallest report that would make him obtain the remaining unit in this case is $\theta_3$, and he therefore pays $\alpha\theta_3$. This yields an expected revenue for the seller equal to $\underline{\theta} + \alpha E[\min\{\theta_2, \theta_3\}]$. Notice that the allocation and payments under the optimal mechanism when $\alpha \geq \hat{\alpha}$ are the *same* as in the sequential mechanism when $\alpha \geq \alpha^*$. We have thus proved the following result:

**Proposition 2** *If $\alpha \geq \max\{\hat{\alpha}, \alpha^*\}$, then the sequential mechanism is optimal.*

Notice that $\hat{\alpha} - \alpha^* > 0$ if and only if

$$1 - \underline{\theta}^2\phi(\underline{\theta})^2 + \overline{\theta}\phi(\underline{\theta})^2 2\int_{\underline{\theta}}^{\overline{\theta}} s(1 - \Phi(s))\phi(s)ds > 0. \tag{25}$$

But $2\int_{\underline{\theta}}^{\overline{\theta}} s(1 - \Phi(s))\phi(s)ds = E[\min\{\theta_2, \theta_3\}] \geq \underline{\theta}$; hence, the left side of (25) is greater than $1 + \underline{\theta}\phi(\underline{\theta})^2(\overline{\theta} - \underline{\theta}) > 0$, thereby proving that $\hat{\alpha} > \alpha^*$. Thus,

**Proposition 3** *If $\alpha^* \leq \alpha \leq \hat{\alpha}$, then the sequential mechanism allocates a unit to buyer 1 more often than the optimal mechanism.*

The analysis reveals that the common procedure used in practice is in fact an *optimal* mechanism for the seller for sufficiently large values of the external effect. For lower values, however, the sequential procedure need not be optimal; there is a range of values for the externality in which the seller can increase her expected revenue by allocating the two units to buyers 2 and 3 with positive probability, thereby raising the amount buyer 1 pays whenever he obtains a unit of the good. It is easy to show that it is still the case that when buyer 1 receives a unit his payment is decreasing in $\alpha$, while the other buyer who obtains the remaining store pays a sum that is an increasing function of the external effect enjoyed.[11] Notice, however, that when buyer 1 does not obtain a store in the sequential mechanism, the payments of buyers 2 and 3 are *independent* of $\alpha$, whereas in the optimal mechanism their payments are still an increasing function of $\alpha$, as the following example illustrates.

**Example 8:** Suppose that $\theta = (\theta_1, \theta_2, \theta_3)$ is such that $y_{[2,3]}(\theta) = 1$. That is, (i) $J(\theta_2) + J(\theta_3) \geq \alpha J(\theta_2) + J(\theta_1)$ and (ii) $J(\theta_2) + J(\theta_3) \geq \alpha J(\theta_3) + J(\theta_1)$. Then $\alpha J(\theta_3) + J(\theta_1) > J(\underline{\theta}) + J(\theta_3) > \alpha J(\underline{\theta}) + J(\theta_1)$, where the first inequality is obvious and the second one holds since otherwise the seller would always prefer $[1,2]$ over $[2,3]$. It follows that $-t_2(\theta) = \hat{\theta}_2 = J^{-1}(J(\theta_1) + (\alpha - 1)J(\theta_3))$, which is an increasing function of $\alpha$ (see Figure 5). A similar analysis holds for $-t_3(\theta_3)$.

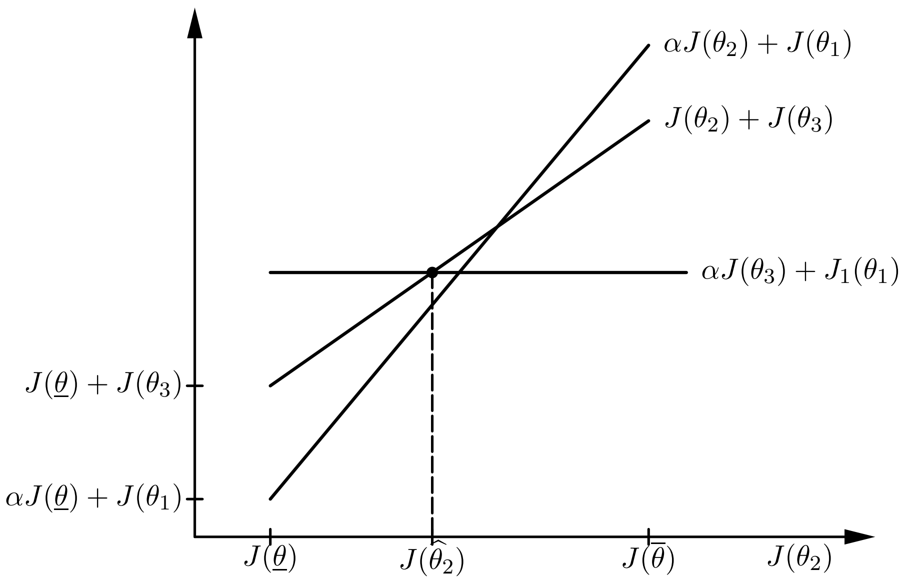


Figure 5

[11] For instance, it follows from Example 6 that if $\alpha \leq \hat{\alpha}$ and buyer 1 obtains a store, then $-t_1(\theta)$ is equal either to $\underline{\theta}$ or to $\hat{\theta}_1$, which is the unique solution to $J(\hat{\theta}_1) = J(\theta_3) - (\alpha - 1)J(\theta_2)$ and is a decreasing function of $\alpha$.

# 5 Generalization of the Results

The analysis of the model and the examples presented above exploit the assumption that a buyer's type and the external effect interact multiplicatively in his utility function. Although this assumption makes the derivation of the results easier and their interpretation more transparent, it precludes the application of the model to natural modifications of the shopping center problem, as the next example illustrates:

**Example 9:** Consider the following variation of Example 2. Suppose that if firm $i$ locates in the shopping mall, then it will face a linear demand $P_i = \alpha_{ij} - Q_i$ for its product, where $j$ denotes the identity of the neighboring firm. Suppose also that $i$ can produce at a constant marginal cost $c_i$ that is private information. It is easy to see that $i$'s profit function is $\frac{(\alpha_{ij}-c_i)^2}{4} = \frac{(\alpha_{ij}+\theta_i)^2}{4}$, where $\theta_i = -c_i$. Notice that profits are increasing in $\alpha_{ij}$ and $\theta_i$, (strictly) convex in $\theta_i$, and (strictly) supermodular in $(\alpha_{ij}, \theta_i)$.[12]

Using this example as motivation, we will show that all the results extend to the case where buyers' utility functions are given by $u_i(\alpha_{ij}, \theta_i) + t_i$ with $u_i(\cdot, \cdot)$ nonnegative, $\frac{\partial u_i}{\partial \alpha_{ij}} > 0$, $\frac{\partial u_i}{\partial \theta_i} > 0$, $\frac{\partial^2 u_i}{\partial \theta_i^2} \geq 0$, and $\frac{\partial^2 u_i}{\partial \alpha_{ij} \partial \theta_i} \geq 0$. Note in passing that $\alpha_{ij}\theta_i$ satisfies all these properties.

Define

$$S_i(\alpha_{ij}, \theta_i) = u_i(\alpha_{ij}, \theta_i) - \frac{(1 - \Phi_i(\theta_i))}{\phi_i(\theta_i)} \frac{\partial u_i(\alpha_{ij}, \theta_i)}{\partial \theta_i}.$$

We will assume that $S_i(\alpha_{ij}, \theta_i)$ is strictly increasing in $\theta_i$ and supermodular in $(\alpha_{ij}, \theta_i)$; in the multiplicative case analyzed in the previous sections, $S_i(\alpha_{ij}, \theta_i) = \alpha_{ij} J_i(\theta_i)$, which clearly satisfies these properties.

For ease of exposition, we have placed in the Appendix the details of the analysis of the general model as well as the proof of the following result.

**Theorem 2** *If $u_i(\alpha_{ij}, \theta_i)$ and $S_i(\alpha_{ij}, \theta_i)$ satisfy the aforementioned assumptions, then the following*

[12] A function of two variables $f(x, y)$ is supermodular if given $(x_1, y_1)$ and $(x_2, y_2)$,

$$f(x_1 \vee x_2, y_1 \vee y_2) + f(x_1 \wedge x_2, y_1 \wedge y_2) \geq f(x_1, y_1) + f(x_2, y_2).$$

If the function is $\mathcal{C}^2$, this is equivalent to $\frac{\partial^2 f(x,y)}{\partial x \partial y} \geq 0$.

*mechanism* $(y^*, t^*)$ *is optimal for the seller:*

$$y^*_{[i,j]}(\theta) = \begin{cases} 1 & \text{if } S_{[i,j]}(\alpha_{ij}, \alpha_{ji}, \theta_i, \theta_j) \geq \max\{0, \max_i S_i(\alpha_{i0}, \theta_i), \max_{[l,k], l \neq k, l,k \geq 1} S_{[l,k]}(\alpha_{lk}, \alpha_{kl}, \theta_l, \theta_k)\} \\ 0 & \text{otherwise,} \end{cases}$$

*and*

$$-t^*_i(\theta_i, \theta_{-i}) = \sum_{j=0}^{I} u_i(\alpha_{ij}, \theta_i) y^*_{[i,j]}(\theta_i, \theta_{-i}) - \int_{\underline{\theta}_i}^{\theta_i} \left( \sum_{j=0}^{I} \frac{\partial u_i(\alpha_{ij}, s)}{\partial \theta_i} y^*_{[i,j]}(s, \theta_{-i}) \right) ds,$$

*where* $S_{[i,j]}(\alpha_{ij}, \alpha_{ji}, \theta_i, \theta_j) = S_i(\alpha_{ij}, \theta_i) + S_j(\alpha_{ji}, \theta_j)$.

Notice that the allocation and payment rules stated in Theorem 2 are suitable generalizations of the ones derived in the multiplicative case. It is evident by inspection that the intuition of the allocation rule is the same as before, and a bit of work reveals that the payment rule also has similar properties as (17) does.

## 6 Concluding Remarks

This paper studies the optimal auction design problem of a seller in the presence of buyers' private information and identity-dependent externalities. We show that the optimal allocation is to give the goods to the set of buyers that generate the largest sum of virtual surpluses; this rule sometimes leads to an ex-post inefficient allocation. We characterize an optimal payment rule that illustrates how the seller can structure payments in such a way that buyers who obtain the goods pay according to the external benefits they enjoy and generate. As an application of the model, we analyze the selling problem faced by a shopping center's developer, and we characterize the main properties of a sequential mechanism commonly used in practice. It turns out that the sequential procedure is optimal when inter-store externalities are large, but it differs from the optimal mechanism in other cases. Finally, we generalize the main results to a larger class of quasi-linear utility functions that contains the multiplicative case as a special one.

# Appendix

In the general case with buyers' utility functions $u_i(\alpha_{ij}, \theta_i) + t_i$, the seller's problem is:

$$\max_{(y_{[i,j]}(\cdot))_{[i,j]\neq[0,0]}, (t_i(\cdot))_{1\leq i\leq I}} E_\theta\left[-\sum_{i=1}^{I} t_i(\theta)\right] \tag{26}$$

$$\text{subject to}\quad U_i(\theta_i) \geq E_{\theta_{-i}}\left[\sum_{j=0}^{I} u_i(\alpha_{ij}, \theta_i) y_{[i,j]}(\hat{\theta}_i, \theta_{-i})\right] + \bar{t}_i(\hat{\theta}_i) \quad \forall (i, \theta_i, \hat{\theta}_i) \tag{27}$$

$$U_i(\theta_i) \geq 0 \quad \forall (i, \theta_i) \tag{28}$$

$$1 \geq y_{[i,j]}(\theta) \geq 0 \quad \forall ([i,j], \theta) \tag{29}$$

$$1 - \sum_{[i,j]\neq[0,0]} y_{[i,j]}(\theta) \geq 0 \quad \forall \theta, \tag{30}$$

The following conditions are necessary and sufficient for incentive compatibility:

(i) $E_{\theta_{-i}}\left[\sum_{j=0}^{I} \frac{\partial u_i(\alpha_{ij}, \cdot)}{\partial \theta_i} y_{[i,j]}(\cdot, \theta_{-i})\right]$ is increasing; and
(ii) $U_i(\theta_i) = U_i(\underline{\theta}_i) + \int_{\underline{\theta}_i}^{\theta_i} E_{\theta_{-i}}\left[\sum_{j=0}^{I} \frac{\partial u_i(\alpha_{ij}, s)}{\partial \theta_i} y_{[i,j]}(s, \theta_{-i})\right] ds$, $\forall \theta_i \in \Theta_i$.

To prove necessity, suppose $(y, t)$ is incentive compatible. Then $(ii)$ follows from an application of the Envelope Theorem (Milgrom and Segal (2002)). Regarding $(i)$, consider without loss of generality $\hat{\theta}_i > \theta_i$; then

$$\begin{aligned} U_i(\theta_i) &\geq U_i(\hat{\theta}_i) + E_{\theta_{-i}}\left[\sum_{j=0}^{I} (u_i(\alpha_{ij}, \theta_i) - u_i(\alpha_{ij}, \hat{\theta}_i)) y_{[i,j]}(\hat{\theta}_i, \theta_{-i})\right] \\ U_i(\hat{\theta}_i) &\geq U_i(\theta_i) + E_{\theta_{-i}}\left[\sum_{j=0}^{I} (u_i(\alpha_{ij}, \hat{\theta}_i) - u_i(\alpha_{ij}, \theta_i)) y_{[i,j]}(\theta_i, \theta_{-i})\right]. \end{aligned}$$

These inequalities yield

$$E_{\theta_{-i}}\left[\sum_{j=0}^{I} (u_i(\alpha_{ij}, \hat{\theta}_i) - u_i(\alpha_{ij}, \theta_i)) y_{[i,j]}(\hat{\theta}_i, \theta_{-i})\right] \geq E_{\theta_{-i}}\left[\sum_{j=0}^{I} (u_i(\alpha_{ij}, \hat{\theta}_i) - u_i(\alpha_{ij}, \theta_i)) y_{[i,j]}(\theta_i, \theta_{-i})\right],$$

which is equivalent to

$$\begin{aligned} E_{\theta_{-i}}\left[\sum_{j=0}^{I} u_i(\alpha_{ij}, \hat{\theta}_i) y_{[i,j]}(\hat{\theta}_i, \theta_{-i})\right] &+ E_{\theta_{-i}}\left[\sum_{j=0}^{I} u_i(\alpha_{ij}, \theta_i) y_{[i,j]}(\theta_i, \theta_{-i})\right] \geq \\ E_{\theta_{-i}}\left[\sum_{j=0}^{I} u_i(\alpha_{ij}, \hat{\theta}_i) y_{[i,j]}(\theta_i, \theta_{-i})\right] &+ E_{\theta_{-i}}\left[\sum_{j=0}^{I} u_i(\alpha_{ij}, \theta_i) y_{[i,j]}(\hat{\theta}_i, \theta_{-i})\right], \end{aligned} \tag{31}$$

thereby showing that $E_{\theta_{-i}}\left[\sum_{j=0}^{I} u_i(\alpha_{ij},\theta_i)y_{[i,j]}(\hat{\theta}_i,\theta_{-i})\right]$ is supermodular in $(\theta_i,\hat{\theta}_i)$ or, equivalently,

$$E_{\theta_{-i}}\left[\sum_{j=0}^{I}\frac{\partial u_i(\alpha_{ij},\theta_i)}{\partial\theta_i}y_{[i,j]}(\cdot,\theta_{-i})\right] \tag{32}$$

is increasing. Since $\frac{\partial^2 u_i}{\partial\theta_i^2}\geq 0$, (32) is increasing in $\theta_i$ as well. Take $\theta_i''>\theta_i'$; then

$$\begin{aligned}
E_{\theta_{-i}}\left[\sum_{j=0}^{I}\frac{\partial u_i(\alpha_{ij},\theta_i'')}{\partial\theta_i}y_{[i,j]}(\theta_i'',\theta_{-i})\right] &\geq E_{\theta_{-i}}\left[\sum_{j=0}^{I}\frac{\partial u_i(\alpha_{ij},\theta_i')}{\partial\theta_i}y_{[i,j]}(\theta_i'',\theta_{-i})\right]\\
&\geq E_{\theta_{-i}}\left[\sum_{j=0}^{I}\frac{\partial u_i(\alpha_{ij},\theta_i')}{\partial\theta_i}y_{[i,j]}(\theta_i',\theta_{-i})\right],
\end{aligned}$$

where the first inequality follows by $\frac{\partial^2 u_i}{\partial\theta_i^2}\geq 0$ and the second by supermodularity. This proves necessity.

Suppose that (*i*) and (*ii*) hold, and let $\hat{\theta}_i>\theta_i$. Then,

$$\begin{aligned}
U_i(\hat{\theta}_i)-U_i(\theta_i) &= \int_{\theta_i}^{\hat{\theta}_i}E_{\theta_{-i}}\left[\sum_{j=0}^{I}\frac{\partial u_i(\alpha_{ij},s)}{\partial\theta_i}y_{[i,j]}(s,\theta_{-i})\right]ds\\
&\geq \int_{\theta_i}^{\hat{\theta}_i}E_{\theta_{-i}}\left[\sum_{j=0}^{I}\frac{\partial u_i(\alpha_{ij},s)}{\partial\theta_i}y_{[i,j]}(\theta_i,\theta_{-i})\right]ds, \qquad (33)
\end{aligned}$$

where the inequality follows from (32). Similarly,

$$\begin{aligned}
E_{\theta_{-i}}\left[\sum_{j=0}^{I}u_i(\alpha_{ij},\hat{\theta}_i)y_{[i,j]}(\theta_i,\theta_{-i})\right]+\bar{t}_i(\theta_i)-U_i(\theta_i) &= E_{\theta_{-i}}\left[\sum_{j=0}^{I}(u_i(\alpha_{ij},\hat{\theta}_i)-u_i(\alpha_{ij},\theta_i))y_{[i,j]}(\theta_i,\theta_{-i})\right]\\
&= \int_{\theta_i}^{\hat{\theta}_i}E_{\theta_{-i}}\left[\sum_{j=0}^{I}\frac{\partial u_i(\alpha_{ij},s)}{\partial\theta_i}y_{[i,j]}(\theta_i,\theta_{-i})\right]ds. \qquad (34)
\end{aligned}$$

Expressions (33) and (34) yield

$$U_i(\hat{\theta}_i)\geq E_{\theta_{-i}}\left[\sum_{j=0}^{I}u_i(\alpha_{ij},\hat{\theta}_i)y_{[i,j]}(\theta_i,\theta_{-i})\right]+\bar{t}_i(\theta_i),$$

which completes the proof of sufficiency.

Let $S_i(\alpha_{ij},\theta_i)=u_i(\alpha_{ij},\theta_i)-\frac{(1-\Phi_i(\theta_i))}{\phi_i(\theta_i)}\frac{\partial u_i(\alpha_{ij},\theta_i)}{\partial\theta_i}$; the seller's problem can then be written as follows:

$$\max_{(y_{[i,j]}(\cdot))_{[i,j]\neq[0,0]}}\sum_{i=1}^{I}E_\theta\left[\sum_{j=0}^{I}S_i(\alpha_{ij},\theta_i)y_{[i,j]}(\theta)\right] \tag{35}$$

subject to (29)-(30) and condition $(i)$.

**Proof of Theorem 2:** Consider the relaxed problem in which condition $(i)$ is ignored. It is immediate to show that the solution to this problem is given by the allocation rule described in the statement of the theorem. If this allocation rule satisfied $(i)$, then it would be optimal for the seller and so would the payment rule given in the statement of the theorem, which can be derived using the same steps that led to (17). The only remaining task is to prove that $y^*_{[i,j]}(\cdot)$ satisfies $(i)$, and it suffices to show that $\sum_{j=0}^{I} \frac{\partial u_i(\alpha_{ij},\cdot)}{\partial \theta_i} y^*_{[i,j]}(\cdot, \theta_{-i})$ is increasing in $\theta_i$.

Take $\theta_i' > \theta_i''$ and suppose $y^*_{[i,l]}(\theta_i'', \theta_{-i}) = 1$ (the other case is trivial); i.e., $\sum_{j=0}^{I} \frac{\partial u_i(\alpha_{ij},\theta_i'')}{\partial \theta_i} y^*_{[i,j]}(\theta_i'', \theta_{-i}) = \frac{\partial u_i(\alpha_{il},\theta_i'')}{\partial \theta_i}$. As in Proposition 1, it is easy to show that $\sum_{j=0}^{I} \frac{\partial u_i(\alpha_{ij},\theta_i')}{\partial \theta_i} y^*_{[i,j]}(\theta_i', \theta_{-i}) > 0$; without loss of generality, suppose that this sum is equal to $\frac{\partial u_i(\alpha_{ik},\theta_i')}{\partial \theta_i}$.

To complete the proof, we need to show that $\frac{\partial u_i(\alpha_{ik},\theta_i')}{\partial \theta_i} \geq \frac{\partial u_i(\alpha_{il},\theta_i'')}{\partial \theta_i}$; by supermodularity, it suffices to show that $\alpha_{ik} \geq \alpha_{il}$. Since $y^*_{[i,l]}(\theta_i'', \theta_{-i}) = 1$ and $y^*_{[i,k]}(\theta_i', \theta_{-i}) = 1$, it follows that

$$\begin{aligned} S_i(\alpha_{il}, \theta_i'') + S_l(\alpha_{li}, \theta_l) &\geq S_i(\alpha_{ik}, \theta_i'') + S_k(\alpha_{ki}, \theta_k) \\ S_i(\alpha_{ik}, \theta_i') + S_k(\alpha_{ki}, \theta_k) &\geq S_i(\alpha_{il}, \theta_i') + S_l(\alpha_{li}, \theta_l). \end{aligned}$$

These inequalities yield

$$S_i(\alpha_{ik}, \theta_i'') + S_i(\alpha_{il}, \theta_i') \leq S_i(\alpha_{ik}, \theta_i') + S_i(\alpha_{il}, \theta_i''). \tag{36}$$

If $\alpha_{ik} < \alpha_{il}$, then (36) would violate the supermodularity of $S_i(\alpha_{ij}, \theta_i)$; thus, $\alpha_{ik} \geq \alpha_{il}$ and the proof is complete. ■

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