Pareto Improving Taxation in Incomplete Markets

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Abstract

When asset markets are incomplete there are almost always many Pareto improving policy interventions. Remarkably, these interventions do not involve adding any new markets.

Focusing on tax policy, I create a framework for proving the existence of Pareto improving taxes, for computing them, and for estimating the size of the Pareto improvement.

The protagonist is the price adjustment following an intervention. Its role is to improve on asset insurance by redistributing endowment wealth across states.

If taxes targeting current incomes are Pareto improving, then they must cause an equilibrium price adjustment. Conversely, if the price adjustment is sufficiently sensitive to risk aversions, then for almost all risk aversions and endowments, Pareto improving taxes exist. I show how to verify this sensitivity test with standard demand theory, which Turner (2003a) extends from complete to incomplete markets.

I show that different policies generically admit Pareto improving taxes, by showing they all pass this same sensitivity test. These include (a) taxes on asset purchases, (b) lump-sum taxes on present income plus one flat tax on asset purchases, (c) asset measurable taxes on capital gains, (d) excise taxes on current commodities.

To numerically identify the Pareto improving taxes, I give a formula for the welfare impact of taxes. The formula requires information about individual marginal utilities and net trades, and about the derivative of aggregate, not individual, demand with respect to prices and taxes.

To bound the rate of Pareto improvement, I define an equilibrium's insurance deficit. Pareto optimality obtains exactly when the insurance deficit is zero. If the tax policy targets only current incomes, then the implied price adjustment determines the best rate, by integration against the covariance of the insurance deficit and net trades across agents.

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1 Introduction

When asset markets are incomplete, there are almost always many Pareto improving policy interventions, if there are multiple commodities and households. Remarkably, these policies do not involve adding any new markets.

Focusing on tax policy, I create a framework for proving the existence of Pareto improving taxes, for computing them, and for estimating the size of the improvement.

The protagonist is the price adjustment following an intervention. Its role is to improve on asset insurance by redistributing endowment wealth across states, as anticipated by Stiglitz (1982). The price adjustment is determined by how taxes and prices affect aggregate, not individual, demand.

If taxes targeting current incomes are Pareto improving, then they must cause an equilibrium price adjustment, Grossman (1975). Conversely, I prove that if the price adjustment is sufficiently sensitive to risk aversion, then for almost all risk aversions and endowments, Pareto improving taxes exist. I show how to verify this sensitivity test with standard demand theory, which Turner (2003a) extends from complete to incomplete markets.

To numerically identify the Pareto improving taxes, I give a formula for the welfare impact of taxes. It requires information on the individual marginal utilities and net trades, and on the derivative of aggregate, but not individual, demand with respect to taxes and prices.

To bound the rate of Pareto improvement, I define an equilibrium's insurance deficit. Pareto optimality obtains exactly when the insurance deficit is zero. If the tax policy targets only current incomes, then the implied price adjustment determines the best rate, by integration against the covariance of insurance deficit and net trades across agents. The equilibrium's insurance deficit arises from the agents' component of marginal utility for contingent income standing orthogonally to the asset span.

Many different tax policies generically support a Pareto improvement, because they all pass this one sensitivity test. These policies include (a) taxes on asset purchases, as in Citanna, Polemarchakis, and Tirelli (2001), (b) lump-sum taxes on current income plus one flat tax on asset purchases, similar to Citanna, Kajii, and Villanacci (1998) and to Mandler (2003), (c) asset measurable taxes on capital gains, and (d) excise taxes on current commodities, similar to Geanakoplos and Polemarchakis (2002), who emphasize consumption externality over asset incompleteness.

Some policies fail the sensitivity test and never improve everyone's welfare. For example, reallocate current incomes lump-sum and force households to keep original asset demands. If utilities are time separable, they keep future commodity demands, inducing utilities for current consumption. The First Welfare Theorem implies this tax policy is not Pareto improving. The example flunks the sensitivity test because the future price adjustment is zero, independently of risk aversion. For another example, for each asset tax purchases and subsidize sales at the same rate. Then each asset's price adjusts to offset the tax, and the final cost of holding a portfolio of assets stays the same. Demand and welfare stay the same. The example flunks the sensitivity test because the price adjustment is the negative of the tax, independently of risk aversion.

To ultimately decide whether a tax policy generically supports a Pareto improvement, I give primitives for the sensitivity of price adjustment. This requires information about the derivatives of aggregate demand with respect to policy and prices. The price adjustment is sensitive to risk aversion if there is (1) Full

Reaction of Demand to Policy, and (2) Sufficient Independence of the Reactions of Demand (to Policy and to Prices). That is, if (1) there is high enough rank in the derivative of aggregate demand with respect to policy, and (2) it is possible to affect the derivative of aggregate demand with respect to prices while preserving the derivative with respect to policy, by perturbations to risk aversion. The first example violates (1); the rank is below the number of households by budget balance. The second example violates (2); the derivatives are each other's inverses, whatever the risk aversion.

The existence result for a tax policy, that it supports a Pareto improvement at any equilibrium, speaks not of every economy but only of a generic economy. At some economies the endowments are Pareto optimal, so that no price adjustment could lead to a Pareto improvement; at equilibria of other economies, everyone has the same marginal propensity to demand, so that no price adjustment exists.

In turn, to decide whether a tax policy meets primitives (1), (2), I invoke an extension of Slutsky theory from complete to incomplete markets.

Turner (2003a) develops the Slutsky theory of demand for commodities and assets in incomplete markets. First, it decomposes the derivative of demand with respect to commodity prices, asset prices, and asset payoffs into an income effect and a Slutsky substitution effect. Next, it identifies the properties that every Slutsky matrix must satisfy, and conversely proves that any matrix satisfying these properties is the Slutsky matrix of some demand. Finally, it shows that the Slutsky matrix can be perturbed arbitrarily, subject only to maintaining these properties, by perturbing the second derivative (risk aversion) of the utility generating the original Slutsky matrix, while preserving demand and the income effect matrix. These results for incomplete markets mirror exactly those for complete markets derived by Geanakoplos and Polemarchakis (1980).

For some economies, the price adjustment function does not admit any Pareto improving interventions, even though the equilibrium allocation is not Pareto optimal. By taking Slutsky perturbations of demand, I show that for almost all nearby economies the price adjustment function does admit them. Slutsky perturbations are thus the key to why there exist almost always Pareto improving taxes.

Geanakoplos and Polemarchakis (1986) began the study of generic improvements with incomplete markets, and introduced the idea of Slutsky perturbations from quadratic utility perturbations. Since they allowed the central planner to decide the agents' asset portfolios, they did not need to go beyond perturbing the Slutsky matrices of commodity demand. To show why weaker interventions may improve welfare, such as anonymous taxes and changes in asset payoffs, it became necessary to take into account how agents' portfolio adjustments caused a further price adjustment. Naturally, this required perturbing asset demand as well as commodity demand. The lack of a Slutsky theory for incomplete markets blocked contributions for over ten years[1], until a breakthrough by Citanna, Kajii, and Villanacci (1998), who analyzed first order conditions instead of Slutsky matrices. Researchers have extended the theory of generic improvements with incomplete markets to many policies by applying this first order approach; Cass and Citanna (1998), Citanna, Polemarchakis, and Tirelli (2001), Bisin et al. (2001), and Mandler (2003).

The Slutsky approach has certain advantages. First, to compute the Pareto improving interventions from my formula the policymaker needs to know the derivative of aggregate, but not individual, demand. In the first order approach the policymaker needs to know the second derivative of every individual's utility, i.e., the derivative of every individual's demand function. Second, to express the economic intuitions the

[1] The sole one is Elul (1995).

economist can keep to the familiar language of demand theory, as in (1), (2), instead of the abstract language of submersions. Third, every time the researcher thinks a new result via Slutsky perturbations, he saves himself the work of implicitly reworking demand theory anew via quadratic utility perturbations.

Turner (2003b) reinterprets the result on the generic existence of Pareto improving financial innovation, by Elul (1995) and Cass and Citanna (1998). It argues that if the price adjustment to financial innovation passes the test of sufficient sensitivity to risk aversion, then generically Pareto improving financial innovation exists. Then Slutsky perturbations reveal that financial innovation passes this test indeed.

These results suggest that the reason any policy would generically admit Pareto improving parameter values, be it fiscal, financial or otherwise, is precisely the passing of the sensitivity test. They also suggest that Slutsky perturbations are useful in discovering which other policies pass this test.

The paper continues as follows. Section 2 presents a general model of tax policy, and details several examples of tax policy. Section 3 has the formula for the welfare impact of taxes. Section 4 obtains the generic existence of Pareto improving taxes from the sensitivity condition on price adjustment, which it then reinterprets in terms of the Reaction of Demand to Prices and to Policy. Section 5 summarizes the demand theory in incomplete markets necessary to check the sensitivity in terms of the Reactions, then section 6 checks it for the several tax policies. Section 7 estimates the rate of Pareto improvement. Section 8 derives the welfare impact formula, and spells out the notation and the parameterization of economies.

2 GEIT model

Households $h = 1, ..., H$ know the present state of nature, denoted 0, but are uncertain as to which among $s = 1, ..., S$ nature will reveal in period 1. They consume commodities $c = 1, ..., C$ in the present and future, and invest in assets $j = 1, ..., J$ in the present only. Each state has commodity C as unit of account, in terms of which all value is quoted. Markets assign to household h an income $w^h \in R^{S+1}_{++}$, to commodity $c < C$ a price $p_{\cdot c} \in R^{S+1}_{++}$, to asset j a price $q^j \in R$ and future yield $a^j \in R^S$. We call $(p_{\cdot c})^C_1 = p = (p_{s\cdot})$ the spot prices, $q = (q^j)$ the asset prices, $(a^j) = a = (a_s)$ the asset structure, and $w = (w^h)$ the income distribution, $\mathbf{P} \equiv R^{(C-1)(S+1)}_{++} \times R^J$.[2] Taxes are $t \in T, T$ some Euclidean space, negative coordinates corresponding to subsidies. The set of **budget variables** is

$$b \equiv (P, a, w, t) \in B \equiv \mathbf{P} \times R^{J \times S} \times R^{(S+1)H}_{++} \times T$$

and has some distinguished nonempty relatively open subset $B' \subset B$. B_0 is B with $T = \{0\}$.

Demand for commodities and assets $d = (x, y) : B' \to R^{C(S+1)}_{++} \times R^J$ is a function on B'. The demand $d^h = (x^h, y^h)$ of household h depends on own income only, $(x^h, y^h)(P, a, w, t) = (x^h, y^h)(P, a, w', t)$ if $w^h = w'^h$. **Tax payment** $\tau : B'_0 \times codom(d) \to R^{S+1 \times \dim(T)}$ is a function such that $\tau(b_0, d)t$ is the actual tax payment, if demand and taxes are d, t. **Tax policy** $(\tau^h)_h$ is **anonymous** if τ^h is independent of h, and **tax revenue** τ is $\tau(b_0, (d^h)_h) \equiv \Sigma \tau^h(b_0, d^h)$.

An **economy** (a, e, t, t_*, d) consists of an asset structure a, endowments e, taxes t, distribution rates t_*, and demands d. For each household h, **endowments** specify a certain number $e^h_{sc} > 0$ of each

[2] The numeraire convention is that unity is the price of $sC, s \geq 0$, which $\mathbf{P}$ therefore omits. The addition to p of the $sC, s \geq 0$ coordinates, bearing value unity, is denoted $\overline{p}$. We use the notation P for $(p, q) \in \mathbf{P}$.

commodity c in each state s, the **distribution rates** specify a fraction $t^h_* > 0$ with $\Sigma t^h_* = 1$, and **demands** specify a demand d^h. Let Ω be the set of (a, e, t, t_*, d).[3]

A list $(P, r; a, e, t, t^*, d) \in \mathbf{P} \times R^{S+1} \times \Omega$ is a **GEIT** $\leftrightarrow$

$$\sum (x^h(b) - e^h) = 0 \qquad \sum y^h(b) = 0 \qquad r - \tau(b_0, (d^h(b))_h)t = 0$$
$$\text{and} \quad b \equiv (P, a, (w^h_s = e^{h\prime}_s \overline{p}_s + t^h_* r_s)^h_s, t) \in B'$$

We say $(a, e, t, t^*, d) \in \Omega$ has **equilibrium** $(P, r) \in \mathbf{P} \times R^S$. A **GEI** is a GEIT with $t = 0$.

Under neoclassical assumptions $(a, e, 0, t_*, d) \in \Omega$ has an equilibrium[4], and then the implicit function theorem gives conditions for a neighborhood of $(a, e, 0, t_*, d)$ to have an equilibrium.

2.1 Neoclassical demand

Consider the **budget** function $\beta^h : B_0 \times R^{C(S+1)} \times R^J \to R^{S+1}$

$$\beta^h(b, x, y) \equiv (\overline{p}'_s x_s - w^h_s)^S_{s=0} - \begin{bmatrix} -q' \\ a' \end{bmatrix} y$$

Demand $d^h = (x^h, y^h)$ is **neoclassical**$_0$ if $T = \{0\}$ and there is a **utility** function $u : R^{C(S+1)}_+ \to R$ with

$$u(x^h(b)) = \max_{X^h_0(b)} u \ \text{ throughout } B' \qquad X^h_0(b) \equiv \{x \in R^{C(S+1)}_+ \mid \beta^h(b, x, y) = 0, \text{ some } y \in R^J\}$$

More generally, demand $d^h = (x^h, y^h)$ is **neoclassical** if there is a **utility** function $u : R^{C(S+1)}_+ \to R$ with

$$u(x^h(b)) = \max_{X^h(b)} u \ \text{ throughout } B' \qquad X^h(b) \equiv \{x \in R^{C(S+1)}_+ \mid \beta^h(b_0, x, y) + \tau^h(b_0, x, y)t_b = 0, \text{ some } y \in R^J\}$$[5]

If taxes $t_b = 0$ are zero, $X^h(b) = X^h_0(b)$. Thus neoclassical demand restricts to neoclassical$_0$ demand.

Neoclassical welfare is $v : B' \to R^H, v(b) = (v^h(b)) \equiv (u^h(x^h(b)))$.

The interpretation of X is that the cost of consumption x in excess of income w is financed by some portfolio $y \in R^J$ of assets, net of taxes. A **portfolio** specifies how much of each asset to buy or sell $(y_j \gtrless 0)$, and a^j_s how much value in state s an asset j buyer is to collect, a seller to deliver.

2.2 Four examples of tax policy

We detail T, B', τ^h for four tax policies.[6]

Tax rates on asset purchases $t \in T = R^J$:

$$\tau = \begin{bmatrix} y'_+ \\ 0 \end{bmatrix}$$
$$B' = \{(p, q, a, w, t) \in B \mid q + t_I \in aR^S_{++} \text{ for all subsets } I \text{ , } a \text{ has linearly independent rows}\}$$[7]

[3] The appendix spells out the parameterization of demand d.

[4] Geanakoplos and Polemarchakis (1986).

[5] The functions $b \to b_0, \to t_b$ are $(p, q, a, w, t) \to (p, q, a, w, 0), \to t$. Here y is defined by x, if a is full rank.

[6] For a vector v of reals, v_+ is defined by $(v_+)_m = \max(0, v_m)$.

[7] For a subset $I \subset \{1, ..., J\}$ of assets, t_I is defined by $(t_I)_j$ being t_j or 0 according as $j \in I$ or not.

Lump-sum taxes on current incomes plus flat tax rate on asset purchases $t' = (l', f') \in T = R^H \times R$:

$$\tau^h = \begin{bmatrix} 1^{h\prime} & 1' y_+ \\ 0 & 0 \end{bmatrix}$$
$$B' = \{(p, q, a, w, t) \in B \mid q + f 1_I \in a R^S_{++} \text{ for all subsets } I, a \text{ has linearly independent rows}\}$$

Asset measurable tax rates on future capital gains $t \in T = a' R^J \subset R^S$. Capital gain is $g^h_s = (\overline{p}'_s x_s - w^h_s)_+$. Measurability has every state's tax rate $t_s = a'_s L$ depending linearly on the asset payoffs:[8]

$$\tau^h = \left[g^h\right]$$
$$B' = \{(p, q, a, w, t) \in B \mid q \in a R^S_{++}, a \text{ has linearly independent rows, } t_s > -1\}$$

Tax rates on net purchases of current commodities $t \in T = R^{C-1}$. *(Excise taxes.)* Given endowments[9]

$$\tau^h = \begin{bmatrix} (x_0 - e^h_0)'_+ \\ 0 \end{bmatrix}$$
$$B' = \{(p, q, a, w, t) \in B \mid q \in a R^S_{++}, a \text{ has linearly independent rows, } p_{0c} + t_c > 0\}$$

Debreu's smooth preferences imply neoclassical demand exists, and is smooth in a neighborhood of b if $y_j, \overline{p}'_s x_s - w_s, x_{0c} - e_{0c} \neq 0$ for all j, s, c. We term **active** a GEI if it satisfies these inequalities for every household, in the context of these four examples, or if all demands are locally smooth, in a general context.

3 Welfare impact of taxes

We think of a smooth path $t = t(\xi)$ of taxes through $t = 0$, and of *infinitesimal taxes* as its initial velocity $\dot{t} = \dot{t}(0)$. Suppose the active GEI $(P, r; a, e, 0, t_*, d)$ is **regular** in that such a path lifts locally to a unique path $(P, r; a, e, t, t_*, d) = (P(\xi), r(\xi); a, e, t(\xi), t_*, d)$ of GEIT through the GEI. Then welfare is $v(b(\xi))$ with $b(\xi) = (P(\xi), r(\xi); a, (w^h_s = e^{h\prime}_s \overline{p}_s(\xi) + t^h_* r_s(\xi))^h_s, t(\xi))$. Thus taxes impact welfare only via the budget variables they imply. By the fundamental theorem of calculus the welfare impact is the integral of $D_b v^h \cdot \dot{b}$, which by abuse we call the *welfare impact*. We compute this product in the appendix, using the envelope theorem for $D_b v^h$ and the chain rule for $\dot{b}$, where the details of the notation appear.

Proposition 1 (Envelope) *The* ***welfare impact*** $\dot{v} \in R^H$ *of infinitesimal taxes* $\dot{t}$ *at a regular GEI is*

$$\dot{v} = (\lambda)' \dot{m} \qquad \dot{m} = \underbrace{(t^h_* \dot{r} - \tau^h \dot{t})_h}_{PRIVATE} \quad \underbrace{-\underline{z} \dot{P}}_{PUBLIC}$$

Here $(\lambda)'$ *collects the households' marginal utilities of income across states, and* $\dot{m}$ *the impact on their incomes, private and public. The private one is the impact* $\dot{r}$ *on revenue distributed at rate* $t_* \in R^H$ *net of the impact* $\tau^h \dot{t}$ *on tax payments, and the public one is the impact on the value of their excess demands* $\underline{z}$ *in all nonnumeraire markets, that implied by the impact* $\dot{P}$ *on prices.*

[8] Occasionally we view g, t as in R^{S+1} with $g_0, t_0 = 0$. For a point $g \in R^{(S+1)k}$, $[g] \in R^{(S+1)k \times S+1}$ denotes the matrix whose s^{th} column is $g_{s\cdot} \in R^k$ in the s^{th} block and zero in all the other k-blocks. If $k = 1$, as here, this is a diagonal matrix with g along the diagonal. See "aggregate notation" in the appendix.

[9] Occasionally we view t as in R^C with $t_C = 0$.

Policy targeting welfare must account for the equilibrium price adjustment it causes. The equilibrium price adjustment undoes the excess aggregate demand that policy causes, and depends on the reactions of aggregate demand to both policy and prices.

Proposition 2 (Revenue Impact) *At a regular GEI* $\dot{r} = \tau\dot{t}$.

This follows from $r = \tau t$, the chain rule, and $t = 0$ at a GEI. At a regular GEI there is a **price adjustment** matrix dP, smooth in a neighborhood of it, such that $\dot{P} = dP\dot{t}$. Thus the welfare impact is

$$\boxed{dv = (\lambda)' \left((t^h_* \tau - \tau^h)_h - \underline{\underline{z}} dP\right)}$$

A policy targeting current incomes is (first order) Pareto improving only if taxes cause a price adjustment. For if $\tau^h_{s\geq 1}\dot{t} = 0, dP\dot{t} = 0$ then $\Sigma \frac{1}{\lambda^h_0}\dot{v}^h = \Sigma \frac{1}{\lambda^h_0}\lambda^{h\prime}\dot{m}^h = \Sigma \dot{m}^h_0 = \Sigma(t^h_*\tau_0 - \tau^h_0) = 0$ so $\dot{v} \gg 0$ is impossible. Next we prove a converse.

4 Framework for generic existence of Pareto improving taxes

We prove the generic existence of Pareto improving taxes, stressing the role of changing commodity prices over the role of the particular tax policy. Existence follows directly from a hypothesis on price adjustment. Thus the tax policy is relevant only insofar as it meets the hypothesis on price adjustment. Then we reinterpret this hypothesis on dP in terms of primitives, the Reaction of Demand to Prices and the Reaction of Demand to Policy.

Pareto improving taxes exist if there exists a solution to $dv\dot{t} \gg 0$. In turn this exists if $dv \in R^{H\times \dim T}$ has rank H, which in turn implies that tax parameters outnumber household types $\dim T \geq H$. The key idea is that if $dv = (\lambda)'(t^h_*\tau - \tau^h)_h - (\lambda)'\underline{\underline{z}}dP$ is rank deficient, then a perturbation of the economy would restore full rank by preserving the first summand but affecting the second one. Namely, if some economy's dP is not appropriate, then almost every nearby economy's dP is.

We have in mind a perturbation of the households' **risk aversion** $(D^2u^h)_h$, which affects nothing but dP in the welfare impact dv. Now, to restore the rank the risk aversion must map into $(\lambda)'\underline{\underline{z}}dP$ richly enough. Since this map keeps $(\lambda)'\underline{\underline{z}}$ fixed, we require that $(\lambda)'\underline{\underline{z}}$ have rank H and that dP be sufficiently sensitive to risk aversion. Cass and Citanna (1998) gift us the first requirement:

Fact 1 (Full Externality of Price Adjustment on Welfare) *Suppose asset incompleteness exceeds household heterogeneity* $S-J \geq H > 1$. *Then generically in endowments every GEI has* $(\lambda^h_s z^h_{s1})^{h\leq H}_{s\leq H-1}$ *invertible.*

Fact 2 *At a regular GEI,* dP *is locally a smooth function of risk aversion; the marginal utilities* λ^i, *tax payments* τ^i, *and excess demands* z^i *are locally constant in risk aversion.*

For $k \in R^{(S+1)(C-1)+J}$ we say that a *commodity coordinate* is one of the first $(S+1)(C-1)$.

Definition 1 *At a regular GEI,* dP *is* k***-Sensitive to risk aversion*** *if for every* $\alpha \in R^{\dim(T)}$ *there is a path of risk aversion that solves* $k'd\dot{P} = \alpha'$.[10] *It is* ***Sensitive to risk aversion*** *if it is k-Sensitive to risk aversion for all* k *with a nonzero commodity coordinate.*

[10]The appendix spells out a path of risk aversion. Here the dot denotes differentiation with respect to the path's parameter.

Figure 1

Assumption 1 (Generic Sensitivity of dP) *If $H > 1$, then generically in endowments and utilities, at every GEI dP is Sensitive to risk aversion.*

Figures 2, 3

This assumption banishes the particulars of the tax policy, leaving only its imprint on dP. Of course, dP is defined only at regular GEI, so implicitly assumed is that regular GEI are generic in endowments.

Theorem 1 (Logic of Pareto Improvement) *Fix the tax policy and the desired welfare impact $\dot{v} \in R^H$. Grant the Generic Sensitivity of dP under $\dim(T), S - J \geq H > 1, C > 1$. Then generically in utilities and endowments, at every GEI $\dot{v}$ is the welfare impact of some $\dot{t} \in T$. Hence a nearby Pareto superior GEIT exists.*

Proof. We fix generic endowments, utilities from the fact, assumption, and apply transversality to

$$\begin{array}{ll} 1 & \text{nonnumeraire excess demand equations} \\ 2 & \gamma'(\lambda)'\left((t_*^h\tau - \tau^h)_h - \underline{\overline{z}}dP\right) = 0 \\ 3 & r - \tau t = 0 \\ 4 & \gamma'\gamma - 1 = 0 \end{array}$$

Suppose this is transverse to zero and the natural projection is proper. By the transversality theorem, for generic endowments and utilities, this system of $(\dim p + \dim q) + \dim(T) + \dim r + 1$ equations is transverse to zero in the remaining endogenous variables, which number $\dim p + \dim q + \dim r + H$. By hypothesis $\dim(T) \geq H$, so for these endowments and utilities the preimage theorem implies that no endogenous variables solve this system–every GEI has dv with rank H.

This is transverse to zero. As is well known, we can control the first equations by perturbing one household's endowment. For a moment, say that we can control the second equations and preserve the top ones. We then perturb the third equations and preserve the top two, by perturbing r as well as numeraire endowments–to preserve incomes $w_s^h = e_s^{h\prime}\overline{p}_s + t_*^h r_s$. We control the fourth equation and preserve the top three, by scalar multiples of γ. So transversality obtains if our momentary supposition on $\gamma' dv$ holds:

Write $k' \equiv \gamma'(\lambda)'\underline{\overline{z}}$. Differentiating $\gamma' dv$ with respect to the parameter of a path of risk aversion,

$$\alpha' =_{def} \frac{d}{d\xi}\gamma'(\lambda)'\left((t_*^h\tau - \tau^h)_h - \underline{\overline{z}}dP\right) = -\gamma'(\lambda)'\underline{\overline{z}}\frac{d}{d\xi}(dP) = -k'd\dot{P}$$

since λ, τ^i, z are locally constant. We want to make α arbitrary, and we can if dP is k-sensitive, which holds by assumption if k has a nonzero commodity coordinate. It has: Full Externality of Price Adjustment on Welfare, $C > 1, \gamma \neq 0$ imply $\gamma'(\lambda)'\underline{\overline{z}}$ is nonzero in the coordinate $m = s1$ for some $s \leq H - 1$.

That the natural projection is proper we omit. (The numeraire asset structure is fixed.) ■

We have seen that tax policy targeting current incomes, such as taxes on asset purchases, on net purchases of current commodities, or lump-sum taxes on current incomes, supports a Pareto improvement only if there is a price adjustment. Conversely, tax policy generically supports a Pareto improvement if the price adjustment is sufficiently sensitive to risk aversion. Therefore price adjustment is pivotal.

4.1 Expression for Price Adjustment

Before we can check whether a particular policy meets the Sensitivity of dP to Risk Aversion, we need an expression for dP. We express dP in terms of the Reaction of Demand to Prices and the Reaction of Demand to Policy, notions which are well defined at an active GEI.

Let an underbar connote the omission of the numeraire in each state, define

$$d : B' \to R_{++}^{(C-1)(S+1)} \times R^J \qquad d = \Sigma \underline{d}^h$$

and the **aggregate demand** of $(a, e, t, t_*) \in \Omega$

$$d_{a,e,t,t_*}(p, q, r) \equiv d(p, q, a, (w_s^h = e_s^{h\prime}\overline{p}_s + t_*^h r)_s^h, t)$$

with domain $\mathbf{P}_{a,e,t,t_*} \equiv \{(p, q, r) \in \mathbf{P} \times R^{S+1} \mid (p, q, a, (w_s^h = e_s^{h\prime}\overline{p}_s + t_*^h r_s)_s^h, t) \in B'\}$.[11]

Now define

$$\begin{array}{ll} J \equiv D_{p,q} d_{a,e,t,t_*} & \text{the } \textbf{Reaction of Demand to Prices} \\ \Delta \equiv D_r d_{a,e,t,t_*} \cdot \tau + D_t d_{a,e,t,t_*} & \text{the } \textbf{Reaction of Demand to Policy}^{12} \end{array} \tag{1}$$

Suppose a path of GEIT $(P(\xi), r(\xi), a, (e_s^{h\prime}\overline{p}_s(\xi) + t_*^h r_s(\xi))_s^h, t(\xi))$ through an active GEI. Then

$$d_{a,e,t,t_*}(P, r) = \begin{bmatrix} \sum \underline{e}^h \\ 0 \end{bmatrix}$$

is an identity in the path's parameter ξ. Differentiating with respect to it,

$$J\dot{P} + D_r d_{a,e,t,t_*} \cdot \dot{r} + D_t d_{a,e,t,t_*} \cdot \dot{t} = 0$$

Substituting for $\dot{r} = \tau \dot{t}$ from the Revenue Impact proposition,

$$J\dot{P} + \Delta \dot{t} = 0$$

An active GEI is **regular** if J is invertible. By the implicit function theorem, a regular GEI lifts a local policy through $t = 0$ to a path of GEIT through itself, such as the one just above.

Proposition 3 (Price Adjustment) *At a regular GEI the Price Adjustment to infinitesimal taxes is*

$$dP = -J^{-1}\Delta \tag{dP}$$

where the Reactions J, Δ are defined in (1).

[11] P_{a,e,t,t_*} is open, as the preimage by a continuous function of the open B'. Recall the notation $P' = (p', q')$.

[12] Clearly $D_r d_{a,e,t,t_*} = \Sigma D_{w^h} \underline{d}^h t_*^h$.

4.2 Primitives for the Sensitivity of Price Adjustment to Risk Aversions

Given the Logic of Pareto improvement, we want to check whether a policy meets the Generic Sensitivity of dP. We provide primitives for the Sensitivity of dP, thanks to expression (dP)[13]:

$$d\dot{P} = -J^{-1}\dot{\Delta} + J^{-1}\dot{J}J^{-1}\Delta$$

Recall equation $k' d\dot{P} = \alpha'$ from definition 1. If $\dot{\Delta} = 0$ and $\tilde{k}' \equiv_{def} k'J^{-1}$ then the equation reads $\tilde{k}'\dot{J}J^{-1}\Delta = \alpha'$. If Δ has rank $\dim(T)$ then there is a solution β to $\beta' J^{-1}\Delta = \alpha'$ so it suffices to solve $\tilde{k}'\dot{J} = \beta'$. Thus dP is k-Sensitive if (1) Δ has rank $\dim(T)$, (2) $\tilde{k}$ is nonzero everywhere, (3) whenever $\tilde{K}$ is nonzero everywhere and $\beta \in R^{(S+1)(C-1)+J}$, there is a path of risk aversion that solves $\dot{\Delta} = 0, \tilde{K}'\dot{J} = \beta'$. (Take $\tilde{k} = \tilde{K}$.) Thus Generic Sensitivity of dP follows from the following (independently of the $\tilde{k}$ defined):

Lemma 1 (Activity) *If $H > 1$, generically in endowments every GEI is active and regular.*[14]

Assumption 2 (Full Reaction of Demand to Policy) *If $C > 1$, generically in utilities and endowments, at every GEI Δ has rank $\dim(T)$.*

Lemma 2 (Mean Externality of Price Adjustment on Welfare is Regular) *Generically in utilities, at every regular GEI, whenever k is nonzero in some commodity coordinate, $\tilde{k}' \equiv k'J^{-1}$ is nonzero everywhere.*

Assumption 3 (Sufficient Independence of Reactions) *If $H > 1$, then generically in endowments and utilities, whenever $\tilde{k} \in R^{(S+1)(C-1)+J}$ is nonzero everywhere and $\beta \in R^{(S+1)(C-1)+J}$, at every GEI there is a path of risk aversion that solves $\dot{\Delta} = 0, \tilde{k}'\dot{J} = \beta'$.*

These primitives for the Generic Sensitivity of dP and the Logic of Pareto Improvement yield

Theorem 2 (Test for Pareto Improvement) *Fix the tax policy and the desired welfare impact $\dot{v} \in R^H$. Say the policy passes the Full Reaction of Demand to Policy and the Sufficient Independence of Reactions under $\dim(T), S - J \geq H > 1, C > 1$. Then generically in utilities and endowments, at every GEI $\dot{v}$ is the welfare impact of some $\dot{t} \in T$. Hence there is a nearby Pareto superior GEIT.*

Next we illustrate how to check whether a tax policy passes this test via demand theory in incomplete markets, as developed by Turner (2003a). We show that the four tax policies in the introduction pass this test, and therefore generically admit Pareto improving taxes, owing to the unifying logic of a sensitive price adjustment. At a GEI J will turn out to be independent of the policy, so we will verify the lemma on the Mean for one and all policies.

[13] Applying the chain rule to $JJ^{-1} = I$ gives $\frac{d}{d\xi}J^{-1} = -J^{-1}(\frac{d}{d\xi}J)J^{-1}$.

[14] We do not argue this relatively simple statement. For these endowments, both Δ and dP are defined.

5 Summary of demand theory in incomplete markets

We must check whether each policy meets the Full Reaction of Demand to Policy and the Sufficient Independence of Reactions. For this we report the theory of demand in incomplete markets as developed by Turner (2003a). The basic idea is to use decompositions of Δ, J in terms of Slutsky matrices, and then to perturb these Slutsky matrices by perturbing risk aversion, while preserving neoclassical demand at the budget variables under consideration. We stress that this theory is applied to, but independent of, equilibrium.

5.1 Slutsky perturbations

Define $H : R^{C^* \times C^*} \to R^{C^*+J+(S+1) \times C^*+J+(S+1)}$ as

$$H(D) = \begin{bmatrix} D & 0 & -[\overline{p}] \\ 0 & 0 & W \\ -[\overline{p}] & W' & 0 \end{bmatrix}$$

where $p, W = [-q : a] \in R^{J \times S+1}$ of rank J are given, and $C^* = C(S+1)$. In other notation,

$$H(D) = \begin{bmatrix} M(D) & -\rho \\ -\rho' & 0 \end{bmatrix} \qquad \text{where } M(D) = \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix}, \rho = \begin{bmatrix} [\overline{p}] \\ -W \end{bmatrix}$$

In showing the differentiability of demand, the key step is the invertibility of $H(D^2u)$. **Slutsky matrices** are $H(D^2u)^{-1}$. If D is symmetric, so are $H(D), H(D)^{-1}$ when defined. Thus we write

$$H(D)^{-1} = \begin{bmatrix} S & -m \\ -m' & -c \end{bmatrix}$$

where S, c are symmetric of dimensions $C^* + J, S + 1$ and $m = (m_x, m_y)$ is $C^* + J \times S + 1$. A **Slutsky perturbation** is $\nabla = H(D)^{-1} - H(D^2u)^{-1}$, for some symmetric $D \approx D^2u$ that is close enough for the inverse to exist. A Slutsky perturbation is a perturbation of Slutsky matrices rationalizable by some perturbation of the Hessian of utility. Being symmetric, we write

$$\nabla = \begin{bmatrix} \dot{S} & -\dot{m} \\ -\dot{m}' & -\dot{c} \end{bmatrix}$$

and view a Slutsky perturbation as a triple $\dot{S}, \dot{m}, \dot{c}$. We identify Slutsky perturbations, without reference to the inversion defining them, in terms of independent linear constraints on ∇ :

on $\dot{S}$	$\rho'\dot{S} = 0$ and $\dot{S}$ is symmetric	
on $\dot{m}$	$\rho'\dot{m} = 0$ and $\dot{m}_x W' = 0$	(constraints)
on $\dot{c}$	$\dot{c}W' = 0$ and $\dot{c}$ is symmetric	

Theorem 3 (Identification of Slutsky perturbations, Turner 2003a) *Given u smooth in Debreu's sense and b in B' with $t = 0$, consider the Slutsky matrices $H(D^2u)^{-1}$. Every small enough Slutsky perturbation ∇ satisfies (constraints). Conversely, every small enough perturbation ∇ that satisfies (constraints) is Slutsky: $H(D^2u)^{-1} + \nabla$ is the inverse of $H(D)$ for some D that is negative definite and symmetric.*

We use only Slutsky perturbations with $\dot{m}, \dot{c} = 0$ by choosing $\dot{S}$ as follows. A matrix $\underline{\dot{S}} \in R^{(C-1)(S+1)+J \times (C-1)(S+1)+J}$ is extendable in a unique way to a matrix $\dot{S} \in R^{C^*+J \times C^*+J}$ satisfying $\rho' \dot{S} = 0$; we call $\dot{S}$ the **extension** of $\underline{\dot{S}}$. It is easy to verify that if $\underline{\dot{S}}$ is symmetric, so is its extension. In sum, any symmetric $\underline{\dot{S}}$ defines a unique Slutsky perturbation with $\dot{m}, \dot{c} = 0$.

5.2 Decomposition of demand

The relevance of Slutsky perturbations is that they allow us to perturb demand functions directly, while preserving their neoclassical nature, without having to think about utility. This is because Slutsky matrices appear in the **decomposition** of demand $D_{p,q}\underline{d}$ at b with $t = 0$:

$$D_{p,q}\underline{d}^h = \underline{S}^h L_+^h - \underline{m}^h \cdot ([\underline{x}^h]' : \overline{y}_0^h) \tag{dec}$$

Here L_+^h a diagonal matrix displaying the marginal utility of contingent income

$$L_+^h \equiv \begin{bmatrix} L^h & 0 \\ 0 & \lambda_0^h I_J \end{bmatrix} \qquad L^h \equiv \begin{bmatrix} \cdot & & 0 \\ & \lambda_s^h I_{C-1} & \\ 0 & & \cdot \end{bmatrix}$$

$m^h = D_{w^h} d^h$, and $([\underline{x}^h]' : \overline{y}_0^h)$ is the transpose of $\underline{d}^h$: [15]

$$[\underline{x}^h]' = \begin{bmatrix} \cdot & 0 & 0 \\ 0 & \underline{x}_s^{h\prime} & 0 \\ 0 & 0 & \cdot \end{bmatrix}_{(S+1)\times(C-1)(S+1)} \qquad \overline{y}_0^h = \begin{bmatrix} y^{h\prime} \\ 0 \end{bmatrix}_{S+1\times J}$$

Writing $(e_s^{h\prime}\overline{p}_s)_s$ as $[e^h]'\overline{p}$, we have $D_{p,q}[e^h]'\overline{p} = ([\underline{e}^h]' : 0)$, so from (1) we have

$$J = \Sigma D_{p,q}\underline{d}^h + D_{w^h}\underline{d}^h \cdot ([\underline{e}^h]' : 0)$$

Inserting decomposition (dec),

$$J = \Sigma \underline{S}^h L_+^h - D_{w^h}\underline{d}^h \cdot ([\underline{x}^h - \underline{e}^h]' : \overline{y}_0^h)$$

Writing $\underline{z}^{h\prime} \equiv ([\underline{x}^h - \underline{e}^h]' : \overline{y}_0^h)$ this reads

$$\boxed{J = \Sigma \underline{S}^h L_+^h - D_{w^h}\underline{d}^h \cdot \underline{z}^{h\prime}} \tag{J}$$

This **decomposition** of the aggregate demand of $(a, e, t, t_*) \in \Omega$ generalizes Balasko 3.5.1 (1988) to incomplete markets.

One implication of the decomposition is that J is independent of the policy. So let us now provide

Proof that Mean Externality of Price Adjustment on Welfare is Regular. Consider the manifold of regular GEI and a k that is nonzero in some commodity coordinate. Fix a coordinate $n \leq (S+1)(C-1) + J$ and apply transversality to

nonnumeraire excess demand equations
$$(k'J^{-1})_n = 0$$

[15] We view p as one long vector, state by state, and p, q as an even longer one; $(* : \#)$ denotes concatenation of $*, \#$.

This is transverse to zero. The burden of the argument is to control the bottom equation independently of the top ones. Consider a Slutsky perturbation with $\dot{m}^1, \dot{c}^1 = 0$ and $\underline{\dot{S}}^1$ symmetric. Then with $\tilde{k}' \equiv k'J^{-1}$

$$\frac{d}{d\xi}(k'J^{-1})_n = -(k'J^{-1}\dot{J}J^{-1})_n = -(\tilde{k}'\dot{J}J^{-1})_n$$

Since J^{-1} is invertible, there is α such that $\alpha' J^{-1}$ is the n^{th} basis vector, so it suffices to solve $\tilde{k}'\dot{J} = \alpha'$. From decomposition (J) $\dot{J} = \underline{\dot{S}}^1 L^1_+$, so we want to solve $\tilde{k}'\underline{\dot{S}}^1 L^1_+ = \alpha'$ or $\tilde{k}'\underline{\dot{S}}^1 = \alpha'(L^1_+)^{-1} \equiv \beta'$ for symmetric $\underline{\dot{S}}^1$. Since $\tilde{k} \neq 0$, say $\tilde{k}_p \neq 0$. Let column $o \neq p$ of $\underline{\dot{S}}^1$ be $1_p \frac{\beta_o}{\tilde{k}_p}$ so that $(\tilde{k}'\underline{\dot{S}}^1)_o = \beta_o$. To preserve symmetry, let column p of $\underline{\dot{S}}^1$ be $\frac{\beta_o}{\tilde{k}_p}$ in coordinate $o \neq p$ and arbitrary x in coordinate p, so that $(\tilde{k}'\underline{\dot{S}}^1)_{\cdot p} = \Sigma_{o \neq p} \tilde{k}_o \frac{\beta_o}{\tilde{k}_p} + \tilde{k}_p x$. We can set this to β_p and solve for x since $\tilde{k}_p \neq 0$.

By the transversality theorem, for generic utilities in Debreu's setting, the system of $\dim p + \dim q + 1$ equations is transverse in the remaining $\dim p + \dim q$ variables. By the preimage theorem, for these generic utilities every regular GEI with nonzero k has $\tilde{k}_n \neq 0$. Taking the intersection over the finitely many coordinates n, for generic utilities every regular GEI with nonzero k has $\tilde{k}$ nonzero everywhere. ■

6 Four policies generically admitting Pareto improving taxes

We check for each policy the Full Reaction of Demand to Policy and the Sufficient Independence of Reactions. In computing

$$\Delta = D_t d_{a,e,t,t_*} + (\Sigma D_{w^h} \underline{d}^h t^h_*) \cdot \tau$$

we use the following notation for $\underline{S}^h$, where A^h, B^h are symmetric of dimensions $(C-1)(S+1), J$:

$$\underline{S}^h = [\underline{S}^h_p : \underline{S}^h_q] = \begin{bmatrix} A^h & P^h \\ P^{h\prime} & B^h \end{bmatrix} \qquad (\underline{S}^h)$$

We can perturb P^h arbitrarily and get a Slutsky perturbation.

Remark 1 *In checking the Sufficient Independence of Reactions, the* $\underline{\dot{S}}^h$ *Slutsky perturbations affect only the Jacobian* $\dot{J} = \Sigma \dot{S}^h L^h_+$ *in (J). Also, we solve* $\tilde{k}'\dot{J} = \beta'$ *piecemeal, solving* $\tilde{k}'\dot{J}_p = \beta'_p, \tilde{k}'\dot{J}_q = \beta'_q$ *by splitting* $\tilde{k}' = (\tilde{k}'_p, \tilde{k}'_q), \beta' = (\beta'_p, \beta'_q)$, $\dot{J} = [\dot{J}_p : \dot{J}_q]$.

6.1 Tax rates on asset purchases

Corollary 1 (Citanna-Polemarchakis-Tirelli 2001) *Fix the desired welfare impact* $\dot{v} \in R^H$. *Assume* $J, S - J \geq H > 1, C > 1$. *Then generically in utilities and endowments, at every GEI* $\dot{v}$ *is the welfare impact of some* $\dot{t} \in T$. *Hence there is a nearby Pareto superior GEIT with tax rates on asset purchases.*

Proof. The next lemmas, $\dim(T) = J$, and the hypothesis $J \geq H$ enable theorem 2. ■

The introduction of tax rates on asset purchases amounts to a household specific change in asset prices. The price of asset j changes for household h exactly when $y^h_j > 0$. So $D_t d_{a,e,t,t_*} = \Sigma D_q \underline{d}^h I^h$ where

$I^h \in R^{J\times J}$ is a diagonal matrix with entry jj equal to one or zero according as $y^h_j > 0$ or not. Specializing to asset prices, (dec) reads

$$D_q \underline{d}^h = \begin{bmatrix} P^h \\ B^h \end{bmatrix} - D_{w^h}\underline{d}^h \cdot \overline{y}^h_0$$

so that

$$\Delta^q = \Sigma\left(\begin{bmatrix} P^h \\ B^h \end{bmatrix} - D_{w^h}\underline{d}^h \cdot \overline{y}^h_0\right) I^h + D_{w^h}\underline{d}^h t^h_* \cdot \tau \qquad (\Delta^q)$$

Lemma 3 (Full Reaction of Demand to Policy) *If* $C > 1$*, generically in utilities and endowments, at every GEI* Δ^q *has rank* $\dim(T)$.

Proof. Fix generic endowments from the Activity lemma and apply transversality to

$$\begin{gathered}\text{nonnumeraire excess demand equations}\\ \hat{\Delta}^q\phi = 0\\ \phi'\phi - 1 = 0\end{gathered}$$

where the hat omits the last J rows of Δ^q. This is transverse to zero. The burden of the argument is to control the middle equations independently of the top and bottom ones. We perturb only the P^h, so that $\frac{d}{d\xi}\hat{\Delta}^q = \Sigma\dot{P}^h I^h$. Say $\phi_j \neq 0$; we make column j of $\frac{d}{d\xi}\hat{\Delta}^q$ arbitrary and preserve the others. The GEI is active and asset markets clear, so fix h with $y^h_j > 0$; the j^{th} column of $P^h I^h = j^{th}$ column of P^h. So let $\dot{P}^h$ be $(\frac{a_k}{\phi_j})_k$ in column j and zero in the others, and $\dot{P}^{i\neq h} = 0$. Then $(\Sigma\dot{P}^h I^h)\phi = a$ is arbitrary.

By the transversality theorem, generically in endowments and utilities the system of $\dim p + \dim q + (S+1)(C-1) + 1$ equations is transverse in the remaining $\dim p + \dim q + J$ variables. By the preimage theorem, for these every GEI is active and has $\hat{\Delta}^q$ (a fortiori Δ^q) with linearly independent columns. ∎

Lemma 4 (Sufficient Independence of Reactions) *If* $H > 1$*, then generically in endowments and utilities, whenever* $\tilde{k} \in R^{(S+1)(C-1)+J}$ *is nonzero in some commodity coordinate and* $\beta \in R^{(S+1)(C-1)+J}$*, at every GEI there is a path of risk aversion that solves* $\dot{\Delta} = 0, \tilde{k}'\dot{J} = \beta'$.

Proof. Fix generic endowments from the Activity lemma, a GEI with $\tilde{k}_m \neq 0$ for some commodity coordinate m, and follow remark 1. For each asset j fix $h(j)$ with $y^{h(j)}_j < 0$, let $\dot{P}^{h(j)}$ be $1_m \frac{\beta_{(S+1)(C-1)+j}}{\tilde{k}_m}$ in column j and zero in the others, and all $\dot{B}^h = 0$. This keeps $\dot{\Delta} = 0$ and equates $\tilde{k}'\dot{J}$ to $\beta_{(S+1)(C-1)+j}$ in coordinate $(S+1)(C-1)+j$. Having dealt with all asset coordinates $j \leq J$ via the $\dot{P}^h$, we turn to the commodity coordinates $n \leq (S+1)(C-1)$. Let $\gamma' = \Sigma\dot{P}^{h\prime}L^h$. From display $(\underline{S}^h)$ it suffices to choose symmetric $\dot{A}^1$ such that $\tilde{k}'_p\dot{A}^1L^1 + \tilde{k}'_q\gamma' = \beta'_p$ or $\tilde{k}'_p\dot{A}^1 = (\beta'_p - \tilde{k}'_q\gamma')(L^1)^{-1} \equiv \alpha'$. Let column $n \neq m$ of $\dot{A}^1$ be $1_m\frac{\alpha_n}{\tilde{k}_m}$ so that $\tilde{k}'_p\dot{A}^1$ equals α_n in coordinate n. To preserve symmetry, column m of $\dot{A}^1$ must be $\frac{\alpha_n}{\tilde{k}_m}$ in row $n \neq m$ and arbitrary x in row m. Then $\tilde{k}'_p\dot{A}^1$ equals $\Sigma_{n\neq m}\tilde{k}_n\frac{\alpha_n}{\tilde{k}_m} + \tilde{k}_m x$ in coordinate m, which we can equate to α_m by solving for x. Having dealt with all coordinates n, this symmetric $\dot{A}^1$ solves $\tilde{k}'\dot{J} = \beta'$. Since A^1 does not appear in Δ, still $\dot{\Delta} = 0$. ∎

6.2 Lump-sum taxes on current income plus flat tax rate on asset purchases

Corollary 2 *Fix the desired welfare impact* $\dot{v} \in R^H$. *Assume* $S - J \geq H > 1, C > 1, J > 0$. *Then generically in utilities and endowments, at every GEI* $\dot{v}$ *is the welfare impact of some* $\dot{t} \in T$. *Hence there is a nearby Pareto superior GEIT with lump-sum taxes on current income plus flat tax rate on asset purchases.*

Proof. The next lemmas and $\dim(T) = H + 1$ enable theorem 2. ■

The part of $D_t d_{a,e,t,t_*}$ relating to the lump-sum taxes $l \in R^H$ on current income is $-\Sigma D_{w_0^h} \underline{d}^h 1^{h\prime}$, and that relating to the flat tax rate f on asset purchases is $\Delta^q 1$ where $1 \in R^J$. Concatenating,

$$D_t d_{a,e,t,t_*} = \left[-\Sigma D_{w_0^h} \underline{d}^h 1^{h\prime} : \Delta^q 1 \right]$$

Since the first row of τ is $[1' : 1'\Sigma y_+^i]$,

$$(\Sigma D_{w^h} \underline{d}^h t_*^h) \cdot \tau = (\Sigma D_{w_0^h} \underline{d}^h t_*^h) \cdot [1' : 1'\Sigma y_+^i]$$

So

$$\Delta^w = \left[\Sigma D_{w_0^h} \underline{d}^h (t_*^h 1 - 1^h)' : \Delta^q 1 + (\Sigma D_{w_0^h} \underline{d}^h t_*^h)(1'\Sigma y_+^i) \right]$$

For convenience, we reexpress the lump-sum part $\Sigma D_{w_0^h} \underline{d}^h (t_*^h 1 - 1^h)' \dot{l}^h = \Sigma_{h \neq H} \nabla^h (t_*^h 1 - 1^h)' \dot{l}^h$ with $\nabla^h \equiv D_{w_0^h} \underline{d}^h - D_{w_0^H} \underline{d}^H$, to think of only $H - 1$ parameters $(l^h)^{h \neq H}$. Now $\dim(T) = H$ and

$$\Delta^w = \left[\Sigma_{h \neq H} \nabla^h (t_*^h 1 - 1^h)' : \Delta^q 1 + (\Sigma D_{w_0^h} \underline{d}^h t_*^h)(1'\Sigma y_+^i) \right] \qquad (\Delta^w)$$

Lemma 5 (Full Reaction of Demand to Policy) *If* $(S + 1)(C - 1) \geq H > 1$, *generically in utilities and endowments, at every GEI* Δ^w *has rank* $\dim(T)$.

Proof. Fix generic endowments from the Activity lemma and apply transversality to

$$\begin{gathered} \text{nonnumeraire excess demand equations} \\ \hat{\Delta}^w \phi = 0 \\ \phi'\phi - 1 = 0 \end{gathered}$$

where the hat omits the last J rows. This is transverse to zero. The burden of the argument is to control the middle equations independently of the top and bottom ones. Write $\phi = (l, f)$ so that $l^i \neq 0$ for some $i \neq H$ or $f \neq 0$.

If $f \neq 0$, then we want $\frac{d}{d\xi} \hat{\Delta}^q 1 = a$ arbitrary but $\frac{d}{d\xi} \hat{\nabla}^h = 0$, where $\hat{\nabla}^h \equiv D_{w_0^h} \underline{x}^h - D_{w_0^H} \underline{x}^H$, and we can by choosing some h with $y_1^h > 0$ and setting $\dot{P}^h$ to be $\frac{a}{f}$ with $a \in R^{(C-1)(S+1)}$ in column 1 and zero in the others, so that $\frac{d}{d\xi} \hat{\Delta}^q 1 = \dot{P}^h I^h 1 = \frac{a}{f}$ and $\frac{d}{d\xi}(\hat{\Delta}^w \phi) = (\frac{d}{d\xi} \hat{\Delta}^q 1) f = a$.

If $l^i \neq 0$, then we want the i^{th} column of $\Sigma_{h \neq H} \hat{\nabla}^h (t_*^h 1 - 1^h)'$ arbitrary:

$$* = \frac{d}{d\xi} \Sigma_{h \neq H} \hat{\nabla}^h (t_*^h 1 - 1^h)' 1^i = \frac{d}{d\xi} \left[(\Sigma_{h \neq H} \hat{\nabla}^h t_*^h) - \hat{\nabla}^i \right] = \frac{d}{d\xi} \left[(\Sigma_{h \neq i, H} \hat{\nabla}^h t_*^h) - (1 - t_*^i) \hat{\nabla}^i \right] = a$$

but $\lozenge = \frac{d}{d\xi} \left(\hat{\Delta}^q 1 + (\Sigma D_{w_0^h} \underline{x}^h t_*^h)(1'\Sigma y_+^i) \right) = 0$. From the identification of Slutsky perturbations, we set $\frac{d}{d\xi} D_{w_0^h} \underline{x}^h = 0$ for all $h \neq i$, and $D_{w_0^i} \underline{x}^i = \frac{a}{1 - t_*^i}$ by setting $\frac{d}{d\xi} D_{w^i} \underline{x}^i = a \frac{\lambda^{i\prime}}{\lambda_0^i}$–a Slutsky perturbation since

$\lambda^{i\prime}W' = 0$ from the FOC–so that $* = -(1-t^i_*)\frac{d}{d\xi}D_{w^i_0}\underline{x}^i = -a$. Any effect on $\Diamond$ we can undo, since as just seen we can make $\frac{d}{d\xi}\hat{\Delta}^q 1$ arbitrary while preserving the $D_{w^h_0}\underline{x}^h$.

By the transversality theorem, generically in endowments and utilities the system of $\dim p + \dim q + (S+1)(C-1)+1$ equations is transverse in the remaining $\dim p + \dim q + H$ variables. By the preimage theorem, for these every GEI is active and has $\hat{\Delta}^w$ (a fortiori Δ^w) with linearly columns. ∎

Lemma 6 (Sufficient Independence of Reactions) *If $H > 1$, then generically in endowments and utilities, whenever $\tilde{k} \in R^{(S+1)(C-1)+J}$ is nonzero in some commodity coordinate and $\beta \in R^{(S+1)(C-1)+J}$, at every GEI there is a path of risk aversion that solves $\dot{\Delta} = 0, \tilde{k}'\dot{J} = \beta'$.*

Proof. The proof of the lemma for Δ^q applies verbatim. ∎

6.3 Asset measurable tax rates on future capital gains

Corollary 3 *Fix the desired welfare impact $\dot{v} \in R^H$. Assume $J, S-J \geq H > 1, C > 1$. Then generically in utilities and endowments, at every GEI $\dot{v}$ is the welfare impact of some $\dot{t} \in T^p$. Hence there is a nearby Pareto superior GEIT with asset measurable tax rates on future capital gains.*

Proof. The next lemmas, $\dim(T) = J$, and the hypothesis $J \geq H$ enable theorem 2. ∎

Capital gain is $g^h_s = (\overline{p}'_s x^h_s - w^h_s)_+$. State contingent taxes are **asset measurable** if they are a linear function of asset payoffs, $t = a'L$ for some L. The introduction of tax rates on capital gains amounts to a household specific proportional change in commodity prices. The prices of state s commodities change in the same proportion exactly when $\overline{p}'_s x^h_s - w^h_s > 0$, i.e. $p^h_{\mathbf{1}} = [p_{\mathbf{1}}](I + [t^h])$.[16] So $D_t d_{a,e,t,t_*} = \Sigma D_{p_{\mathbf{1}}}\underline{d}^h[p_{\mathbf{1}}]I^h$ where $I^h \in R^{S\times S}$ is a diagonal matrix with entry ss equal to one or zero according as $\overline{p}'_s x^h_s - w^h_s > 0$ or not. Specializing to period 1 commodity prices, (dec) reads

$$D_{p_{\mathbf{1}}}\underline{d}^h = \begin{bmatrix} A^h_{\mathbf{1}} \\ P'^{\,h}_{\mathbf{1}} \end{bmatrix} - D_{w^h_{\mathbf{1}}}\underline{d}^h \cdot [\underline{x}^h_{\mathbf{1}}]'$$

so that with the parameterization $\dot{t} = a'\dot{L}$

$$\Delta^p = \left\{ \Sigma\left(\begin{bmatrix} A^h_{\mathbf{1}} \\ P'^{\,h}_{\mathbf{1}} \end{bmatrix} - D_{w^h_{\mathbf{1}}}\underline{d}^h \cdot [\underline{x}^h_{\mathbf{1}}]' \right)[p_{\mathbf{1}}]I^h + D_{w^h}\underline{d}^h t^h_* \cdot \tau \right\} a' \qquad (\Delta^p)$$

Note that at an active GEI for every s there are h, i with $\overline{p}'_s x^h_s - w^h_s > 0 > \overline{p}'_s x^i_s - w^i_s$. For with $t = 0$ the budget equation is $\overline{p}'_s x^h_s - w^h_s = a'_s y^h$ for all h, so $\Sigma\overline{p}'_s x^h_s - w^h_s = 0$ by asset market clearing.

Lemma 7 (Full Reaction of Demand to Policy) *If $C > 1$, generically in utilities and endowments, at every GEI Δ^p has rank $\dim(T)$.*

[16] For $\phi_s = \overline{p}'_s x_s - w^h_s + g_s t - a'_s y$. If $g_s = 0$ then ϕ_s reduces to the GEI ϕ_s. If $g_s \neq 0$ then $g_s = \overline{p}'_s x_s - w^h_s$ so $\phi_s = (1+t_s)(\overline{p}'_s x_s - w^h_s) - a'_s y$. At a GEI $w^h_s = \overline{p}'_s e^h_s + t^h_* r_s$ with $r = 0$, so that $\phi_s = (1+t_s)\overline{p}'_s(x_s - e^h_s) - a'_s y$, as if now prices $p_s(t) = (1+t_s)p_s$. In sum, for every $s \geq 1$ $\phi_s(t) = \overline{p}_s(t)'(x_s - e^h_s) - a'_s y$ with $p_s(t) = (1+t^h_s)p_s$ with $t^h_s = t_s, 0$ according as $\overline{p}'_s x^h_s - w^h_s > 0$ or not.

Proof. Fix generic endowments from the Activity lemma and apply transversality to

$$\text{nonnumeraire excess demand equations}$$
$$\{\hat{\cdot}\}\,\phi = 0$$
$$\phi'\phi - 1 = 0$$

where the hat selects the $(s1)_{s\geq 1}$ rows in the bracketed matrix $\{\}$, omitting the $(sc)_{s\leq S, c\neq 1}$ and asset rows. This is transverse to zero. The burden of the argument is to control the middle equations independently of the top and bottom ones. We perturb only the $(A^h_{\mathbf{1}})_{s1} \in R^{S(C-1)}$, so that $(\frac{d}{d\xi}\{\hat{\cdot}\})_s = \Sigma(\dot{A}^h_{\mathbf{1}})_{s1}[p_{\mathbf{1}}]I^h$. Say $\phi_s \neq 0$; the GEI is active so fix $h = h(s)$ with $g^h_s > 0$; for it, the s^{th} column of $[p_{\mathbf{1}}]I^h = s^{th}$ column of $[p_{\mathbf{1}}]$. Now let $(\dot{A}^h_{\mathbf{1}})_{s1}$ be $(...0 : \frac{a_s}{\phi_s p_{s1}} 1'_{c=1} : 0...)$, so that $(\dot{A}^h_{\mathbf{1}})_{s1}[p_{\mathbf{1}}]I^h\phi = a_s$, and $(\dot{A}^h_{\mathbf{1}})_{t1} = 0$ for $t \neq s$, so that $\dot{A}^h_{\mathbf{1}}[p_{\mathbf{1}}]I^h\phi = 1_s a_s$. Note that $\dot{A}^h$ is symmetric. Finding such $h(s)$ for each s, $\Sigma_s \dot{A}^h_{\mathbf{1}}[p_{\mathbf{1}}]I^h\phi = a$ is arbitrary. Thus let $\dot{A}^i = 0$ for those i distinct from every $h(s)$ to get $\frac{d}{d\xi}\{\hat{\cdot}\}\,\phi = a$.

By the transversality theorem, generically in endowments and utilities the system of $\dim p + \dim q + S + 1$ equations is transverse in the remaining $\dim p + \dim q + S$ variables. By the preimage theorem, for these every GEI is active and has $\{\hat{\cdot}\}$ (a fortiori $\{\}$ and $\{\}\,a'$) with linearly independent columns. ■

Lemma 8 (Sufficient Independence of Reactions) *If $H > 1$, then generically in endowments and utilities, whenever $\tilde{k} \in R^{(S+1)(C-1)+J}$ is nonzero everywhere and $\beta \in R^{(S+1)(C-1)+J}$, at every GEI there is a path of risk aversion that solves $\dot{\Delta} = 0, \tilde{k}'\dot{J} = \beta'$.*

Proof. Consider generic endowments from the Activity lemma, and follow remark 1.

To solve $\tilde{k}'\dot{J}_p = \beta'_p$ we set the $\dot{A}^h = 0$ so that we seek $\tilde{k}'_q \Sigma \dot{P}^{h\prime} L^h = \beta'_p$ or $(\Sigma L^h \dot{P}^h)\tilde{k}_q = \beta_p$. For each s there is $i = i(s)$ with capital loss $0 > \overline{p}'_s x^i_s - w^i_s$, so we can fix $\dot{P}^{i(s)}$ in coordinates $(sc, j)_{c,j}$ and still preserve $\dot{\Delta} = 0$; fix $\dot{P}^{i(s)}$ in coordinates $(sc, J)_c$ to be $\frac{\beta_{sc}}{\tilde{k}_J \lambda^i_s}$ and zero in coordinates $(s'c, J)_c$ for $s' \neq s$, and zero in columns $j < J$. Then $L^{i(s)}\dot{P}^{i(s)}\tilde{k}_q$ equals β_{sc} in coordinates $(sc)_c$ and zero in $(s'c)_{s'\neq s,c}$, so $(\Sigma_s L^{i(s)}\dot{P}^{i(s)})\tilde{k}_q = \beta_p$. We let $\dot{P}^h = 0$ for those h distinct from any $i(s)$, so $(\Sigma L^h \dot{P}^h)\tilde{k}_q = \beta_p$. Recall $\dot{\Delta} = 0$ so far.

To solve $\tilde{k}'\dot{J}_q = \beta'_q$, having fixed the $\dot{P}^h$, we want to solve $\tilde{k}'_q \Sigma \dot{B}^h = \beta'_q - \tilde{k}'_p \Sigma \dot{P}^h \equiv \gamma' \in R^J$ with the $\dot{B}^h$ being symmetric. Since the latter do not figure in Δ, such as solution will complete $\tilde{k}'\dot{J} = \beta'$ with $\dot{\Delta} = 0$. Set $\dot{B}^1$ to be diagonal with j^{th} diagonal element $\frac{\gamma_j}{\tilde{k}_j}$ and the other $\dot{B}^{h\neq 1} = 0$. ■

6.4 Excise taxes on current commodities

Corollary 4 *Fix the desired welfare impact $\dot{v} \in R^H$. Assume $C - 1, S - J \geq H > 1$. Then generically in utilities and endowments, at every GEI $\dot{v}$ is the welfare impact of some $\dot{t} \in T$. Hence there is a nearby Pareto superior GEIT with tax rates on net purchases of current commodities.*

Proof. The next lemmas, $\dim(T) = C - 1$, and the hypothesis $C - 1 \geq H$ enable theorem 2. ■

The introduction of tax rates on net purchases of commodities, given endowments, amounts to a household specific change in commodity prices. The price of commodity $0c$ changes to $p_{0c} + t_c > 0$ exactly when $x^h_{0c} - e^h_{0c} > 0, c < C$. So $D_t d_{a,e,t,t_*} = \Sigma D_{p_0} \underline{d}^h I^h$ where $I^h \in R^{C-1\times C-1}$ is a diagonal matrix with coordinate cc one or zero according as $x^h_{0c} - e^h_{0c} > 0$ or not. Specializing to period 0 commodity prices, (dec) reads

$$D_{p_0}\underline{d}^h = \begin{bmatrix} A^h_0 \\ P^{h\prime}_0 \end{bmatrix} \lambda^h_0 - D_{w^h_0}\underline{d}^h \cdot \underline{x}^{h\prime}_0$$

so that

$$\Delta^c = \Sigma\left(\begin{bmatrix} A^h_0 \\ P^{h\prime}_0 \end{bmatrix} \lambda^h_0 - D_{w^h_0}\underline{d}^h \cdot \underline{x}^{h\prime}_0\right) I^h + D_{w^h}\underline{d}^h t^h_* \cdot \tau \qquad (\Delta^p)$$

Lemma 9 (Full Reaction of Demand to Policy) *If $C > 1$, generically in utilities and endowments, at every GEI Δ^c has rank $\dim(T)$.*

Proof. Fix generic endowments from the Activity lemma and apply transversality to

nonnumeraire excess demand equations

$$\hat{\Delta}^p \phi = 0$$

$$\phi'\phi - 1 = 0$$

where the hat selects the $(Sc)_{c<C}$ rows in (Δ^p). This is transverse to zero. The burden of the argument is to control the middle equations independently of the top and bottom ones. We perturb only the $(\dot{A}^h_0)_{Sc,\cdot} \in R^{C-1}$, so that $(\frac{d}{d\xi}\hat{\Delta}^p)_c = \Sigma\lambda^h_0(\dot{A}^h_0)_{Sc,\cdot}I^h$. Say $\phi_c \neq 0$; since the GEI is active fix $h = h(c)$ with $x^h_{0c} - e^h_{0c} > 0$; we set row $(\dot{A}^h_0)_{Sc,\cdot}$ to be $\frac{\alpha_c 1'_c}{\lambda^h_0}$ so that $\lambda^h_0(\dot{A}^h_0)_{Sc,\cdot}I^h\phi = \alpha_c$. To preserve the symmetry of $\dot{A}^h$, we set $(\dot{A}^h_S)_{\cdot,0c}$ to be $\frac{\alpha_c 1_c}{\lambda^h_0}$ but this does not appear in $\hat{\Delta}^p$. Setting $(\dot{A}^h_0)_{Sc',\cdot} = 0$ for rows $c' \neq c$, we get $\lambda^h_0(\dot{A}^h_0)I^h\phi = 1_c\alpha_c$. Doing so for each $c < C, \Sigma_c\lambda^{h(c)}_0(\dot{A}^{h(c)}_0)I^{h(c)}\phi = \alpha$ is arbitrary. Now set $\dot{A}^i = 0$ for those i distinct from all the $h(c)$. Then $\frac{d}{d\xi}\hat{\Delta}^p\phi = \alpha$ is arbitrary with all $\dot{A}^k$ symmetric.

By the transversality theorem, generically in endowments and utilities the system of $\dim p + \dim q + (C-1) + 1$ equations is transverse in the remaining $\dim p + \dim q + (C-1)$ variables. By the preimage theorem, for these every GEI is active and has $\hat{\Delta}^p$ (a fortiori Δ^p) with linearly independent columns. ■

Lemma 10 (Sufficient Independence of Reactions) *If $H > 1$, then generically in endowments and utilities, whenever $\tilde{k} \in R^{(S+1)(C-1)+J}$ is nonzero everywhere and $\beta \in R^{(S+1)(C-1)+J}$, at every GEI there is a path of risk aversion that solves $\dot{\Delta} = 0, \tilde{k}'\dot{J} = \beta'$.*

Proof. Fix generic endowments from the Activity lemma, a GEI with $\tilde{k}_m \neq 0$ for every coordinate m, and follow remark 1. Fix a commodity coordinate $m = sc$. Pick $h(m)$ with $x^{h(m)}_m - e^h_m < 0$, let $\dot{P}^{h(m)}$ be $1_m\frac{\beta_{(S+1)(C-1)+j}}{\tilde{k}_m}$ in column j, so that $\tilde{k}'\dot{J}$ equals $\beta_{(S+1)(C-1)+j}$ in coordinate $(S+1)(C-1)+j$, for all $j \leq J$. This $\dot{P}^{h(m)}$ keeps $\dot{\Delta}^p = 0$ because $h(m)$ is a net seller in commodity market m. Having dealt with all asset coordinates via the $\dot{P}^h$, we turn to the commodity coordinates $n \leq (S+1)(C-1)$. Let $\gamma' = \Sigma\dot{P}^{h\prime}L^h$. From display $(\underline{S}^h)$ it suffices to choose symmetric $\dot{A}^h$ such that $\tilde{k}'_p\Sigma\dot{A}^hL^h + \tilde{k}'_q\gamma' = \beta'_p$ or $\tilde{k}'_p\Sigma\dot{A}^hL^h = (\beta'_p - \tilde{k}'_q\gamma') \equiv \alpha'$. For column $n = s'c'$ pick $h(n)$ with $x^{h(n)}_n - e^h_n < 0$ and let $\dot{A}^{h(n)}$ be zero everywhere but $\frac{\alpha_n}{\lambda^{h(n)}_{s'}\tilde{k}_n}$ in coordinate nn, and $\dot{A}^{i\neq h(n)}$ be zero in column n, so that $(\tilde{k}'_p\Sigma\dot{A}^hL^h)_n = \alpha_n$ and still $\dot{\Delta}^p = 0$ because $h(n)$ is a net seller in commodity market n. Doing so simultaneously for all n, we get $\tilde{k}'_p\Sigma\dot{A}^hL^h = \alpha'$. This keeps the symmetry of the $\dot{A}^h$ and $\dot{\Delta}^p = 0$. ■

7 The insurance deficit bound on the rate of improvement

We bound the rate of Pareto improvement by the equilibrium's *insurance deficit*, which vanishes exactly at Pareto optimality. The bound turns out to be the covariance of the insurance deficit with the marginal purchasing power.

Recall that the welfare impact is $\dot{v}^h = \lambda^{h\prime} dm^h$ where dm^h is marginal purchasing power, for some matrices $\Sigma dm^h = 0$. ($dm^h = (t^h_* \tau - \tau^h) - \underline{z}^h dP$.) Converting marginal welfare from utils to the numeraire at time 0, marginal utility becomes $\frac{\lambda^h}{\lambda^h_0}$, which we rewrite as λ^h with $\lambda^h_0 = 1$. In this common unit,

$$dW = \frac{1}{H} \Sigma \lambda^{h\prime} dm^h \qquad \text{the } \textbf{mean welfare impact}$$

Every household's marginal utility of future income projects to a common point in the asset span,

$$\lambda^h_{\mathbf{1}} = \delta^h + c \in a^{\perp} \oplus a$$

by the first order condition, being unique only in its **insurance deficit** δ^h. If the **mean insurance deficit** is $\overline{\delta} = H^{-1} \Sigma \delta^h$, then the GEI's **insurance deficit** is

$$\Delta = [\delta^1 - \overline{\delta} : ... : \delta^H - \overline{\delta}]_{S \times H}$$

Note that the GEI is Pareto optimal exactly when $\Delta = 0$[17]. Computing the mean welfare impact,

$$\begin{aligned} H \cdot dW &= \Sigma \lambda^h_0 dm^h_0 + \Sigma \lambda^{h\prime}_{\mathbf{1}} dm^h_{\mathbf{1}} \\ &= \Sigma dm^h_0 + \Sigma (\delta^h + c)' dm^h_{\mathbf{1}} \\ &= 0 + \Sigma \delta^{h\prime} dm^h_{\mathbf{1}} + c' \Sigma dm^h_{\mathbf{1}} \\ &= \Sigma \delta^{h\prime} dm^h_{\mathbf{1}} \\ &= \Sigma (\delta^h - \overline{\delta})' dm^h_{\mathbf{1}} \\ &= H \cdot cov(\Delta, dm_{\mathbf{1}}) \end{aligned}$$

since $\Sigma dm^h = 0$. The **rate of Pareto improvement** is the norm of the functional $dW \mid_{dv \geq 0}$.

Remark 2 *At a regular GEI, the mean welfare impact equals the covariance across households of the insurance deficit and the marginal purchasing power, $dW = cov(\Delta, dm_{\mathbf{1}})$. So the rate of Pareto improvement is bounded above by the norm of this covariance.*

If the tax policy targets only current income, i.e. $\tau^h_{\mathbf{1}}, \tau_{\mathbf{1}} = 0$, then $dm^h_{\mathbf{1}} = -\underline{z}^h_{\mathbf{1}} dP_{\mathbf{1}}$ and

$$dW = -cov(\Delta, \underline{z}_{\mathbf{1}}) dP_{\mathbf{1}}$$

The sole control is the future price adjustment, since the GEI sets the insurance deficit and net trade. In a nutshell, the mean welfare impact of the sole control is minus the covariance of insurance deficit and net trade.

[17] Also, a household's commodity demand is as though asset markets were complete exactly when $\delta^h = 0$.

8 Appendix

8.1 Derivation of formula for welfare impact

It is standard how Debreu's smooth preferences, linear constraints, and the implicit function theorem imply the smoothness of neoclassical$_0$ demand. In fact, the implicit function theorem implies smoothness of neoclassical demand in a neighborhood $\tilde{b} \approx b \in B$, if neoclassical$_0$ demand is active at $b \in B_0$. It is standard also that the envelope property follows from the value function's local smoothness, which is the case for v^h as the composition of smooth functions:

$$D_b v^h = D_b L(x, y, \lambda^h) \mid_{(x^h, y^h)(b)}$$

where $b = (p, q, a, w^h, t)$ and

$$L(x, y, \lambda^h) \equiv u^h(x) - \lambda^{h\prime} \left([\overline{p}]' x - w^h - \begin{bmatrix} -q' \\ a' \end{bmatrix} y + \tau^h(b_0, x, y) t \right)$$

Thus

$$D_b v^h = -\lambda^{h\prime} \left([\underline{x}^h]' + D_p \tau^h t : \overline{y}_0^h + D_q \tau^h t : * : -I + D_{w^h} \tau^h t : \tau^h \right) \qquad \text{where } \overline{y}_0^h = \begin{bmatrix} y^{h\prime} \\ 0 \end{bmatrix}$$

If $t = 0$

$$D_b v^h = -\lambda^{h\prime} \left([\underline{x}^h]' : \overline{y}_0^h : * : -I : \tau^h \right)$$

So much for demand theory. Recalling regular GEI from the subsection on the Expression for the Price Adjustment, $dP' = (dp', dq')$ exists and

$$\begin{aligned} w^h &= [\overline{p}]' e^h + t_*^h r \Rightarrow \\ dw^h &= [\underline{e}^h]' dp + t_*^h dr \\ &= ([\underline{e}^h]' : 0) dP + t_*^h \tau \end{aligned}$$

using $dr = \tau$ from the Revenue Impact proposition.

Thus the welfare impact at a regular GEI is

$$\begin{aligned} dv^h &= D_b v^h \cdot db \\ &= -\lambda^{h\prime} \left(([\underline{x}^h]' : \overline{y}_0^h) : * : -I : \tau^h \right) \cdot \left(dP : 0 : ([\underline{e}^h]' : 0) dP + t_*^h \tau : I \right) \\ &= -\lambda^{h\prime} \left(([\underline{x}^h]' : \overline{y}_0^h) dP - ([\underline{e}^h]' : 0) dP - t_*^h \tau + \tau^h \right) \\ &= -\lambda^{h\prime} \left(\underline{z}^{h\prime} dP - t_*^h \tau + \tau^h \right) \end{aligned}$$

where $\underline{z}^{h\prime} \equiv ([\underline{x}^h - \underline{e}^h]' : \overline{y}_0^h)$ by definition. In sum,

$$\boxed{dv^h = \lambda^{h\prime} \left((t_*^h \tau - \tau^h) - \underline{z}^h dP \right)}$$

8.2 Aggregate notation

We collect marginal utilities of contingent income, and denote stacking by an upperbar

$$(\lambda)' \equiv \begin{bmatrix} \cdot & & 0 \\ & \lambda^{h\prime} & \\ 0 & & \cdot \end{bmatrix}_{H \times H(S+1)} \qquad \underline{\overline{z}} \equiv \begin{bmatrix} \cdot \\ \underline{z}^{h\prime} \\ \cdot \end{bmatrix}_{H(S+1)\times(S+1)(C-1)+J}$$

Thus

$$dv = (\lambda)' \left((t_*^h \tau - \tau^h)_h - \underline{\overline{z}} dP \right)$$

To visualize the bracket notation $[\cdot]$ defined in footnote 7, it staggers state contingent vectors:

$$[p] \equiv \begin{bmatrix} \cdot & & & & \\ & p_{s-1} & & 0 & \\ & & p_s & & \\ & 0 & & p_{s+1} & \\ & & & & \cdot \end{bmatrix}_{C(S+1)\times S+1}$$

8.3 Transversality

A function $F : M \times \Pi \to N$ defines another one $F_\pi : M \to N$ by $F_\pi(m) = F(m, \pi)$. Given a point $0 \in N$ consider the "equilibrium set" $E = F^{-1}(0)$ and the natural projection $E \to \Pi, (m, \pi) \mapsto \pi$. A function is *proper* if it pulls back sequentially compact sets to sequentially compact sets.

Remark 3 (Transversality) *Suppose F is a smooth function between finite dimensional smooth manifolds. If 0 is a regular value of F, then it is a regular value of F_π for almost every $\pi \in \Pi$. The set of such π is open if in addition the natural projection is proper.*

A subset of Π is **generic** if its complement is closed and has measure zero. Write $C^* = C(S+1)$. Here the set of parameters is

$$\Pi = O \times O' \times (0, \epsilon)$$

where O, O' are an open neighborhoods of zero in $R^{C^* H}, R^{\frac{C^*(C^*+1)}{2} H}$ relating to endowments and symmetric perturbations of the Hessian of utilities. We have in mind a fixed assignment of utilities, which we perturb by $O' \times (0, \epsilon)$. Specifically, given an equilibrium commodity demand $\overline{x}$ by some household and $\Box \in R^{\frac{C^*(C^*+1)}{2}}, \alpha \in (0, \epsilon)$ we define $u_{\Box,\alpha}$ as

$$u_{\Box,\alpha}(x) \equiv u(x) + \frac{\omega_\alpha(\|x - \overline{x}\|)}{2} (x - \overline{x})' \Box (x - \overline{x})$$

where $\omega_\alpha : R \to R$ is a smooth bump function, $\omega_\alpha |_{(-\frac{\alpha}{2}, \frac{\alpha}{2})} \equiv 1$ and $\omega_\alpha |_{R \setminus (-\alpha, \alpha)} \equiv 0$. In a neighborhood $x \approx \overline{x}$ we have

$$\begin{aligned} u_{\Box,\alpha}(x) &= u(x) + \frac{1}{2}(x - \overline{x})' \Box (x - \overline{x}) \\ Du_{\Box,\alpha}(x) &= Du(x) + (x - \overline{x})' \Box \Rightarrow Du_{\Box,\alpha}(\overline{x}) = Du(x) \\ D^2 u_{\Box,\alpha}(x) &= D^2 u(x) + \Box \end{aligned}$$

So in an α-neighborhood the Hessian changes, by $\square$, but the gradient, demand do not. For small enough $\alpha, \square$ this utility remains in Debreu's setting, so neoclassical demand is defined and smooth when active.

In the Sufficient Independence of Reactions, the path of risk aversion is identified with a linear path $(\square^h, \alpha^h)(\xi) \equiv (\square^h \xi, \frac{\|\overline{x}^h\|}{2})$ for each household, so that $\frac{d}{d\xi} D^2 u^h_{\square,\alpha}(x) = \square^h$.

References

[1] Balasko, Y.,1988, *Foundations of the Theory of General Equilibrium.* New York: Academic Press.

[2] Bisin, A., J. Geanakoplos, P. Gottardi, E. Minelli, and H. Polemarchakis, 2001, "Markets and Contracts," Working Paper No. 01-xx, Department of Economics, Brown University.

[3] Cass, D. and A. Citanna, 1998, "Pareto Improving Financial Innovation in Incomplete Markets," *Economic Theory* 11: 467-494.

[4] Citanna, A., A. Kajii and A. Villanacci, 1998, "Constrained Suboptimality in Incomplete Markets: A General Approach and Two Applications," *Economic Theory* 11: 495-522.

[5] Citanna, A., H. Polemarchakis and M. Tirelli, 2002, "The Taxation of Trades in Assets," Working Paper No. 01-21, Department of Economics, Brown University.

[6] Debreu, G., 1972, "Smooth Preferences," *Econometrica* 40(4): 603-615.

[7] Debreu, G., 1976, "Smooth Preferences: a Corrigendum," *Econometrica* 44(4): 831-832.

[8] Duffie, D. and W. Shafer, 1985, "Equilibrium in Incomplete Markets: I A Basic Model of Generic Existence," *Journal of Mathematical Economics* 14, 285-300.

[9] Elul, R., 1995, "Welfare Effects of Financial Innovation in Incomplete Markets Economies with Several Consumption Goods," *Journal of Economic Theory* 65: 43-78.

[10] Geanakoplos, J., 2000, *Lecture Notes on Incomplete Markets.* Yale University.

[11] Geanakoplos, J. and H. Polemarchakis, 1980,"On the Disaggregation of Excess Demand Functions," *Econometrica* 48(3): 315-335.

[12] Geanakoplos, J. and H. Polemarchakis, 1986, "Existence, Regularity and Constrained Suboptimality of Competitive Allocations when the Asset Market is Incomplete." In Heller, W., Starr, R., Starrett, D. (eds.) *Uncertainty, Information and Communication: Essays in Honor of Kenneth J. Arrow, Vol.III.* Cambridge: Cambridge University Press, 1986.

[13] Geanakoplos, J. and H. Polemarchakis, 2002, "Pareto Improving Taxes," Working Paper No. 02-xx, Department of Economics, Brown University.

[14] Guillemin, V. and A. Pollack, 1974, *Differential Topology.* Englewood Cliffs, NJ: Prentice Hall.

[15] Hart, O., 1975, "On the Optimality of Equilibrium when the Market Structure is Incomplete," *Journal of Economic Theory* 11: 418-443.

[16] Lipsey, R.G. and K. Lancaster, 1956, "The General Theory of the Second-Best." *Review of Economic Studies* 24: 11-32.

[17] Magill, M. and M. Quinzii, 1996, *Theory of Incomplete Markets*. Cambridge, MA: The MIT Press.

[18] Mandler, M., 2003, "Policy Effectiveness," Working Paper, Department of Economics, Royal Holloway.

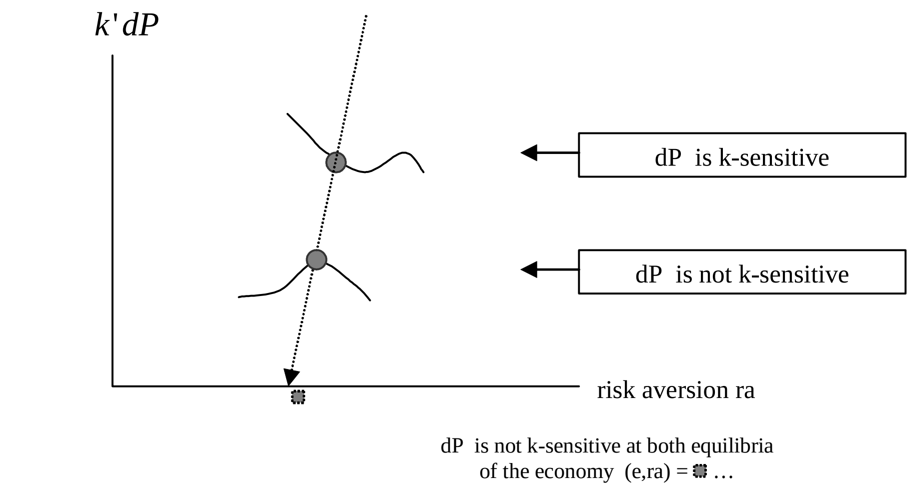
[19] Salanie, B., 2003, *The Economics of Taxation*. Cambridge, MA: The MIT Press.

[20] Stiglitz, J., 1982, "The Inefficiency of Stock Market Equilibrium," *Review of Economic Studies* 49: 241-261.

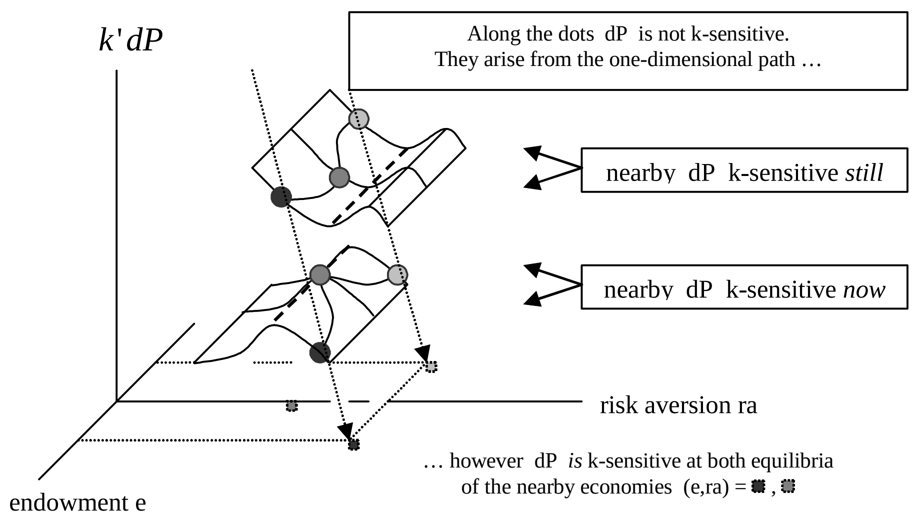
[21] Turner, S., 2003a, "Theory of Demand in Incomplete Markets," Yale University.

[22] Turner, S., 2003b, "Welfare Impact of Financial Innovation," Yale University.

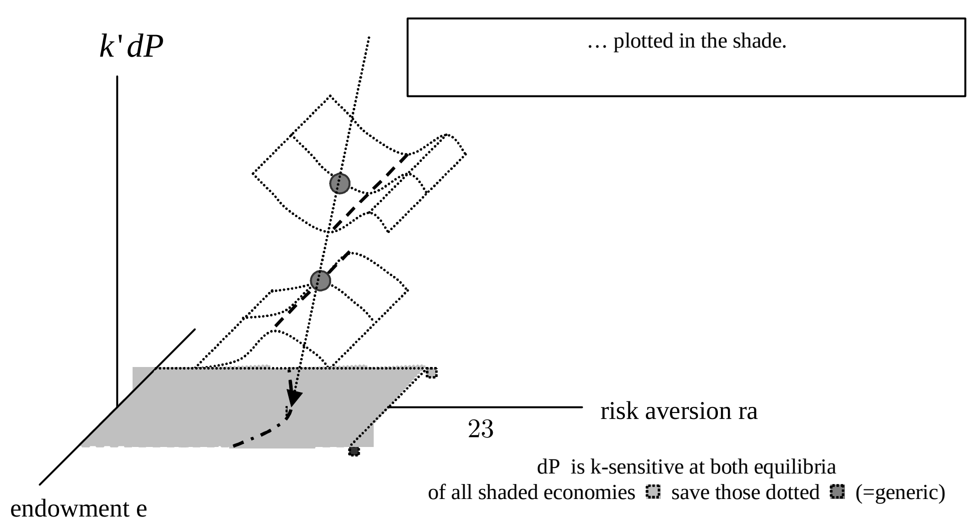
9 Figures



k' dP
dP is k-sensitive
dP is not k-sensitive
risk aversion ra
dP is not k-sensitive at both equilibria
of the economy (e,ra) = …



k' dP
Along the dots dP is not k-sensitive.
They arise from the one-dimensional path …
nearby dP k-sensitive *still*
nearby dP k-sensitive *now*
risk aversion ra
endowment e
… however dP *is* k-sensitive at both equilibria
of the nearby economies (e,ra) = ,



k' dP
… plotted in the shade.
risk aversion ra
endowment e
dP is k-sensitive at both equilibria
of all shaded economies save those dotted (=generic)