Social Interactions in a Synchronization Game *

Áureo de Paula †

This version: July 8, 2004

Abstract

This paper analyzes a synchronization game. Agents take part in an activity and benefit from the participation of others. Coordinated actions are fruit of correlated effects as well as endogenous interactions. Standard tools applied in optimal stopping problems for continuous parameter stochastic processes are used but the processes under study are endogenized by making their distribution dependent on the participation of the group. Under certain conditions, this setup allows for identifiability and separation of correlated and endogenous influences.
JEL Codes: C10, C70, D70.

*I am particularly indebted to José Scheinkman, Bo Honoré and Elie Tamer for their encouragement, guidance and support throughout the development of this project. I would like to thank Savas Dayanik for many invaluable suggestions and I also benefited from comments by participants at the Inter-University Student Conference (2004). Financial support from the Fellowship of Woodrow Wilson Scholars and NSF Grant SES-0001647 is gratefully acknowledged. All remaining errors are my own.

†Department of Economics, Princeton University, Princeton, NJ 08544-1021. E-mail: apaula@princeton.edu

1 Introduction

Social interactions that are not mediated through markets and their consequences to the economic arena have commanded increasing consideration from researchers in the recent past. Whereas economists have naturally paid much more attention to social relations that can be incorporated by market institutions, the existence of non-market interactions is nevertheless fundamental for the study of many issues in various fields of economics and the accurate identification and measurement of the types of interactions underlying these phenomena can be helpful in designing appropriate policies.

It is usually difficult to disentangle why agents behave similarly when they do so. Manski [31] labeled this as the "reflection problem" and categorized the forces that generate similarities in group behavior into three main types: endogenous, contextual (or exogenous) and correlated effects. In order to understand this taxonomy, Manski provides a concrete example. One can enumerate many elements affecting a child's achievement in school. Aside from those that are inherent to the individual we could cite 1. the performance of his or her classmates, as well as 2. their socio-economic background and maybe 3. the teacher or the infra-structure provided by the school itself. The first and second are inherently social effects whereas the third arises only because individuals are subject to the same institutional environment or are very similar in their individual characteristics. For this reason the last effect is qualified by Manski as a "correlated effect". Although the first two factors are social effects, they are distinct in nature. The first effect, that of the group's achievement on a student's achievement, will typically generate a feedback phenomenon: a student will have higher achievement when their classmates have higher achievement and these on the other hand will also follow the same prescription, creating a virtuous circle. This "bootstrap" effect will blow up the impact of other factors, generating a "social multiplier", and might even allow for multiple equilibria if strong enough (see, for instance, Glaeser and Scheinkman [15]). In this case, parameter identifiability and estimation become even more problematic as pointed out in Bresnahan and Reiss [5] but nonetheless feasible under certain circumstances as indicated in Tamer [40] among others. Because of this feedback, the effect that the group's actions (achievement) have on one's action is classified as an endogenous effects. The remaining social effects are labeled as contextual, or exogenous, effects.

It is important to distinguish these elements because they usually imply very different policy prescriptions. If exogenous effects exist, for instance, change of group membership is bound to have an impact in the outcomes. On the other hand, the existence of endogenous influences would indicate that application of the prescribed interventions only to a subset of the group would likely be sufficient to affect all the members. Even though it is important to properly identify the nature of the social influence channel, this might not be an easy task. As explained in Manski [31] the empirical problems faced in this area are akin to the fact that the observation of equilibrium prices and quantities is not enough to separate the market interactions of consumers and suppliers (demand and supply curves). A simultaneity problem is latent and might hamper any attempts to pin down the nature of the social environment under consideration. As a reaction, many studies have been recently produced to advance techniques and illuminate the empirical work on the field. Areas in which statistical identification of these phenomena has been shown to be possible include, for instance, non-linear econometric models and the econometric analysis of games.

In light of the above discussion, imagine an individual faced with the decision of whether to join a certain welfare program or not. It has been documented that a significant portion of eligible candidates choose not to take part in these programs. The participation rate in the Food Stamps Program (FSP) among eligible individuals for example declined from 74% in 1994 to 59% in 1999 (*The Decline in Food Stamps Participation: A Report to*

Congress, USA/FNS, 2001). Similar phenomena also take place in other programs such as the Temporary Assistance to Needy Families (TANF), which replaced the Aid to Families with Dependent Children (AFDC) program. This behavior is usually ascribed to the existence of a stigma attached to the decision of participation. Under these circumstances an interesting feedback phenomenon may be latent: communities of high participation would induce lower stigmatization, which would itself favor increased participation, whereas a group that starts out with a low level of participation would imply higher stigma, which would in turn favor low levels of participation. Since the decision of whether to join a program is not a once-and-for-all choice, the timing coordination is an important element in the analysis of this problem and we can see basically as a synchronization problem. In this example, undoubtedly a relevant practical situation, as in other similar circumstances, policy interventions that operate directly on this "bootstrap" channel may be extremely efficient.

In the first part of this paper we develop a theoretical model capturing this synchronization phenomenon while allowing for correlated as well as endogenous effects. The model describes the issue through optimal stopping decisions in the presence of interactions (externalities). In Economics, standard optimal stopping problems arise naturally in investment models and in financial derivatives pricing (see for example Dixit and Pindyck [10]). The general idea in these models is that a certain flow of benefits is described by a stochastic process and the decision-maker is to devise a rule (i.e. stopping time) in order to extract the maximum benefit (i.e. maximize a certain gains function in expected terms). The usual model assumes that the underlying stochastic process is unaffected by the decision of the agents in the population under consideration. The modification suggested incorporates the possibility that the stopping decisions by a certain group of agents affect the evolution of the benefit flow and ultimately the decision of other individuals in the population contemplated. In other words, a stopping rule would typically be thought of as a "set of instructions" telling the decision maker to quit the activity as soon as the state variable reaches a certain threshold level. The difficulty in the problem analyzed is that the decision by other community members will affect the relevant state variable and affect one individual's stopping rule formulation and the other community members' decision rules are themselves endogenous to the problem. Circumstances that are likely to be described by such a model involve all those that require coordination on the timing of acts, such as the welfare program participation example cited above. Others may include stock market participation (Hong, Kubik and Stein [19] and Ivković and Weibenner [21]), bank runs (Kelly and Gráda [25]), South-North migration (Orrenius [32]), marriage decisions (Goldin and Katz [17]) and even crime recidivism (see, for instance, the empirical investigation by Sirakaya [39] where social interactions are found to meaningfully affect recidivism among individuals on probation).

One particular application is the situation studied in a recent paper by Costa and Kahn [8] (see also [9]). The purpose of the article is to investigate the effect of group homogeneity on shirking. In order to do so, the authors use a dataset comprising detailed individual records for soldiers of the Union Army in the American Civil War. The dataset allows Costa and Kahn to build proxies for group homogeneity as well as to control for a series of other potential determinants of "loyalty", which is captured by events such as desertion, arrest, AWOL (absence without leave) or promotion. In connection with the model alluded above, the stopping decision here is represented by the timing of each of these events, especially desertion. Using a standard statistical duration model, the researchers find that company homogeneity indeed decreases shirking within groups. This might not seem very surprising once one considers that a more homogeneous group facilitates communication and increases the chances of social sanction to those who shirk. On the other hand, a more uniform environment might also facilitate coordination among the agents involved and spark mass desertions, which seems to have been the case in other situations and especially among confederate soldiers toward the end of the war (see, for instance, Bearman [4] and Lonn [27]).

Whereas standard statistical duration models could be employed to identify the existence of duration dependence among agents (as indeed is done in Costa and Kahn [8] and Sikaraya [39] and suggested in Brock and Durlauf [6]) it is still unclear what is the source of such effects: endogenous influences or correlated unobservables. In contrast, our model clearly separates both channels and lays out the circumstances under which each of these sources is separately identifiable, thus presenting a substantial contribution to this increasing field. Furthermore, our model bears empirical consequences that are not incorporated in the more standard econometric duration models such as the positive probability of concomitant exits from the game and points to the necessity of utilizing other methods in such estimation exercises.

The next section presents a review of the relevant literature. A general model is outlined in the following two sections and a specialization explored in the subsequent one. Section 6 deals with the empirical implications and the final section concludes.

2 Literature Review

To be added. Cite Mamer [29] and Lakari, Solan and Vieille [26]. Fudenberg and Tirole [13], Ch.13 (differential and stochastic games, MPE). Literature on game estimation.

3 The Model

Consider a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t\in\mathbb{R}_+})$ (the filtration is assumed to satisfy the usual conditions). Assume also that there are I agents and that these agents take part in a certain activity (I will loosely use I

to denote the set of agents and its cardinality). Each individual's utility is captured by a gain function ($u_i : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}$) evaluated at the value taken by an individual state variable $x_t^i \in \mathbb{R}_+$ and instant $t \in \mathbb{R}_+$ of a chosen stopping point. An agent $i \in I$ can choose to abandon the activity at any time $t \in \overline{\mathbb{R}}_+$ (where $\overline{\mathbb{R}}_+ = [0, \infty]$), so that the decision to stop can be represented by τ_i, a (possibly infinite) stopping time with respect to the individual filtration $(\mathcal{F}_t^i)_{t \in \mathbb{R}_+}$[1] representing agent i's flow of information. Throughout the paper we will focus on the filtration generated by his or her state variable process and whether or not the other individuals have stopped. In other words, $\mathcal{F}_t^i = [\vee_{j \neq i} \sigma(t \wedge \tau^j)] \vee \sigma(\{x_s^i : 0 \leq s \leq t\})$ (with $\sigma_A \vee \sigma_B$ denoting the σ-algebra generated by the union of the σ-algebras σ_A and σ_B). This assures that each subject's information relies only on each one's individual latent utility up to that particular point in time and the observation of the others' decisions instead of having knowledge of all the other agents' state variables evolution. I assume that the state variable evolves as a process (adapted to the above filtration) which may depend on the participation of the remaining individuals in the group (thus the reference to externalities). Let θ_t^i be the process representing the fraction of the population (excluding agent i) that has abandoned the activity before time t. In other words, $\theta_t^i = \sum_{s=1, s \neq i}^{I} \mathbb{I}_{\{\tau_s < t\}} / I$ (with $\mathbb{I}_{\{A\}}$ as the indicator function for the event $A \subset \Omega$). This process will be determined endogenously as individuals choose the stopping times in consonance with their preferences. Each individual state variable x_t^i is assumed Markovian and is characterized by a transition function $\{P_t^i\}_{t \in \mathbb{R}_+}$ where $P_t^i : \mathbb{R}_+^I \times [0,1] \times \mathbb{R}_+ \times \mathcal{B}(\mathbb{R}_+) \to [0,1]$ is a kernel such that, for $s \leq t \in \mathbb{R}_+, \Gamma \in \mathcal{B}(\mathbb{R}_+), \mathbf{x}_s \in \mathbb{R}_+$,

$$\mathbb{P}(x_t^i \in \Gamma | \mathcal{F}_s) = P_{t-s}^i(\mathbf{x}_s, \theta_s^i, s; \Gamma)$$

[1] A random variable $\tau : \Omega \to \mathbb{R}_+$ is a stopping time with respect to $(\mathcal{F}_t)_{t \in \mathbb{R}_+}$ if, for each $t \in \mathbb{R}_+$, $\{\omega : \tau(\omega) \leq t\} \in \mathcal{F}_t$. Some authors use the term Markov time for this definition and refer to stopping times as finite Markov times. In this paper we use infinite and finite stopping times respectively for these objects. Intuitively they represent stopping strategies that rely solely on past information.

The superscript i reminds the reader that the state variable process may differ across individuals. The structure for the multi-person decision problem is presented in the following definition.

Definition 1 (Stopping Game with Externalities) *A* Stopping Game with Externalities *is a tuple* $\langle I, (\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t\in\mathbb{R}_+}),\ (u_i)_{i\in I}, (x^i)_{i\in I}, (T_i)_{i\in I}\rangle$ *where* I *is the set of agents;* $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t\in\mathbb{R}_+})$*, a filtered probability space;* $u_i : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}$*, an individual gain (utility) function;* x^i*, an individual process adapted to the filtration* $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$ *and having as state space* $\mathbb{R}_+$*; and* T_i*, a set of stopping strategies* $\tau : \Omega \to \overline{\mathbb{R}}_+$ *such that* $\{\omega \in \Omega : \tau(\omega) \le t\} \in \mathcal{F}^i_t, \forall t$ *where* $\mathcal{F}^i_t \subset \mathcal{F}_t, \forall i, t$ *(i.e.,* τ *is an* $(\mathcal{F}^i_t)_{t\in\mathbb{R}_+}$ *stopping time).*

Having defined the basic structure of the problem, the idea is that each person i is confronted with a decision problem that is mathematically represented by the following (individual) optimal stopping problem (where τ generically denotes a stopping time with respect to $(\mathcal{F}^i_t)_{t\in Z}$):

$$\left\{ \begin{array}{lll} & & \{P^i_t\}_{t\in Z} \\ V_i(x_i) = \sup_{\tau\in T_i} \mathbb{E}_{x_i}[u_i(x_\tau, \tau)] & \text{s.t.} & \theta^i_t = \sum_{s=1, s\neq i}^{I} \mathbb{I}_{\{\tau_s < t\}}/I \\ & & x^i_0 = x_i \end{array} \right. \tag{1}$$

In the above definition, $\mathbb{E}_{x_i}[u_i(x^i_\tau, \tau)] = \int_\Omega \mathbb{P}(d\omega) u_i(x^i_{\tau(\omega)}(\omega), \tau(\omega))$ with initial condition given by x_i. We assume that $u_i(x_\infty(\omega), \infty) = \limsup_{t\in Z} u_i(x_t(\omega), t)$.

In this paper, the state variable is assumed to obey a transition law given by the following stochastic differential equation:

$$dx^i_t = \alpha^i(x^i_t, \theta^i_t, t)dt + \sigma^i(x^i_t, \theta^i_t, t)dW^i_t, \qquad x^i_0 \sim F^i_0 \tag{2}$$

where W^i_t is a Wiener process defined in the particular probability space we are considering and the drift and dispersion coefficients are assumed to be positive Borel-measurable functions. The initial distribution F^i_0 is furthermore independent of the Brownian motion W^i_t. In order to assure that this stochastic differential equation has a strong solution given a profile of stopping times for each player, we impose the following assumptions on the drift and dispersion coefficients:

Assumption 1 (Lipschitz and Growth Conditions) *The coefficients $\alpha^i(x,\theta,t)$ and $\sigma^i(x,\theta,t)$ satisfy the global Lipschitz and linear growth conditions:*

$$\|\alpha^i(x,\theta,t)-\alpha^i(y,\theta,t)\|+\|\sigma^i(x,\theta,t)-\sigma^i(y,\theta,t)\|\leq K\|x-y\| \tag{3}$$

$$\|\alpha^i(x,\theta,t)\|^2+\|\sigma^i(x,\theta,t)\|^2\leq K^2(1+\|x\|^2) \tag{4}$$

for every $t\in\mathbb{R}_+, x,y\in\mathbb{R}, \theta\in[0,1]$ and $i\in I$, where K is a positive constant.

Notice that $\theta_t^i=\sum_{s=1,s\neq i}^{I}\mathbb{I}_{\{\tau_s<t\}}/I$ is adapted since θ is the aggregation of indicator functions of events such as $\{\tau<t\}$, where τ is a stopping time. By Theorem I.1 in Protter [34], $\{\tau<t\}\in\mathcal{F}_t$. Given the Borel-measurability conditions on the drift and dispersion coefficients, this guarantees that, for fixed x, $(t,\omega)\mapsto\alpha^i(x,\theta_t^i(\omega),t)$ and $\sigma^i(x,\theta_t^i(\omega),t)$ are adapted. The above assumptions guarantee the existence of a strong solution for the stochastic differential equation (2). A sketch for the proof is presented in the Appendix. The following section analyzes the existence of equilibria for this game.

4 Equilibrium: Existence

The solution concept we seek for this group situation is that of mutual best responses, a standard Nash Equilibrium point: a collection of individual optimal stopping times indexed by the set I such that each individual stopping time is optimal given the stopping rules adopted by the other agents. Denoting by $\tau=(\tau_i)_{i\in I}$ a stopping time profile, let $U_i(\tau)=\mathbb{E}_{x_i}[u_i(x_{\tau_i},\tau_i)]$ subject to the above transition laws and initial conditions and evaluated at the strategy profile τ. We also adopt the convention of using τ_{-i} as shorthand notation for $(\tau_s)_{s\in I-\{i\}}$. A Nash Equilibrium [2] for the above game is then:

[2]Since the strategies depend on information generated by the state variables and these are Markovian and since optimization follows Bellman's principle of optimatility in dy-

Definition 2 (Equilibrium) *A* Nash Equilibrium *for the* Stopping Game with Externalities *is a stopping time profile* $\tau^* = (\tau_i^*)_{i \in I}$ *such that:*

$$U_i(\tau^*) \geq U_i(\tau_i, \tau_{-i}^*), \forall i, \tau_i \text{ stopping time.}$$

In order to proceed with the analysis of equilibrium, we make the following assumptions:

Assumption 2 (Exponential Discounting) *Let* $u_i(x,t) = e^{-\gamma_i t} g_i(x), \gamma_i > 0, g_i : \mathbb{R}_+ \to \mathbb{R}, \forall i \in I$ *We refer to* $g_i(\cdot)$ *as the reward function.*

Assumption 3 (Reward Function) *The individual reward functions* $g_i(\cdot)$, $\forall i \in I$ *are assumed to satisfy:*

- *Monotonicity.* $g_i(\cdot)$ *is increasing.*
- *Convexity.* $g_i(\cdot)$ *is convex.*
- $\mathbb{E}[\sup_{t \in Z} |e^{-\gamma_i t} g_i(x_t^i)|] < \infty$.
- *Twice differentiability.* $g_i(\cdot)$ *is twice differentiable.*
- *Bounded derivative. The derivative* $g'(\cdot)$ *is bounded.*

Assumption 4 (Bounded volatility) *For each* $t < \infty$ *and feasible profile of stopping strategies the dispersion coefficient is assumed to satisfy:*

$$\mathbb{E}[\int_0^t (e^{-\rho s} \sigma(x_s, \theta_s, s))^2 ds] < \infty.$$

Assumption 5 (Complementarity) *The drift and the dispersion coefficients are assumed to be decreasing on their second argument.*

---

namic programming — whatever the initial state and decisions are, the remaining decisions must be optimal with regard to the state resulting from the first decision — these are also Markov Perfect Equilibria. For a discussion of MPE, see Fundenberg and Tirole [13], chapter 13.

The Exponential Discounting Assumption (2) significantly simplifies the manipulation and is fairly standard in the field. The set of assumptions regarding the reward functions, (3), encompasses monotonicity and convexity, which are not very controversial either; bounded range, which is employed to assert the existence of a solution for the optimal stopping problem, and technical assumptions that facilitate the application of existing results in the comparison of solutions of stochastic differential equations. The Bounded Volatility Assumption (4) will imply that changes in the profile of stopping decisions will affect the objective function only through the drift of the discounted gain function. Finally, the Complementarity Assumption (5) expresses the idea that higher participation makes the activity more attractive as well as increases the volatility of the returns. We are now ready to state the following result [3]:

Theorem 1 (Existence) *Under Assumptions 1-5, the* Stopping Game with Externalities *has a nonempty set of equilibrium points and this set possesses a maximal element.*

Proof. See Appendix.

Under such general conditions, very little can be said regarding uniqueness and other properties of the model. In fact, unless more stringent conditions are imposed on the information structure, the setup will admit multiple equilibria, as will be delineated in the following section.

5 The Desertion Game

We will frame this particular specialization of the model in terms of the strategic situation present in Costa and Kahn's dataset (see [8] and [9]): that of military desertion. Consider initially the hypothetical army where

[3] Mamer [29] obtains existence of equilibria in a similar (but more restrictive) game in discrete time through similar techniques.

soldiers contemplate the possibility of desertion. We use a state variable x (which is assumed to evolve according to a certain stochastic process) to represent the latent utility a soldier derives from remaining in the front. At desertion, he or she pays a cost C. Such individual has to devise a timing rule dictating his or her desertion decision. Given a discount rate γ, the objective of the agent is then to maximize the following reward function:

$$\mathbb{E}^x[e^{-\gamma t}(x_t - C)]$$

One should expect that the stopping decision of a solder directly affects the decision of the other one. If no one deserts, the social sanctions attached to desertion tend to be high; whereas if there is mass desertion, such sanctions tend to be minimized as well as the effectiveness of the military company. Our strategy is to model such external effects through a change in the drift of the latent utility process, x.

* * * * *

At an initial stage though consider the individual problem where the state variable, x, changes according to the following law:

$$\log x_t = \begin{cases} (\alpha - \frac{\sigma^2}{2})t + \sigma W_t + \log x_0 \text{ if } t < \nu \\ (\alpha - \Delta\alpha)(t-\nu) + \alpha\nu - \frac{\sigma^2}{2}t + \sigma W_t + \log x_0 \text{ if } t \geq \nu \end{cases}$$

where $\Delta\alpha > 0$ and ν is an $\mathcal{F}_t$-stopping time (assume, as usual, a filtered probability space $(\Omega, \mathbb{P}, \mathcal{F}, (\mathcal{F}_t)_{t\in\mathbb{R}_+})$). The initial condition is drawn from an independent distribution F_0 as in equation (2). Notice that the break point for the drift here is exogenously given. At a later stage we will endogenize this stopping time. For there to be a well-defined solution to this problem, we assume that $\gamma > \alpha$.

Let $\overline{x}$ be the process corresponding to $\nu(\omega) = \infty, \forall\omega \in \Omega$ (i.e. a geometric Brownian motion with drift and diffusion coefficients αx and σx) and $\underline{x}$ be the process corresponding to $\nu(\omega) = 0, \forall\omega \in \Omega$ (i.e. a geometric

Brownian motion with drift and diffusion coefficients ($(\alpha - \Delta\alpha)x$ and σx). By standard dynamic programming calculations, the optimal stopping times for these two processes are characterized by threshold levels $\overline{z} = z(\alpha, \sigma, C, \gamma)$ and $\underline{z} = z(\alpha - \Delta\alpha, \sigma, C, \gamma)$, where

$$z(\alpha, \sigma, C, \gamma) = \frac{\beta(\alpha, \sigma, \gamma)}{\beta(\alpha, \sigma, \gamma) - 1} C$$

and

$$\beta(\alpha, \sigma, \gamma) = 1/2 - \alpha/\sigma^2 + \sqrt{\left[\alpha/\sigma^2 - 1/2\right]^2 + 2\gamma/\sigma^2} > 1$$

(see Dixit and Pindyck [10], p.140-144). For notational convenience, we omit the parameter dependence in the remainder of the section. Given an arbitrary ν we propose the stopping rule characterized by the following continuation region:

$$\begin{array}{l} \{x \leq \overline{z}\} \text{ if } t < \nu \\ \{x \leq \underline{z}\} \text{ if } t \geq \nu \end{array} \tag{5}$$

If we let $\overline{\tau} \equiv \inf_t \{t : \overline{x}_t \geq \overline{z}\}$ and $\underline{\tau} \equiv \inf_t \{t : \underline{x}_t \geq \underline{z}\}$, the stopping time associated with this region is

$$\tau = \overline{\tau} \mathbb{I}_{\overline{\tau} \leq \nu} + \inf\{t > \nu : x_t > \overline{z}\} \mathbb{I}_{\overline{\tau} > \nu}$$

where $\mathbb{I}_A$ is the indicator function for the event A. [4] We are then ready to state the following proposition.

Proposition 1 *The continuation region (5) defines an optimal stopping time for the stated problem.*

[4] To see that τ is indeed an $\mathcal{F}_t$-stopping time, notice that $\overline{\tau}, \nu$ being stopping times implies that $\overline{\tau} \wedge \nu$ is also a stopping time (see Karatzas and Shreve [23], Lemma I.2.9). This in turn implies that $\{\nu \wedge \overline{\tau} \leq t\} \in \mathcal{F}_t \Leftrightarrow (\{\overline{\tau} \leq \nu\} \cap \{\overline{\tau} \leq t\}) \cup (\{\nu < \overline{\tau}\} \cap \{\nu \leq t\}) \in \mathcal{F}_t, \forall t$. On the other hand, $\mathcal{F}_\nu \subset \mathcal{F}_{\nu + \underline{\tau}}$ by Lemma I.2.15 in Karatzas and Shreve [23]. This means that $(A \cap \{\nu \leq t\} \in \mathcal{F}_t \Rightarrow A \cap \{\nu + \underline{\tau} \leq t\} \in \mathcal{F}_t)$. Then, from above, $(\{\overline{\tau} \leq \nu\} \cap \{\overline{\tau} \leq t\}) \cup (\{\nu < \overline{\tau}\} \cap \{\nu + \underline{\tau} \leq t\}) \in \mathcal{F}_t, \forall t$. This is equivalent to $\{\tau \leq t\} \in \mathcal{F}_t, \forall t$ which establishes that τ is a stopping time.

Proof. See Appendix.

The optimal stopping time is thus given by

$$\tau = \overline{\tau}\mathbb{I}_{\overline{\tau}\leq\nu} + \inf\{t > \nu : x_t > \overline{z}\}\mathbb{I}_{\overline{\tau}>\nu}$$

and the value of this decision rule can be deduced to be

$$V^0(x) = \mathbb{E}^x[e^{-\gamma\overline{\tau}}(x_{\overline{\tau}} - I)\mathbb{I}_{\overline{\tau}\leq\nu} + e^{-\gamma\inf\{t>\nu:x_t>\overline{z}\}}(x_{\inf\{t>\nu:x_t>\overline{z}\}} - I)\mathbb{I}_{\overline{\tau}>\nu}]$$

The above proposition can easily be extended to processes with multiple breaks at increasing stopping times (as we do later on) and delivers a stopping rule where the agent switches progressively to lower threshold levels as the drift breaks take place.

* * * * *

Now consider a hypothetical army with two soldiers indexed by $i = 1, 2$. They both contemplate a desertion decision that will cost them $I_i, i = 1, 2$ in return for a value $x_i, i = 1, 2$, just as in the previous setup. The difference is that the latent utility process for one soldier is negatively affected once the other soldier decides to leave the front.

In particular, consider all the above parameters indexed by i and the latent utility process, given by:

$$\log x_t^i = \begin{cases} (\alpha^i - \frac{\sigma^{i2}}{2})t + \sigma^i W_t^i + \log x_0^i \text{ if } t < \tau^j \\ (\alpha^i - \Delta\alpha^i)(t - \tau^j) + \alpha^i\tau^j - \frac{\sigma^{i2}}{2}t + \sigma^i W_t^i + \log x_0^i \text{ if } t \geq \tau^j \end{cases}$$

where $i, j = 1, 2$, $i \neq j$ and τ^j is the stopping time adopted by the other soldier in the game. Notice that the dependence between the Brownian motions is left unconstrained and that $\Delta\alpha^i$ measures the external effect of the other agent's decision on i. This reveals the two major aspects of group behavior under consideration in this study: correlated and endogenous social effects. Individuals might behave similarly in response to associated (unobservable) shocks, which are reflected in the possibly non-null cross-variation of the Brownian motions driving individual latent utilities. These are correlated effects. On the other hand, agents may directly affected by other agents' actions as well and this would appear as a decrease in the implicit utility of a soldier for remaining in the front after another soldier leaves the army. This is the endogenous effect. We assume that each player only knows the evolution of his or her own latent utility process and the timing of previous desertions. In other words, $\mathcal{F}_t^i = [\vee_{j \neq i} \sigma(t \wedge \tau^j)] \vee \sigma(\{x_s^i : 0 \leq s \leq t\}), \forall t, i$.

The previous analysis establishes that each soldier will use the "high drift" optimal stopping rule characterized by the threshold $\overline{z}_i$ until the stopping time τ^j, at which she switches to the "low drift" stopping rule, characterized by the threshold $\underline{z}_i$. In this case though we need to handle the fact that τ^j is not exogenously given, but determined within the strategic situation at hand. It is illustrative to portray this interaction graphically.

Consider the $X_1 \times X_2$ space where the evolution of the vector-valued process (x_1, x_2) is represented. Since $\Delta\alpha^i > 0$, we should have $\overline{z}_i > \underline{z}_i, i = 1, 2$. As in the previous analysis, soldiers start out under threshold $\overline{z}_i$. If the other soldier stops, the threshold level drops to $\underline{z}_i$. For instance, in Figure 1 the process fluctuates in rectangle $(0, \overline{z}_1) \times (0, \underline{z}_2)$ and reaches the barrier $\overline{z}_1$ causing soldier 1 to stop. Once this happens, soldier 2's threshold drops to $\underline{z}_2$, which, once reached, provokes soldier 2 to stop. A symmetric situation occurs if we interchange the soldiers roles.

A more interesting situation is depicted in Figure 2. Here, the vector process sample path attains the upper threshold for soldier 1 at a point where $x^2 \geq \underline{z}^2$. The second soldier's threshold moves down immediately and both stop simultaneously. So, if a soldier's latent utility process is above the subsequently lower threshold when the other one drops out, there will be clustering and they move out concomitantly.

If agents are allowed to base their rules on "enough" data, much leverage is gained and multiple equilibria are possible. For the sake of illustration, assume that $\mathcal{F}_t^i = \mathcal{F}_t, \forall i$ and t. In this case, it is licit for soldier 1 to follow a stopping rule that dictates stopping once the process reaches the diagonal line joining $(\overline{z}_1, \underline{z}_2)$ and $(\underline{z}_1, \overline{z}_2)$ (both soldier observe the two latent processes). As shown in Figure 3, as soon as the process reaches this barrier soldier 2's threshold moves and leaves him or her in the stopping region, causing this soldier to stop as well. If we use the same barrier though to characterize soldier 2's stopping rule and observe soldier 1's response to it, we also get the same result. This curve then characterizes an equilibrium for this game. But there is indeed nothing special about this curve and we could have used any other shape in the square $(\underline{z}_1, \overline{z}_1) \times (\underline{z}_2, \overline{z}_2)$ connecting the NW and SE corners. This rectangle supports multiple equilibria. The following proposition states the equilibrium for this situation:

Proposition 2 *Assume $\mathcal{F}_t^i = \mathcal{F}_t, \forall i, t$. Let*

$$S = \left\{ (x^1, x^2) : \begin{array}{l} \exists i \text{ such that } x^i \geq \overline{z}^i; \text{ or} \\ x^2 \geq f(x^1) \text{ where } x^1 \in (\underline{z}_1, \overline{z}_1), f(\cdot) \text{ is continuous,} \\ f(\underline{z}_1) = \overline{z}_2, f(\overline{z}_1) = \underline{z}_2 \text{ and } f((\underline{z}_1, \overline{z}_1)) \subset (\underline{z}_2, \overline{z}_2) \end{array} \right\}$$

and let $\tau_S = \inf\{t > 0 : (x_t^1, x_t^2) \in S\}$ denote the hitting time for this set. The equilibrium strategies for the desertion game are given by

$$\tau_i^* = \tau_S \mathbb{I}_{x^i_{\tau_S} = \overline{z}_i} + \inf\{t > \tau_S : x_t^i > \underline{z}_i\} \mathbb{I}_{x^i_{\tau_S} \neq \overline{z}_i}, \qquad i = 1, 2.$$

Proof. See Appendix.

The area of the rectangle with vertices in $\{(x^1, x^2) : x^i = \overline{z}_i \text{ or } x^i = \underline{z}_i, i = 1, 2\}$ could be seen as a "measure of multiplicity" in the model. Noticing that $\partial(\overline{z}_i - \underline{z}_i)/\partial\cdot = \int(\partial^2 z_i/\partial\alpha\partial\cdot)d\alpha$, it can be seen that the area of this rectangle varies positively with the intensity of the external effects $\Delta\alpha^i = \overline{\alpha}^i - \underline{\alpha}^i, i = 1, 2$, and the uncertainty in the latent utility process

$\sigma^i, i = 1, 2$. So, as long as there are external effects ($\Delta\alpha^i > 0, i = 1, 2$), there will be multiple equilibria in this game.

Is multiplicity a reasonable outcome? The achievement of multiple equilibria seems to require a (very fragile) omniscience by the players involved. If one restricts the information set accessible by each soldier to his or her own state variable, the result breaks down and uniqueness is achieved. Since at each moment an individual is unable to pinpoint the exact location of the other player's state variable and this will, with positive probability, lie below the lower stopping threshold, an "early" stopping decision may not elicit desertion by the other player and risk sacrificing potentially profitable expected rewards by staying put in the game. In the proposition below we restrict each agents information set to his or her own state variable and whether or not the other agent has stopped[5].

Proposition 3 *Assume* $\mathcal{F}_t^i = [\vee_{j \neq i} \sigma(t \wedge \tau^j)] \vee \sigma(\{x_s^i : 0 \leq s \leq t\}), \forall t, i$. *Let*

$$S = \{(x^1, x^2) : \exists i \text{ such that } x^i \geq \overline{z}^i\}$$

and let $\tau_S = \inf\{t > 0 : (x_t^1, x_t^2) \in S\}$ *denote the hitting time for this set. The equilibrium for the desertion game is unique and the equilibrium strategies are given by*

$$\tau_i^* = \tau_S \mathbb{I}_{x_{\tau_S}^i = \overline{z}_i} + \inf\{t > \tau_S : x_t^i > \underline{z}_i\} \mathbb{I}_{x_{\tau_S}^i \neq \overline{z}_i}, \qquad i = 1, 2.$$

Proof. See Appendix.

Even if access to more data is allowed, epistemological frictions may be seen to render uniqueness. If for instance one reasonably assumes that the other agent's desertion and state variable are perceived with delay by a soldier, dropping out may not elicit the other player's desertion as in the

[5]Notice that the filtration is itself endogenously generated since the stopping times are decision variables.

previous situation. This "synchronization risk" is inherent in many similar situations as the following quote in one of the earliest discussions of this problem asserts:

> It is usually the essence of mob formation that the potential members have to know not only where and when to meet but just when to act so that they act in concert. (...) In this case the mob's problem is to act in unison without overt leadership, to find some common signal that makes everyone confident that, if he acts on it, he'll not be acting alone. (Schelling [37])

For this reason we restrict our attention to the above equilibrium which is robust to such perturbations.

* * * * *

We now generalize the analysis for a military company comprising I soldiers. As in the previous case, we begin by extending the analysis for a situation with multiple (random) breaks in the drift coefficient.

Proposition 4 *Let* $\log x_t = \alpha t - \Delta\alpha \sum_{k=1}^{n}(t-\nu_k)\mathbb{I}_{t\geq\nu_k} - \frac{\sigma^2}{2}t + \sigma W_t$ *where* $\Delta\alpha, \alpha, \sigma > 0, t \in \mathbb{R}_+, n \in \mathbb{I}, W$ *is a standard Brownian motion and* $\{\nu_k\}_{k=1,\ldots,n}$ *is an increasing sequence of stopping times. The optimal continuation region for the stopping problem is given by*

$$\{x \leq z_{k-1}\} \text{ if } t < \nu_k, k = 1, \ldots, n$$
$$\{x \leq z_n\} \text{ if } t \geq \nu_n$$

where z_k *is the threshold level associated to the problem with* $\log x_t = (\alpha - k\Delta\alpha)t + \sigma W_t$.

Proof. See Appendix.

* * * * *

As before, a desertion decision is assumed to cost a soldier $C_i, i \in I$ in return for a payoff $x^i, i \in I$. The latent utility process is now given by:

$$\log x_t^i = \alpha^i t - \Delta\alpha^i \sum_{j:j\neq i}(t-\tau^j)\mathbb{I}_{t\geq\tau^j}/(I-1) - \frac{\sigma^{i2}}{2}t + \sigma^i W_t^i, \quad i \in I$$

where τ^j is the stopping time adopted by the soldier j. Notice that the external effect of other soldiers on i is given by $\Delta\alpha^i > 0$ and is considered to be homogeneous across soldiers, i.e. the amount by which the drift α^i decreases with each stopping decision is the same regardless of who deserts.

In order to generalize Proposition 2, a few definitions are convenient.

z_m^i : $z(\alpha^i - \Delta\alpha^i(\frac{m-1}{I-1}), \sigma^i, C^i, \gamma^i)$ where $i, m \in I$

S_m : $\{(x^1, x^2, \ldots, x^I) \in \mathbb{R}_+^I : \exists i$ such that $x^i \geq z_m^i\}$where $m \in I$

τ_0 : 0 (meaning $\tau_0(\omega) = 0, \forall\omega$)

A_0 : I_I (identity matrix of order I)

τ_m : $\inf\{t > \tau_{m-1} : A_{m-1}x_t \in A_{m-1}S_{I+1-\mathbf{1}'A_{m-1}\mathbf{1}}\}$ where $A_{m-1}S_{I+1-\mathbf{1}'A_{m-1}\mathbf{1}}$ denotes the set formed by operating the matrix A_{m-1} on each element of $S_{I+1-\mathbf{1}'A_{m-1}\mathbf{1}}$, $\mathbf{1}$ is an $I \times 1$ vector of ones and $m \in I$

A_m : $[a_{kl}^m]_{I\times I}$ where $a_{kl}^m = \mathbb{I}_{x_{\tau_m}^i < z_m^i}$ if $k = l = i$ and $a_{kl}^m = 0$ otherwise and $m \in I$

An inequality sign relating two vectors is understood as a relation that holds component by component. The stopping times defined above are essentially hitting times.

The idea is that the military company starts out with no defection and desertions occur at the random times $\tau_1 \leq \tau_2 \leq \tau_3 \ldots$. Our interest at first is on the hitting time for the set S_1, which is a generalization for the two dimensional S in Proposition 2. As the vector process reaches this set, one or more agents will quit. This will shift the stopping threshold $S_\cdot$. In order to do so, we need to take into account the soldiers that have defected. This is done by use of the matrices $A_\cdot$. Defections will occur at stopping times

τ. and $\mathbf{1}'A.\mathbf{1}$ basically records the number of agents that have not stopped after that stage. This goes on until all agents have stopped. The following proposition summarizes this intuition:

Proposition 5 *The equilibrium strategies for the desertion game with I soldiers are given by:*

$$\tau_i^* = \sum_{k=1}^{I} (\Pi_{j=1}^{k-1} \mathbb{I}_{x^i_{\tau_j} < z^i_j}) \mathbb{I}_{x^i_{\tau_k} \geq z^i_k} \tau_k$$

Proof. See Appendix.

In the next section, we initiate the discussion on the econometrics of this model.

6 Empirical Implications

This section analyzes the empirical content of the model. We remind the reader that by endogenous effects we mean the effect of other agents' participation (represented in the model by θ_t) on the transition law for the individual state variables. More specifically we say that there are endogenous effects when the drift and dispersion coefficients in equation (2) are affected by θ_t. Correlated effects refer to the possible association among the Brownian motions that drive each individual's latent utility process.

It is agreed that each agent's latent utility is given by the following stochastic process:

$$\log x_t^i = \alpha^i t - \Delta\alpha \sum_{j:j \neq i} (t - \tau^j) \frac{\mathbb{I}_{t \geq \tau^j}}{I-1} - \frac{\sigma^2}{2} t + \sigma W_t^i + \log x_0^i, \quad i \in I$$

where τ^j is the stopping time adopted by the soldier j. The cross-variation process for the Brownian motions is given by $\langle W^i, W^j \rangle_t = \rho t, i \neq j$ and the initial condition $\mathbf{x}_0 = (x_0^i)_{i \in I}$ follows a probability law F_0^i. The individual initial drift coefficient is potentially a function of an l-dimensional vector of individual covariates $w_{i(1\times l)}$ which is independent of the Brownian motion. Let $F_{\mathbf{w}}$ denote the distribution of $\mathbf{w} = (w_i)_{i\in I}$. For simplicity, assume that

$$\alpha^i = \alpha(w_i) = \exp(\beta' w_i).$$

In benefit of readability we suppress the argument and denote the drift by α^i. In what follows all the statements are conditional on $\mathbf{w} = (w_i)_{i\in I}$. The parameter $\Delta\alpha$ measures the external effect of the other agents decision on i and introduces endogenous social effects and ρ represents correlated social effects. In addition to the above parameters, each agent pays a cost C to leave and discounts the future at the exponential rate γ. Let's denote by $\overline{z}^i = z(\alpha^i, \sigma, \gamma, C), i \in I$ the upper threshold for each agent.

If there are no social interactions or correlated effects ($\Delta\alpha = 0$ and $\rho = 0$), the individual Brownian motions are independent and each agent's latent utility evolves as a geometric Brownian motion with drift α^i, diffusion coefficient σ and initial position x_i. As a consequence the desertion times τ_i^* are independent inverse Gaussian random variables [6]. Below are some of the features of this distribution with the parameters of our model:

$$\begin{aligned}
\text{PDF}: \quad & q(t; x_i, z, \alpha^i, \sigma) = \tfrac{\log(z^i/x_i)}{\sigma\sqrt{2\pi t^3}} \exp\left[- \tfrac{(\log(z^i/x_i) - (\alpha^i - \sigma^2/2)t)^2}{2\sigma^2 t} \right] \mathbb{I}_{t>0} \\
\text{CDF}: \quad & IG(t; x_i, z, \alpha^i, \sigma) = N\left(\tfrac{\log(z^i/x_i) - (\alpha^i - \sigma^2/2)t}{\sigma\sqrt{t}} \right) - \\
& - e^{\frac{2(\alpha^i - \sigma^2/2)(\log(z^i/x_i))}{\sigma^2}} N\left(\tfrac{-\log(z^i/x_i) - (\alpha^i - \sigma^2/2)t}{\sigma\sqrt{t}} \right) \\
\text{Moments}: \quad & \mathbb{E}[t] = \tfrac{\log(z^i/x_i)}{(\alpha^i - \sigma^2/2)} \text{ and } \mathbb{E}[t^{-1}] = \mathbb{E}[t]^{-1} + \sigma^2/(\log(z^i/x_i))^2
\end{aligned}$$

where

$$z = z(\alpha^i, \sigma, C, \gamma) = \frac{\beta(\alpha^i, \sigma, \gamma)}{\beta(\alpha^i, \sigma, \gamma) - 1} C$$

[6]The Inverse Gaussian is the distribution of the hitting time of a Brownian motion on a given barrier $\log z$. In our case, the initial position is $\log x_i$; the drift coefficient, $\alpha^i - \sigma^2/2$ and the diffusion coefficient, σ. For an economic application of the Inverse Gaussian distribution, see Lancaster [28]. Chhikara and Folks [7] provide an extensive characterization of this distribution.

with

$$\beta(\alpha^i, \sigma, \gamma) = 1/2 - \alpha^i/\sigma^2 + \sqrt{\left[\alpha^i/\sigma^2 - 1/2\right]^2 + 2\gamma/\sigma^2} > 1.$$

All the moments of this distribution are functions of $\mathbb{E}[t]$ and $\mathbb{E}[t^{-1}]$ (see Chhikara and Folks [7]). Furthermore, the sample mean and harmonic mean are sufficient statistics and MLE estimators for the distributional parameters above.

We are now in shape to start looking at the outcomes in the presence of interactions and correlated effects. The next proposition states that simultaneous desertions only occurs in the presence of endogenous effects.

Proposition 6 $\mathbb{P}[\tau^i = \tau^j, i \neq j, i, j \in I] > 0$ *if and only if there are endogenous effects.*

Proof. See Appendix.

This is a desirable feature of the model since it seems to hold in the Union Army data, whereas traditional econometric models in duration analysis typically do not generate clustering in timing — the probability of simultaneous exit is zero. The result relies basically on the continuity of the sample paths for the stipulated process. If discontinuities are allowed, this would not hold any longer[7]. What events could possibly provoke discontinuities in the latent utility process? In the military example that motivates this exercise, one could think of the advent of battles, for instance. But the problem would be diluted if one knows the timing of such shocks. If one observes clustering in other moments, this is evidence in favor of endogenous effects. As a matter of fact the timing of the battles fought by each individual in the Civil War data is known and could be controlled for.

[7]One way to introduce such discontinuities is to insert an exogenous jump component dQ^i in equation (2).

Another implication of the model is that the game size, or the military company size, in our case, does affect outcomes. This is stated in the next proposition.

Proposition 7 *The size of the game I affects the equilibrium if and only if there are endogenous effects.*

Proof. In preparation.

We present below a first characterization for the probability distribution of observable outcomes. If we represent by $G(t, \mathbf{x})$ the probability that the players will abandon the activity *after* time t when the vector of initial conditions is given by $\mathbf{x}$, the following result then holds:

Proposition 8 *Let $G(t, \mathbf{x}) = \mathbb{P}[\tau_i^* > t, i \in I | \mathbf{x}_0 = \mathbf{x}]$. Then G is the unique solution to*

$$\begin{aligned} &\partial G / \partial t = \mathcal{A}((\alpha^i)_{i \in I}, \rho, \sigma) G \text{ in } S^c, t > 0 \\ &G(0, \mathbf{x}) = 1, \mathbf{x} \in S^c \\ &G(t, \mathbf{x}) = 0, \mathbf{x} \in \partial S \text{ and } t \geq 0 \end{aligned}$$

where $S = S((\alpha^i)_{i \in I}, \sigma, \gamma, C) = \{\mathbf{x} \in \mathbb{R}_+^I : \exists i \text{ such that } x^i \geq z(\alpha^i, \sigma, \gamma, C)\}$ and

$$\mathcal{A}((\alpha^i)_{i \in I}, \rho, \sigma) f = \sum_{i \in I} \alpha^i x_i \frac{\partial f}{\partial x_i} + \frac{1}{2} \sigma^2 \sum_{i \in I} x_i^2 \frac{\partial^2 f}{\partial x_i^2} + \rho \sigma^2 \sum_{\substack{i,j \in I \\ i \neq j}} x_i x_j \frac{\partial^2 f}{\partial x_i \partial x_j}$$

is the infinitesimal generator for the I-dimensional diffusion representing the latent utility vector process with killing time at $\tau_S : \{\mathbf{x}_t : t \leq \tau_S\}$.

Proof. See Appendix.

In the same fashion, one can obtain an expression for the probability that the agents will abandon the game *after* time t *and* there is simultaneous exit given an initial condition $\mathbf{x}$. We denote this probability by

$H(t, \mathbf{x}) = \mathbb{P}[\tau_i^* > t, i \in I$ and $\tau_S \in \partial S_1 | \mathbf{x}_0 = \mathbf{x}]$ (where ∂S_1 is defined below). The following characterization follows:

Proposition 9 *Let* $H(t, \mathbf{x}) = \mathbb{P}[\tau_i^* > t, i \in I$ *and* $\tau_S \in \partial S_H | \mathbf{x}_0 = \mathbf{x}]$*. Then* H *is the unique solution to*

$$\begin{aligned} &\partial H / \partial t = \mathcal{A}((\alpha^i)_{i \in I}, \rho, \sigma) H \text{ in } S^c, t > 0 \\ &H(0, \mathbf{x}) = u(\mathbf{x}), \mathbf{x} \in S^c \\ &H(t, \mathbf{x}) = 0, \mathbf{x} \in \partial S \text{ and } t \geq 0 \end{aligned}$$

with

$$\begin{aligned} &\mathcal{A}((\alpha^i)_{i \in I}, \rho, \sigma) u = 0 \text{ in } S^c \\ &u(\mathbf{x}) = 1, \mathbf{x} \in \partial S_H \\ &u(\mathbf{x}) = 0, \mathbf{x} \in \partial S \backslash \partial S_H \end{aligned}$$

where $S = S((\alpha^i)_{i \in I}, \sigma, \gamma, C) = \{\mathbf{x} \in \mathbb{R}_+^I : \exists i \text{ such that } x^i \geq z(\alpha^i, \sigma, \gamma, C)\}$, $\partial S_H = \partial S_H((\alpha^i)_{i \in I}, \sigma, \gamma, C, \Delta\alpha) = \{\mathbf{x} \in \partial S | x^i \geq z(\alpha^i - \Delta\alpha/(I-1), \sigma, \gamma, C)$ *and*

$$\mathcal{A}((\alpha^i)_{i \in I}, \rho, \sigma) f = \sum_{i \in I} \alpha^i x_i \frac{\partial f}{\partial x_i} + \frac{1}{2} \sigma^2 \sum_{i \in I} x_i^2 \frac{\partial^2 f}{\partial x_i^2} + \rho \sigma^2 \sum_{\substack{i,j \in I \\ i \neq j}} x_i x_j \frac{\partial^2 f}{\partial x_i \partial x_j}$$

is the infinitesimal generator for the I*-dimensional diffusion representing the latent utility vector process with killing time at* $\tau_S : \{\mathbf{x}_t : t \leq \tau_S\}$.

Proof. See Appendix.

One question that arises naturally is the possibility of disentangling correlated and endogenous effects in the data. The econometrician presumably observes the equilibrium exit strategies $(\tau_1^*, \ldots, \tau_I^*)$ for a certain number of realizations of the game and would be interested in knowing what parameters of the model can be retrieved given data on the situation under analysis. Could two different parameter vectors generate the same distribution for the data? This is the typical problem of statistical identification of a parameter vector. Similar problems arise in natural sciences, where a researcher is confronted with a distribution of exit times and is interested in reconstructing aspects of an unobserved stochastic process (see, for instance, Bal and Chou [3] for related problems in chemical and neurological studies). Following Hsiao [20] (see also Manski [30]), we define the parameters in a model to be identified if two different parametric specifications are not observationally equivalent. Letting τ denote some outcome variables observed by the researcher, $\mathbf{w}$ some observable covariates and ψ a parameter (of arbitrary finite dimension) lying in a certain set Ψ and governing the probability distribution $P(\cdot|\mathbf{w};\psi)$ of the outcome variables, the following defines identification.

Definition 3 (Identification) *The parameter $\psi \in \Psi$ is identified relative to $\hat{\psi}$ if $(\hat{\psi} \notin \Psi)$ or $(P(\cdot|\mathbf{w};\psi) = P(\cdot|\mathbf{w};\hat{\psi}), F_{\mathbf{w}}\text{-a.e.} \Rightarrow \psi = \hat{\psi})$.*

We say that ψ is globally identified if it is identified relative to any parameter vector in the parameter space and that it is locally identified if it is identified relative to any parameter vector in a neighborhood of ψ. In what follows we analyze the identification for the desertion model. Let $g(t;\psi,\mathbf{w})$ denote the probability density function for the first desertion time under the parameters $\psi = (\mathbf{x}, \beta, \sigma, \rho, \gamma, C)$ and conditioned on the observable covariates $\mathbf{w}$. The following statement establishes sufficient conditions for the identification of ψ.

Theorem 2 *Let w be a set of continuous random covariates and, for some i and some covariate l,*

$$\partial_{w_{il}} \int \log\Big[\frac{g(t;\psi,\mathbf{w})}{g(t;\hat{\psi},\mathbf{w})}\Big] g(t;\psi,\mathbf{w})dt \neq 0 \tag{6}$$

then ψ is identified relative to $\hat{\psi}$.

Proof. See Appendix.

In order to check condition (6) one should obtain g from the solution to the partial differential equation in Proposition 8 [8]. In certain special cases this solution may be available analytically. This is so for instance if $I = 2$, in which case Theorem 3.5.2 in Rebholz [35] delivers (see alternatively He, Keierstead and Rebholz [18], Remark 2.2(ii)):

$$\mathbb{P}(\tau_1^* \wedge \tau_2^* \geq t) = e^{a_1 \log(z^1/x_1)+a_2 \log(z^2/x_2)+bt} f(r', \theta', t) \tag{7}$$

where

$$f(r', \theta', t) = \frac{2}{\lambda' t} \sum_{n=1}^{\infty} \sin(\frac{n\pi\theta'}{\lambda'}) e^{-\frac{r'^2}{2t}} \int_0^{\lambda'} \sin(\frac{n\pi\theta}{\lambda'}) g_n(\theta) d\theta$$

with

$$g_n(\theta) = \int_0^{\infty} r e^{-\frac{r^2}{2t}} e^{-b_1 r \cos(\theta-\lambda)-b_2 r \sin(\theta-\lambda)} I_{\frac{n\pi}{\lambda}}(\frac{rr'}{t}) dr$$

and

$$\begin{aligned}
\tan\lambda' &= -\tfrac{\sqrt{1-\rho^2}}{\rho} \\
\lambda &= \lambda' - \tfrac{\pi}{2} \\
r' &= \tfrac{1}{\sqrt{1-\rho^2}} (\tfrac{\log(z^1/x_1)^2 - 2\rho \log(z^1/x_1)\log(z^2/x_2) + \log(z^2/x_2)^2}{\sigma^2})^{\frac{1}{2}} \\
\theta' &= \tfrac{log(z^1/x_1)}{\sigma r'} \\
a_1 &= \tfrac{(\alpha^1-\sigma^2/2)-\rho(\alpha^2-\sigma^2/2)}{(1-\rho^2)\sigma^2} \\
a_2 &= \tfrac{(\alpha^2-\sigma^2/2)-\rho(\alpha^1-\sigma^2/2)}{(1-\rho^2)\sigma^2} \\
b &= \tfrac{\sigma^2}{2}(a_1^2 + 2\rho a_1 a_2 + a_2^2) - (\alpha^1 - \sigma^2/2)a_1 - (\alpha^2 - \sigma^2/2)a_2 \\
b_1 &= (a_1 + a_2\rho)\sigma \\
b_2 &= a_2\sigma\sqrt{1-\rho^2}
\end{aligned}$$

with $z^i \equiv z(\alpha^i, \sigma, C, \gamma)$. Iyengar [22], which also derives an expression for the above function, hints that the above is also generalizable for higher dimensions in our specific situation.

The following statement provides an alternative identification result for the main parameters in our model under the assumption that the initial condition is observed and can be controlled for.

[8] This PDF can be obtained as $-dG(\cdot)/dt$ where $G(\cdot)$ is the solution to the PDE in Proposition 8.

Theorem 3 *Assume that the initial condition is observed, its probability distribution F_0 has support with non-empty interior and $I > 2$. The parameter vector $((\alpha^i)_{i\in I}, \Delta\alpha, \rho, \sigma, \gamma, C)$ is then identified.*

Proof. See Appendix.

The key to this result is to notice that, whereas the existence of endogenous effects has no impact before the exit of the first player, the correlation coefficient ρ and drift level α^i do affect the timing of the first dropout. Given the nonlinear character of the model, these two parameters can be identified by the exit distribution of the first deserter. The impact of the endogenous effect parameter on the probability of clustering of agents on the other hand helps identify $\Delta\alpha$.

The statement rests on the assumption that observer can condition on $\mathbf{x}_0$, the initial latent utility level. One could imagine the stock market participation application mentioned in the introduction with the latent utility level representing some measure of relative portfolio performance among the players in a certain reference group. In this case one could conceive of datasets recording initial stock allocations and entry and exit decisions but not interim portfolios and apply the above result. In other applications though, where the state variables x^i stand for some subjective measure of satisfaction, this assumption loses much of its appeal.

Notwithstanding the difficulties attached to the solution of the partial differential equation in Proposition 8, a few cases allow some investigation. We finish the section with a few remarks about the case of two agents.

1. The cumulative distribution function (CDF) for the time of first desertion is given by Theorem 3.5.2 in Rebholz [35] — expression (7).

   Remark 1: If there are no endogenous effects ($\Delta\alpha = 0$) each desertion time follows a univariate inverse gaussian which is (potentially) correlated across soldiers in a given company. Expression (7) then gives the following probability: $\mathbb{P}(\tau_1 \geq t, \tau_2 \geq t)$.

REMARK 2: If there are no correlated effects, the expression (7) reduces to $(1 - IG(t; x_1, \overline{z}^1, \alpha^1, \sigma))(1 - IG(t; x_2, \overline{z}^2, \alpha^2, \sigma))$ where IG is the CDF for the Inverse Gaussian distribution given in the opening paragraphs of this section.

2. From Theorem 2.2(iii) in He, Keierstead and Rebholz [18] one obtains an expression for $\mathbb{P}(x^1(t) \in dx^1, x^2(t) \in dx^2, \tau_S \geq t) = \mathcal{P}(x^1, x^2, t)$. Since $(x^1(t), x^2(t)) \in \partial S_1 \Rightarrow \tau_S \leq t$, one obtains that the probability density function (PDF) for joint exit at t is:

$$\int_{(x^1,x^2)\in\partial S_1} \mathcal{P}(x^1, x^2, t) d(x^1, x^2).$$

3. Accordingly, one can deduce that the PDF for agent 2's exit at s and agent 1's exit at t ($s < t$) is:

$$\int_0^{z(\alpha^1-\Delta\alpha,\sigma,\gamma,C)} q(t; x^1, \text{other parameters})\mathcal{P}(x^1, z(\alpha^2, \sigma, \gamma, C), s)dx^1$$

where $t > s$ and q is the PDF for the Inverse Gaussian given previously.

7 Discussion

This paper analyzes a synchronization game which allows for endogenous and correlated effects among players. Agents participate in an activity and benefit from the participation of others. If the group leaves *en masse*, an agent will be more likely to depart; whereas if the agents stay, an individual will be inclined to stay. Standard tools of optimal stopping problems for continuous parameter stochastic processes are used but the processes are endogenized by making their distribution dependent on the participation of the group.

This is a problem of great importance in many settings. Social welfare program participation, bank runs, South-North migration, marriage and divorce decisions are only a few of the possibilities. Disentangling endogenous and correlated effects is thus fundamental not only to illuminate economic research but also to enlighten policy. The setup delineated in this paper allows us to separately identify the endogenous and correlated effects associated with each individual's decision. Whereas this problem is unfeasible in simpler settings (see Manski [31]), the separation is not clear in other approaches that deal with similar situations (as in Brock and Durlauf [6]). Empirical consequences that are not present in conventional methods of estimation are also obtained and point to the application of richer estimation schemes in the analysis of these phenomena.

Appendix

Sketch of Proof for Existence of a Strong Solution

The proof that there exists a strong solution for equation 2 follows from a slight modification of the proof provided in Karatzas and Shreve [23], p.289. The key is to note that the iterative construction of a solution follows through if we replace $b(s,x)$ and $\sigma(s,x)$ by $b(s,x,\omega)$ and $\sigma(s,x,\omega)$ in the definition of $X^{(k)}$. If, for fixed x, $(s,\omega) \mapsto b(s,x,\omega)$ and $(s,\omega) \mapsto \sigma(s,x,\omega)$ are adapted processes, the resulting process is still adapted. The remainder of the proof is identical. (See also Protter [34], Theorem V.7)

Proof of Theorem 1

Consider a player $i \in I$. Let the stopping strategies for $I - \{i\}$ be given by the following profile of stopping times $\tau_{-i} = (\tau_s)_{s \in I - \{i\}}$. Given Assumption 3, according to Theorem 4 in Fakeev [12], there exists a solution for the optimal stopping time. Let the individual i's best response function $b_i(\cdot)$ map a stopping time profile τ_{-i} onto one such optimal stopping solution. Given this, consider $b(\cdot)$ defined as the following mapping $\tau = (\tau_s)_{s \in I} \mapsto b(\tau) = (b_i(\tau_{-i}))_{i \in I}$. A Nash Equilibrium is then simply a fixed point for the mapping $b(\cdot)$. In order to establish the existence of such a result we use the Knaster-Tarski Fixed Point Theorem, reproduced below from Aliprantis and Border [1], p.6:

> **Knaster-Tarski Fixed Point Theorem**: Let $(X, \geq)$ be a partially ordered set with the property that every chain in X has a supremum. Let $f : X \to X$ be increasing, and assume that there exists some a in X such that $a \leq f(a)$. Then the set of fixed points of f is nonempty and has a maximal fixed point.

In the following discussion we consider the set of stopping time profiles and identify two stopping times that are $\mathbb{P}$-almost everywhere identical. We proceed by steps:

<u>Step 1</u>: (Partial order) The set of stopping times endowed with the relation $\geq$ defined as: $\tau \geq \upsilon$ if and only if $\mathbb{P}(\tau(\omega) \geq \gamma(\omega)) = 1$ is partially ordered. In other words, $\geq$ is reflexive, transitive and anti-symmetric.

<u>Step 2</u>: (Every chain has a supremum) Given a set of stopping times T, we should be able to find a stopping time $\overline{\tau}$ such that 1. $\overline{\tau} \geq \tau, \forall \tau \in T, \mathbb{P}$-a.s. and 2. if $\upsilon \geq \tau, \mathbb{P}$-a.s., $\tau \in T$ then $\upsilon \geq \overline{\tau}, \mathbb{P}$-a.s.. If T is countable $\sup_{\tau \in T} \tau$ is a stopping time and satisfies conditions 1 and 2 (see Karatzas and Shreve, Lemma 1.2.11). If not, first notice that, since the only structure that matters for this property is the ordering in $\overline{\mathbb{R}}_+$, we can always assume that the stopping times take values on $[0, 1]$ (otherwise, pick an increasing mapping from $\overline{\mathbb{R}}_+$ onto $[0, 1]$). Let $\mathcal{C}$ be the collection of all countable subsets $C \subset T$. For each such C, define:

$$l_C = \sup_{\tau \in C} \tau \text{ and } \upsilon = \sup_{C \in \mathcal{C}} \mathbb{E}(l_C) < \infty$$

By the previous reasoning, l_C is a stopping time. Then, there is a sequence $\{C_n\}_n \subset \mathcal{C}$ such that $\upsilon = \lim_{n \to \infty} \mathbb{E}(l_{C_n})$. Now define $\overline{C} = \cup_{n=1}^{\infty} C_n \in \mathcal{C}$. To show that $l_{\overline{C}}$ satisfies condition 1., first notice that $\overline{C} \in \mathcal{C}, \upsilon \geq \mathbb{E}(l_{\overline{C}})$. On the other hand, since $C_n \subset \overline{C}$, $\mathbb{E}(l_{\overline{C}}) \geq \mathbb{E}(l_{C_n}) \to_n \upsilon$. These two imply that $\upsilon = \mathbb{E}(l_{\overline{C}})$.

For an arbitrary $\tau \in T$, set $\overline{C}_\tau = \{\tau\} \cup \overline{C} \in \mathcal{C}$. Now, $l_{\overline{C}_\tau} \geq l_{\overline{C}}$. This renders $\upsilon \geq \mathbb{E}(l_{\overline{C}_\tau}) \geq \mathbb{E}(l_{\overline{C}}) = \upsilon \Rightarrow \mathbb{E}(l_{\overline{C}_\tau} - l_{\overline{C}}) = 0 \Rightarrow l_{\overline{C}_\tau} = l_{\overline{C}}, \mathbb{P}$-a.s. This and $l_{\overline{C}_\tau} \geq \tau, \mathbb{P}$-a.s. in turn imply that $l_{\overline{C}} \geq \tau, \mathbb{P}$-a.s.

To see that 2. is satisfied, notice that, if $\upsilon \geq \tau, \forall \tau \in T$, in particular, $\upsilon \geq \tau, \forall \tau \in \overline{C}$. This implies that $\upsilon \geq \sup_{\tau \in \overline{C}} \tau = l_{\overline{C}}$.

<u>Step 3</u>: ($\exists a$ such that $a \leq f(a)$) Pick a as the profile of stopping times

that are identically zero.

<u>Step 4</u>: ($b(\cdot)$ is increasing) This is the case if each individual best response function $b_i(\cdot)$ is increasing. By the version of Itô's Lemma for twice differentiable functions (see Revuz and Yor [36], p.224, remark 3), and the fact that $u_i(x,t) = e^{-\gamma_i t} g_i(x)$ is twice differentiable (since $g_i(\cdot)$ is twice differentiable), $e^{-\gamma_i t} g_i(x)$ obeys the following stochastic differential equation (given a profile of stopping times τ_{-i}):

$$d[e^{-\gamma_i s} g_i(x_s^i)] = \underbrace{e^{-\gamma_i t}[g_i'(x_t^i)\alpha^i(x_t^i,\theta_t,t) + \frac{1}{2}\sigma^{i2}(x_t^i,\theta_t,t)g_i''(x_t^i) - \gamma_i g_i(x_t^i)]}_{\equiv \alpha^i(x^i,\theta_t^i,t)} dt + \\ + \underbrace{e^{-\gamma_i t} g_i'(x_t^i)\sigma^i(x_t^i,\theta_t,t)}_{\equiv \beta^i(x_t^i,\theta_t^i,t)} dW_t^i$$

where the $\alpha(\cdot,\cdot,\cdot)$ and $\beta(\cdot,\cdot,\cdot)$ denote the drift and dispersion coefficients of $e^{-\gamma_i t} g_i(x_t^i)$. If $g_i(\cdot)$ is increasing and convex and if $\alpha_i(\cdot,\cdot,\cdot)$ and $\sigma_i(\cdot,\cdot,\cdot)$ are decreasing in θ, the above drift is decreasing in θ.

Now consider a profile of stopping times τ_{-i} and υ_{-i} such that τ_{-i} dominates $\upsilon_{-i}, \mathbb{P}$- a.s. Moving from one profile to another will impact θ and this will have effects on both the drift and the dispersion coefficients of $e^{-\gamma_i t} g_i(x_t^i)$.

The effect on the dispersion coefficient does not affect the objective function of an individual agent. This obtains from the fact that $g'(\cdot)$ is bounded and the Bounded Volatility Assumption. These assumptions deliver that, for each $t < \infty$:

$$\mathbb{E}[\int_0^t (e^{-\gamma s} g'(x_s)\sigma(x_s,\theta_s,s))^2 ds] < K\mathbb{E}[\int_0^t (e^{-\gamma s}\sigma(x_s,\theta_s,s))^2 ds] < \infty$$

for some $K \in \mathbb{R}$. This in turn implies that $z_t = \int_0^t (e^{-\gamma s} g'(x_s)\sigma(x_s,\theta_s,s))dW_s^i$ is a martingale (see Karatzas and Shreve [23], p.139) and by the Optional

Sampling Theorem, $\mathbb{E}[\int_0^\tau (e^{-\gamma s} g'(x_s)\sigma(x_s, \theta_s, s))dW_s^i] = 0, \forall \tau$ where τ is an $(\mathcal{F}_t)$-stopping time (see Karatzas and Shreve [23], p.19).

Given τ_{-i} and υ_{-i}, we know that $\theta_t^{i,\tau} \leq \theta_t^{i,\upsilon}$, $\mathbb{P}$-almost surely, $\forall t$ (where $\theta_t^{i,\tau}$ and $\theta_t^{i,\upsilon}$ aggregate the stopping decisions for the profiles τ_{-i} and υ_{-i}) we will have $\alpha(x, \theta_t^{i,\upsilon}, t) \leq \alpha(x, \theta_t^{i,\tau}, t)$, $\mathbb{P}$-almost surely, $\forall x, t$. Letting $y_t^{i,\tau}$ be the process given by

$$dy_t^{i,\tau} = \alpha^i(x_t^i, \theta_t^{i,\tau}, t)dt + \beta(x_t^i, \theta_t^{i,\tau}, t)dW_t^i$$

and $y_t^{i,\upsilon}$ be the process given by

$$dy_t^{i,\upsilon} = \alpha^i(x_t^i, \theta_t^{i,\upsilon}, t)dt + \beta(x_t^i, \theta_t^{i,\tau}, t)dW_t^i$$

using a slight variation of Proposition 5.2.18 in Karatzas and Shreve [23], we get:

$$\mathbb{P}[y_t^{i,\tau} \geq y_t^{i,\upsilon}, \forall 0 \leq t < \infty] = 1$$

Again, a slight variation of the proof of this proposition can be repeated using this fact and focusing on $y_t^{i,\tau} - y_s^{i,\tau} - (y_t^{i,\upsilon} - y_s^{i,\upsilon}), t \geq s$ instead of simply $y_t^{i,\tau} - y_t^{i,\upsilon}$. This is enough to achieve the following result:

$$\mathbb{P}[(y_t^{i,\tau} - y_s^{i,\tau}) - (y_t^{i,\upsilon} - y_s^{i,\upsilon}) \geq 0, \forall 0 \leq s \leq t < \infty] = 1$$

This suffices to show that it is not profitable for agent i to stop earlier when the profile is τ_{-i} than when the profile is υ_{-i}. Suppose not. Then, let $A = \{b_i(\tau_{-i}) < b_i(\upsilon_{-i})\}$. According to Lemma 1.2.16 in Karatzas and Shreve [23], $A \in \mathcal{F}_{b_i(\tau_{-i})} \cap \mathcal{F}_{b_i(\upsilon_{-i})}$. By the above result we can then see that $\mathbb{E}\{\mathbb{I}_A[y^{i,\tau}_{b_i(\upsilon_{-i})} - y^{i,\tau}_{b_i(\tau_{-i})}]\} \geq \mathbb{E}\{\mathbb{I}_A[y^{i,\upsilon}_{b_i(\upsilon_{-i})} - y^{i,\upsilon}_{b_i(\tau_{-i})}]\}$. The RHS expression in this inequality is positive because $A \in \mathcal{F}_{b_i(\tau_{-i})} \cap \mathcal{F}_{b_i(\upsilon_{-i})} = \mathcal{F}_{b_i(\tau_{-i}) \wedge b_i(\upsilon_{-i})}$ which implies that the agent would do better by picking $b_i(\tau_{-i}) \wedge b_i(\upsilon_{-i})$ if the RHS were negative. But this would contradict the fact that $b_i(\upsilon_{-i})$ is a best response. So, if

$A \neq \emptyset$, delaying the response by choosing $b_i(\upsilon_{-i}) \vee b_i(\tau_{-i})$ would improve the agent's payoff given that the remaining agents are playing τ_{-i}.

■

Proof of Proposition 1

A comparison result such as the one in Karatzas and Shreve [23], Proposition V.2.18, or Protter [34], Theorem V.54, may be established to show that:

$$e^{-\gamma t}(x_t - I) \leq e^{-\gamma t}(\overline{x}_t - I), \forall t \quad \mathbb{P}\text{-a.s.}$$

This in turn implies that

$$e^{-\gamma \overline{\tau}(\omega)}(x(\omega)_{\overline{\tau}(\omega)} - I) \leq e^{-\gamma \overline{\tau}(\omega)}(\overline{x}(\omega)_{\overline{\tau}(\omega)} - I), \quad \mathbb{P}\text{-a.s.}$$

The same comparison result can also be used to show that, on $\{\omega \in \Omega : \overline{\tau}(\omega) \leq \nu(\omega)\}$, we indeed have

$$e^{-\gamma \overline{\tau}(\omega)}(x(\omega)_{\overline{\tau}(\omega)} - I) = e^{-\gamma \overline{\tau}(\omega)}(\overline{x}(\omega)_{\overline{\tau}(\omega)} - I), \quad \mathbb{P}\text{-a.s.}$$

So, we can do no better than to use $\overline{\tau}$ on the set $\{\overline{\tau} \leq \nu\}$.

On the complementary set, $\{\overline{\tau} \geq \nu\}$, think of the process $\log y_t = \log x_{\nu+t}, t \in \mathbb{R}_+$. Using the fact that the process satisfies the Strong Markov Property, this process is given by

$$\log y_t = (\alpha - \Delta\alpha)t + \sigma \tilde{W}_t + \log y_0$$

where $\tilde{W}_t = W_{t+\nu}$ and $y_0 = x_\nu$. In other words, the process starts "afresh" at ν with initial value given by x_ν and obeying the stochastic process with low drift. As ascertained before, the optimal stopping time for this process is $\inf\{t > \nu : x_t^i > \underline{z}_i\}$. So, on $\{\omega \in \Omega : \overline{\tau}(\omega) \geq \nu(\omega)\}$ we might as well choose $\inf\{t > \nu : x_t^i > \underline{z}_i\}$.

■

Proof of Proposition 2

Set $\nu = \tau_j^*$ in Proposition 1. In this case, i should use $\overline{\tau}_i$ on $\{\overline{\tau} \leq \tau_j^*\}$ and $\inf\{t > \tau_j^* : x_t^i > \underline{z}_i\}$ on the complementary set.

Now notice that:

$$x_{\tau_S}^i = \overline{z}_i \Rightarrow \overline{\tau}_i = \tau_S$$

When the vector process hits S on the subset where $x^i = \overline{z}_i$, the hitting times for the vector process to reach S and for the component process to hit $\overline{z}_i$ coincide. Since $\tau_j^* \geq \tau_S$ by construction, we should also conclude that:

$$\{x_{\tau_S}^i = \overline{z}_i\} \subset \{\overline{\tau}_i \leq \tau_j^*\}$$

Agent i should then use $\overline{\tau}_i$ (which coincides with τ_S on this set).

On the other hand,

$$x_{\tau_S}^i \neq \overline{z}_i \Rightarrow \begin{cases} \overline{\tau}_i > \tau_S \\ (x_{\tau_S}^j > \underline{z}_j \Rightarrow \tau_j^* = \tau_S) \end{cases} \Rightarrow \overline{\tau}_i > \tau_j^*$$

So, we are in the complementary set, in which is sensible to use $\inf\{t > \tau_j^* : x_t^i > \underline{z}_i\} = \inf\{t > \tau_S : x_t^i > \underline{z}_i\}$.

■

Proof of Proposition 3

<u>Step 1</u>: ($(\tau_i^*)_{i \in I}$ is an equilibrium) The proof basically reproduces the previous proposition.

<u>Step 2</u>: (Uniqueness) Suppose there is another equilibrium profile $(\nu_i^*)_{i \in I}$. Let $\theta(\omega) \equiv \nu_1^*(\omega) \wedge \nu_2^*(\omega)$. Uniqueness is proved if we establish that $(x_\theta^1, x_\theta^2) \in \overline{z}_1 \times [0, \overline{z}_2] \cup [0, \overline{z}_1] \cup \overline{z}_2$.

First one should notice that optimality requires $x^i_{\nu^*_i} \geq \underline{z}_i$. This in turn means that $x^i_\theta < \underline{z}_i \Rightarrow \nu^*_i > \theta = \nu^*_j$ and consequently $x^j_\theta = \overline{z}_j$. Otherwise, agent j would be stopping early and would do better by delaying this decision.

Can both agents stop simultaneously when $x^i \in (\underline{z}_i, \overline{z}_i)$ and $x^j \in (\underline{z}_j, \overline{z}_j)$? It is optimal for an agent i to stop when $x^i \in (\underline{z}_i, \overline{z}_i)$ if by stopping he or she elicits a similar decision by the other player. This happens only when $x^j > \underline{z}_j$. Although in equilibrium such information is endogenously generated, for an arbitrary stopping time ν, there is not enough information at ν for i to tell whether j is below or above the lower threshold. In other words, $\sigma(x^j_\nu) \nsubseteq \mathcal{F}^i_\nu$ (unless $\rho = 1$). This in turn implies that $\sigma(x^i_\nu, x^j_\nu) \nsubseteq \mathcal{F}^i_\nu$ (unless $\rho = 1$). But such information (about the opposing agent's latent utility location) is necessary to avoid stopping when the other agent's state variable is below the lower threshold. Thus they cannot stop simultaneously at this region and this should be enough to complete the proof.

■

Proof of Proposition 4

The proof is by induction. For $n = 1$, Proposition 1 establishes the result.

For a generic n, assume that the statement holds for $n - 1$. The same comparison argument used in Proposition 1 delivers that

$$e^{-\gamma\overline{\tau}(\omega)}(x(\omega)_{\overline{\tau}(\omega)} - I) = e^{-\gamma\overline{\tau}(\omega)}(\overline{x}(\omega)_{\overline{\tau}(\omega)} - I), \mathbb{P}\text{-a.s.}$$

where $\log \overline{x}_t = \alpha t + \sigma W_t$ and $\overline{\tau}$ denotes the optimal stopping time associated with this process. So, we can do no better than to use $\overline{\tau}$ on the set $\{\overline{\tau} \leq \nu_1\}$.

On the complementary set, $\{\overline{\tau} \geq \nu_1\}$, think of the process $\log y_t = \log x_{\nu+t}, t \in \mathbb{R}_+$. Using the Strong Markov Property, it is seen that the process starts "afresh" at ν with initial value given by x_ν and obeying the stochastic process with $n - 1$ drift breaks. The induction argument takes care of this situation and we achieve the result desired.

■

Proof of Proposition 5

We divide the proof in three steps:

<u>Step 1</u>: (Stopping times are an increasing sequence) Notice that, by definition, $\tau_0 \leq \tau_1 \leq \cdots \leq \tau_I$ and consequently form an increasing sequence of stopping times.

<u>Step 2</u>: (At each stage at least one agent stops) $\forall k \in I, \exists j : \tau_j^* = \tau_k$.

Take a stopping time τ_k. There are two possibilities, represented by two disjoint subsets of Ω, say Ω_1 and Ω_2:

1. Ω_1. The vector process $A_{k-1}x_t$ hits $A_{k-1}\Psi_{I+1-\mathbf{1}'A_{k-1}\mathbf{1}}$ where ($\exists i \in I$: $x^i \geq z_m^i$ and $\forall j \neq i, x^j \leq z_{m-1}^j$). In this case, $\tau_i^*(\omega) = \tau_k(\omega)$ (provided i hasn't stopped yet), $\forall \omega \in \Omega_1$.

2. Ω_2. The above does not happen. In this case, $\exists j : z_{m+1}^j \leq x_{\tau_k}^j$ (provided j hasn't stopped yet). Also in this case it can be seen that $\tau_{k+1} = \tau_k$. But then, $x_{\tau_k}^j = x_{\tau_{k+1}}^j \geq z_{k+1}$ and this implies that $\tau_j^*(\omega) = \tau_{k+1}(\omega) = \tau_k(\omega), \forall \omega \in \Omega_2$.

This means that, at each stopping time τ_k, the drift of x^i drops by $\Delta\alpha^i$.

<u>Step 3</u>: Apply Proposition 3.

■

Proof of Proposition 6

Let $S = \{\mathbf{x} \in \mathbb{R}_+^I : \exists i \text{ such that } x^i \geq z_1^i = z(\alpha^i, \sigma^i, C^i, \gamma^i)\}$ and $\tau_S = \inf\{t > 0 : \mathbf{x}_t \in S\}$ [9]. Since the sample paths are continuous $\mathbb{P}$-almost surely, by

[9] S would correspond to S_1 in the I-agent setup of the desertion game.

Theorem 2.6.5 in Port and Stone [33] the distribution of x_{τ_S} will be concentrated on ∂S. Also,, it is easily seen that $\mathbb{P}(\tau_S < \infty) = 1$.

(Sufficiency) If there are endogenous effects, $z_1^i = z(\alpha^i, \sigma^i, C^i, \gamma^i) > z_2^i = z(\alpha^i - \Delta\alpha^i, \sigma^i, C^i, \gamma^i), i \in I$. There will be simultaneous exit whenever $z_1^i \geq x_{\tau_S}^i \geq z_2^i, i \in I$. This has positive probability as long as $z_1^i > z_2^i, i \in I$. In order to see this, first notice that the latent utilities process can be represented as the following diffusion process with killing time at τ_S:

$$dx_t^i = \alpha^i x_t^i dt + \sum_{j \in I} \sigma_{ij} dB_t^j, \quad i = 1, \ldots, I$$

where $\mathbf{B}_t$ is an I-dimensional Brownian motion (with independent components) and $\sigma_{I \times I} = [\sigma_{ij}]$. Let $\partial S_1 = \{\mathbf{x} \in \partial S : z_1^i \geq x^i \geq z_2^i\}$. By Corollary II.2.11.2 in Gihman and Skorohod [16] (p.308), one gets that $\mathbb{P}(\mathbf{x}_{\tau_S} \in \partial S_1) = u(\mathbf{x})$ is an $\mathcal{A}$-harmonic function in S^c. In other words,

$$\begin{aligned}
&\mathcal{A}u(\mathbf{x}) = 0 \text{ in } S^c \\
&u(\mathbf{x}) = 1 \text{ if } \mathbf{x} \in \partial S_1 \\
&u(\mathbf{x}) = 0 \text{ if } \mathbf{x} \in \partial S \backslash \partial S_1
\end{aligned}$$

where

$$\mathcal{A}f = \sum_{i \in I} \alpha^i x_i \frac{\partial f}{\partial x_i} + \frac{1}{2} \sum_{\substack{i,j \in I \\ i \neq j}} (\sigma\sigma')_{ij} x_i x_j \frac{\partial^2 f}{\partial x_i \partial x_j}.$$

is the infinitesimal generator associated with the above diffusion. By the Minimum Principle for elliptic operators (see Proposition 4.1.3 in Port and Stone [33] or Section 6.4 in Evans [11]), if u attains a minimum (which in this case would be zero) on S^c, it is constant on S^c. This would in turn imply that $\forall \mathbf{x} \in S^c, u(\mathbf{x}) = \mathbb{P}[\mathbf{x}_{\tau_S} \in \partial S_1 | \mathbf{x}_0 = \mathbf{x}] = 0$. But by Proposition 2.3.6 in Port and Stone [33], one can deduce that $u(\mathbf{x}) = \mathbb{P}[\mathbf{x}_{\tau_S} \in \partial S_1 | \mathbf{x}_0 = \mathbf{x}] \neq 0$.

(Necessity) If there are no endogenous effects, one agent's drift is never affected by the exit of other agents. Each agent's decision is given by $\tau_i^* = \inf\{t \in \mathbb{R}_+ : x_t^i > z^i = z(\alpha^i, \sigma^i, C^i, \gamma^i)\}$. There will be clustering only if $\tau_i^* = \tau_j^*, i \neq j$. The state-variable vector can be represented as above until the killing time τ_S. Then, there will be clustering only if $\mathbf{x}_t$ hits S at the point $(z^i)_{i\in I}$. But in $I \geq 2$ dimensions any one-point set A is polar with respect to a Brownian motion, i.e., $\mathbb{P}[\tau_A < \infty] = 0$ where τ_A is the hitting time for A (Proposition 2.2.5 in Port and Stone [33]). So, $\mathbb{P}[\tau_i^* = \tau_j^*, i \neq j] = 0$.

■

Proof of Proposition 7

In preparation.

■

Proof of Proposition 8

Notice that (for $t \in [0, \tau_S]$) the vector process with the latent utilities can be represented as the following diffusion process with killing at time τ_S:

$$dx_t^i = \alpha^i x_t^i dt + \sigma x_t^i dW_t^i, \qquad i \in I$$

Denote by $\mathcal{A}((\alpha^i)_{i\in I}, \rho, \sigma)$ the infinitesimal generator associated with the above diffusion (where the argument reminds the reader of the dependence of the operator on the parameters). In other words, $\mathcal{A}((\alpha^i)_{i\in I}, \rho, \sigma)$ is the following differential operator:

$$\mathcal{A}((\alpha^i)_{i\in I}, \rho, \sigma)f = \sum_{i\in I} \alpha^i x_i \frac{\partial f}{\partial x_i} + \frac{1}{2}\sigma^2 \sum_{i\in I} x_i^2 \frac{\partial^2 f}{\partial x_i^2} + \rho\sigma^2 \sum_{\substack{i,j\in I\\ i\neq j}} x_i x_j \frac{\partial^2 f}{\partial x_i \partial x_j}$$

for f in the appropriate domain (see Karatzas and Shreve [23], p.281).

Let $G(t, \mathbf{x})$ be the probability that the diffusion will reach $S(\alpha)$ after t. In other words, $G(t, \mathbf{x}) = \mathbb{P}[\tau_S > t | \mathbf{x}_0 = \mathbf{x}]$ and represents the survival function for the exit time distribution of the first deserter. Following Gardiner [14], Subsection 5.4.2, this probability can be conveniently written as the solution to the following (parabolic) partial differential equation (Kolmogorov backward equation):

$$\begin{aligned}
&G_t = \mathcal{A}((\alpha^i)_{i\in I}, \rho, \sigma)G \text{ in } S^c((\alpha^i)_{i\in I}, \sigma, \gamma, C), t > 0 \\
&G(0, \mathbf{x}) = 1, \mathbf{x} \in S^c((\alpha^i)_{i\in I}, \sigma, \gamma, C) \\
&G(t, \mathbf{x}) = 0, \mathbf{x} \in S((\alpha^i)_{i\in I}, \sigma, \gamma, C) \text{ and } t \geq 0
\end{aligned}$$

where the initial value condition holds since $G(0, \mathbf{x}) = \mathbb{P}[\tau_S < \infty | \mathbf{x}_0 = \mathbf{x}] = 1, \forall \mathbf{x} \in S^c((\alpha^i)_{i\in I}, \sigma, \gamma, C)$ and the boundary condition follows since $\partial S((\alpha^i)_{i\in I}, \sigma, \gamma, C) \subset S((\alpha^i)_{i\in I}, \sigma, \gamma, C)$ and because 0 is an absorbing boundary for $x^i, i \in I$.

Uniqueness is obtained in Theorem 4, Section 7.1.2 in Evans [11].

■

Proof of Proposition 9

The proof follows the same techniques as above (see Gardiner [14], Section 5.4.2). The (parabolic) partial differential equation is now subject to the following initial value condition:

$$H(0, \mathbf{x}) = \mathbb{P}[\mathbf{x}_{\tau_S} \in \partial S_1 | \mathbf{x}_0 = \mathbf{x}] = u(\mathbf{x})$$

and u follows the following (elliptic) differential equation by Corollary II.2.11.2 in Gihman and Skorohod [16] (p.308):

$$\begin{aligned}
&\mathcal{A}((\alpha^i)_{i\in I}, \rho, \sigma)u = 0 \text{ in } S^c((\alpha^i)_{i\in I}, \sigma, \gamma, C) \\
&u(\mathbf{x}) = 1, \mathbf{x} \in \partial S_1 \\
&u(\mathbf{x}) = 0, \mathbf{x} \in \partial S \backslash \partial S_1.
\end{aligned}$$

■

For the next theorem we will make use of the following result (Theorem 1 in Araújo and Mas-Colell [2]), which we cite as a lemma.

Lemma 1 *Let* Ψ *be a topological space,* $E \subset \mathbb{R}^n, 1 \leq n \leq \infty$ *and* ν *denote a Borel probability measure on* $\mathbb{R}^n$*. Assume the following:*

1. $(\Psi \times \Psi) \backslash \Delta$ *is a Lindelöf space (i.e. any open cover has a countable subcover), where* $\Delta = \{(x, y) \in \Psi \times \Psi : x = y\}$*.*

2. $F : \Psi \times E \to \mathbb{R}$ *is a continuous function.*

3. $\forall i, \psi \in \Psi$ *and* $a \in E, \partial_{a_i} F(\psi, a)$ *exists and depends continuously on* x *and* a*.*

4. ν *is a product probability measure, each factor being absolutely continuous with respect to the Lebesgue measure.*

5. *(Sondermann Condition) If* $F(\psi, a) = F(\hat{\psi}, a), \psi \neq \hat{\psi}$*, then* $\partial_{a_i}(F(\psi, a) - F(\hat{\psi}, a)) \neq 0$ *for some* i*.*

Then, for ν*-a.e.* $a \in E$*, the function* $F(\cdot, a) : \Psi \to \mathbb{R}$ *has at most one maximizer.*

Proof of Theorem 2

<u>Step 1</u>: Consider the expected log-likelihood function conditioned on $\mathbf{w}$:

$$\int \log[g(t; \hat{\psi}, \mathbf{w})] g(t; \psi, \mathbf{w}) dt$$

From the properties of the Kullback-Leibler divergence or relative entropy for two probability distributions, it is obtained that $\hat{\psi}$ is maximizes the expected log-likelihood if and only if $g(t; \hat{\psi}, \mathbf{w}) = g(t; \psi, \mathbf{w})$. In particular, $\hat{\psi} = \psi$ is one such maximizer.

<u>Step 2</u>: Take $\Psi = \{\psi, \hat{\psi}\}$. By Lemma 1, there is at most one maximizer for the expected log-likelihood function $F_{\mathbf{w}}$-a.e. and we know that ψ is maximizes it.

■

Proof of Theorem 3

Take two (potentially different) parameter vectors $\psi = ((\alpha^i)_{i\in I}, \Delta\alpha, \rho, \sigma, \gamma, C)$ and $\hat{\psi} = ((\hat{\alpha}^i)_{i\in I}, \widehat{\Delta\alpha}, \hat{\rho}, \hat{\sigma}, \hat{\gamma}, \hat{C})$. Consider $\mathbf{x} \in \text{int}[\text{supp}(F_0)]$. We are interested in showing that

$$\mathbb{P}[(\tau_i^*)_{i\in I} \in A | \mathbf{x}_0 = \mathbf{x}; \psi]$$

(where $A \in \mathcal{B}(\mathbb{R}_+^I)$) differs for any such $\mathbf{x}$ unless these two parameter vectors are identical. A few things are noteworthy before we proceed. First, the threshold $\overline{z}^i = z(\psi) = z(\alpha^i, \sigma, \gamma, C), i \in I$ depends on α, σ, γ and C, but not on $(\Delta\alpha, \rho)$. Also, at least one player will quit at $\tau_{S_1} = \inf\{t > 0 : \mathbf{x}_t \in S_1\}$ (i.e., $\tau_{S_1} = \wedge_{i\in I}\tau_i^*$). Remark also that the event "at least one player leaves", which occurs at the hitting time τ_{S_1}, is not affected by $\Delta\alpha$, but is nevertheless influenced by ρ.

We sequentially identify the parameters and thus break the proof into three steps:

<u>Step 1</u>: $((\alpha^i)_{i\in I}, \rho, \sigma)$. As in previous propositions, the vector process with the latent utilities can be represented as the following diffusion process with killing at time τ_{S_1}:

$$dx_t^i = \alpha^i x_t^i dt + \sigma x_t^i dW_t^i, \qquad i \in I.$$

The infinitesimal generator associated with the above diffusion process is given by:

$$\mathcal{A}((\alpha^i)_{i\in I}, \rho, \sigma)f = \sum_{i\in I} \alpha^i x_i \frac{\partial f}{\partial x_i} + \frac{1}{2}\sigma^2 \sum_{i\in I} x_i^2 \frac{\partial^2 f}{\partial x_i^2} + \rho\sigma^2 \sum_{\substack{i,j\in I \\ i\neq j}} x_i x_j \frac{\partial^2 f}{\partial x_i \partial x_j}$$

for f in the appropriate domain (see Karatzas and Shreve [23], p.281).

Letting $G(t; \mathbf{x}, \psi) = \mathbb{P}[\tau_{S_1} > t | \mathbf{x}_0 = \mathbf{x}, \psi]$, one obtains that for $\mathbf{x} \in \widehat{S}_1 \cap$

$S_1 \cap \text{int}[\text{supp}(F_0)]$ (where hatted elements are those defined for $\hat{\psi}$ instead of ψ),

$$\frac{\partial G(t;\mathbf{x},\psi)}{\partial t} = \mathcal{A}((\alpha^i)_{i\in I},\rho,\sigma)G(t;\mathbf{x},\psi)$$

and

$$\frac{\partial G(t;\mathbf{x},\hat{\psi})}{\partial t} = \mathcal{A}((\hat{\alpha}^i)_{i\in I},\hat{\rho},\hat{\sigma})G(t;\mathbf{x},\hat{\psi}).$$

Let $h(t;\mathbf{x},\psi,\hat{\psi}) = G(t;\mathbf{x},\psi) - G(t;\mathbf{x},\hat{\psi})$. Using the equations above, one gets that

$$\frac{\partial h(t;\mathbf{x},\psi,\hat{\psi})}{\partial t} = \mathcal{A}((\alpha^i)_{i\in I},\rho,\sigma)h(t;\mathbf{x},\psi,\hat{\psi}) + g(t;\mathbf{x},\psi,\hat{\psi})$$

where $g(t;\mathbf{x},\psi,\hat{\psi}) = [\mathcal{A}((\alpha^i)_{i\in I},\rho,\sigma) - \mathcal{A}((\hat{\alpha}^i)_{i\in I},\hat{\rho},\hat{\sigma})]G(t;\mathbf{x},\hat{\psi})$.

If we assume that, for $\mathbf{x} \in \widehat{S_1} \cap S_1 \cap \text{int}[\text{supp}(F_0)]$,

$$G(t;\mathbf{x},\psi) = G(t;\mathbf{x},\hat{\psi}), \quad \forall t > 0.$$

we shall have that

$$\frac{\partial h(t;\mathbf{x},\psi,\hat{\psi})}{\partial t} = \frac{\partial G(t;\mathbf{x},\psi)}{\partial t} - \frac{G(t;\mathbf{x},\hat{\psi})}{\partial t} = 0$$

and

$$\mathcal{A}((\alpha^i)_{i\in I},\rho,\sigma)h(t;\mathbf{x},\psi,\hat{\psi}) = \mathcal{A}((\alpha^i)_{i\in I},\rho,\sigma)[G(t;\mathbf{x},\psi) - G(t;\mathbf{x},\hat{\psi})] = 0,$$

$\forall t > 0$. This in turn implies that $g(t;\mathbf{x},\psi,\hat{\psi}) = 0$ or, in other words,

$$[\mathcal{A}((\alpha^i)_{i\in I},\rho,\sigma) - \mathcal{A}((\hat{\alpha}^i)_{i\in I},\hat{\rho},\hat{\sigma})]G(t;\mathbf{x},\hat{\psi}) = 0.$$

Since (having fixed t) $G(t;\cdot,\hat{\psi})$ is not constant, one must have

$$\mathcal{A}((\alpha^i)_{i\in I},\rho,\sigma) - \mathcal{A}((\hat{\alpha}^i)_{i\in I},\hat{\rho},\hat{\sigma}) = 0.$$

This means that $((\alpha^i)_{i\in I},\rho,\sigma) = ((\hat{\alpha}^i)_{i\in I},\hat{\rho},\hat{\sigma})$.

STEP 2: $(\Delta\alpha)$. Assume then that $(\alpha^i)_{i\in I} = (\hat{\alpha}^i)_{i\in I}, \rho = \hat{\rho}$ and $\sigma = \hat{\sigma}$ and suppose $C \neq \hat{C}, \gamma \neq \hat{\gamma}$ or $\Delta\alpha \neq \widehat{\Delta\alpha}$. Remember that

$$z(\alpha, \sigma, C, \gamma) = \frac{\beta(\alpha, \sigma, \gamma)}{\beta(\alpha, \sigma, \gamma) - 1} C$$

where

$$\beta(\alpha, \sigma, \gamma) = 1/2 - \alpha/\sigma^2 + \sqrt{\left[\alpha/\sigma^2 - 1/2\right]^2 + 2\gamma/\sigma^2} > 1.$$

Since $(\alpha^i)_{i\in I} = (\hat{\alpha}^i)_{i\in I}, \rho = \hat{\rho}$ and $\sigma = \hat{\sigma}$, we must have that

$$z(\alpha^i, \sigma, C, \gamma) = z(\alpha^i, \sigma, \hat{C}, \hat{\gamma}), \quad i \in I.$$

If this were not the case, one would have either

$$z(\alpha^i, \sigma, C, \gamma) > z(\alpha^i, \sigma, \hat{C}, \hat{\gamma}), \quad i \in I.$$

or

$$z(\alpha^i, \sigma, C, \gamma) < z(\alpha^i, \sigma, \hat{C}, \hat{\gamma}), \quad i \in I.$$

and, consequently, either

$$S_1((\alpha^i)_{i\in I}, \sigma, \gamma, C) \equiv S_1 \subset \hat{S}_1 \equiv S_1((\hat{\alpha}^i)_{i\in I}, \hat{\sigma}, \hat{\gamma}, \hat{C})$$

or

$$S_1((\alpha^i)_{i\in I}, \sigma, \gamma, C) \equiv S_1 \supset \hat{S}_1 \equiv S_1((\hat{\alpha}^i)_{i\in I}, \hat{\sigma}, \hat{\gamma}, \hat{C}).$$

Considering that the infinitesimal generators for the latent utility process coincide (being given by $\mathcal{A}((\alpha^i)_{i\in I}, \rho, \sigma)$) we would have $G(\cdot, \mathbf{x}; \psi) \neq G(\cdot, \mathbf{x}; (\hat{\psi})$ (since then either $\tau_{S_1} < \tau_{\hat{S}_1}$ or $\tau_{S_1} > \tau_{\hat{S}_1}, \mathbb{P}$-a.s.).

Taking then into account that $S_1 = \hat{S}_1$ (and $\tau_{S_1} = \tau_{\hat{S}_1}, \mathbb{P}$-a.s.), it should be also the case that

$$z(\alpha^i - \frac{\Delta\alpha}{I-1}, \sigma, C, \gamma) = z(\alpha^i - \frac{\widehat{\Delta\alpha}}{I-1}, \sigma, \hat{C}, \hat{\gamma}), \quad i \in I.$$

If this were not true, one could show similarly that either

$$\partial S_H((\alpha^i)_{i\in I},\sigma,\gamma,C,\Delta\alpha)\equiv\partial S_H\subset\widehat{\partial S_H}\equiv\partial S_H((\hat{\alpha}^i)_{i\in I},\hat{\sigma},\hat{\gamma},\hat{C},\widehat{\Delta\alpha})$$

or

$$\partial S_H((\alpha^i)_{i\in I},\sigma,\gamma,C,\Delta\alpha)\equiv\partial S_H\supset\widehat{\partial S_H}\equiv\partial S_H((\hat{\alpha}^i)_{i\in I},\hat{\sigma},\hat{\gamma},\hat{C},\widehat{\Delta\alpha})$$

where $\partial S_H((\alpha^i)_{i\in I},\sigma,\gamma,C,\Delta\alpha)$ is defined in Proposition 9. This will then imply that the probability of simultaneous exits will be larger in one parametric configuration than in other.

Define $y_t^i=\tilde{x}^i_{\tau_{S_1}+t}=\log x^i_{\tau_{S_1}+t}$ and $\tilde{W}_t^i=W^i_{\tau_{S_1}+t}$. It is easy to observe that

$$dy_t^i=(\alpha^i-\Delta\alpha/(I-1))dt+\sigma d\tilde{W}_t^i \quad y_0^i=\log x^i_{\tau_{S_1}}, \quad \forall i\in I$$

provided $\mathbf{x}_{\tau_{S_1}}\notin\partial S_H$ (i.e., there are no simultaneous exits in the first desertion round). But then one can apply the results in Step 1 for the identifiability of the drift and diffusion coefficients to obtain that, unless $\Delta\alpha=\widehat{\Delta\alpha}$, the probability distribution for the timing of the second round of desertions given that there was no clustering in the first round of desertions will differ for the two parameters.

<u>Step 3</u>: (C,γ). Notice that, as opposed to the previous set of parameters, C and γ only affect the thresholds $z(\alpha,\sigma,C,\gamma)$ but do not interfere with the probability law governing the latent utilities process. With more than two agents it is then easy to see that for a given pair C,γ, the parameters $\hat{C},\hat{\gamma}$ would have to satisfy the set of equations

$$z(\alpha^i-k\Delta\alpha/(I-1),\sigma,C,\gamma)=z(\alpha^i-k\Delta\alpha/(I-1),\sigma,\hat{C},\hat{\gamma})$$

for $k=0,\ldots,I-1$ and $\forall i\in I$, where

$$z(\alpha,\sigma,C,\gamma)=\frac{\beta(\alpha,\sigma,\gamma)}{\beta(\alpha,\sigma,\gamma)-1}C$$

and

$$\beta(\alpha, \sigma, \gamma) = 1/2 - \alpha/\sigma^2 + \sqrt{\left[\alpha/\sigma^2 - 1/2\right]^2 + 2\gamma/\sigma^2} > 1.$$

If the above system of equations hold we have that

$$h(\alpha, k\Delta\alpha, \sigma, \gamma, \hat{\gamma}) \equiv \frac{\beta(\alpha - k\Delta\alpha/(I-1), \sigma, \gamma)}{\beta(\alpha - k\Delta\alpha/(I-1), \sigma, \gamma) - 1} \times$$
$$\times \frac{\beta(\alpha - k\Delta\alpha/(I-1), \sigma, \hat{\gamma}) - 1}{\beta(\alpha - k\Delta\alpha/(I-1), \sigma, \hat{\gamma})} = \frac{C}{\hat{C}}, \quad k = 0, \ldots, I-1 \quad \forall i \in I.$$

Given that $I > 2$ this relation is impossible since h can be checked not to be homogeneous of degree zero with respect to the argument $\Delta\alpha$. This in turn implies that not all the thresholds can coincide, which means that the probability distribution over exit times will change if one modifies C or γ.

■

References

[1] Aliprantis, C. and K. Border (1999) *Infinite Dimensional Analysis: A Hitchhiker's Guide*. Springer-Verlag.

[2] Araújo, A. and A. Mas-Colell (1978) "Notes on the Smoothing of Aggregate Demand", *Journal of Mathematical Economics* **5**(2).

[3] Bal, G. and T. Chou (2004) "On the Reconstruction of Diffusions from First-Exit Time Distributions", *Inverse Problems*, **20**(4).

[4] Bearman, P. (1991) "Desertion as Localism: Army Unit Solidarity and Group Norm in the U.S. Civil War", *Social Forces*, **70**(2).

[5] Bresnahan, T. and P. Reiss (1991) "Empirical Models of Discrete Games", *Journal of Econometrics*, **48**.

[6] Brock, W. and S. Durlauf (2002) "Interactions-Based Models" **in** J. Heckman and E. Leamer (eds.), *Handbook of Econometrics*, v.5, Elsevier Science.

[7] Chhikara, R. and J.L. Folks (1989) *The Inverse Gaussian Distribution*, Marcel Dekker, Inc.

[8] Costa, D. and M. Kahn (2003) "Cowards and Heroes: Group Loyalty during the American Civil War", *Quarterly Journal of Economics*, **118**(2).

[9] Costa, D. and M. Kahn (2004) "Shame and Ostrascism: Union Army Deserters Leave Home", mimeo., Department of Economics, MIT.

[10] Dixit, A. and R. Pindyck (1994) *Investment Under Uncertainty*, Princeton University Press.

[11] Evans, L. (1998) *Partial Differential Equations*, American Mathematical Society.

[12] Fakeev, A.G. (1970) "On Optimal Stopping of Random Processes with Continuous Time", *Theory Prob. Applications*, **15**(2).

[13] Fudenberg, D. and J. Tirole (1991) *Game Theory*, The MIT Press.

[14] Gardiner, C.W. (2002) *Handbook of Stochastic Methods for Physics, Chemistry and the Natural Sciences*. Second Edition. Springer-Verlag.

[15] Glaeser, E. and J. Scheinkman (2002) "Non-Market Interactions" **in** M. Dewatripont, L.P. Hansen and S. Turnovsky (eds.), *Advances in Economics and Econometrics: Theory and Applications, Eight World Congress*, Cambridge University Press.

[16] Gihman, I.I. and A.V. Skorohod (1972) *Stochastic Differential Equations*. Springer-Verlag.

[17] Goldin, C. and L. Katz (2002) "The Power of the Pill: Oral Contraceptives and Women's Career and Marriage Decisions", *Journal of Political Economy*, **110**(4).

[18] He, H., W. Keierstead and J. Rebholz (1998) "Double Lookbacks", *Mathematical Finance*, **8**(3).

[19] Hong, H., J. Kubik and J. Stein (2004) "Social Interactions and Stock-Market Participation", *Journal of Finance*, forthcoming.

[20] Hsiao, C. "Identification" **in** Z. Griliches and M. Intrilligator, *Handbook of Econometrics*, V.1, Elsevier Science.

[21] Ivković, Z. and S. Weisbenner (2004) "Information Diffusion Effects in Individual Investors' Common Stock Purchases: Covet Thy Neighbors' Investment Choices", NBER Working Paper 10436.

[22] Iyengar, S. (1985) "Hitting Lines with Two-Dimensional Brownian Motion", *SIAM Journal of Applied Mathematics*, **45**(6).

[23] Karatzas, I. and S. Shreve (1991) *Brownian Motion and Stochastic Calculus*. Springer-Verlag.

[24] Karlin, S. and H. Taylor (1981) *A Second Course in Stochastic Processes*, Academic Press.

[25] Kelly, M. and C. Gráda (2000) "Market Contagion: Evidence from the Panics of 1854 and 1857", *The American Economic Review*, **90**(5).

[26] Laraki, R., E. Solan and N. Vieille (2002) "Continuous-time Games of Timing", MEDS Northwestern University, Discussion Paper.

[27] Lonn, E. (1928) *Desertion During the Civil War*, The Century Co.

[28] Lancaster, T. (1972) "A Stochastic Model for the Duration of a Strike" *Journal of the Royal Statistical Society, Series A (General)*, **135**(2)

[29] Mamer, J. (1987) "Monotone Stopping Games", *Journal of Applied Probability*, **24**.

[30] Manski, C. (1988) "Identification of Binary Response Models", *Journal of the American Statistical Association*, **83**(403).

[31] Manski, C. (1993) "Identification of Endogenous Social Effects: The Reflection Problem", *Review of Economic Studies*, **60**(3).

[32] Orrenius, P. (1999) "The Role of Family Networks, Coyote Prices and the Rural Economy in Migration from Western Mexico: 1965-1994", Discussion Paper, Federal Reserve Bank of Dallas.

[33] Port, S. and C. Stone (1978) *Brownian Motion and Classical Potential Theory*, Academic Press.

[34] Protter, P. (1991) *Stochastic Integration and Differential Equations. A New Approach.*, Springer- Verlag.

[35] Rebholz, J. (1994) *Planar Diffusions with Applications to Mathematical Finance*. Ph.D. Dissertation, University of California, Berkeley.

[36] Revuz, D. and M. Yor (1999) *Continuous Martingales and Brownian Motion*, Springer-Verlag.

[37] Schelling, T. (1960) *The Strategy of Conflict*, Harvard University Press.

[38] Shiryayev, A.N. (1978) *Optimal Stopping Rules*, Springer-Verlag.

[39] Sirakaya, S. (2002) "Recidivism and Social Interactions", mimeo., Department of Economics, University of Wisconsin-Madison

[40] Tamer, E. (2003) "Incomplete Simultaneous Discrete Response Model with Multiple Equilibria", *Review of Economic Studies*, **70**(1).

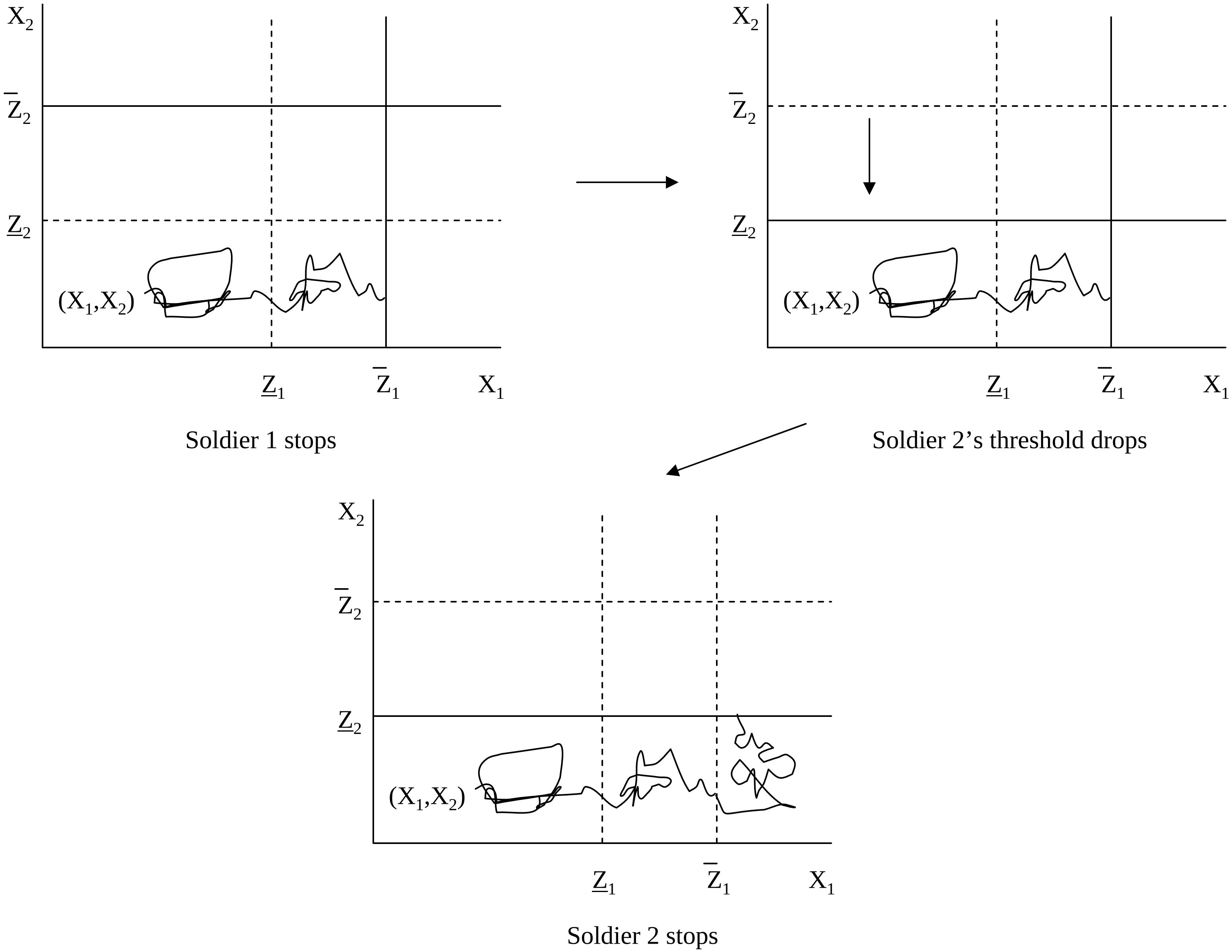
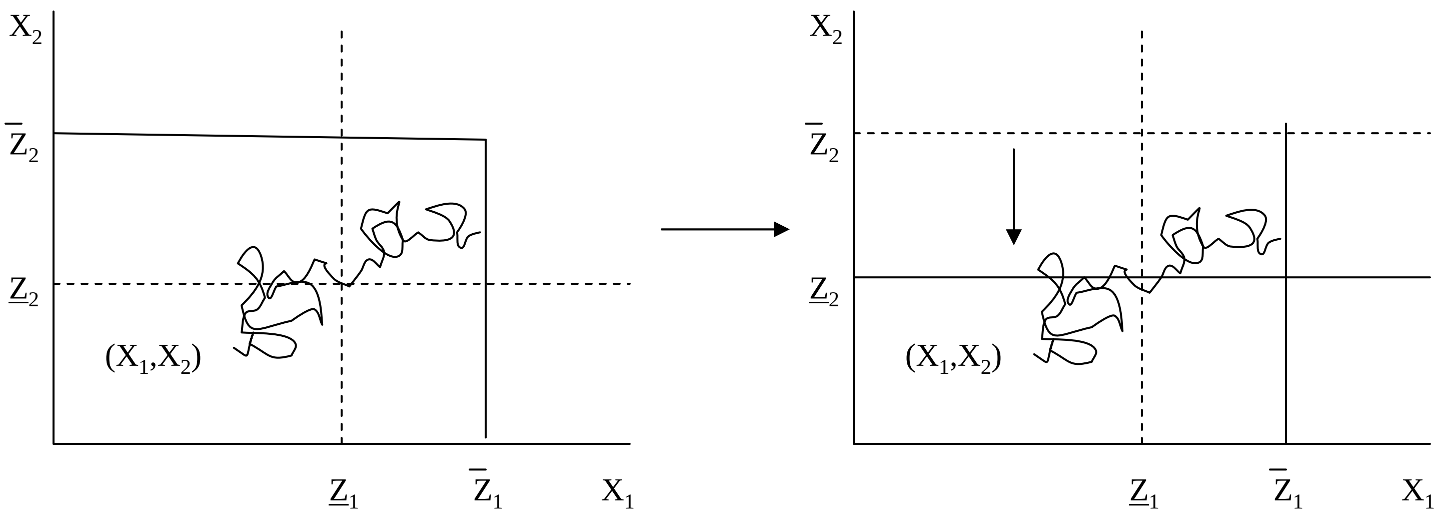

X_2
$\bar{Z}_2$
$\underline{Z}_2$
(X_1,X_2)
$\underline{Z}_1$
$\bar{Z}_1$
X_1
Soldier 1 stops
X_2
$\bar{Z}_2$
$\underline{Z}_2$
(X_1,X_2)
$\underline{Z}_1$
$\bar{Z}_1$
X_1
Soldier 2's threshold drops
X_2
$\bar{Z}_2$
$\underline{Z}_2$
(X_1,X_2)
$\underline{Z}_1$
$\bar{Z}_1$
X_1
Soldier 2 stops

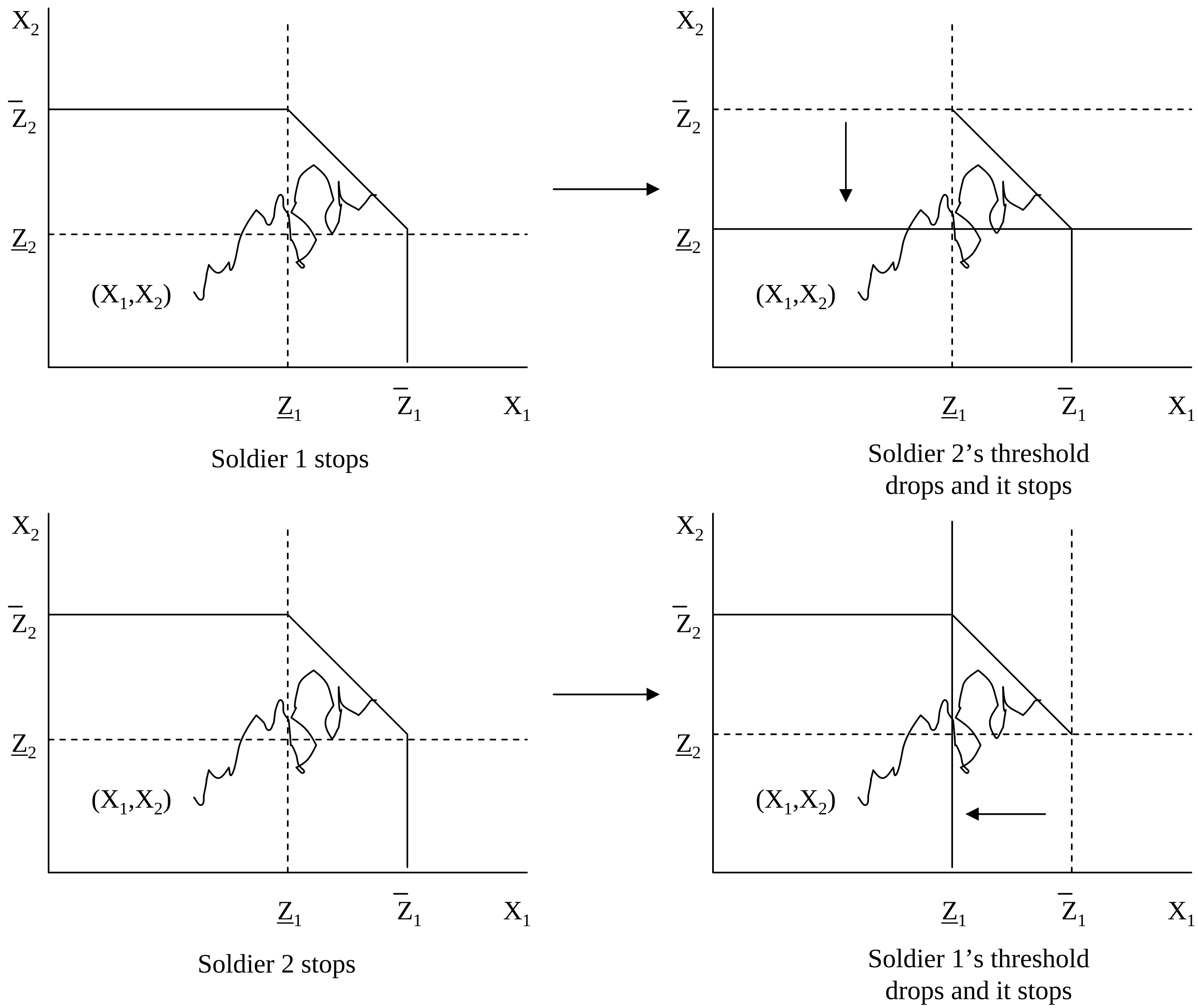
Figure 1



Soldier 1 stops

Soldier 2's threshold drops and it stops

Figure 2



X2
Z2
Z2
(X1,X2)
Z1
Z1
X1
Soldier 1 stops
Soldier 2's threshold drops and it stops
Soldier 2 stops
Soldier 1's threshold drops and it stops

Figure 3