Optimal Auction Design For Multiple Objects with Externalities

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Abstract

In this paper we characterize the optimal allocation mechanism for N objects, (permits), to I potential buyers, (firms). Firms' payoffs depend on their costs, the costs of competitors and on the final allocation of the permits, allowing for externalities, substitutabilities and complementarities. Firms' cost parameter is private information and is independently distributed across firms. Externalities are type dependent. This has two consequences: first, even though the private information of each firm is one dimensional (its cost), an allocation's virtual valuation (the natural generalization of the virtual valuation introduced in (Myerson (1981), [12] depends on the cost parameters of all firms. Second, the "critical"type of each buyer, (the type for which participation constraint binds) is not exogenously given but depends on the particular mechanism selected. This is not as in the papers by Jehiel, Moldovanu and Stacchetti 1996, 2001 [6], [7], and makes the characterization of the optimum intricate, since the objective function is altered. However, the feasibility constraints remain tractable, which makes the use of variational methods possible. A further consequence of having type-dependent externalities, which does not arise in the

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previous work, is that not only payments, but also the revenue maximizing allocation is different from the optimum derived without taking into account the existence of externalities. Our model captures key features of many important multi-object allocation problems like the allocation of time slots for TV commercials, landing slots in airports, privatization and firm takeovers. Keywords: *Optimal Auctions, Multiple Objects, Externalities, Mechanism Design; JEL D44, C7, C72.*

1 Introduction

In this paper we characterize the optimal allocation mechanism for N objects (permits), to I potential buyers (firms). Firms' payoffs depend on their costs, the costs of competitors and on the final allocation of the permits, allowing for externalities, substitutabilities and complementarities. A firm cares not only whether it obtains a particular set of permits, but also cases about who obtained which licence. Firms' cost parameter is private information and is independently distributed across firms. Externalities are type dependent.

In a large variety of multi-object allocation problems the presence of externalities is of central role. Our model with small modifications can help address the following problems.

• **Firm Take-overs:** Externalities are of huge importance in firm take-overs: Recently (February 2004), Cingular bought AT&T wireless for $41 billion after a bidding war with Vodafone. Some perceive that the big winner of this sale will be Verizon even though it was not a participant in the auction (NY Times February 17, 2004 "Verizon Wireless May Benefit From Results of Auction").

• **Allocation of Airport Take-Off and Landing Slots.** Airport take-off and landing slots are a scarce resource yet not priced! There are important externalities since for instance if two airlines are fierce competitors in a big airport say United and American at O'Hare, then if United obtains critical landing slots in LAX, (Los Angeles International Airport), this may well affect its market position in O'Hare vis-a-vis American.

• **Auctioning of time slots for advertisements on TV, radio.** In reality airtime for advertisements is priced using conventional mechanisms, whereas if networks take-into account the presence of externalities and auction off the time slots, we might end up with less (even zero) airtime of advertisement yet higher revenue. How much would a firm pay so that its fiercest competitor does not advertise in the intermission of Super-ball? One can imagine a network asking this question to Miller, Budweiser, Bud etc. Taking this to an extreme there may be a potential for a lot of revenue with actually no one airing a spot. In other words strongly opposed interest may permit the seller to extract payments just for doing nothing [1]!

• **Privatization - Mechanism Design with Endogenous Market Structure.** The model of this paper can be thought as follows: There is revenue-maximizing seller [2](the government) trying to sell permits for operating in a certain market to some potential buyers. These permits represent a right to participate in the market, and the profits a given firm can make out of it depend on three things: their own marginal cost, which is private information, the market structure and the marginal costs of the competitors that also participate in the market. After the permits have been allocated, the firms that got one or more of them face a perfectly anticipated demand and engage in some sort of oligopolistic competition. We also allow for the possibility that these firms are already competing in different markets, so even if they do not get permits assigned, who gets the permits will affect their profits. The presence of such externalities allows the seller can to extract extra payments from any given firm, just by threatening to setup a very damaging market structure in case it does not participate in the process. Our model captures such scenarios and generalizes previous work by [2], who examine whether the government should sell a firm in one piece or cut it into two. (for a discussion on this, see [11]).[3] In the work of [2] the outcome of the mechanism depends heavily on the weight that the government assigns to revenue versus efficiency and on the type of competition that prevails in

[1] This is similar to the common agency problem, as identified for example in [3]

[2] The objective function can be modified to take into account the possibility that the governement cares also for consumer surplus.

[3] Gale([4]) also considers a variation of this problem but because he imposes a very strong superadditivity condition to the profit function, he shows that an optimal mechanism always gives all the "permits" to at most one buyer, so the market structure is always the one of a monopoly.

the market after privatization.

• **Selling licences for cellular networks, TV or radio broadcasting.**

• **Optimal Bundling:** Since we allow for complementarities and substitutability one can think of applications for cases of bundling of goods, (bundles like telecommunication services - internet - cable TV, computers-printers-software-digital cameras etc).

This paper is related to the optimal auction literature for multiple objects: [10] analyze the case of single dimensional private information and continuously divisible goods, [1] allows for multidimensional uncertainty but there are only two buyers and two types. [6] study optimal auction design in the presence of externalities in a *single* unit environment where externalities are type *independent*. Because of the presence of externalities the seller can extract payment for the losers but the revenue maximizing allocation of the object is the same as in the case of the revenue maximizing auction without taking into account the presence of externalities. [7] consider again the design of optimal auctions of a single object in the presence of externalities. Here the externalities are type dependent: the type of each buyer is a vector of numbers that determines his/her utility as a function of who gets the object. The multi-dimensionality makes the solution of the general problem intractable: it is almost impossible to verify that the set of conditions that are implied by incentive compatibility are satisfied (the allocation rule has to be monotonic and conservative - or path independent). Our innovation is to allow for multiple objects, general payoff functions that allow for complementarities and substitutabilities and type dependent externalities among buyers, but because private information is single-dimensional we can solve the problem. The first consequence of approach is that even though the private information of each firm is one dimensional (its cost), an allocation's virtual valuation depend on the cost parameters of all other firms. This captures nicely the existence of externalities among buyers: how much money the seller can extract from firm A depends on the technology of firm B, which captures together with other parameters how strong of a competitor firm B is. As in [6] and [7] the critical type of the buyer is not exogenously given but depends on the range of the externalities. But in our approach, since we allow for more general payoff functions, the critical type (where the participation constraint binds) also depends on the actual mechanism. This critical type of each agent determines how much money the seller can extract from the players. Hence the characterization of the optimum becomes intricate: given a mechanism there is a vector of critical types and a amount of payments that the seller can extract from the buyers: the mechanism depends not only on the virtual valuations, but also on which is the critical type. Moreover the vector of critical types is mechanism specific. A consequence of this interrelationship between the critical types and the mechanism is that the optimal allocation of the object in the presence of externalities is different from the one we would obtain with no externalities. In contrast, the presence of externalities in [6] affect only the payment that the seller can extract from the buyers and not the allocation of the object. General models allowing for type dependent externalities like those ([5], [9]) are concerned with the design of efficient mechanisms.

2 The model

There are I risk-neutral firms trying to buy N permits. Each firm has a marginal cost $c_i \in [\underline{c_i}, \overline{c_i}] \equiv C_i$ that is drawn independently from a distribution with density f_i (with cumulative distribution F_i). We assume $f_i(c_i) > 0$ for all $c_i \in [\underline{c_i}, \overline{c_i}]$. This cost is private information of each firm. Let $C = \prod_{i=1}^{I} [\underline{c_i}, \overline{c_i}]$ and $C_{-i} = \prod_{j \neq i} [\underline{c_j}, \overline{c_j}]$. The set of possible allocations is given by $Z = \{(z_1, ..., z_I) \in \mathbb{N}^I | \sum_{i=1}^{I} z_i \leq N\}$ (notice that Z is finite). For any given firm i, its profits are represented by a function $\pi_i(z, c_i, c_{-i})$ with the following characteristics:[4]

- π_i is decreasing and convex in c_i

[4]Gale's (1990) condition on profit function, would in our notation read as follows:

$$(\forall z \in \partial Z)(\forall z' \in Z/\partial Z)(\forall c \in [\underline{c_i}, \overline{c_i}]) \sum_{i=1}^{I} \pi_i(z, c_i, c_{-i}) \geq \sum_{i=1}^{I} \pi_i(z', c_i, c_{-i})$$

where $\partial Z = \{z \in Z | (\exists i \in \{1, ..., I\}) z_i = N\}$.

- $\pi_i(z, \cdot, c_{-i})$ is differentiable for all z and c_{-i}

This specification makes clear that we are in the context of an auction with externalities, since each buyer cares not only for the permits that are assigned to him, but also for the distribution of the remaining ones. Notice also that we allow $\pi_i(z, c_i, c_{-i}) \neq 0$ even when the allocation z does not include any rights for firm i, so we can include the cases when the bidders are already competing in a different market, and whatever happens in this one will affect their profits in those.
As a matter of notational convenience, let's define

$$f_{-i}(c_{-i}) = f_1(c_1)f_2(c_2)...f_{c_{i-1}}(c_{i-1})f_{c_{i+1}}(c_{i+1})...f_I(c_I)$$

and

$$f(c) = f_1(c_1)...f_I(c_I).$$

A mechanism will be then $M = (\{S_i\}_{i=1}^I, p, x, \{\rho_i\}_{i=1}^I)$, where S_i stands for the set of messages available to firm i, $p : \prod_{i=1}^I S_i \longrightarrow \Delta(Z)$ specifies the probability of each allocation for a given message, $x : \prod_{i=1}^I S_i \longrightarrow \mathbb{R}_+^I$ is the specification of the payments for each message and $\rho_i : \prod_{i=1}^I S_i \longrightarrow \Delta(Z)$ specifies the probability of each allocation when firm i decides not to participate in the auction. This is the threat allocation rule: because there externalities the seller can threat i that in the event that i fails to participate, he will face a very unfavorable allocation. In our model this unfavorable allocation is not only buyer and type-specific but depends also on the allocation p that the seller desires to implement. We denote by $p^z(s)$ the probability that allocation z is implemented when the message tuple is s. It's easy to check that in this context the revelation principle holds, so without loss of generality we can consider $S_i = [\underline{c_i}, \overline{c_i}]$ and restrict our attention to mechanisms where each agent truthfully reveals its type. With this in mind, we drop the strategy spaces from the definition of a mechanism and consider $M = (p, x, \{\rho_i\})$.
For a fixed mechanism $M = (p, x, \{\rho_i\}_{i=1}^I)$,the ex-ante utility of a firm of type c_i when

it participates and declares c_i' is:

$$U_i(c_i, c_i'; p, x) = \int\limits_{C_{-i}} \sum_{z \in Z} (p^z(c_i', c_{-i})\pi_i(z, c_i, c_{-i}) - x(c_i', c_{-i}))f_{-i}(c_{-i})dc_{-i}$$

The utility of the same firm if it decides not to participate in the auction is:

$$\int\limits_{C_{-i}} \sum_{z \in Z} \rho_i^z(c_i', c_{-i})\pi(z, c_i, c_{-i})f_{-i}(c_{-i})dc_{-i}$$

Since, as we said before, the revelation principle holds and we assume firms satisfy the constraint of individual rationality, we introduce the following definition:

Definition 1. *We say that a mechanism (p, x) is feasible iff*

$$\begin{array}{ll} U_i(c_i, c_i; (p, x)) \geq U_i(c_i, c_i'; (p, x)) & \text{for all } c_i, c_i' \in C_i \text{ and } c_{-i} \in C_{-i} \\ U_i(c_i, c_i; (p, x)) \geq \int\limits_{C_{-i}} \sum\limits_{z \in Z} \rho_i(c_i', c_{-i})\pi(z, c_i, c_{-i})f_{-i}(c_{-i})dc_{-i} & \text{for all } c_i \in C_i \\ \sum\limits_{z \in Z} p^z(c) \leq 1, p^z(c) \geq 0 & \text{for all } c \in C \end{array}$$

The first set of constraints are the incentive compatibility constraints, the second set of constraints are the voluntary participation constraints and the third set of constraints impose the requirements that probabilities sum up to one and are non-negative numbers. Then the problem of a revenue maximizing seller can be written as:

$$\begin{array}{ll} \max\limits_{p,x} & \int\limits_C \sum\limits_{i=1}^{I} x_i(c)f(c)dc \\ s.t.\ (p, x) \text{ feasible.} & \end{array}$$

We will solve this problem in steps. Before examining the complete model where threats will play a crucial role in the optimum, we first look at the case that the worst punishment that the seller can impose on a firm is to leave it out of the market which leaves this firm a payoff of zero. We call this case the single market case.

3 The single market case

We consider first the case when the outcome of the process under question does not have any effect on the profits of the firms outside of the current market. This is summarized in the following assumption:

Assumption 1. *The worst punishment that the seller can impose on a buyer guarantees that buyer a payoff of zero.*[5]

Under this assumption $\{\rho_i\}_{i=1}^I$ becomes completely irrelevant, since the maximal punishment that can be imposed to a firm is to be left outside of the market. The problem for the seller then becomes:

$$\begin{array}{ll}
\max\limits_{p,x} \int\limits_C \sum\limits_{i=1}^I x_i(c) f(c) dc & \\
s.t. & \\
U_i(c_i, c_i; (p, x)) \geq U_i(c_i, c_i'; (p, x)) & \text{for all } c_i \in C_i \text{ and } c_{-i} \in C_{-i} \\
U_i(c_i, c_i; (p, x)) \geq 0 & \text{for all } c_i \in C_i \\
\sum\limits_{z \in Z} p^z(c) \leq 1, p^z(c) \geq 0 & \text{for all } c \in C
\end{array}$$

Now we will completely characterize the structure of the revenue maximizing mechanism for this case. Let's define[6]

$$P(c_i) = \int\limits_{C_{-i}} \sum_{z \in Z} p^z(c_i, c_{-i}) \frac{\partial \pi_i(z, c_i, c_{-i})}{\partial c_i} f_{-i}(c_{-i}) dc_{-i}$$

and

$$V_i(c_i) = \max_{c_i'} \int\limits_{C_{-i}} (\sum_{z \in Z} p^z(c_i', c_{-i}) \pi_i(z, c_i, c_{-i}) - x(c_i', c_{-i})) f_{-i}(c_{-i}) dc_{-i}$$

[5] A sufficient condition for this is $\pi(z, c_i, c_{-i}) \geq 0$ and $\pi_i(z, c_i, c_{-i}) = 0$ for all $z \notin Z_i$, where $Z_i = \{z \in Z | z_i \neq 0\}$ is the set of allocations where agent i receives some participation in the market.

[6] In the case with one object, no externalities and a valuation of the object equal to the type this reduces to the familiar expression $R(c_i) = -\int\limits_{C_{-i}} p^i(c_i, c_{-i}) dc_{-i}$

Lemma 1. *A mechanism (p, x) is feasible iff*

$$P_i(c_i') \geq P_i(c_i) \quad \textit{where } c_i' > \quad c_i \tag{1}$$

$$V_i(c_i) = V_i(\overline{c}) - \int_{c_i}^{\overline{c_i}} P(s)ds \qquad \textit{for all } c_i \in C_i \tag{2}$$

$$V_i(\overline{c_i}) \quad \geq \quad 0 \tag{3}$$

$$p^z(c) \geq 0 \qquad \sum_{z \in Z} p^z(c) \leq 1 \tag{4}$$

Proof. By the convexity of $\pi_i(z, \cdot, c_{-i})$ we have V is a maximum of convex functions, so it is convex, and therefore differentiable a.e. It's also easy to check that the following are equivalent:

(a) (p, x) is incentive compatible

(b) $R(c_i) \in \partial V(c_i)$

(c) $U(c_i, c_i; (p, x)) = V(c_i)$

($\Longrightarrow$) Since the mechanism is incentive compatible, from the previous characterization we get that a feasible mechanism must satisfy (b). A result in [8] then implies (2). By the convexity of V, we know ∂V is monotone, so:

$$(R(c_i) - R(c_i'))(c_i - c_i') \geq 0$$

This immediately implies (1). Finally, using (2) and individual rationality we get (3). ($\Longleftarrow$)Individual rationality is immediately implied by (2) and (3). To prove incentive compatibility it's enough to show that $R(c) \in \partial V(c)$. By (1) and (2),

$$\begin{aligned} V(c') - V(c) &= \int_c^{c'} R(s)ds \\ &\geq R(c)(c' - c) \end{aligned}$$

which shows $R(c) \in \partial V(c)$. $\square$

Lemma 2. *The expected payment of an agent can be written as*

$$Y_i = \int\limits_{C_{-i}} \sum_{z\in Z} p^z(c_i, c_{-i})[\pi_i(z, c_i, c_{-i}) + \frac{F_i(c_i)}{f_i(c_i)}\frac{\partial \pi_i(z, c_i, c_{-i})}{\partial c_i}]f(c)dc$$

Proof.

$$\begin{aligned} Y_i &= \int\limits_{C_{-i}} x_i(c)f(c)dc \\ &= \int\limits_{C_{-i}} [\sum_{z\in Z} p^z(c_i, c_{-i})\pi_i(z, c_i, c_{-i})]f(c)dc - \int\limits_{C_i} V(c_i)dc_i \end{aligned}$$

But because of (5), and using changing the order of integration we get:

$$\begin{aligned} \int\limits_{C_i} V(c_i)dc_i &= \int\limits_{C_i} [V(\overline{c_i}) - \int\limits_{c_i}^{\overline{c_i}} R(s)ds]f_i(c_i)dc_i \\ &= V(\overline{c_i}) - \int\limits_{C_i} R(s)[\int\limits_{\underline{c_i}}^{s} f_i(c_i)dc_i]ds \\ &= V(\overline{c_i}) - \int\limits_{C_i} R(c)F_i(c)dc \\ &= V(\overline{c_i}) - \int\limits_{C_i} (\int\limits_{C_{-i}} \sum_{z\in Z} p^z(c_i, c_{-i})\frac{\partial \pi_i(z, c_i, c_{-i})}{\partial c_i} f_{-i}(c_{-i})dc_{-i})F_i(c_i)dc_i \\ &= V(\overline{c_i}) - \int\limits_{C_{-i}} \sum_{z\in Z} p^z(c_i, c_{-i})\frac{\partial \pi_i(z, c_i, c_{-i})}{\partial c_i}\frac{F_i(c_i)}{f_i(c_i)} f(c)dc \end{aligned}$$

and since in an optimal mechanism $V(\overline{c_i}) = 0$, the result follows. □

The two previous lemmas allow us to fully characterize the problem in terms of the assignment function p:

Proposition 1. *If in a mechanism $(\widehat{p}, \widehat{x})$ the assignment function $\widehat{p}$ solves:*

$$\begin{aligned} &\max_p \quad \int\limits_C \sum_{z\in Z} p^z(c) \sum_{i=1}^{I} \left[\pi_i(z, c_i, c_{-i}) + \tfrac{F_i(c_i)}{f_i(c_i)}\tfrac{\partial \pi_i(z,c_i,c_{-i})}{\partial c_i}\right] f(c)dc \\ &s.t. (4), (7) \end{aligned}$$

and the payment function $\widehat{x}$ satisfies:

$$\widehat{x}_i(c) = \sum_{z \in Z} \widehat{p}^z(c)\pi_i(z, c_i, c_{-i}) + \int_{c_i}^{\overline{c_i}} \sum_{z \in Z} \frac{\partial \pi_i(z, s, c_{-i})}{\partial s} \widehat{p}^z(s, c_{-i}) f_i(s) ds$$

then the mechanism is optimal

Proof. Just apply lemmas 1-2, and notice that with this definition of $x^i(c)$, conditions (5) and (6) are immediately satisfied. □

Assumption 2. [7] *Let z_1,$z_2 \in Z$ be any two allocations. For a given cost realization (c_i, c_{-i}), define $J_z(c) = \sum_{i=1}^{I} [\pi_i(z, c_i, c_{-i}) + \frac{F_i(c_i)}{f_i(c_i)} \frac{\partial \pi_j(z, c_i, c_{-i})}{\partial c_i}]$.*
If $z_1 \in aPg \max_{z \in Z} J_z(c_i^-, c_{-i})$ and $z_2 \in aPg \max_{z \in Z} J_z(c_i^+, c_{-i})$, then

$$\frac{\partial \pi_i(z_2, c_i, c_{-i})}{\partial c_i} \geq \frac{\partial \pi_i(z_1, c_i, c_{-i})}{\partial c_i}$$

In the case of no externalities, this condition is implied by $J_z(c_i)$ decreasing in c_i, which is the equivalent to the regularity condition in [12].

Based on the previous proposition we can characterize the optimal auction for the regular case:

Proposition 2. *Suppose that assumption 2 is satisfied. Then the optimal allocation $\widehat{p}$ is given by:*

$$\widehat{p}^{z^*}(c) = \begin{cases} 1 & \text{if } z^* \in aPg \max_z \sum_{i=1}^{I} [\pi_i(z, c_i, c_{-i}) + \frac{F_i(c_i)}{f_i(c_i)} \frac{\partial \pi_i(z, c_i, c_{-i})}{\partial c_i}] \\ 0 & \text{if not} \end{cases}$$

when $\max_z \sum_{i=1}^{I} [\pi_i(z, c_i, c_{-i}) + \frac{F_i(c_i)}{f_i(c_i)} \frac{\partial \pi_j(z, c_i, c_{-i})}{\partial c_i}] \geq 0$ and $p^z(c) \equiv 0$ in other case.

[7] In the case of no externalities, this condition is implied by $J_z(c_i)$ decreasing in c_i, which is the equivalent to the regularity condition in [12]

Proof. The solution proposed corresponds to pointwise maximization, so the only possibility that is not optimal is that is not feasible. To check that feasibility is satisfied notice that

$$R(c_i) = \int\limits_{C_{-i}} \sum_{z\in Z} p^z(c_i, c_{-i}) \frac{\partial \pi_i(z, c_i, c_{-i})}{\partial c_i} f_{-i}(c_{-i}) dc_{-i}$$

and consider a fixed c_{-i}. In a region $[\underline{c}, \overline{c}]$ where $\overline{z} \in arg \max_{z\in Z} J_z(c)\ p(c_i, c_{-i})$ does not change ($p^{\overline{z}} = 1$) and $R(c_i)$ is nondecreasing by the convexity of $\pi_i(z, \cdot, c_{-i})$. For a given c^* where $z_1 \in arg \max_{z\in Z} J_z(c_i^-, c_{-i})$ and $z_2 \in arg \max_{z\in Z} J_z c_i^+, c_{-i})$, $p^{z_1}(c_i^{*-}, c_{-i}) = 1$ and $p^{z_2}(c_i^{*+}, c_{-i}) = 1$, so $R(c_i)$ is nondecreasing because of Assumption 2. □

Now we consider the complete model.

4 The multiple markets case

The problem of the seller then becomes:

$$\begin{array}{lll}
\max\limits_{p,x,\rho} & \int\limits_C \sum\limits_{i=1}^{I} x_i(c) f(c) dc & \\
 & U_i(c_i, c_i; (p,x)) \geq U_i(c_i, c_i'; (p,x)), \text{for all } c_i, c_i' \in C_i & \text{(IC)} \\
s.t. & U_i(c_i, c_i; (p,x)) \geq \int\limits_{C_{-i}} \sum\limits_{z\in Z} \rho_i(c_i', c_{-i}) \pi(z, c_i, c_{-i}) f_{-i}(c_{-i}) dc_{-i} & \text{(PC)} \\
 & \sum\limits_{z\in Z} p^z(c) \leq 1, p^z(c) \geq 0 &
\end{array}$$

Before we proceed with our solution a few remarks are in place. In the standard problem without externalities the worst that the seller can do to a buyer is not to assign him the object, hence the worse that the seller can do is to enforce a payoff of zero: the payment function is then

$$x_i(c) = \sum_{z\in Z} \left[(p^z(c_i', c_{-i}) \pi(z, c_i, c_{-i}) - \int_{c_i}^{\bar{c}_i} \left[\sum_{z\in Z} p^z(s, c_{-i}) \frac{\partial \pi(z, s, c_{-i})}{\partial c_i} \right] ds \right] - U_i(\bar{c}_i),$$

where $U_i(\bar{c}_i)$ is determined by the fact that the buyer always has the option not to participate which implies that

$$U_i(\bar{c}_i) = 0.$$

In [6], where the externalities are type independent, the seller can threaten buyer i that she will assign the object to the buyer that imposes the "worst" externality on buyer i in the event he does not participate. Suppose that this buyer is j, then the seller chooses (for buyer i), the buyer who imposes the worst externality on him

$$j^*(i) \in aPg \max_{j \in I} -\alpha_j(i).$$

Now using as a threat that the object will be assigned to $j^*(i)$ in the event that i does not participate, the seller not only gets i to participate but also extracts higher payments from him: in this case we have that

$$x_i(c) = \sum_{z \in Z} (p^z(c_i', c_{-i})\pi(z, c_i, c_{-i}) - \int_{c_i}^{\bar{c}_i} \left[\sum_{z \in Z} p^z(s, c_{-i}) \frac{\partial \pi(z, s, c_{-i})}{\partial c_i} \right] ds - V_i(\bar{c}_i),$$

where

$$V_i(\bar{c}_i) = -\alpha_j(i).$$

Hence compared to the no-externality the seller can extract higher payments. The difference in the payments is $-\alpha_j(i)$. Notice that the threat is independent of the mechanism.

Now, in our environment where there are many objects to be allocated and many ways that the seller can bundle the objects, there are many potential allocations that can be used for punishments. An optimal punishment depends on the mechanism that the seller wants to implement, on the identity of the buyer that the seller wants to "punish" and on his type. Let us examine what is going on step-by-step: A given allocation rule p determines up to a constant the expected payoff for each type of a buyer, which is given by the familiar expression

$$V_i(c_i) = V_i(\bar{c}_i) - \int_{c_i}^{\bar{c}_i} \int_{C_{-i}} \left[\sum_{z \in Z} p^z(s, c_{-i}) \frac{\partial \pi(z, s, c_{-i})}{\partial c_i} \right] f_{-i}(c_{-i}) dc_{-i} ds,$$

the constant $V_i(\bar{c}_i)$ is determined by the optimal threat that the seller can design. Let us call $\hat{V}(c_i)$ the payoff of type c_i of buyer i net of the constant that is

$$\hat{V}_i(c_i) = - \int_{c_i}^{\bar{c}_i} \int_{C_{-i}} \left[\sum_{z \in Z} p^z(s, c_{-i}) \frac{\partial \pi(z, s, c_{-i})}{\partial c_i} \right] f_{-i}(c_{-i}) dc_{-i} ds,$$

which is, as we have shown, a decreasing and convex function of c_i. Now for each such expression there exists a "worst punishment". The worst punishment is identified in two steps. First given an allocation rule p which determines $\hat{V}(c_i)$ and given an allocation $z$we determine the critical type. Define $\overline{\pi}_i(z, c_i) = \int\limits_{C_{-i}}]\pi(z, c_i, c_{-i})dc_{-i}$, the expected payoff given an allocation z for agent i if his type is c_i. Suppose that the seller is contemplating what would be the largest constant that he could reduce $i's$ payoff by given a proposed allocation p and a threat z : this constant is going to be determined by the type where $\hat{V}_i$ would hit π_i first if we were to shift it down, we call this type $c_i^*(z)$. Now the constant by which the seller can reduce i's payoff is given by the difference

$$\hat{V}_i(c_i^*(z)) - \pi_i(c_i^*(z), z)$$

Formally $c_i^*(z)$ solves the following program:

$$c^*(z) \in \arg\min_{c_i} \left[\hat{V}_i(c_i) - \overline{\pi}_i(z, c_i)\right]$$

Given the convexity of $V(\cdot)$ and $\overline{\pi}_i(\cdot, z)$, the problem can be written as:

$$\begin{aligned} c^*(z) &\in \arg\min_{c_i} \left[\hat{V}_i(c_i) - \overline{\pi}_i(z, c_i)\right] \\ s.t. \overline{\pi}_i'(c_i, z) &\in \partial V(c_i). \end{aligned}$$

This characterization, even if it looks more difficult, is extremely useful when the "threat" functions $\overline{\pi}_i(\cdot, z)$ are linear, since the set of types where $\overline{\pi}_i'(c_i, z) \in \partial V(c_i)$ is a singleton. Suppose for example that $\overline{\pi}_i(\cdot, z) = a_z c_i + b_z$. Then

$$c^*(z) = \begin{cases} \underline{c} & \mathbf{if} \, P(\underline{c}) \geq a_z \\ \overline{c} & \mathbf{if} \, P(\overline{c}) \leq a_z \\ P^{-1}(a_z) & \mathbf{otherwise} \end{cases}$$

After finding the critical type for a particular "threat" z, we can compute the value of $V(\overline{c})$, since from the characterization of incentive compatible mechanisms we have that $V(\overline{c}) = V(c^*(z)) + \int\limits_{c^*(z)}^{\overline{c}} P(s)ds$, so we can write $V(\overline{c}) = \overline{\pi}(c^*(z), z) + \int\limits_{c^*(z)}^{\overline{c}} P(s)ds$.

The next step for the seller is to find the worst threat, which solves the following program

$$z^*(p) \in \min_z \pi_i(c^*(z), z) + \int_{c^*(z)}^{\bar{c}} P(s)ds$$

and the optimal threat is thus

$$\rho^z(s) = 1 \text{ for } z = z^*(p).$$

The objective function of the seller can now be written as:

$$\int_C \sum_{z\in Z} p^z(c_i, c_{-i}) \sum_{i\in I} [\pi_i(z, c_i, c_{-i}) + \frac{F_i(c_i)}{f_i(c_i)} \frac{\partial \pi_i(z, c_i, c_{-i})}{\partial c_i}] f(c)dc - \sum_{i=1}^{I} V_i(\bar{c}_i)$$

where $V_i(\bar{c}_i)$ explicitly shows that the payments that can be extracted from the firms depend on the assignment function, by

$$V_i(\bar{c}_i) = \pi_i(c^*(z)) + \int_{c^*(z)}^{\bar{c}} P(s)ds$$

The two previous lemmas allow us to fully characterize the problem in terms of the assignment function p:

Proposition 3. *If in a mechanism $(\widehat{p}, \widehat{x})$ the assignment function $\widehat{p}$ solves:*

$$\begin{array}{ll} \max\limits_p & \int\limits_C \sum\limits_{z\in Z} p^z(c) \sum\limits_{i=1}^{I} \left[\pi_i(z, c_i, c_{-i}) + \frac{F_i(c_i)}{f_i(c_i)} \frac{\partial \pi_i(z, c_i, c_{-i})}{\partial c_i}\right] f(c)dc - \sum\limits_{i=1}^{I} V_i(\bar{c}_i) \\ s.t. & P_i \text{ increasing}, \sum\limits_{z\in Z} p^z(c) \leq 1 \text{ and } p(c) \geq 0 \end{array}$$

and the payment function $\widehat{x}$ satisfies:

$$\widehat{x}_i(c) = \sum_{z\in Z} \widehat{p}^z(c)\pi_i(z, c_i, c_{-i}) + \int\limits_{c_i}^{\overline{c_i}} \sum_{z\in Z} \frac{\partial \pi_i(z, s, c_{-i})}{\partial s} \widehat{p}^z(s, c_{-i})g(s)ds - V_i(\bar{c}_i),$$

where

$$V_i(\bar{c}_i) = \int_{C_{-i}} \left[\pi_i(c^*(z^*)) + \int_{c^*(z^8)}^{\bar{c}} P(s)ds\right] f_{-i}(v_{-i})dv_{-i},$$

and where in turn $c_i^*(z^*)$ solves the following program: For any given z we find the "critical type" which depends on p via $V_i(c_i)$

$$c^*(z) \in \arg\min_{c_i} \left[\hat{V}_i(c_i) - \pi_i(z, c_i)\right]$$

and then we find the optimal z

$$z^*(p) \in \min_{z} \pi_i(c^*(z)) + \int_{c^*(z)}^{\bar{c}} P(s)ds,$$

and the optimal threat is given by:

$$\rho^z(s) = 1 \text{ for } z = z^*(p),$$

then the mechanism is optimal.

This program is not anymore linear in p, and as we demonstrate below via an example, the usual approach of maximizing the objective function pointwise ignoring the constraint set will in our case lead to infeasible mechanisms. Fortunately the problem has enough structure[8]to allow the use of variational methods without imposing any restrictions, such as differentiability on the mechanism.

[8]In particular enough differentiability on π will guarantee enough regularity on $c^*(z)$ as a function of the mechanism.

5 Example

As an illustration of our pervious analysis we present an example. Consider 2 firms fighting for a single slot to advertise their products. The value of actually airing a spot depends on the actual cost c_i of the firm, which is private information. The cost is uniformly and independently distributed in $[0, 1]$. We denote by $z = 0$ the allocation when the object is not sold and $z = i$ the allocation when the object is given to agent i.

5.1 No externalities

Suppose that firms care only about getting the object. This case is one that can be just solved as in [12]. For example, profit functions for agent 1 are given by:

$$\begin{aligned} \pi_1(0, c_1, c_2) &= 0 \\ \pi_1(1, c_1, c_2) &= 1 - c_1 \\ \pi_1(2, c_1, c_2) &= 0 \end{aligned}$$

and for agent 2 are given by:

$$\begin{aligned} \pi_2(0, c_1, c_2) &= 0 \\ \pi_2(1, c_1, c_2) &= 0 \\ \pi_2(2, c_1, c_2) &= 1 - c_2 \end{aligned}$$

In this case the virtual valuations are

$$\begin{aligned} J_1(c_1, c_2) &= \sum_{i=1}^{2} [\pi_i(1, c_i, c_{-i}) + \frac{F_i(c_i)}{f_i(c_i)} \frac{\partial \pi_i(1, c_i, c_{-i})}{\partial c_i}] \\ &= 1 - 2c_1 \end{aligned}$$

Analogously we get $J_2(c_1, c_2) = 1 - 2c_2$ and $J_0(c_1, c_2) = 0$.
The solution is then given by:

$$\begin{array}{ll} p^1(c) = 1 & \textbf{if } c_1 \leq c_2 \textbf{ and } c_1 \leq \frac{1}{2} \\ p^2(c) = 1 & \textbf{if } c_1 \geq c_2 \textbf{ and } c_2 \leq \frac{1}{2} \\ 0 & \quad \textbf{otherwise} \end{array}$$

This assignation is illustrated in figure 2.

5.2 Type Independent Externalities

Now, let's suppose that firms also care about the competitor not getting the advertisement slot. But suppose that the cost of a competitor winning the auction is independent of their own cost. For example, profit functions for agent 1 are given by:

$$\begin{aligned}\pi_1(0, c_1, c_2) &= 0 \\ \pi_1(1, c_1, c_2) &= 1 - c_1 \\ \pi_1(2, c_1, c_2) &= -\alpha\end{aligned}$$

and for agent 2 are given by:

$$\begin{aligned}\pi_2(0, c_1, c_2) &= 0 \\ \pi_2(1, c_1, c_2) &= -\alpha \\ \pi_2(2, c_1, c_2) &= 1 - c_2\end{aligned}$$

The optimal allocation is exactly the same as in the previous case, that is:

$$\begin{array}{ll} p^1(c) = 1 & \textbf{if } c_1 \leq c_2 \textbf{ and } c_1 \leq \frac{1}{2} \\ p^2(c) = 1 & \textbf{if } c_1 \geq c_2 \textbf{ and } c_2 \leq \frac{1}{2} \\ 0 & \textbf{otherwise} \end{array}$$

The only difference is in the payments. Now the seller can extract an extra payment of α from each bidder.

5.3 Type Dependent Externalities

Now, let's suppose that firms also care about the competitor not getting the advertisement slot. Even more, the cost of a competitor winning the auction is higher when their own cost realization is higher. For example, profit functions for agent 1 are given by:

$$\begin{aligned}\pi_1(0, c_1, c_2) &= 0 \\ \pi_1(1, c_1, c_2) &= 1 - c_1 \\ \pi_1(2, c_1, c_2) &= -\alpha c_1\end{aligned}$$

and for agent 2 are given by:

$$\begin{aligned}
\pi_2(0, c_1, c_2) &= 0 \\
\pi_2(1, c_1, c_2) &= -\alpha c_2 \\
\pi_2(2, c_1, c_2) &= 1 - c_2
\end{aligned}$$

With this we can write the virtual valuations associated to each allocation:

$$\begin{aligned}
J_1(c_1, c_2) &= \sum_{i=1}^{2} [\pi_i(1, c_i, c_{-i}) + \frac{F_i(c_i)}{f_i(c_i)} \frac{\partial \pi_i(1, c_i, c_{-i})}{\partial c_i}] \\
&= (1 - c_1 - c_1) + (-\alpha c_2 - \alpha c_2) \\
&= 1 - 2c_1 - 2\alpha c_2
\end{aligned}$$

Analogously we get:

$$\begin{aligned}
J_2(c_1, c_2) &= 1 - 2c_2 - 2\alpha c_1 \\
J_0(c_1, c_2) &= 0
\end{aligned}$$

The seller's problem can be written as:

$$\begin{aligned}
\max_{p} \quad & \int_{[0,1]} \int_{[0,1]} [p^1(c)[1 - 2c_1 - 2\alpha c_2] + p^2(c)[1 - 2c_2 - 2\alpha c_1]] dc_1 dc_2 - V_1(1) - V_2(1) \\
s.t. \quad & \int [p^1(c)[-1] + p^2(c)[-\alpha]] dc_2 \textbf{ is nondecreasing} \\
& \int [p^1(c)[-\alpha] + p^2(c)[-1]] dc_1 \textbf{ is nondecreasing} \\
& p_1(c) + p_2(c) \leq 1
\end{aligned}$$

5.4 A "Naive" Solution

If we do not consider the terms $V_1(1)$ and $V_2(1)$ at all, we can try pointwise maximization, which gives us a solution of the form:

$$\begin{array}{ll}
p^1(c) = 1 & \textbf{if } 1 - 2c_1 - 2\alpha c_2 \geq 1 - 2c_2 - 2\alpha c_1 \textbf{ and } 1 - 2c_1 - 2\alpha c_2 \geq 0 \\
p^2(c) = 1 & \textbf{if } 1 - 2c_2 - 2\alpha c_1 \geq 1 - 2c_1 - 2\alpha c_2 \textbf{ and } 1 - 2c_2 - 2\alpha c_1 \geq 0 \\
0 & \textbf{otherwise}
\end{array}$$

For the moment being, we concentrate on the case $\alpha < 1$, and we rewrite the mechanism as:

$$\begin{array}{ll} p^1(c) = 1 & \textbf{if } c_1 \leq c_2 \textbf{ and } 1 - 2c_1 - 2\alpha c_2 \geq 0 \\ p^2(c) = 1 & \textbf{if } c_2 \leq c_1 \textbf{ and } 1 - 2c_2 - 2\alpha c_1 \geq 0 \\ 0 & \textbf{otherwise} \end{array}$$

This allocation can be seen in figures 3 and 4. Feasibility is satisfied since for a fixed $\overline{c}_2$ the function $c_1 \longrightarrow -p_1(c_1, \overline{c}_2) - \alpha p_2(c_1, \overline{c}_2)$ is nondecreasing. The same is true for a fixed $\overline{c}_1$ and the function $c_2 \longrightarrow -\alpha p_1(\overline{c}_1, c_2) - p_2(\overline{c}_1, c_2)$ (see figures 3 and 4).

What is the value of this solution? We will directly compute it for the case when $\alpha < \frac{1}{2}$:

$$P = \int_0^1 \int_0^1 p_1^*(c) J_1(c) + \int_0^1 \int_0^1 p_2^*(c) J_2(c) - V_1(1) - V_2(1)$$

Claim: the critical type satisfies $c_1^* \in \left[\frac{1-2\alpha}{2}, \frac{1}{2(1+\alpha)}\right]$ and is given by $c_1^* = \frac{1-2\alpha^2}{2(1+\alpha-\alpha^2)}$

Proof. Notice that in this interval $P(c_1) = (1-\alpha)c_1 + \frac{2c_1-1}{2\alpha}$, setting it equal to $-\alpha$ the result follows. Now it is left to check that $c_1^* \in \left[\frac{1-2\alpha}{2}, \frac{1}{2(1+\alpha)}\right]$. For that:

$$\begin{aligned} c_1^* &\geq \frac{1-2\alpha}{2} \\ \frac{1-2\alpha^2}{2(1+\alpha-\alpha^2)} &\geq \frac{1-2\alpha}{2} \\ 1-2\alpha^2 &\geq 1-2\alpha+\alpha-2\alpha^2-\alpha^2+2\alpha^3 \\ 2\alpha^2-\alpha-1 &\leq 0 \end{aligned}$$

Which is always true since solving the quadratic equation we get that the critical types are $\alpha = \frac{1+\sqrt{9}}{4}$ which implies $\alpha_1 = -\frac{1}{2}$ and $\alpha_2 = 1$.
Note: $1+\alpha-\alpha^2$ is always positive in the interval, so the step between the second and the third line is well justified.

And:

$$\begin{aligned}
c_1^* &\leq \frac{1}{2(1+\alpha)} \\
\frac{1-2\alpha^2}{2(1+\alpha-\alpha^2)} &\leq \frac{1}{2(1+\alpha)} \\
1-2\alpha^2+\alpha-2\alpha^3 &\leq 1+\alpha-\alpha^2 \\
-2-\alpha &\leq -1 \\
\alpha &\geq -1
\end{aligned}$$

which is also always true. $\square$

Now, noticing that $V_1(1) = -\alpha c_1^* + \int\limits_{c_1^*}^{1} P(c_1) dc_1$ we can compute:

$$\begin{aligned}
\int\limits_{c_1^*}^{1} P(c_1) dc_1 &= \int\limits_{c_1^*}^{\frac{1}{2(1+\alpha)}} \left[(1-\alpha)c_1 + \frac{2c_1 - 1}{2\alpha}\right] dc_1 + \int\limits_{\frac{1}{2(1+\alpha)}}^{1} \left[c_1 - \frac{1}{2}\right] dc_1 \\
&= \int\limits_{c_1^*}^{\frac{1}{2(1+\alpha)}} \left[\left[1 - \alpha + \frac{1}{\alpha}\right] c_1 - \frac{1}{2\alpha}\right] dc_1 + \int\limits_{\frac{1}{2(1+\alpha)}}^{1} \left[c_1 - \frac{1}{2}\right] dc_1 \\
&= \left[1 - \alpha + \frac{1}{\alpha}\right] \left[\frac{1}{8(1+\alpha^2)} - \frac{c_1^{*2}}{2}\right] - \frac{1}{4\alpha(1+\alpha)} + \frac{c_1^*}{2\alpha} + \frac{1}{2} - \frac{1}{8(1+\alpha)^2} - \frac{1}{2} + \frac{1}{4(1+\alpha} \\
&= \left[-\alpha + \frac{1}{\alpha}\right] \frac{1}{8(1+\alpha)^2} - \left[1 - \alpha + \frac{1}{\alpha}\right] \frac{c_1^{*2}}{2} + \frac{\alpha - 1}{4\alpha(1+\alpha)} + \frac{c_1^*}{2\alpha} \\
&= \frac{1-\alpha}{8\alpha(1+\alpha)} + \frac{\alpha - 1}{4\alpha(1+\alpha)} - \left[1 - \alpha + \frac{1}{\alpha}\right] \frac{c_1^{*2}}{2} + \frac{c_1^*}{2\alpha} \\
&= \frac{\alpha - 1}{8\alpha(1+\alpha)} \left[1 - \alpha + \frac{1}{\alpha}\right] \frac{c_1^{*2}}{2} + \frac{c_1^*}{2\alpha} \\
&= \frac{1}{2\alpha} \left[\frac{\alpha - 1}{4(1+\alpha)} - (1 + \alpha - \alpha^2) c_1^{*2} + c_1^*\right] \\
&= \frac{1}{2\alpha} \left[\frac{\alpha - 1}{4(1+\alpha)} + c_1^* \left[-(1 + \alpha - \alpha^2) c_1^* + 1\right]\right] \\
&= \frac{1}{2\alpha} \left[\frac{\alpha - 1}{4(1+\alpha)} + c_1^* \left[\frac{1 - 2\alpha^2}{2} + 1\right]\right] \\
&= \frac{1}{2\alpha} \left[\frac{\alpha - 1}{4(1+\alpha)} + \frac{(2\alpha^2 - 1)(1 + 2\alpha^2)}{4(1 + \alpha - \alpha^2)}\right]
\end{aligned}$$

That gives us the expression:

$$V_1(1) = -\alpha \frac{1 - 2\alpha^2}{2(1 + \alpha - \alpha^2)} + \frac{1}{2\alpha} \left[\frac{\alpha - 1}{4(1+\alpha)} + \frac{(2\alpha^2 - 1)(1 + 2\alpha^2)}{4(1 + \alpha - \alpha^2)}\right]$$

By symmetry we get that $V_1(1) = V_2(1)$.

Now for the rest of the expression:

$$
\begin{aligned}
S_1 &= \int_0^1\int_0^1 p_1(c)[1-2c_1-2\alpha c_2]dc \\
&= \int_0^{\frac{1-2\alpha}{2}} [\int_{c_1}^1 [1-2c_1-2\alpha c_2]dc_2]dc_1 + \int_{\frac{1-2\alpha}{2}}^{\frac{1}{2(1+\alpha)}} [\int_{c_1}^{\frac{1-2c_1}{2\alpha}} [1-2c_1-2\alpha c_2]dc_2]dc_1 \\
&= \int_0^{\frac{1-2\alpha}{2}} [(1-2c_1)(1-c_1)-\alpha+\alpha c_1^2]dc_1 + \int_{c_1}^{\frac{1-2c_1}{2\alpha}} [(1-2c_1)(\frac{1-2c_1}{2\alpha}-c_1)-\alpha\frac{(1-2c_1)^2}{4\alpha^2}+\alpha c_1^2]dc_1 \\
&= \int_0^{\frac{1-2\alpha}{2}} [(1-\alpha)-3c_1+(2+\alpha)c_1^2]dc_1 + \int_{c_1}^{\frac{1-2c_1}{2\alpha}} [\frac{(1-2c_1)^2}{2\alpha}-c_1+(2+\alpha)c_1^2]dc_1 \\
&= \frac{(1-\alpha)(1-2\alpha)}{2}-\frac{3(1-2\alpha)^2}{8}+\frac{(2+\alpha)(1-2\alpha)^3}{24} \\
&\quad + \int_{c_1}^{\frac{1-2c_1}{2\alpha}} [\frac{(1-2c_1)^2}{2\alpha}-c_1+(2+\alpha)c_1^2]dc_1 \\
&= \frac{1-2\alpha}{2}\left[\frac{1}{4}+\frac{\alpha}{2}+\frac{(2+\alpha)(1-2\alpha)^2}{12}\right] + \int_{c_1}^{\frac{1-2c_1}{2\alpha}} [\frac{1}{2\alpha}-\frac{(2+\alpha)c_1}{\alpha}+\frac{(2+2\alpha+\alpha^2)c_1^2}{\alpha}]dc_1 \\
&= \frac{1-2\alpha}{2}\left[\frac{1}{4}+\frac{\alpha}{2}+\frac{(2+\alpha)(1-2\alpha)^2}{12}\right] \\
&\quad -\frac{1+2\alpha}{4(1+\alpha)}+\frac{2+\alpha}{8\alpha}[\frac{1}{(1+\alpha)^2}-(1-2\alpha)^2]+\frac{2+2\alpha+\alpha^2}{24\alpha}[\frac{1}{(1-\alpha)^3}-(1+2\alpha)^3]
\end{aligned}
$$

By the symmetry of the problem we have that $S_2 = \int_0^1\int_0^1 p_2^*(c)J_2(c) = S_1$.

[To be continued.]

Figure 1

The computation of $c_i*(z)$

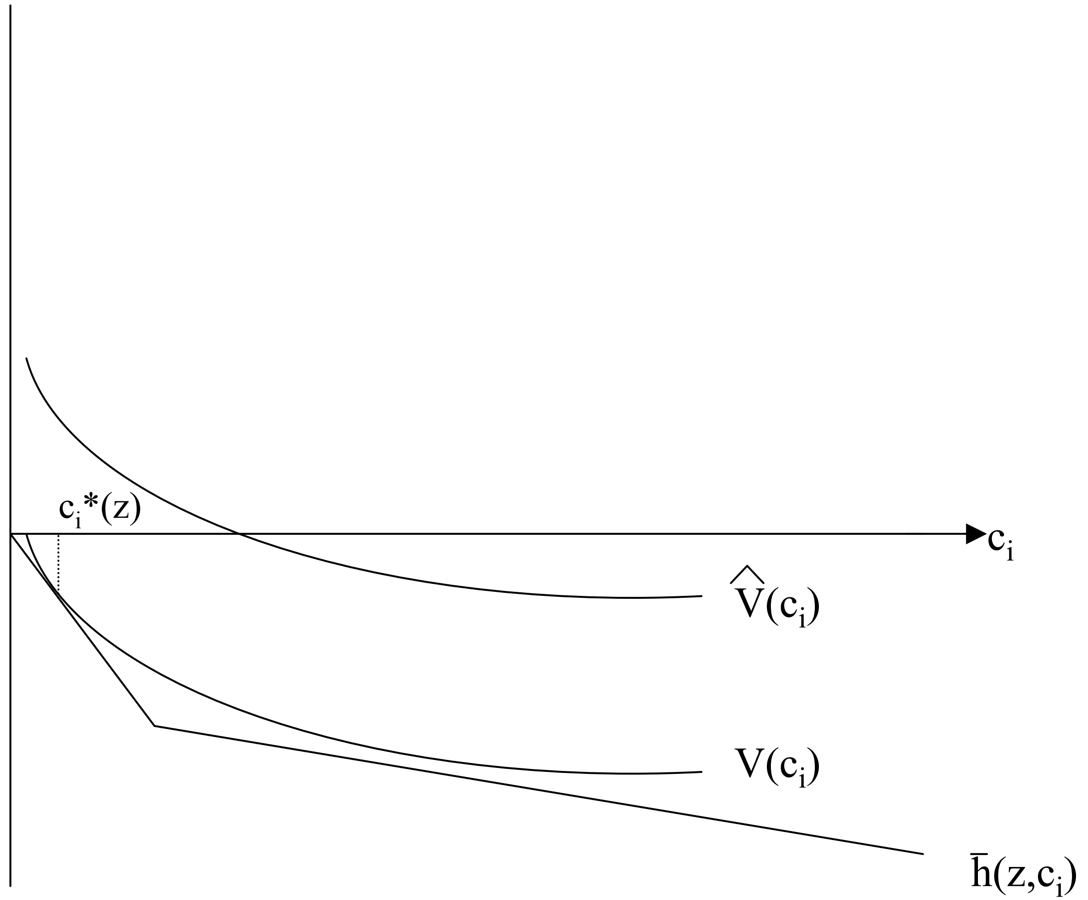


Figure 2

The no externality case

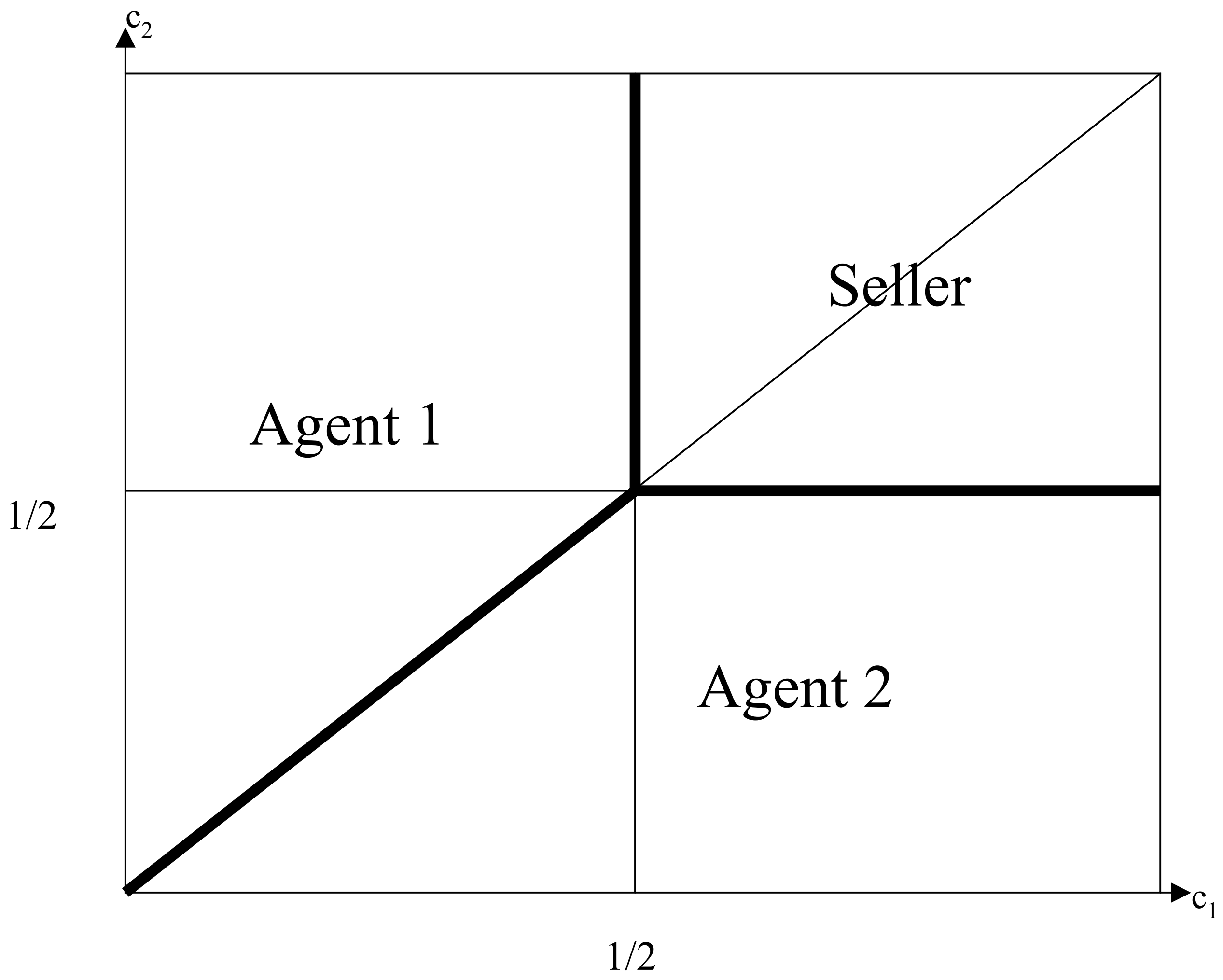


Figure 3

$1>\alpha>1/2$

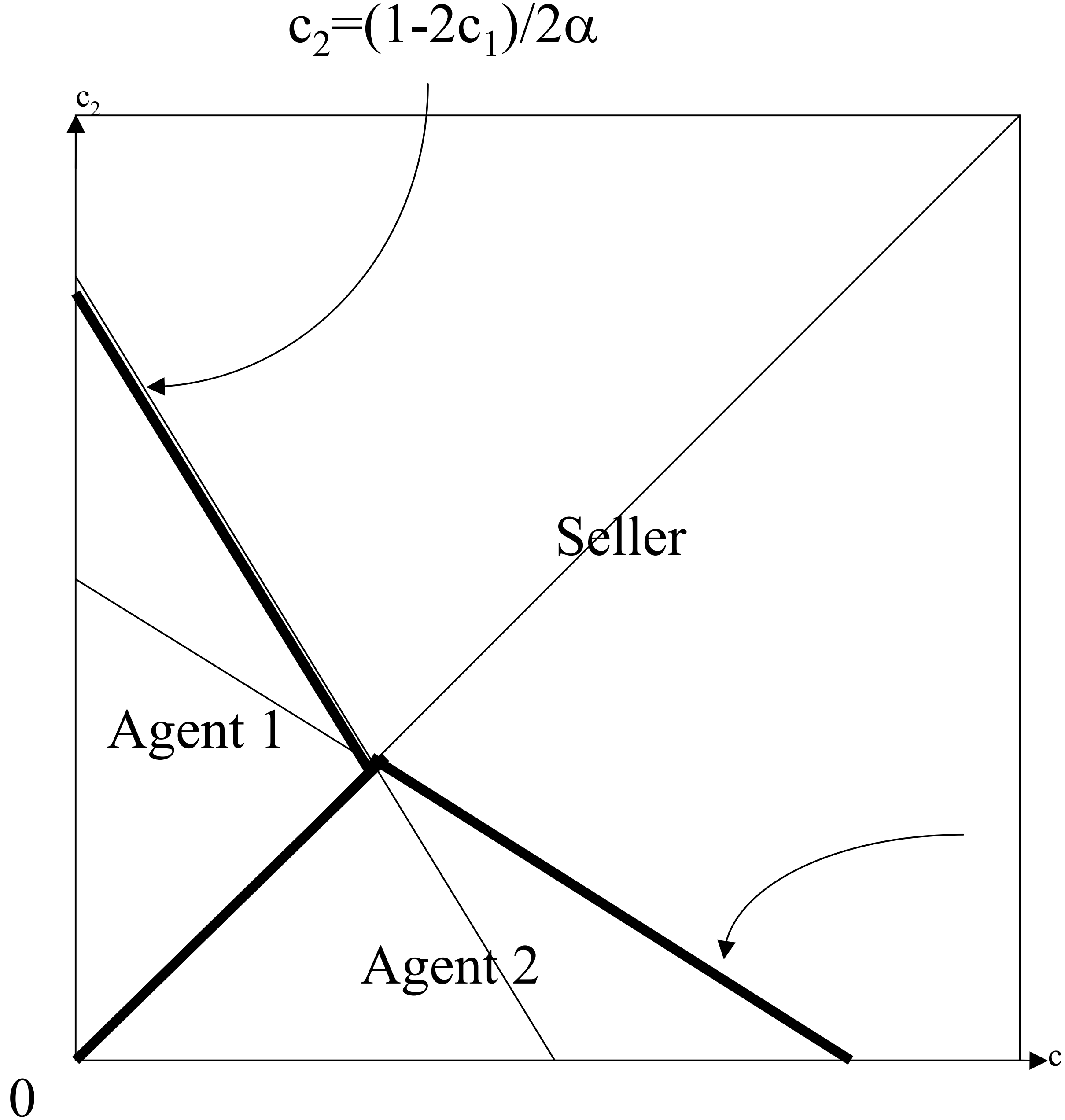
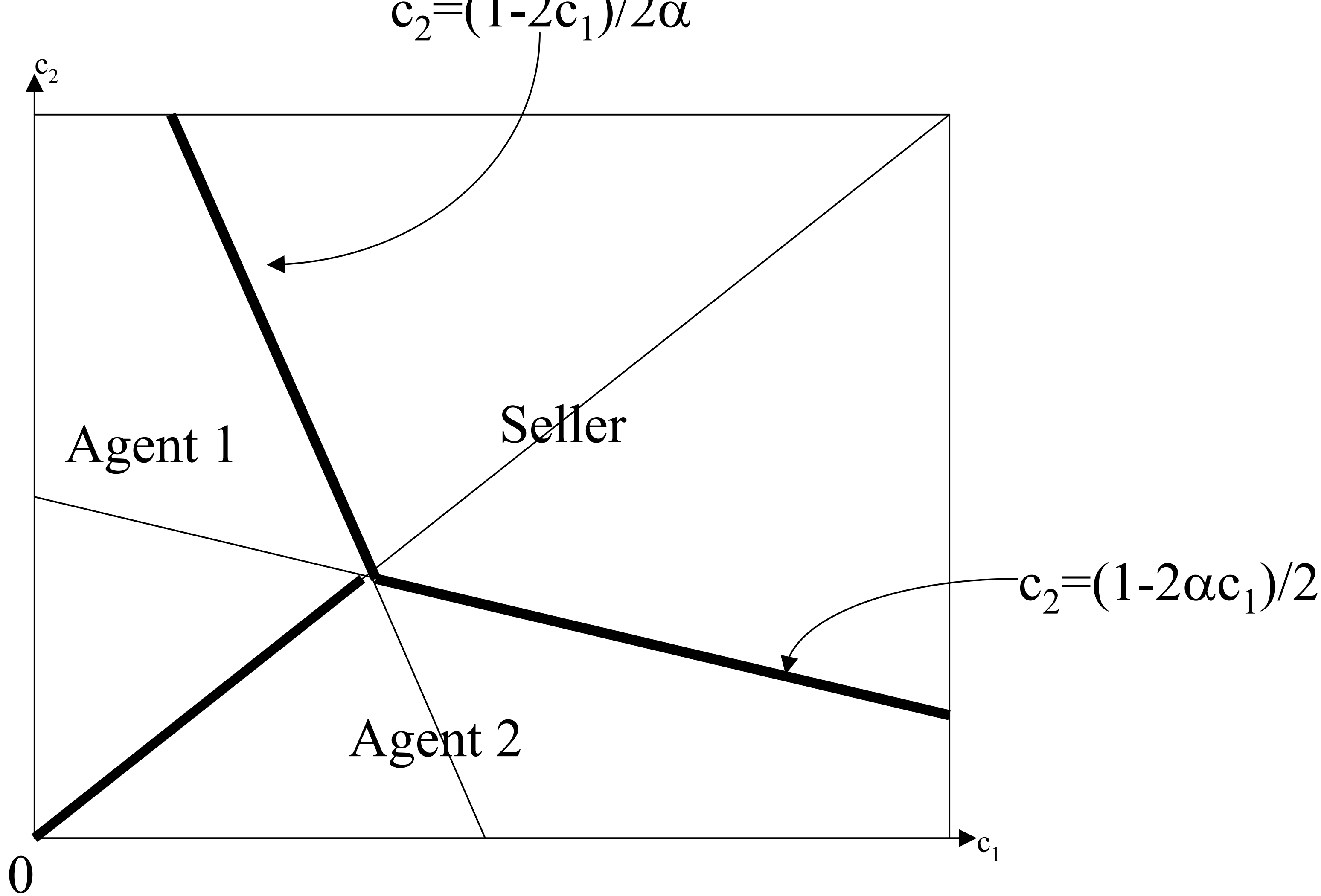


Figure 4
$1>\alpha>1/2$



References

[1] M. Armstrong. Optimal multi-object auctions. *Review of Economic Studies*, (67):455–181, 2000.

[2] J. Dana, Jr and K. Spier. Designing an industry. government auctions with endogenous market structure. *Journal of Public Economics*, (53):127–147, 1994.

[3] A. Dixit. Power of incentives in government bureaucracies. *American Economic Review*, 2(87):378–382, 1997.

[4] I. Gale. A multiple-object auction with superadditive values. *Economic Letters*, (34):323–328, 1990.

[5] P. Jehiel and B. Moldovanu. Efficient design with interdependent values. *Econometrica*, (69):1237–1259, 2001.

[6] P. Jehiel, B. Moldovanu, and E. Stacchetti. How (not) to sell nuclear weapons. *The American Economic Review*, 86.

[7] P. Jehiel, B. Moldovanu, and E. Stacchetti. Multidimensional mechanism design for auctions with externalities. *Journal of Economic Theory*, (85):258–293, 2001.

[8] V. Krishna and E. Maenner. A path independence result for convex functions with an application to mechanism design. *Mimeo Penn State University*, 1998.

[9] V. Krishna and M. Perry. Efficient mechanism design. *Mimeo Penn State University*, 2001.

[10] E. Maskin and J. Riley. Optimal multi-unit auctions. In F. Hahn, editor, *The Economics of Missing Markets, Information and Games*, pages 312–335, 1989.

[11] P. Milgrom. Procuring universal service: Putting auction theory to work. Lecture at the Royal Academy of Sciences, 1996.

[12] R. Myerson. Optimal auction design. *Mathematics of Operations Research*, 6(1):58–73, 1981.