Obsolescence of Durable Goods and Optimal Consumption*

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April 9, 2004

Abstract

We study a model with a durable good subject to abrupt, periodic obsolescence, and characterize the optimal purchasing policy. Consumers optimally synchronize new purchases with the arrival of new durable models. Hence, some agents use a "flexible" optimal replacement rule that switches between two adjacent replacement frequencies at irregular intervals. These agents react to wealth shocks by changing the timing of future purchases.

The model has distinct comparative statics on obsolescence and durability and can explain how durables with high depreciation rates may have more volatile expenditure. The model also predicts how demand fluctuations respond to a change in product variety. These predictions match the observed changes in volatility of the US auto sales after the introduction of smaller foreign cars in the 1970s.

1 Introduction

Obsolescence is the major reason for depreciation of durables in markets with technological innovation.[1] Since much of this innovation is incorporated in new durables, modeling obsolescence of durable goods is vital for our understanding of macroeconomic effects.

Depreciation is usually modeled as gradual wear and tear, but obsolescence is different in two important respects. First, obsolescence affects all durables at the same time. For example, all analog TVs, no matter how new, will depreciate at the same time when the broadcasting switches to digital format (HDTV). Second, obsolescence does not happen at a constant rate; rather, it is periodic and abrupt. One reason for this is costly development of new products.[2] For example, car bodies are redesigned every 4-5 years, and new generations of Intel processors appear, on average, every 3 years. Another reason is that obsolescence is

*This material is based upon work supported by the National Science Foundation under Grant No. 0241640. Any opinions, findings and conclusions or recomendations expressed in this material are those of the authors and do not necessarily reflect the views of the National Science Foundation (NSF).

[1] One of the most dramatic examples of obsolescence is computers, whose quality-adjusted price has been falling at an average rate of 23.5% a year during 1960-2000. The average annual obsolescence rate for communication equipment is 8.7%, and for automobiles is 2.5% (Cummins and Violante, 2002, Table II).

[2] Fishman and Rob (2000) show that it is optimal for the durable goods producer to introduce new models periodically.

related to the periodic arrival of innovations shared by many goods, such as, for example, the LCD display, the compact disc or the lithium-ion battery. This means that obsolescence is a function of the good's *technological*, rather than physical, age. Computers and many other types of equipment have the same pattern of obsolescence. Since designs for durables start aging immediately, consumers who purchase a certain model late in its life cycle will enjoy a lower service flow. Hence, obsolescence gives consumers an incentive to coordinate their replacement decisions with the introductions of new models. This coordination may have important effects on demand fluctuations and propagation of shocks.

We consider an economy with infinitely lived agents who consume a durable and a non-durable good. Agents differ in their permanent income level, and can borrow and lend at an exogenously given interest rate. The durable is produced by a competitive industry with CRS technology. There are no secondary markets for used durables; units that are replaced are thrown away. Consequently, durables are purchased infrequently because the service from a current unit acts as a fixed opportunity cost of adjustment.

We solve analytically for the optimal consumption paths of individuals. Consumers optimally synchronize their new durable purchases with the design cycle. Although durables can be replaced at any time, consumers only purchase them at dates when new models are introduced. That is, agents only choose holding periods that are multiples of the design cycle length.[3] Since the relevant choices of holding periods are discrete, the consumers smooth consumption by alternating between two holding periods from time to time.

Consumers endogenously partition themselves into classes according to their wealth and the age of their durable goods, with each class following a different durable replacement rule. Two types of rules are optimal. One type, which we term a "fixed" rule, is an (s, S) policy with a constant replacement frequency. The other type is a "flexible" rule that alternates between two adjacent fixed rules at irregular intervals.

A key difference between the two types of rules is how the agents react to unexpected changes in wealth. Consumers that follow a fixed rule adjust *only* their non-durable consumption in response to a marginal windfall. By contrast, consumers that follow flexible rules adjust *only* the timing of the durable purchases. This dichotomy in response to shocks gives our model the flexibility to match several empirical regularities in consumption.

Empirically, aggregate durable consumption is more volatile than non-durable consumption (e.g. Attanasio (1999), p. 746). The usual explanation for this regularity is that durables have more volatile consumption because they have a lower depreciation rate than non-durables. But, this logic also predicts that consumption of durables with *lower* depreciation rate should exhibit *higher* volatility. The empirical evidence seems to contradict this prediction. We compare the consumption of furniture, computers and cars, and find that consumption is *more* volatile for durables with *higher* rates of economic depreciation. Our model can match both pieces of evidence because it predicts separate effects for obsolescence and durability. We show that the total mass of consumers in flexible rule classes grows with the length of the good's life and with the rate of obsolescence. Accordingly, our model predicts that consumption of longer-lived durables is more variable, and that consumption of durables with higher obsolescence rate is also more variable. The former prediction agrees with the evidence on aggregate consumption, and the latter one helps explain the data on different durable markets.

[3]This strong form of synchronization is partly due to perfect competition which makes the relative price of durables constant and uncorrelated with demand. If prices of new models were falling over time, some consumers would have incentives to buy in the middle of the design cycle.

According to Caballero (1990), aggregate durable purchases are slow to respond to aggregate wealth shocks. This effect takes a very strong form in our model because no one buys durables in the middle of the design cycle. All changes in durable expenditure are delayed until the next model is introduced. The evidence on this delay comes from household purchases of computers. In particular, computer expenditure exhibits sharp accelerations (and subsequent fast fall-offs) around the dates when new Intel processors became available. This feature of aggregate demand is consistent with households waiting to buy or replace their computer until the new model arrives.

In our model, durable demand fluctuates because consumers accelerate or delay durable purchases as a response to wealth shocks. Empirically, the number of automobiles purchased is much more variable than expenditure per auto (Bar-Ilan and Blinder, 1992, p. 263), and very little of this variation is due to the echo effects from past automobile sales (Adda and Cooper, 2000). It is difficult for an (S, s) model with an unconstrained menu of durable sizes to match this observation. In the (s, S) model all consumers choose the same replacement frequency and vary the purchase size to smooth consumption.[4] This means that all fluctuations in the demand for cars must be a result of echo effects from past sales, and not because of shifts in purchase timing. We show how this prediction can be reversed. We extend our model to include the choice of purchase size from some feasible interval. In the optimum, there is a group of consumers who are constrained by purchase size. These agents react to wealth shocks as if there was only one size available and adjust either non-durable consumption or purchase timing. In contrast, a consumer that is unconstrained by size does not modify the timing of future purchases and adjusts the size. The total mass of unconstrained consumers grows with the range of available sizes. Therefore, one should expect purchase timing to respond to shocks less and purchase size to respond more in markets with more variety of sizes. The history of the auto industry in the US offers a natural test. The rapid penetration of smaller foreign autos on the US market in the 1970s can be viewed as an increase in variety. In line with our predictions, we find that the number of new autos per adult became less volatile and the purchase size became more volatile in the 1980s and 90s.

Our work is related to a large literature that studies models with infrequent replacement of durable goods. Most of this literature considers optimal (s, S) replacement policies. There are three broad categories of related (s, S) models. The first category includes representative agent models with a budget constraint (e.g. Grossman and Laroque (1990), Eberly (1994)). These models have only one good, the durable, and thus look at durable consumption separately. The second category includes replacement models with aggregate dynamics (e.g. Caballero and Engel (1999), Caplin and Leahy (1999), Adda and Cooper (2000)). These papers consider a replacement problem without an inter-period budget constraint. The model of Adda and Cooper (2000) includes durables and non-durables, but does not allow borrowing and lending. The third body of literature (e.g. Caballero (1993), Attanasio (2000)) does not consider the optimal replacement problem but *assumes* that the optimal replacement policy for the durable is an (s, S) rule. Apart from having a different model, we develop a solution methodology that can be used in other replacement problems with indivisibilities.

Our work also contributes to a broader macroeconomic literature that studies the interaction of durable and non-durable consumption and the propagation of income and wealth shocks. In our model, periodic obsolescence determines the optimal timing of durable purchases, and this, in turn, affects how shocks propagate. Leahy and Zeira (2000) derive a

[4] This result is general as long as the depreciation rate for the durable is not a function of the purchase size. See Appendix 2 for details.

closely related result in a framework where consumers buy the durable only once in their lifetime. They find what they call an "insulation effect": both non-durable consumption and the size of the durable are unaffected by wealth shocks, but the timing of purchases is. Our analysis offers a few caveats for the insulation effect. When the size is fixed, an aggregate shock affects the timing of purchases selectively, depending on the type of replacement rule. Only flexible-rule consumers adjust the timing of future purchases; fixed-rule consumers adjust their non-durable consumption instead. When the size is variable, the insulation effect depends on whether or not there are binding constraints on sizes. If size is unconstrained and the durable can be bought repeatedly, the insulation effect is reversed: the wealth shock is fully absorbed by adjustments of size and non-durable consumption. Then aggregate unit sales are completely insulated from wealth shocks.

Section 2 describes the model. Section 3 separately solves the durable consumption problem. We construct optimal policies using a very simple geometric argument. Section 4 determines the optimal allocation of wealth between durable and non-durable consumption and derives our key comparative statics. In Section 5 we match our results with observations on aggregate consumption behavior. Section 6 extends the model to allow variable purchase size and discusses its empirical implications. Section 7 concludes.

2 Model

We consider a dynamic economy with two goods, a durable and a non-durable good, and a continuum of agents that differ in their permanent income $y \in \left[\underline{y}, \bar{y}\right]$. Incomes are given exogenously, and they stay constant over time.

Goods, technology and preferences: The durable good is indivisible and is produced by a constant returns to scale technology that uses p_0 units of the non-durable good for each unit of the durable good. New durables (new models) are introduced regularly into the market at times $\tau \in \mathbf{N} = \{0, 1, \ldots\}$. Without loss of generality, we have normalized to 1 the length of a design cycle. We refer to the durable introduced at time τ as " model τ". The *technological age* of a durable good is the number of new models introduced since it was produced. The consumers are infinitely-lived and have a (common) discount rate ρ and a (common) separable flow utility function $v(\alpha, c) = x_\alpha + u(c)$, where $\alpha \in \{0, 1, \ldots, T\}$ denotes the technological age of the durable good, and c is the consumption flow for the non-durable. Durable goods of any age less than T are perfect substitutes and each agent consumes at most one unit (additional units provide no utility). We think of the non-durable good as money for the consumption of other goods, and of u as an indirect utility function. We assume that $u' > 0$, $u'' < 0$, $u'(0) = \infty$, and $x_0 \geq x_1 \geq \cdots x_{T-1} > x_T = 0$.

Obsolescence is the only form of depreciation in our model. A durable becomes useless when its technological age is T or more. A new model τ provides a flow service of x_0 in the period $[\tau, \tau + 1)$. When a new model is introduced at time $\tau + 1$, model τ's flow service decreases to x_1, and so on. The consumers can buy a new durable at any moment, but the durable is depreciated as soon as the new model is introduced. Thus, if a consumer buys a new durable at time $t \in [\tau, \tau + 1)$, he gets the flow service x_0 in the interval $[t, \tau + 1)$, and then the flow service x_1 in the interval $[\tau + 1, \tau + 2)$, and so on, as long as he doesn't replace the durable.

Our model assumes that a durable good becomes less useful as soon as a new model appears in the market. But, in Appendix 1 we show that in a model where the flow service of a durable remains constant for its lifetime, we can re-normalize utility and define x_α as the *relative utility* of a durable with respect to the latest model. In the example of Appendix 1, $x_\alpha = g(T-\alpha)$, where g is the average rate of technical progress in durables. In this case x falls with α simply because better goods become available at the same price.

The consumers can borrow and lend, but there are no secondary markets for used durables.

PRICES: Since the production technology is CRS, the price ratio of the durable good to the non-durable good is equal to p_0 at all times. We will assume that the interest rate is fixed and equal to the discount rate: $r(t) = \rho$ for all $t \geq 0$. We therefore perform a partial equilibrium analysis. We think of the market for durables as being a small part of the aggregate economy and hence ignore the effect of durable demand on the interest rate. Our choice of interest rate is consistent with stationary equilibria. In a general equilibrium model where income (resource) flow and production technology are constant over time, a stationary equilibrium would imply a constant interest rate equal to the discount rate. If $q(t)$ and $p(t)$ denote, respectively, the prices of the non-durable and durable goods at time t, our assumption of a constant interest rate implies that $q(t) = e^{-\rho t}$ and $p(t) = p_0 q(t)$ for all $t \geq 0$, where we have normalized so that $q(0) = 1$. Define the total discount rate for one period $\beta = e^{-\rho}$.

CONSUMER PROBLEM: Given his initial *state* (α, w), where $\alpha \in \{1, \ldots, T\}$ is the age of his endowed durable and w is his total wealth, a consumer chooses a sequence of durable purchase dates and a non-durable consumption path to maximize his discounted lifetime utility, $\int_0^\infty e^{-\rho t}[x_{\alpha_t} + u(c_t)]dt$, subject to a lifetime budget constraint. An agent's current wealth is equal to the present discounted value of all his future earnings, y/ρ, minus the present discounted value of his debts (past borrowing minus lending).

Since $r(t) = \rho$ for all t and utility is additively separable, optimally, non-durable consumption must be constant over time. Indeed, the (necessary and sufficient) first-order condition for non-durable consumption is in this case $e^{-\rho t}u'(c(t)) = \lambda e^{-\rho t}$ for all t, where $\lambda > 0$ is the Lagrange multiplier on the budget constraint. This implies that $c(t) = c(0)$ for all $t > 0$.

Let $\hat{u}(c)$ be the discounted non-durable consumption utility over one period (of length 1) in which a consumer spends (optimally) a budget c. This budget affords the constant consumption flow $c\rho/(1-\beta)$. Hence

$$\hat{u}(c) = \int_0^1 e^{-\rho t} u\left(\frac{\rho c}{1-\beta}\right) dt = \left[\frac{1-\beta}{\rho}\right] u\left(\frac{\rho c}{1-\beta}\right).$$

Let the consumer spend a constant non-durable budget c per period. Then, his lifetime non-durable discounted utility and total budget are respectively $\hat{u}(c)/(1-\beta)$ and $c/(1-\beta)$, and his residual budget for the consumption of durables is $b = w - c/(1-\beta)$.

Let $V_\alpha(b)$ denote the optimal durable consumption utility of a consumer that is endowed with a good of age α and spends a total budget b on durables. Then the problem of an agent with initial state (α, w) is

$$U_\alpha(w) = \max_{c\in[0,w]} \frac{\hat{u}(c)}{1-\beta} + V_\alpha\left(w - \frac{c}{1-\beta}\right). \tag{1}$$

In Section 3, we explicitly construct the functions V_α, $\alpha \in \{1, \ldots, T\}$, and in Section 4 we obtain the full solution for problem (1).

3 Durable consumption problem

Discrete Time: As a preliminary step in analyzing the durable consumption problem, we study a discrete time problem where the consumers are arbitrarily constrained to make new purchases only at the beginning of every period, that is, at times $t \in \mathbf{N}$. We subsequently show that removing this restriction does not change the optimal durable purchasing policy.

A consumer must choose the periods when he purchases a (new) unit of the durable good. A *durable purchasing policy* $\delta = \{\delta_t\}_{t\geq 0}$ specifies the periods in which the agent buys a new unit ($\delta_t = 1$) or keeps the old unit he has ($\delta_t = 0$). For any $i, j \in \mathbf{N}$, let $i \oplus j = \min\{i+j, T\}$ and $i \ominus j = \max\{i-j, 0\}$. Given an initial unit of age α_{-1},[5] a purchasing policy determines the age of the unit consumed in every period $t \geq 0$ recursively as follows: $\alpha_t = 0$ if $\delta_t = 1$ and $\alpha_t = \alpha_{t-1} \oplus 1$ if $\delta_t = 0$.

The optimization problem of an agent that initially has a good of age α and durable budget b is

$$\begin{aligned} V_\alpha(b) = \max & \sum_{t\geq 0} \beta^t \hat{x}_{\alpha_t} \\ \text{s.t.} \quad & \alpha_{-1} = \alpha - 1, \quad \delta_t \in \{0,1\} \text{ and } \alpha_t = (1-\delta_t)[\alpha_{t-1} \oplus 1],\ t \geq 0 \\ & b = p_0 \sum_{t\geq 0} \beta^t \delta_t, \end{aligned}$$

where $\hat{x}_\alpha = x_\alpha(1-\beta)/\rho$ denotes the total discounted utility from the consumption of a durable of age α over one period.

We solve the potentially difficult integer programming problem above using a direct geometric argument focusing on a particularly simple class of policies.

Definition: For each $R = 1, \ldots, T$, a policy δ that replaces the durable every time it reaches age R is called an *R-fixed rule*. That is, δ is an R-fixed rule if for all t, $\delta_t = 1$ if and only if $\alpha_{t-1} = R - 1$. A $(T+1)$-fixed rule is to never replace the durable: $\delta_t = 0$ for all t.

Let $X_{\alpha,R}$ denote the total discounted utility from holding a durable from age α until age R:

$$X_{\alpha,R} = \begin{cases} \sum_{t=\alpha}^{R-1} \beta^{t-\alpha} \hat{x}_t & \alpha < R \\ 0 & \alpha \geq R. \end{cases}$$

For $R \leq T$, the value of following the R-fixed rule starting with a useless durable ($\alpha = T$) equals $v_{T,R} = X_{0,R}/(1-\beta^R)$, and its corresponding budget is $b_{T,R} = p_0/(1-\beta^R)$. The value and budget of the $(T+1)$-fixed rule are both zero.

Construct a piecewise linear function by joining the adjacent points $(b_{T,R+1}, v_{T,R+1})$ and $(b_{T,R}, v_{T,R})$ $(1 \leq R \leq T)$ with straight lines. Theorem 1 below states that this piecewise linear function is V_T. Moreover, V_T is concave (see the left frame of Figure 1 below).

Assume that $\alpha = T$ and for an arbitrary purchasing policy δ, group purchases by their "replacement age". That is, for each $R = 1, \ldots, T$, let L_R be the purchase dates of all durable that are used for R periods and then replaced at age R. Compute the weight $\lambda_R = (1-\beta^R)\sum_{t\in L_R} \beta^t$ and let $\lambda_{T+1} = 1 - \sum_{R=1}^{T} \lambda_R$. Roughly, the weight λ_R corresponds to the fraction of purchases that result in the replacement of a durable at age R. For example, if the policy is an R-fixed rule with $R < T+1$, then L_R contains all the periods t where

[5] To deal with period 0 as with any other period, we specify the age that the endowed durable would have been in the "previous period".

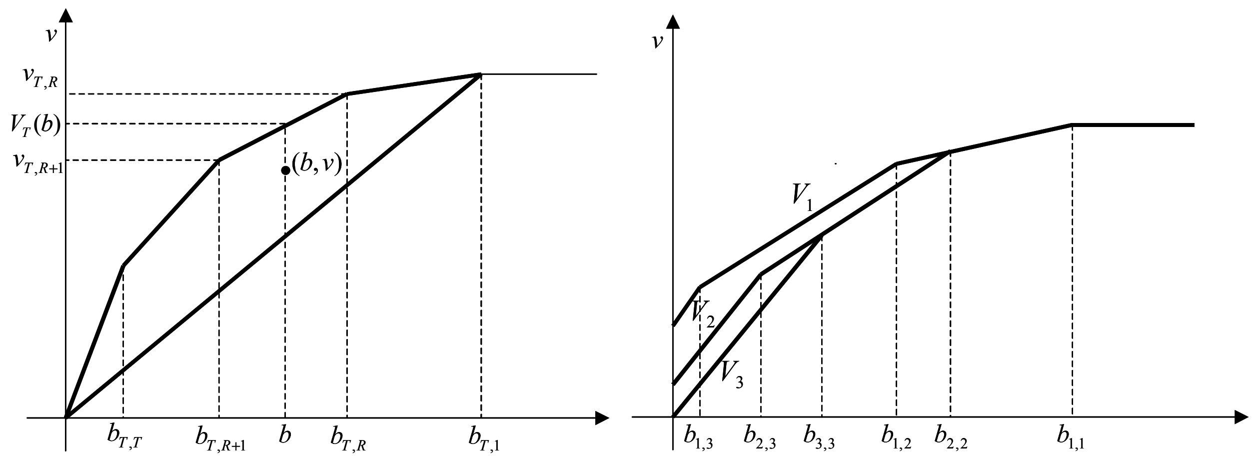

Figure 1: Optimal value function

$\delta_t = 1$, so that $\lambda_R = 1$ and $\lambda_k = 0$ for all $k \neq R$. Let (b, v) denote the budget and value of policy δ. It turns out that:

$$\begin{bmatrix} b \\ v \end{bmatrix} = \sum_{R=1}^{T+1} \lambda_R \begin{bmatrix} b_{T,R} \\ v_{T,R} \end{bmatrix}.$$

Since the weights λ_R are nonnegative and add up to 1, the right-hand side is a convex combination of the points $\{(v_{T,R}, b_{T,R})\}_{R=1}^{T+1}$. That is, the point (b, v) must be in the convex hull of $\{(v_{T,R}, b_{T,R})\}_{R=1}^{T+1}$, as depicted in Figure 1. Note that the upper frontier of this set coincides with the graph of the posited optimal value function V_T. Hence, $v \leq V_T(b)$. The upper bound $V_T(b)$ is attained by a particular type of policy. Suppose R is such that $b \in [b_{T,R+1}, b_{T,R}]$, and let δ^* be a policy that replaces durables at age R or $R+1$ only. Such a policy is called an R-flexible rule. Its corresponding weights satisfy $\lambda_k^* = 0$ for all $k \notin \{R, R+1\}$. By appropriately choosing the periods when durables of age R or age $R+1$ are replaced, we can also ensure that $b = \lambda_R^* b_{T,R} + \lambda_{R+1}^* b_{T,R+1}$ (as we explain later, this is always possible provided that β is sufficiently large). Then, the value of δ^* is $\lambda_R^* v_{T,R} + \lambda_{R+1}^* v_{T,R+1} = V_T(b)$. That is, δ^* is optimal for the budget b.

For an arbitrary α now, let $b_{\alpha,R}$ and $v_{\alpha,R}$ denote the cost and the value of following the R-fixed rule when the endowed durable is of age α. Then

$$\begin{bmatrix} b_{\alpha,R} \\ v_{\alpha,R} \end{bmatrix} = \begin{bmatrix} 0 \\ X_{\alpha,R} \end{bmatrix} + \frac{\beta^{R \ominus \alpha}}{1 - \beta^R} \begin{bmatrix} p_0 \\ X_{0,R} \end{bmatrix} \quad \text{for all } R \leq T$$

and $(b_{\alpha,T+1}, v_{\alpha,T+1}) = (0, X_{\alpha,T})$. It is also convenient to define $b_{T+1,T+1} = p_0$ and $b_{0,1} = \beta p_0/(1-\beta)$. Rules that replace goods more frequently require bigger budgets and have higher values. Hence $b_{\alpha,R} > b_{\alpha,R+1}$ and $v_{\alpha,R} > v_{\alpha,R+1}$.

The piecewise linear function obtained by joining the adjacent points $(b_{\alpha,R+1}, v_{\alpha,R+1})$ and $(b_{\alpha,R}, v_{\alpha,R})$ $(1 \leq R \leq T)$ with straight lines is the optimal value function V_α (see Theorem 1 below). Figure 1 (right frame) presents simultaneously the optimal value functions V_1, V_2 and V_3 for the case when $T = 3$.

Definition: Let $1 \leq R \leq T-1$ and $b \geq 0$. A policy δ is an (R, b)-*flexible rule* if it replaces durables only when they are of age R or age $R+1$ and spends the budget b exactly. If δ is an (R, b)-flexible rule then for all t, $\delta_t = 1$ implies that $\alpha_{t-1} \in \{R-1, R\}$.

Since an (R,b)-flexible rule sometimes replaces goods at age R, and sometimes at age $R+1$, it costs more than an $(R+1)$-fixed rule but less than an R-fixed rule. Hence, when the endowed good is of age α, b must be in the interval $[b_{\alpha,R+1}, b_{\alpha,R}]$. As we will see, for $b \in (b_{\alpha,R+1}, b_{\alpha,R})$, there are multiple (R,b)-flexible rules. The R-fixed and the $(R+1)$-fixed rules are both special cases of the (R,b)-flexible rule for $b = b_{T,R}$ and $b = b_{T,R+1}$, respectively.

For $1 \le \alpha, R \le T$, let

$$A_R = \frac{v_{\alpha,R} - v_{\alpha,R+1}}{b_{\alpha,R} - b_{\alpha,R+1}} = \frac{1}{p_0}\left[X_{0,R} - \hat{x}_R\left[\frac{1-\beta^R}{1-\beta}\right]\right].$$

Note that A_R is independent of α and equals the slope of V_α on $[b_{\alpha,R+1}, b_{\alpha,R}]$. It is easy to check that $A_T > A_{T-1} > \cdots > A_1 > 0$, and therefore V_α is indeed a concave function.

Theorem 1: *Assume that*

$$\beta^{T-1}(1+\beta) > 1. \tag{2}$$

For each $\alpha = 1, \ldots, T$, the optimal value function V_α is

$$V_\alpha(b) = v_{\alpha,R+1} + A_R(b - b_{\alpha,R+1}), \quad b \in [b_{\alpha,R+1}, b_{\alpha,R}],\ R = T, \ldots, 1,$$

and for any budget $b \ge 0$, a corresponding optimal purchasing policy is an (R,b)-flexible rule, where R is such that $b \in [b_{\alpha,R+1}, b_{\alpha,R}]$ (when $b = b_{\alpha,R}$, this policy coincides with the R-fixed rule). More precisely, the optimal purchasing policy is given by

$$\delta^*_\alpha(b) = \begin{cases} 0 & \text{for } b < b_{\alpha+1,\alpha+1} \\ \{0,1\} & \text{for } b_{\alpha+1,\alpha+1} \le b \le b_{\alpha-1,\alpha} \\ 1 & \text{for } b > b_{\alpha-1,\alpha}. \end{cases} \tag{3}$$

Proof: See Appendix 1.

Assumption (2) is equivalent to $\beta > \bar{\beta}$, where $\bar{\beta}$ is the (unique) root of $\beta^{T-1}(1+\beta) = 1$. This is the same as assuming that $\rho < \bar{\rho}$, where $\bar{\rho} = e^{-\bar{\beta}}$. When β is relatively small, there are budgets b that do not correspond to any durable purchasing policy. The intuition is clear. Suppose β is close to 0. Then the durable budget is almost fully determined by the timing of the first purchase. Let $\alpha < R$, δ be an R-flexible rule, and b be its corresponding budget. If the first purchase happens when the good is of age R, then $b \sim b_{\alpha,R}$ (even if all subsequent purchases replace durables of age $R+1$), and if it happens at age $R+1$, then $b \sim b_{\alpha,R+1}$ (even if all subsequent purchases replace durables of age R). Hence, budgets around the middle of the interval $(b_{\alpha,R+1}, b_{\alpha,R})$ are unattainable.

An agent that follows an R-flexible rule replaces goods of age R or $R+1$, but he is not always indifferent between these replacement ages. To follow an R-flexible rule requires that in each period the agent maintain a budget that is compatible with this rule. Assume that the durable has reached age R in the current period. Then, the current budget b must be in the interval $[b_{R,R+1}, b_{R,R}]$. Suppose b is close to $b_{R,R}$. If the agent keeps the good this period, his budget next period would be $b/\beta > b_{R+1,R}$, too large to follow the R-flexible rule from that point onward. Therefore, the agent can keep the durable this period only if $b \in [b_{R,R+1}, b_{R-1,R}]$; if $b > b_{R-1,R}$, the agent must replace now at age R. Now assume that b is close to $b_{R,R+1}$. If the agent replaces the durable now, his budget next period would be $(b - p_0)/\beta < b_{1,R+1}$, too small to follow the R-flexible rule from that point onward. Therefore, the agent can replace his durable of age R this period only if $b \in [b_{R+1,R+1}, b_{R,R}]$;

if $b < b_{R+1,R+1}$, the agent must keep the durable for one more period. Assumption (2) also guarantees that $b_{R+1,R+1} < b_{R-1,R}$, and for $b \in [b_{R+1,R+1}, b_{R-1,R}]$ both keeping and replacing the durable this period are consistent with the R-flexible rule. For this interval of budgets, the agent is indifferent between replacing the durable now at age R and next period at age $R+1$.

Continuous Time: We now allow consumers to purchase durables at times other than $t \in \mathbf{N}$ and show that this does not change the optimal value function. For the continuous time replacement problem, we need a more detailed representation of the durable purchasing policy. Let τ_k denote the period (or, equivalently, the model number) and $d_k \in [0, 1)$ be the "delay" of the k-th purchase, so the time of the k-th purchase is $\tau_k + d_k$. The following theorem states that it is optimal to set $d_k = 0$ for all k.

Theorem 2: *For each $\alpha = 1, \ldots, T$, the optimal value function is*

$$V_\alpha(b) = v_{\alpha,R+1} + A_R(b - b_{\alpha,R+1}), \quad b \in [b_{\alpha,R+1}, b_{\alpha,R}],\ R = T, \ldots, 1.$$

For any budget $b \geq 0$, the corresponding optimal purchasing policy $\{(\tau_k, d_k)\}_{k\geq 1}$ has $d_k = 0$ for all k and is an (R, b)-flexible rule, where R is such that $b \in [b_{\alpha,R+1}, b_{\alpha,R}]$.

Proof: See Appendix 1.

The idea of the proof is as follows. When the consumer decides whether to delay by d a durable purchase, he weights the loss of service flow against the financial gain of paying for the durable later. When $r = \rho$, the financial gain is less than the corresponding loss of service (in fact, the result holds as long as the interest rate is not too high relative to ρ). An arbitrary policy with delays can be modified recursively by eliminating one delay at a time while maintaining the same budget and improving its value

4 Optimal budget allocation

We now solve problem (1) for the optimal consumption of non-durables as a function of α and w. An agent with wealth w that spends b on durables optimally spends $c = (1-\beta)(w-b)$ per period on non-durables. Optimally, the agent should pick c (or, equivalently, b) so as to equate the marginal utility of consumption $\hat{u}'(c)$ and the marginal utility of wealth $V'_\alpha(b)$. Figure 2 depicts the marginal utility of wealth (the falling step-function because V_α is a concave piecewise linear function) and the marginal utility of consumption as functions of b (for given values of α and w). In the figure, $\hat{u}'$ crosses V'_α at a point of discontinuity. This depicts the situation when the optimal durable budget equals $b_{\alpha,R}$ and the corresponding durable purchasing policy is the R-fixed rule. Now decrease w by a small amount. The graph of $\hat{u}'((1-\beta)(w-b))$ will shift to the left, but it will still cross V'_α at $b = b_{\alpha,R}$. In other words, there is an *interval* of wealths w for which it is optimal to follow the R-fixed rule in the state (α, w). If we further decrease w, $\hat{u}'$ will eventually cross V'_α at a point where V'_α is flat and equal to A_R. This is the case when it is optimal to choose a budget corresponding to an R-flexible rule and pick the non-durable budget c_R, where $\hat{u}'(c_R) = A_R$. Hence, there is also an interval of wealths w for which it is optimal to follow the R-flexible rule and spend c_R in non-durables every period. For that range of wealths, the optimal non-durable budget remains *constant* and variations of wealth affect the durable consumption path only (higher wealths afford replacing durables at age R more frequently, while lower

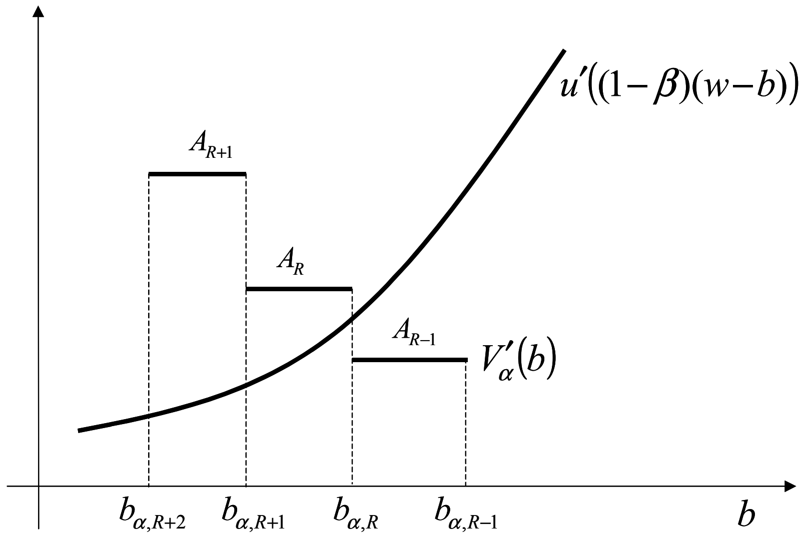

Figure 2: Marginal utilities of consumption and wealth as functions of durable budget.

wealths require replacing durables at age $R+1$ more often). In contrast, when a fixed rule is optimal, a higher wealth leads to a higher level of non-durable consumption.

For a fixed α, if w varies continuously from infinity to zero, the intersection of $\hat{u}'$ with V'_α in Figure 2 moves monotonically to the left and maps out the optimal durable replacement rule (as a function of w). The wealthiest consumers use a 1-fixed rule. Next comes a group of consumers that follow 1-flexible rule, and then a group that follows the 2-fixed rule, and so on. The intervals of wealth where agents follow fixed rules are interlaced with the intervals of wealth where they follow flexible rules. The bounds of these intervals can be computed explicitly. Fix α and let

$$w_{\alpha,R}(c) = \frac{c}{1-\beta} + b_{\alpha,R}$$

be the wealth required to follow the R-fixed rule and spend a constant non-durable budget c per period when the initial durable is of age α. The wealthiest person that follows the R-flexible rule replaces his durable every R periods and consumes c_R. Hence his wealth is $w_{\alpha,R}(c_R)$. The poorest person that follows the $(R-1)$-flexible rule also replaces his durable every R periods but consumes $c_{R-1} > c_R$, so that his wealth is $w_{\alpha,R}(c_{R-1}) > w_{\alpha,R}(c_R)$. In between, there are consumers with wealth $w \in [w_{\alpha,R}(c_R), w_{\alpha,R}(c_{R-1})]$ that follow the R-fixed rule. Each one spends the same durable budget $b_{\alpha,R}$ and the non-durable budget per period

$$c_{\alpha,R}(w) = (1-\beta)(w - b_{\alpha,R}).$$

A consumer with more wealth than $w_{1,1}(c_1) = (c_1 + p_0)/(1-\beta)$ will replace his durable every period and spend more than c_1 per period in non-durables. We will assume that $\bar{y}/\rho \geq w_{1,1}(c_1)$, and define $\bar{w} = \bar{y}/\rho$ and $c_0 = (1-\beta)\bar{w} - p_0$. Similarly, a consumer with less wealth than $c_T/(1-\beta)$ will spend all his wealth in non-durable consumption. We will assume that $\underline{y}/\rho \leq c_T/(1-\beta)$, and define $\underline{w} = \underline{y}/\rho$ and $c_{T+1} = (1-\beta)\underline{w}$.

We can also express the optimal purchasing policy (3), stated in Theorem 1, as a function of wealth (and with abuse of notation denote this function by the same symbol δ^*_α). The following theorem states these results formally.

Theorem 3: *Let* $c_0 = (1-\beta)\bar{w} - p_0$, $c_{T+1} = (1-\beta)\underline{w}$, *and for each* $R = 1, \ldots, T$, *let* c_R *be such that* $\hat{u}'(c_R) = A_R$. *Denote by* $c^*_\alpha(w)$ *the optimal solution of problem (1). Then, for* $\alpha = 1, \ldots, T$,

$$c^*_\alpha(w) = \begin{cases} c_{\alpha,R}(w) & \text{for } w \in [w_{\alpha,R}(c_R), w_{\alpha,R}(c_{R-1})],\ R = T+1, \ldots, 1 \\ c_R & \text{for } w \in [w_{\alpha,R+1}(c_R), w_{\alpha,R}(c_R)],\ R = T, \ldots, 1, \end{cases}$$

and

$$\delta^*_\alpha(w) = \begin{cases} 0 & \text{for } w < w_{\alpha+1,\alpha+1}(c_\alpha) \\ \{0,1\} & \text{for } w_{\alpha+1,\alpha+1}(c_\alpha) \leq w \leq w_{\alpha-1,\alpha}(c_\alpha) \\ 1 & \text{for } w > w_{\alpha-1,\alpha}(c_\alpha). \end{cases} \tag{4}$$

Proof: See Appendix 1.

Over time, a consumer that follows an R-fixed rule has a constant holding time R and revisits the same points in the state space (α, w) every R periods. His wealth trajectory is cyclical. While the consumer keeps the current good, both α and w increase, as the consumer "saves" for the next purchase. When the new durable is purchased, both α and w go down, and the holding cycle starts again.

The time path for wealth of a consumer that follows an R-flexible rule is more erratic. Usually, his wealth trajectory is not cyclical: each time the durable is of age R, he has a different wealth level. For example, suppose that the consumer starts with a durable of age R and wealth level $w_0 \in (w_{R-1,R}(c_R), w_{R,R}(c_R))$. Then, he must replace the durable now, and the next time his good reaches age R, his wealth will be $w_R = [w_0 - p_0]\beta^R < w_0$. If $w_R > w_{R-1,R}$, he will have to replace the durable again. But eventually, if he continues to replace each time the durable reaches age R, he will reach a state (R, w), where $w < w_{R+1,R+1}(c_R)$. At this point, he is forced to wait one more period. Thus, the agent will switch replacement frequencies erratically, as each time that his state is of the form (R, w), his wealth level w is in a different region of the interval $[w_{R+1,R}(c_R), w_{R,R}(c_R)]$.

4.1 Consumption classes

The optimal policies partition the state space (α, w) into disjoint classes, with each class corresponding to a different durable replacement rule. All individuals in a class follow the same rule and the trajectories of their states stay forever in the same class. For every $R \in \{1, \ldots, T+1\}$ and $\alpha \in \{1, \ldots, R\}$, let

$$W^\alpha_R = [w_{\alpha,R}(c_R), w_{\alpha,R}(c_{R-1})]$$

be the wealth levels of consumers that follow an R-fixed rule and currently have a durable of age α. Similarly, for every $R \in \{1, \ldots, T\}$ and $\alpha \in \{1, \ldots, \min\{R+1, T\}\}$ let

$$W^\alpha_{R,R+1} = (w_{\alpha,R+1}(c_R), w_{\alpha,R}(c_R))$$

be the wealth levels of consumers that follow an R-flexible rule and currently have a durable of age α. Note that for each α, $\{W^\alpha_R\}^{T+1}_{R=1} \cup \{W^\alpha_{R,R+1}\}^T_{R=1}$ forms a partition of $[\underline{w}, \bar{w}]$. At the beginning of every period, agents with a state in $C_R = \bigcup^R_{\alpha=1}\{\alpha\} \times W^\alpha_R$ follow the R-fixed rule, and with a state in $C_{R,R+1} = \bigcup^{R+1}_{\alpha=1}\{\alpha\} \times W^\alpha_{R,R+1}$ follow the R-flexible rule. Note that after the initial period, nobody visits the states $\{\alpha\} \times W^\alpha_R$, $\alpha > R$, or the states $\{\alpha\} \times W^\alpha_{R,R+1}$, $\alpha > R+1$. A consumer with one of these initial states has been endowed with a durable

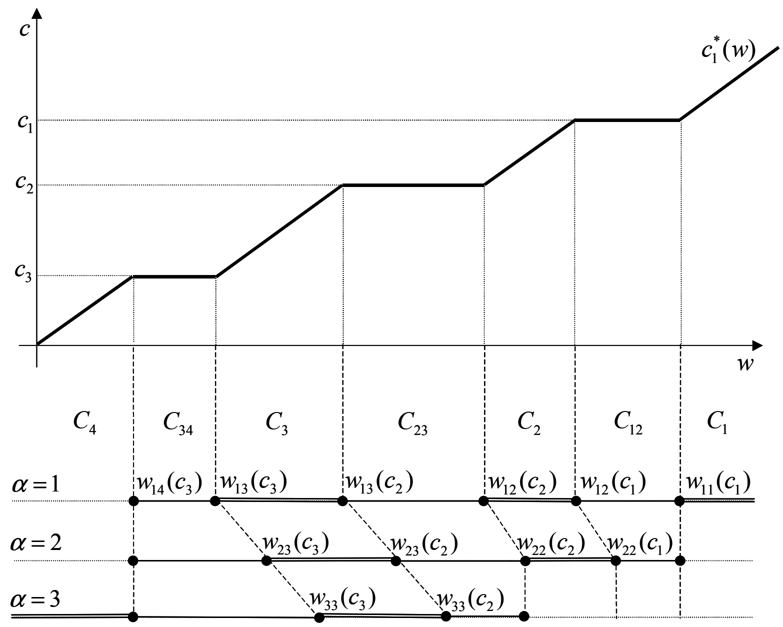

Figure 3: Consumption classes; optimal consumption function $c_1^*(w)$.

that is "too old" for his initial wealth level. The classes C_R and $C_{R,R+1}$ are closed: if an agent follows the R-fixed rule (R-flexible rule) and his initial state is in C_R ($C_{R,R+1}$), then his state remains in C_R (in $C_{R,R+1}$) forever. Figure 3 illustrates consumption classes for the case $T = 3$ and one of the corresponding optimal consumption function $c_1^*(w)$ described in Theorem 3. Three horizontal lines on the lower panel of figure 3 represent the state space $\{1, 2, 3\} \times [\underline{w}, \bar{w}]$. Double lines indicate wealth intervals that belong to fixed rule classes, and single solid lines indicate flexible rule classes. Class boundaries are marked by dashed lines. Dotted lines indicate the intervals in the state space that are empty in the long run.

4.2 Durability and obsolescence

Aggregate durable and non-durable consumption both respond to aggregate changes in wealth. Consumers in class C_R have a fixed durable budget and a positive marginal propensity to consume non-durables (see figure 3). Therefore, if any such consumer receives windfall income, he will spend it all on non-durable consumption. By contrast, consumers in a class $C_{R,R+1}$ have a zero marginal propensity to consume non-durables and a variable durable budget. The magnitude of the overall response of durable consumption to a change in wealth will depend on the mass of consumers in fixed and flexible rule classes. These masses, of course, are functions of the wealth distribution. To isolate the effect of the model's parameters on the sensitivity of durable consumption, we assume a uniform distribution over the set of recurrent states (i.e. the states marked by solid lines on figure 3). Then the mass of consumers in classes C_R and $C_{R,R+1}$ ($R = 1, \ldots, T$) are respectively

$$\mu(C_R) = \sum_{\alpha=1}^{R} [w_{\alpha,R}(c_{R-1}) - w_{\alpha,R}(c_R)]$$
$$\mu(C_{R,R+1}) = \sum_{\alpha=1}^{\min\{R+1,T\}} [w_{\alpha,R}(c_R) - w_{\alpha,R+1}(c_R)].$$

Also, define $\mu(C_{T+1}) = w_{T,T+1}(c_T) - \underline{w}$. Then, the fraction of consumers that follow flexible rules is

$$\theta = \frac{\Sigma_{R=1}^{T} \mu(C_{R,R+1})}{\Sigma_{R=1}^{T+1} \mu(C_R) + \Sigma_{R=1}^{T} \mu(C_{R,R+1})}.$$

Given a small change in wealth, approximately[6] θ consumers will adjust only their durable consumption and $1-\theta$ consumers will adjust only their non-durable consumption. The larger is θ, the more sensitive is durable consumption to changes in wealth.

Assume that $x_\alpha = g(T - \alpha)$, $\alpha = 0, \ldots, T$. This is the obsolescence pattern that arises in the detrended model of Appendix 1. In this case, g represents the obsolescence rate – the speed at which service flow decays – and T represents the durability of the good – the length of its useful life. The following proposition states that under some restrictions on preferences faster obsolescence and higher durability both make durable consumption more sensitive to changes in wealth.

Proposition 1 *Let $x_\alpha = g(T - \alpha)$, $\alpha = 0, \ldots, T$. Then*

(i) An increase in the rate of obsolescence increases (decreases) θ if u has decreasing (increasing) absolute risk aversion.

(ii) Assume that $\beta^T(1+\beta) > 1$ (so (2) is satisfied for T and $T+1$) and $u(c) = \frac{1}{1-\gamma} c^{1-\gamma}$ with $0 < \gamma \leq \gamma^* = 1.36$. Then, θ increases with durability (for any g and p_0).

Proof: See Appendix 1.

The critical value $\gamma^* = 1.36$ has been computed numerically (and the proof explains how γ^* is defined). A typical assumption on preferences is decreasing absolute risk aversion, which implies that θ increases with the rate of obsolescence.

Proposition 1 separates the effects of durability and obsolescence. Higher obsolescence rate makes the service flow decline more steeply with age. As a result, the endpoints of all consumption classes shift downward. In contrast, expanding the lifetime of durables does not shift class boundaries but changes the optimal replacement rule for consumers at the bottom of the wealth distribution (that used to follow the $(T+1)$-fixed rule).

5 Empirical implications

5.1 Volatility of durable consumption

It is well-known that aggregate durable consumption is more volatile than aggregate non-durable consumption. The standard PIH model (e.g. Mankiw, 1982, see also Appendix 2 for details) can explain this. In the standard model, the short-run wealth elasticity of demand for

[6] A small mass of consumers will change their consumption class as a result of change in wealth.

a durable is inversely proportional to its rate of economic depreciation. Therefore, demand for durables (with depreciation rates less than 100%) should be more volatile than demand for non-durables (with depreciation rate of 100%). However, the model also implies that the smaller is the rate of depreciation of a good, the more volatile is its demand. The data seems to contradict this. Figure 4 shows the year-on-year percentage change in investment rate (i.e. the ratio of expenditure to stock) for three categories of durable goods: computers, furniture and autos.[7] Furniture has the lowest economic depreciation rate (0.1 annually) and the least variable investment rate, computers have the highest economic depreciation (0.45) and the most variable investment rate, and automobiles (depreciation rate 0.18) are in the middle. The evidence on figure 4 suggests that durables with higher obsolescence rates have a higher volatility of expenditure.

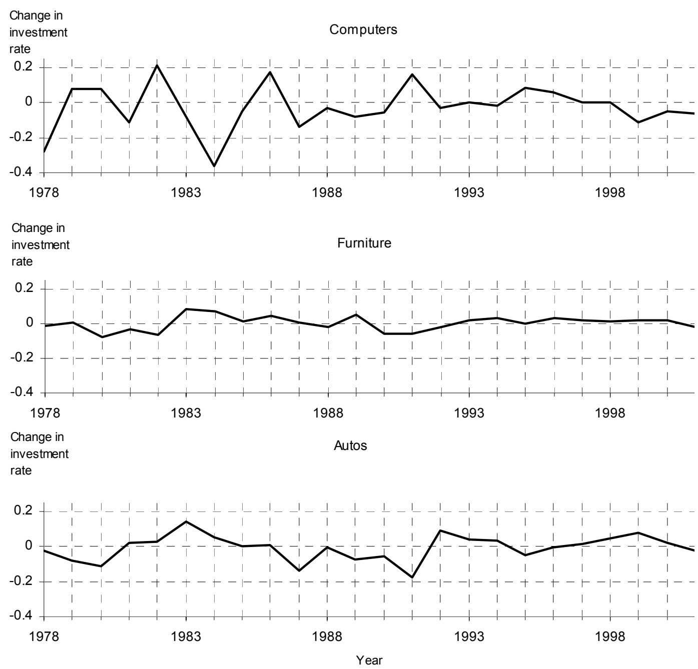

Figure 4: Changes in investment rate for three categories of durables.

Our model can simultaneously match both pieces of evidence, because it separates the effect of higher durability from that of slower obsolescence. In Proposition 1, we show that consumption of longer-lived durables exhibit a stronger response to a wealth shock. This makes our model consistent with the aggregate data. A higher rate of obsolescence also makes durable consumption more volatile, which explains the pattern on figure 4.

5.2 Synchronization of purchases

Our model predicts that demand for durables is concentrated around the dates when the new models come out, and we can look for such coordination using the data on household purchases of computers. Figure 5 reproduces the plot for computers on the previous figure,

[7] Source: BEA. Consumption expenditures are taken from NIPA Table 2.6, lines 45 (new autos), 59 (furniture) and 73 (computers). The corresponding stocks of durable goods are from NIPA Table 8.1, lines 3, 7 and 12.

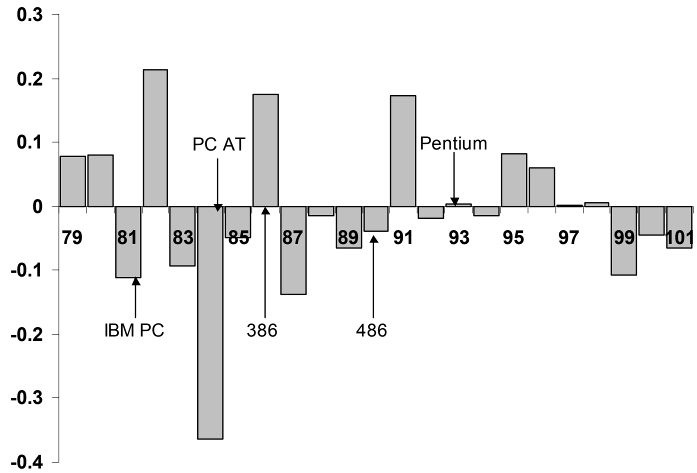

Figure 5: Accelerations in computer investment rate with the introduction of new models.

but adds the dates when the new generations of personal computers came on the market. The general pattern on the figure is one-year expenditure spikes (in 1982, 86, 91 and 95) followed by several years of falling investment rates. Apparently, computer expenditure is concentrated around certain dates. If computer purchases were simply driven by the business cycle, one would expect to see a positive correlation between investment rates across durable categories. The data for computers show exactly the opposite. While investment rates for autos and furniture are positively correlated, both exhibit a negative correlation with computer investments.

Durable categories	Correlation of investment rates, 1978-2001
Autos and furniture	0.49
Computers and autos	-0.27
Computers and furniture	-0.40

One dramatic example of this is the 1990-1991 recession when final sales of autos plummeted 21% in real terms while final sales of computers have doubled over the same period.

The spikes in computer expenditure seem to closely follow the introduction of new models of PCs. The 1982 spike corresponds to the introduction of the IBM PC in the summer of 1981. The 1986 spike probably corresponds to mass purchases of 286 PCs. The first one of those was the IBM PC AT in late 1984, with most of the "AT compatibles" by Compaq and other manufacturers becoming available in 1985. In the summer of 1986, Compaq also introduced the first 386 PC, but it probably was not a mass market model at the time. PC magazine (Nov 25, 1986, p. 157) wrote: "Compaq says it knows perfectly well that this is not a machine that will sell in huge volumes this year, nor, probably next." The next generation 486 processor was formally announced in the spring of 1989, but the initial chip had bugs and a slower clock speed than the existing 386. The 486 50-MHz chip that offered significant performance advantage over the 386 was not produced until October of 1990 (PC Magazine, September 11, 1990, p. 100). The 1991 spike is consistent with pent-up demand as advanced users were waiting for the faster 486 PC. The next Intel processor, the Pentium, came out in 1993, however the purchase spike did not follow until 1995 for two reasons. The initial Pentium cost twice as much as a 486 with the same speed, and this price premium was "inconsistent with the additional performance" (PC magazine, July 1993, p. 126). More importantly, users may have been postponing computer upgrades until the arrival of the new Windows 95 operating system.

It is interesting that the pattern of computer investment seems to have changed after 1995. We think this is because in the 1990s innovations in software and hardware became more staggered and less synchronized with innovation in processors. For example, early on machine language changed from one generation of processor to the next (e. g. PC magazine, November 25, 1986, p. 154), and software innovations had to happen simultaneously to take full advantage of new processors.

6 Variable sizes

In our model, wealth shocks affect the timing of future durable purchases. Evidence of this can be found in the data on auto sales, as the next subsection details. However, so far we assumed that there is only one "size" for the durable. We now explore a model where consumers can purchase durables of any size. We assume that a new durable of size S costs $p_0 S$, and a durable of age α and size S provides a utility flow $S^\eta x_\alpha$, where $\eta \in (0,1)$.

Since a consumer can now adjust durable expenditure by changing the durable sizes, he will only use fixed rules. Moreover, Proposition 2 below shows that the *same* fixed rule is optimal for *all* the consumers. That is, when durables are replaced is no longer a function of a consumer's wealth.

Proposition 2: *For all consumers, the optimal purchasing policy is the R^*-fixed rule, where*

$$R^* = \arg\max_R \, [v_{T,R}/b_{T,R}^\eta],$$

and the optimal purchase size for a consumer with a durable of age R^* and a budget b is $S^*(b) = b/b_{T,R^*}$.

Proof: See Appendix 1.

In this model, wealth shocks are absorbed by changes in non-durable consumption and durable size. The same result holds for a broader class of models, if we continue to assume that depreciation rate of the durable does not depend on the purchase size. An (s,S) replacement model with variable purchase size also predicts that the replacement frequency is independent of wealth (see Appendix 2). Leahy and Zeira (2000) do find that the dates of durable purchases are a function of wealth, but their model assumes that consumers buy durables at most once in their lifetime, an assumption that is inappropriate for cars.

We now introduce feasibility constraints. Specifically, assume that there is a maximum size $\bar{S}$. That is, consumers can choose durable sizes $S \in [0, \bar{S}]$. Consumers with budgets $b \leq \bar{S} b_{T,R^*}$ (at the moment they replace the durable) will follow the R^*-fixed rule, because for them the constraint $S \leq \bar{S}$ does not bind. Consumers with budgets $b > \bar{S} b_{T,R^*}$ will partition into fixed-rule and flexible-rule classes in much the same way as with fixed purchase size. As we will show below, for a range of values of η close to 1, a consumer with budget $b > \bar{S} b_{T,R^*}$ and a useless durable will choose every future durable purchase to be of maximum size. Therefore, his total durable consumption utility is $\bar{S}^\eta V_T(b/\bar{S})$, where V_T is the function defined earlier (for the case where durables are of size 1). More precisely, such a consumer faces the same optimization problem as in Section 3, but where the durable costs $p_0 \bar{S}$ and provides utility flow $\bar{S}^{\eta} x_{\alpha}$ for $\alpha = 0, \ldots, T$. Thus, for such a consumer, his optimal policy is a flexible rule. Proposition 3 states this result formally.

Proposition 3: *Assume that $v_{T,R}/b_{T,R}^{\eta}$ is single-peaked in R and that $\eta \in [\underline{\eta}, 1)$ where*

$$\underline{\eta} = \max_{1 \leq R \leq T-1} \frac{A_R b_{T,R}}{v_{T,R}} < 1.$$

Assume a consumer's current state is (T, b) where $b \in [b_{T,R+1}\bar{S}, b_{T,R}\bar{S}]$ for some $1 \leq R < R^*$. Then, his optimal policy is an R-flexible rule with every durable purchase of size $\bar{S}$.

Proof: See Appendix 1.[8]

Thus, consumers with relatively high budgets buy durables of size $\bar{S}$ only. These consumers react to wealth shocks as if there is only one size available – those following a fixed rule, for example, will adjust their non-durable consumption only. In contrast, consumers with relatively small budgets all follow an R^*-fixed rule with *variable* purchase size. They respond to a wealth shock by changing the *sizes* of their durables and their non-durable consumption. Relaxing the size constraint by increasing $\bar{S}$ will make the latter group larger, so that fewer people will adjust the dates of future durable purchases. Hence, the model predicts that for durables with a broad range of sizes, aggregate unit sales follow a deterministic path (that depends on initial condition) as wealth shocks are absorbed by size variations. To be clear, the time path for unit sales may not be stationary, but is not affected by wealth shocks either.

A minimum purchase size produces similar results. Relatively wealthy consumers are not constrained by size and follow the R^*-fixed rule. Those consumers that cannot even afford the R^*-fixed rule with the minimum size follow R-flexible rules with $R > R^*$.

EMPIRICAL EVIDENCE: Figure 6 presents the year-by-year growth rates for the number of new autos and the average real expenditure per auto between 1950 and 2001.[9] Over that period, the number of cars sold is much more variable than the expenditure per car (which is a good proxy for "size"). In theory, the number of autos sold can vary because the distribution of auto stocks evolves over time, and each year there is a different number of cars that need to be replaced. Adda and Cooper (2000, section 4.3.2) estimated an (s, S) model with auto sales data and simulated the time path for the distribution of auto stocks. They found that the variations in this distribution are not a major source of fluctuations in sales. Then the shifts in purchase timing must the main reason why auto sales fluctuate.

As explained above, constraints on sizes determine the relative importance of purchase size and timing in the transmission of shocks. The history of the automobile market in the US allows us to test this hypothesis. The market share of foreign cars has skyrocketed from 14% in 1972 to 35% in 1980, and has stabilized afterwards. Foreign cars initially cost about

[8] The solution can be more complex if $\eta < \underline{\eta}$. For η in this range there may exist $R < R^*$ and $B_R^* \in (b_{T,R+1}\bar{S}, b_{T,R+1}\bar{S})$ such that for $b \in [b_{T,R+1}\bar{S}, B_R^*]$ the optimal policy is to follow a modified R-flexible rule that switches between buying a durable of size $\bar{S}$ and holding it for $R+1$ periods and buying a durable of size $B_R^*/b_{T,R} < \bar{S}$ and holding it for R periods. For $b \in [B_R^*, b_{T,R}\bar{S}]$ the optimal policy is the R-fixed rule with variable purchase size $S_R(b) = b/b_{T,R}$.

[9] Final sales of new autos are taken from NIPA table 8.8A line 2 (1949-1966) and table 8.8U line 4 (1967-2002). The price index for new autos is from NIPA table 7.5 line 4. Auto sales for 1949-1966 are taken from Ward's Automotive Yearbook (1958 and 1967) and from NIPA table 8.8M, line 1, for 1967-2002. The time series for the civilian non-institutional population over age 16 is from the FRED database (series CNP16OV).

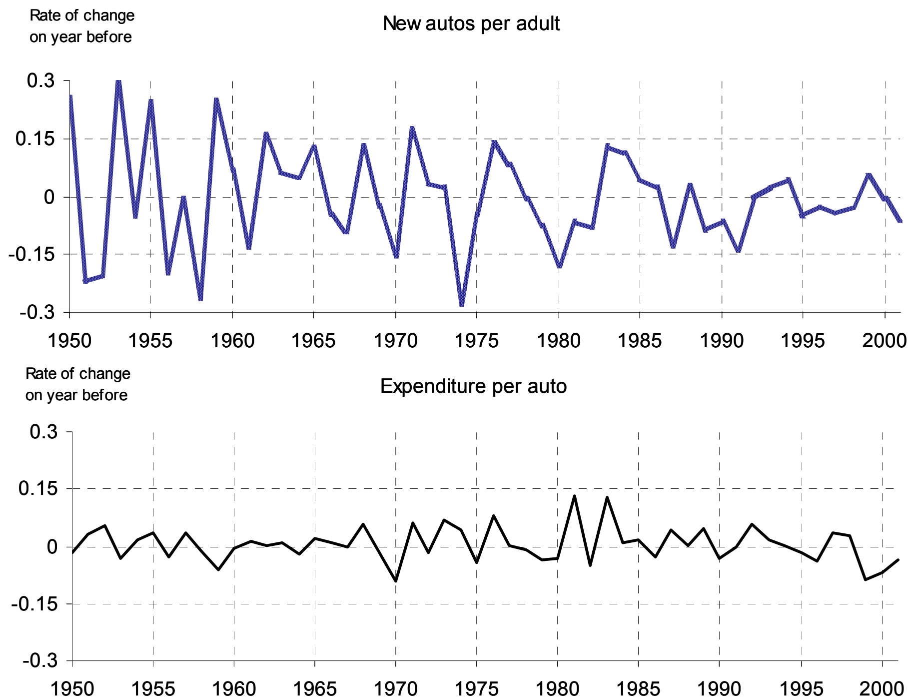

Figure 6: Changes in number and real purchase size for autos, 1950-2001.

20% less, on average[10], than domestic models (for comparison, a foreign car in 2002 cost 45% more, on average). Therefore, the mass introduction of foreign cars in the US market must have expanded the range of available sizes. According to our model, we should expect that in the 1980s and 1990s the number of autos sold became less variable and, at the same time, the purchase size became more variable. To see if this is the case, we split the sample into two intervals, before and after the introduction of foreign cars, and allow the cutoff year to vary from 1972 to 1980. Let σ_1 and σ_2 be the standard deviations of the time series $\ln(N_t/N_{t-1})$ for the periods 1950-1979 and 1980-2001, respectively, where N_t denotes the unit auto sales per adult. For the same periods, let σ_3 and σ_4 be, respectively, the standard deviations of $\ln(E_t/E_{t-1})$, where E_t denotes the average real expenditure per car. Then

$$\sigma_1 = 16.0\%, \quad \sigma_2 = 7.7\%, \quad \sigma_3 = 4.0\%, \quad \text{and} \quad \sigma_4 = 5.5\%.$$

The difference between σ_1 and σ_2 is highly significant (P-value of the F-test is 0.0005), as is the difference between σ_3 and σ_4 (P-value 0.053).[11] These values do not change much as we move the cutoff from 1980 back to 1972.[12] In the 1980s and 90s there is a volatility

[10] The average is taken over 1967-1980.

[11] The time series for $\ln(N_t/N_{t-1})$ and $\ln(E_t/E_{t-1})$ do not show significant autocorrelation. The first series has the AR(1) coefficient of -0.18 (P-value 0.203) and the AR(2) coefficient of -0.14 (P-value 0.308). For the $\ln(E_t/E_{t-1})$ series, the AR(1) coefficient is -0.21 (P-value 0.160) and AR(2) coefficient is -0.01 (P-value 0.941).

[12] σ_1 changes monotonically from 16.0 to 17.2 and σ_2 varies almost monotonically in the range $[7.5, 9.2]$. The P-value for the F-test is in $[0.0001, 0.0024]$. Similarly, σ_3 stays in the interval $[3.7, 4.1]$ while σ_4 stays in the interval $[5.2, 5.3]$. The corresponding P-value for the F-test is in $[0.042, 0.117]$.

moderation in the number of cars per adult, and a somewhat weaker increase in the volatility of purchase size.

It is interesting that the volatility moderation in the number of autos is stronger than the accompanying increase in the volatility of purchase size, and there may be an explanation for it. There is a literature (e. g. Stock and Watson, 2002 and references therein) that presents evidence of a general decline in the cyclical volatility of the economic activity in the US since the early 1980s. Such a general decline would magnify the any volatility moderation and counteract any volatility increase.

7 Conclusions

We have presented a model of durable goods that highlights the difference between obsolescence and physical wear and tear. The model is simple and it can be solved analytically. It also has the flexibility to simultaneously match empirical regularities on consumption behavior that other models cannot. We identify periodic obsolescence as a distinct source of aggregate fluctuations, and explain why purchase timing is a major channel for the transmission of wealth shocks.

Our model offers a building block for a general equilibrium analysis of an investment problem with capital obsolescence. Periodic obsolescence makes investment spiky even at the aggregate level, although interest rate adjustments will partially smooth out these spikes. Our framework can generate cyclical investment patterns and suggests a relationship between technological innovations in capital goods and the business cycle.

References

[1] Adda, J. and R. Cooper, "The Dynamics of Car Sales: A Discrete Choice Approach", NBER Working Paper No 7785, July 2000.

[2] Attanasio, O., "Consumer Durables and Inertial Behaviour: Estimation and Aggregation of (S, s) Rules for Automobile Purchases", *Review of Economic Studies* 67(4), 667-96, 2000.

[3] Attanasio, O., "Consumption", Handbook of Macroeconomics. Volume 1B, p. 741-812, 1999

[4] Bar-Ilan, A. and A. Blinder, "Consumer Durables: Evidence on the Optimality of Usually Doing Nothing", *Journal of Money, Credit, and Banking* 24(2), 258-72, 1992.

[5] Caballero, R. J. and E. Engel, "Explaining Investment Dynamics in U.S. Manufacturing: A Generalized (S, s) Approach", *Econometrica* 67(4), 783-826, 1999.

[6] Caballero, R. J., "Durable Goods: An Explanation for Their Slow Adjustment", *Journal of Political Economy* 101(2), 351-84, 1993.

[7] Caballero, R. J., "Expenditure on Durable Goods: A Case for Slow Adjustment", *Quarterly Journal of Economics* 105(3), 727-43, 1990.

[8] Caplin, A. and J. V. Leahy, "Durable Goods Cycles", National Bureau of Economic Research Working Paper #6987, February 1999.

[9] Carroll, Christopher D., and Wendy E. Dunn, "Unemployment Expectations, Jumping (s,S) Triggers, and Household Balance Sheets", in *NBER Macroeconomics Annual*, 1997, ed. by Benjamin S. Bernanke and Julio Rotemberg, pp. 165-229. MIT Press, Cambridge, MA.

[10] Cummins, J. G. and G. L. Violante, "Investment-Specific Technical Change in the United States (1947-2000): Measurement and Macroeconomic Consequences", *Review of Economic Dynamics* 5(2), 243-84, 2002.

[11] Eberly, J. C., "Adjustment of Consumers' Durables Stocks: Evidence from Automobile Purchases", *Journal of Political Economy* 102(3), 403-36, 1994.

[12] Fishman, A. and R. Rob, "Product Innovation by a Durable-Good Monopoly", *RAND Journal of Economics* 31(2), 237-52, 2000.

[13] Grossman, S. J. and G. Laroque, "Asset Pricing and Optimal Portfolio Choice in the Presence of Illiquid Durable Consumption Goods", *Econometrica* 58(1), 25-51, 1990.

[14] Leahy, J. V. and J. Zeira, "The Timing of Purchases and Aggregate Fluctuations", NBER Working Paper No. 7672, April 2000.

[15] Mankiw, G., "Hall's Consumption Hypothesis and Durable Goods", *Journal of Monetary Economics* 10(3), 417-25, 1982.

[16] Stock, James H., Mark W. Watson, "Has the Business Cycle Changed and Why?", NBER Working Paper No. 9127, August 2002.

8 Appendix 1: Proofs

Detrending: Our model can be viewed as the detrended version of a fully dynamic model with a constant rate of technical progress. Suppose that a model τ provides a *constant* service flow z_τ for the duration of its useful life, in the interval $[\tau, \tau+T)$, and that $z_\tau = e^{g\tau}$, where g is the rate of technical progress, or, equivalently, the rate of decrease of the quality-adjusted price for the durable. Now assume that the consumers' utility function is $\hat{v}(z,c) = \ln(z) + u(c)$, where z is the service flow of the durable good and c is the flow of non-durable consumption. This dynamic model corresponds to the stationary model we propose when

$$x_\alpha = g(T-\alpha) \quad \text{for } \alpha = 0, \ldots, T.$$

Indeed, let $\alpha : \mathbf{R}_+ \to \{0, \ldots, T\}$ and $c : \mathbf{R}_+ \to \mathbf{R}_+$ be two measurable functions representing the consumption trajectory of a consumer (where $\alpha(t)$ is the technological age of the durable being consumed at time t). For any $r \in \mathbf{R}$, let $\lfloor r \rfloor$ denote the largest integer less than or equal to r. Note that along that trajectory, the model being consumed at time t is $\tau(t) = \lfloor t \rfloor - \alpha(t)$. Thus, the total discounted utility for the trajectory (α, c) is

$$\begin{aligned} U(\alpha,c) &= \int_0^\infty e^{-\rho t}[\ln(z_{\tau(t)}) + u(c(t))]dt = \int_0^\infty e^{-\rho t}[g(\lfloor t \rfloor - T + T - \alpha(t)) + u(c(t))]dt \\ &= K + \int_0^\infty e^{-\rho t}[x_{\alpha(t)} + u(c(t))]dt, \end{aligned}$$

where

$$K = \int_0^\infty e^{-\rho t} g(\lfloor t \rfloor - T)dt = \sum_{k=0}^\infty gk \int_0^1 e^{-\rho(k+t)}dt - \frac{gT}{\rho} = \frac{g}{\rho}\left[\frac{e^{-\rho}}{1-e^{-\rho}} - T\right].$$

Arbitrarily, we can re-normalize utility to set $K = 0$ without changing the consumer's preferences over consumption paths. Then, the total discounted utility coincides with that of a consumer with utility function $v(\alpha, c) = x_\alpha + u(c)$.

Proof of Theorem 1: Suppose the agent is endowed with a durable of age α and follows an arbitrary purchasing policy $\tau = \{\tau_k\}_{k=1}^\infty$. We first show that the total cost and value (b, v) of policy τ can be represented as a convex combination of the points $\{(b_{T,R}, v_{T,R})\}_{R=1}^{T+1}$. Let $\tau_0 = -\alpha$ and $r_k = \min\{\tau_{k+1} - \tau_k, T\}$ for all $k \geq 0$. Then

$$b = p_0 \sum_{k\geq 1} \beta^{\tau_k} \quad \text{and} \quad v = X_{\alpha, r_0} + \sum_{k \geq 1} \beta^{\tau_k} X_{0, r_k}.$$

Define $K_R = \{k \geq 1 \mid r_k = R\}$ for $R = 1, \ldots, T$. Then

$$\beta^{\tau_1} = [\beta^{\tau_1} - \beta^{\tau_2}] + [\beta^{\tau_2} - \beta^{\tau_3}] + \cdots \geq \sum_{R=1}^{T} \sum_{k \in K_R} \beta^{\tau_k}(1-\beta^R),$$

where the inequality is strict if for some $k \in K_T$, $\tau_{k+1} - \tau_k > T$. Let $\lambda_R = \sum_{K_R} \beta^{\tau_k}(1-\beta^R)$ for $R = 1, \ldots, T$, and let $\lambda_{T+1} = 1 - \sum_{R=1}^T \lambda_R$. Thus $\lambda_R \geq 0$ for all R, $\sum_{R=1}^{T+1} \lambda_R = 1$, and since $b_{T,T+1} = v_{T,T+1} = 0$,

$$b = p_0 \sum_{R=1}^{T} \sum_{k \in K_R} \beta^{\tau_k} = \sum_{R=1}^{T} \sum_{k \in K_R} \beta^{\tau_k}(1-\beta^R) b_{T,R} = \sum_{R=1}^{T+1} \lambda_R b_{T,R}$$

$$v - X_{\alpha,r_0} = \sum_{R=1}^{T} X_{0,R} \sum_{k \in K_R} \beta^{\tau_k} = \sum_{R=1}^{T} \frac{X_{0,R}}{1-\beta^R} \sum_{k \in K_R} \beta^{\tau_k}(1-\beta^R) = \sum_{R=1}^{T+1} \lambda_R v_{T,R}.$$

Put differently, $(b, v - X_{\alpha,r_0}) = \sum_R \lambda_R (b_{T,R}, v_{T,R})$ is a convex combination of the two-dimensional vectors $(b_{T,R}, v_{T,R})$. Note that when $\alpha = T$, $X_{\alpha,r_0} = 0$ for all r_0.[13]

We next deduce an optimal policy for the case where $\alpha = T$ (i.e., when the agent is endowed with a useless durable). If $b \geq b_{T,1}$, the agent can afford to replace the durable every period and $V_T(b) = v_{T,1}$ (moreover, if $b > b_{T,1}$, it is not possible for the agent to spend the budget b in durables). For what follows assume that $b < b_{T,1}$. Let R and $\lambda_R^* \in [0,1]$ be such that $b = \lambda_R^* b_R + (1-\lambda_R^*) b_{R+1}$. Since $(b, V_T(b)) = \sum \lambda_R (b_{T,R}, v_{T,R})$ for some nonnegative weights λ_R adding to 1, we have that $V_T(b) \leq \lambda_R^* v_{T,R} + (1-\lambda_R^*) v_{T,R+1}$. To conclude, we only need to show that this bound is attained. For this we need to show that there exists a policy τ such that $\sum_{k \in K_R} \beta^{\tau_k}(1-\beta^R) = \lambda_R^*$ and $\sum_{k \in K_{R+1}} \beta^{\tau_k}(1-\beta^{R+1}) = 1 - \lambda_R^*$. Put differently, we need to show that there exists an R-flexible rule with budget b.

Assume that $R < T$ and let B_R^* denote the set of budgets $b(\tau)$ corresponding to policies τ that are R-flexible rules and satisfy $\tau_1 = 0$ (that is, τ makes a purchase in the first period). Let τ be such a policy and τ' be its continuation policy from the period of the second purchase onward: $\tau'_t = \tau_{t+1} - \tau_1$ for all $t \geq 1$. Then, τ' is also an R-flexible rule and $\tau'_1 = 0$ and its corresponding budget $b(\tau') \in B_R^*$. Now, either $b(\tau) = p_0 + \beta^R b(\tau')$ (if $\tau_2 = R$) or $b(\tau) = p_0 + \beta^{R+1} b(\tau')$ (if $\tau_2 = R+1$). Therefore, B_R^* is the largest set B such that $B = [p_0 + \beta^R B] \cup [p_0 + \beta^{R+1} B]$. Observe that $p_0 + \beta^{R+1} b_{T,R+1} = b_{T,R+1}$ and $p_0 + \beta^R b_{T,R} = b_{T,R}$, and that $p_0 + \beta^R b_{T,R+1} < p_0 + \beta^{R+1} b_{T,R}$ when $\beta^{T-1}(1+\beta) > 1$. Therefore $B = [b_{T,R+1}, b_{T,R}]$ is a fixed point of the above equation. Since $p_0 + \beta^R d < d$ for all $d > b_{T,R}$ and $p_0 + \beta^{R+1} d > d$ for all $d < b_{T,R+1}$, B is also the largest such fixed point, and thus $B_R^* = B$. That is, for each budget $b \in B_R^* = [b_{T,R+1}, b_{T,R}]$ there exists a (R,b)-flexible rule (that spends the budget b exactly). The proof for $R = T$ is similar (here $b_{T,T+1} = 0$ and we must consider policies τ where $\tau_{k+1} - \tau_k > T+1$ for some k).

Finally, observe that if (T,b) is the initial state and τ and $\hat{\tau}$ are two (R,b)-flexible rules (they spend the same budget b), then their corresponding λ_R (and $1 - \lambda_R$) must coincide, and therefore they must have the same value as well. In particular, if $b \in [b_{T,R+1}, b_{T,R}]$, then *any* R-flexible rule that spends the budget b exactly is an optimal policy.

By construction, the value of following an (R,b)-flexible rule starting from a durable of age T is given by

$$V_T(b) = v_{T,R+1} + A_R (b - b_{T,R+1}), \; b \in [b_{T,R+1}, b_{T,R}], \; R = T, \ldots, 1.$$

When the endowed durable is of age $\alpha < T$, the corresponding optimal value function $V_\alpha(b)$ can be deduced from $V_T(b)$ from the observation that the continuation of an optimal policy is an optimal policy for the subproblem that arises in the second period after following the policy in the first period.

[13] For each $R = 1, \ldots, R$, we could define instead $L_R = \{\tau_k \mid k \in \mathbf{N} \text{ and } r_k = R\}$, as we did in Section 3. Then, $\lambda_R = \sum_{t \in L_R} \beta^t$. While K_R contains the purchase numbers, L_R contains the purchase periods of durables that are disposed at age R. However, for other purposes, the set K_R is more convenient.

If starting with a budget $b \in [p_0, b_{T,1}] = [b_{T+1,T+1}, b_{T,1}]$, a consumer buys a durable in the first period and then keeps it for the next $\alpha - 1$ periods, his budget at the beginning of period $\alpha \geq 1$ is $\theta_\alpha(b) = (b - p_0)/\beta^\alpha$. Moreover, for any $1 \leq R \leq T+1$ and $1 \leq \alpha \leq \min\{R, T\}$, $\theta_\alpha(b_{T,R}) = b_{\alpha,R}$.

Assume that the initial state is (α, b), where $1 \leq \alpha < T$ and $b \in [b_{\alpha,R+1}, b_{\alpha,R}]$ for some $\alpha \leq R \leq T$. Let $\tilde{b} = p_0 + \beta^\alpha b$. Then $b = \theta_\alpha(\tilde{b})$. Since $b \in [b_{\alpha,R+1}, b_{\alpha,R}]$, it must be that $\tilde{b} \in [b_{T,R+1}, b_{T,R}]$. Therefore, starting at state $(T, \tilde{b})$, it is optimal to follow an R-flexible rule. Assume he does so. Then, after α periods his state becomes (α, b), and from state (α, b) he must be following an R-flexible rule as well. Hence, the agent must keep the durable for another $R - \alpha$ periods (at least). At that point, he arrives at state $(R, b/\beta^{R-\alpha})$. Note that $(1/\beta^{R-\alpha})[b_{\alpha,R+1}, b_{\alpha,R}] = [b_{R,R+1}, b_{R,R}]$ and that $\beta^{R-\alpha} b_{R+1,R+1} = b_{\alpha+1,R+1} \in (b_{\alpha,R+1}, b_{\alpha,R})$. Hence, if $b/\beta^{R-\alpha} \in [b_{R,R+1}, b_{R+1,R+1})$ he must keep the durable this period and buy a new durable next period, so his continuation value is $\hat{x}_R + V_T(b/\beta^{R+1-\alpha})$. If $b/\beta^{R-\alpha} \in [b_{R+1,R+1}, b_{R,R}]$ he can optimally buy a new durable this period, and his continuation value is $V_T(b/\beta^{R-\alpha})$. Therefore

$$V_\alpha(b) = \begin{cases} X_{\alpha,R+1} + \beta^{R+1-\alpha} V_T(b/\beta^{R+1-\alpha}) & \text{for } b \in [b_{\alpha,R+1}, b_{\alpha+1,R+1}) \\ X_{\alpha,R} + \beta^{R-\alpha} V_T(b/\beta^{R-\alpha}) & \text{for } b \in [b_{\alpha+1,R+1}, b_{\alpha,R}]. \end{cases}$$

Suppose that $b \in [b_{\alpha,R+1}, b_{\alpha+1,R+1})$. Then $b/\beta^{R+1-\alpha} \in [b_{R+1,R+1}, b_{R+1,R+1}/\beta) \subset [b_{T,R+1}, b_{T,R}]$. Therefore, $V_T(b/\beta^{R+1-\alpha}) = v_{T,R+1} + A_R(b/\beta^{R+1-\alpha} - b_{T,R+1})$, and

$$X_{\alpha,R+1} + \beta^{R+1-\alpha} V_T(b/\beta^{R+1-\alpha}) = v_{\alpha,R+1} + A_R(b - b_{\alpha,R+1}).$$

Now suppose that $b \in [b_{\alpha+1,R+1}, b_{\alpha,R}]$. Then $b/\beta^{R-\alpha} \in [b_{R,R+1}, b_{R,R}/\beta) \subset [b_{T,R+1}, b_{T,R}]$. Therefore, $V_T(b/\beta^{R-\alpha}) = v_{T,R+1} + A_R\left(b/\beta^{R-\alpha} - b_{T,R+1}\right)$, and tedious algebra shows again that

$$X_{\alpha,R} + \beta^{R-\alpha} V_T(b/\beta^{R-\alpha}) = v_{\alpha,R+1} + A_R(b - b_{\alpha,R+1}).$$

Therefore, for all $\alpha \leq R \leq T$ and $b \in [b_{\alpha,R+1}, b_{\alpha,R}]$, $V_\alpha(b) = v_{\alpha,R+1} + A_R(b - b_{\alpha,R+1})$. It remains to find $V_\alpha(b)$ for $b > b_{\alpha,\alpha}$. We claim that $V_\alpha(b) = V_T(b)$ for all $b > b_{\alpha,\alpha}$. Since $b_{\alpha,R} = b_{T,R}$ for all $R \leq \alpha$, we have that $V_T(b) = v_{\alpha,R+1} + A_R(b - b_{\alpha,R+1})$ for all $b \in [b_{\alpha,R+1}, b_{\alpha,R}]$ and $1 \leq R < \alpha$, and the claim would complete the proof. To prove our claim, we show that

$$V_T(b) > X_{\alpha,s+\alpha} + \beta^s V_T(b/\beta^s) \text{ for all } s > 0 \text{ and } b > b_{\alpha,\alpha}.$$

That is, when $b > b_{\alpha,\alpha}$, the consumer strictly prefers to replace the durable immediately than to replace it at any later time. One can check that the above inequality holds when $b = b_{\alpha,\alpha}$. Also, since V_T is concave, the function $V_T(b)$ has a higher slope than the function on the right hand side for any $b > 0$. Hence, the inequality holds for every $b > b_{\alpha,\alpha}$. ■

Proof of Theorem 2: Consider an arbitrary purchasing policy $\{(\tau_k, d_k)\}_{k=1}^\infty$, where $\tau_k + d_k$ denotes the time of the k-th purchase and $\tau_k \in \mathbf{N}$ its corresponding period (so $d_k \in [0, 1)$ denotes its "delay"). Let $\tau_0 = -\alpha$ and $r_k = \min\{\tau_{k+1} - \tau_k, T\}$ for all $k \geq 0$. The budget and value of such a policy are respectively

$$b = p_0 \sum_{k \geq 1} \beta^{\tau_k} e^{-\rho d_k} = p_0 \sum_{k \geq 1} \beta^{\tau_k} - p_0 \sum_{k \geq 1} \beta^{\tau_k}(1 - e^{-\rho d_k})$$

$$v = X_{\alpha,r_0} + \sum_{k \geq 1} \beta^{\tau_k} X_{0,r_k} - \sum_{k \geq 1} \beta^{\tau_k}(x_0 - x_{r_{k-1}})(1 - e^{-\rho d_k})/\rho.$$

For $1 \leq R \leq T$, let $K_R = \{k \geq 1 \mid r_k = R\}$,

$$\lambda_R = \sum_{k \in K_R} \beta^{\tau_k}(1-\beta^R), \quad \bar{\gamma}_R = \sum_{k-1 \in K_R} \beta^{\tau_k} \quad \text{and} \quad \gamma_R = \sum_{k-1 \in K_R} \beta^{\tau_k} \left[\frac{1-e^{-\rho d_k}}{1-\beta}\right], \tag{5}$$

so that

$$\begin{bmatrix} b \\ v \end{bmatrix} = \begin{bmatrix} \hat{b} \\ \hat{v} \end{bmatrix} - \sum_{R=1}^{T} \gamma_R \begin{bmatrix} p_0(1-\beta) \\ \hat{x}_0 - \hat{x}_R \end{bmatrix}, \quad \text{where} \quad \begin{bmatrix} \hat{b} \\ \hat{v} \end{bmatrix} = \begin{bmatrix} 0 \\ X_{\alpha,r_0} \end{bmatrix} + \sum_{R=1}^{T} \lambda_R \begin{bmatrix} b_{T,R} \\ v_{T,R} \end{bmatrix}.$$

Observe that for $1 \leq R \leq T-1$, $k-1 \in K_R$ implies that $\tau_k = \tau_{k-1} + R$ (if $k-1 \in R_T$ then $\tau_k \geq \tau_{k-1} + T$, where strict inequality holds when a useless good is not replaced for one or more periods). Therefore

$$\bar{\gamma}_R = \left[\frac{\beta^R}{1-\beta^R}\right] \lambda_R \quad \text{for all } 1 \leq R \leq T-1, \text{ and}$$
$$\sum_{R=1}^{T} \frac{\lambda_R}{1-\beta^R} = \sum_{k \geq 1} \beta^{\tau_k} = \sum_{R=1}^{T} \bar{\gamma}_R = \sum_{R=1}^{T-1} \frac{\beta^R}{1-\beta^R} \lambda_R + \bar{\gamma}_T.$$

Thus, $\bar{\gamma}_T = \sum_{R=1}^{T-1} \lambda_R + \lambda_T/(1-\beta^T)$. Let $\mu = (1-e^{-\rho S})/(1-\beta)$ (note that $\mu < 1$), $\Lambda = \{\lambda \in \mathbf{R}_+^T \mid \sum_{R=1}^{T} \lambda_R \leq 1\}$, and Γ be the set of all $(\lambda, \gamma) \in \Lambda \times \mathbf{R}_+^T$ such that

$$\gamma_R \leq \mu \lambda_R \left[\frac{\beta^R}{1-\beta^R}\right] \quad \text{for } 1 \leq R \leq T-1, \text{ and} \quad \gamma_T \leq \mu \Big[\sum_{R=1}^{T-1} \lambda_R + \frac{\lambda_T}{1-\beta^T}\Big].$$

CLAIM 1: Let $\{(\tau_k, d_k)\}$ be an arbitrary purchasing policy and (λ, γ) be the weights defined by (5). Then $(\lambda, \gamma) \in \Gamma$. Conversely, for any $(\lambda, \gamma) \in \Gamma$, there exists a purchasing policy $\{(\tau_k, d_k)\}$ that satisfies (5). Though this policy is usually not unique, all such policies have the same budget and value. Thus, with abuse of notation we will also refer to a $(\lambda, \gamma) \in \Gamma$ as a purchasing policy.

CLAIM 2: Suppose that the policy corresponds to an R-flexible rule where $\tau_1 = 0$ and the replacement of durables of age $R+1$ is never delayed but the replacement of durables of age R is sometimes delayed. Then, the policy is suboptimal: there exists another R-flexible rule without delays that costs the same and has a strictly higher value.

Proof: For such a policy, $\lambda_R + \lambda_{R+1} = 1$, $\gamma_R > 0$, $\gamma_{R+1} = 0$, and $\lambda_k = \gamma_k = 0$ for all $k \notin \{R, R+1\}$. Moreover, since $\gamma_R < \lambda_R \beta^R/(1-\beta^R)$, we also have $\lambda_R > 0$. In this case, $(\hat{b}, \hat{v})$ is on the "Pareto frontier" (i.e., $\hat{v} = V_\alpha(\hat{b})$). The vector $(\hat{b}, \hat{v}) - (b, v) = (p_0(1-\beta), \hat{x}_0 - \hat{x}_R)$ has "slope" $\sigma = [\hat{x}_0 - \hat{x}_R]/[p_0(1-\beta)]$, and

$$A_R = \frac{1}{p_0}\left[X_{0,R} - \hat{x}_R \frac{1-\beta^R}{1-\beta}\right] \leq (1-\beta^R)\frac{\hat{x}_0 - \hat{x}_R}{p_0(1-\beta)} < \sigma.$$

So, as the delays increase (γ_R increases), (b, v) moves away of $(\hat{b}, \hat{v})$, below the Pareto frontier. But, if $\sigma < A_{R+1}$, the delays may eventually take (b, v) back above the Pareto frontier. This could happen only if $b < b_{T,R+1}$. But even if every durable of age R is replaced with delay, the cost of the policy is more than replacing the durables at age $R+1$ all the time. That is,

$b \geq b_{T,R+1}$. Therefore $b_{T,R+1} \leq b \leq b_{T,R}$ and $v < V_T(b)$, and there exists another R-flexible rule with no delays that costs b and has value $V_T(b)$.

CLAIM 3: Suppose that the policy $\{(\tau_k, d_k)\}$ is such that $\gamma_k > 0$ for some k. Then the policy is suboptimal: there exists another policy without delays that uses the same budget but has strictly higher value.

Assume that the policy has delays. We now recursively modify the policy by eliminating delays while maintaining the same budget and improving its value in every step. Let $h = \lambda_1 + \lambda_2$, $\hat{\lambda}_k = \lambda_k/h$ for $k = 1, 2$, and $\hat{\gamma}_1 = \gamma_1/h$. Then

$$\begin{bmatrix} b \\ v \end{bmatrix} = h \left[\hat{\lambda}_1 \begin{bmatrix} b_{T,1} \\ v_{T,1} \end{bmatrix} + \hat{\lambda}_2 \begin{bmatrix} b_{T,2} \\ v_{T,2} \end{bmatrix} - \hat{\gamma}_1 \begin{bmatrix} p_0(1-\beta) \\ \hat{x}_0 - \hat{x}_1 \end{bmatrix} \right] + \sum_{R=2}^{T} \left[\lambda_R \begin{bmatrix} b_{T,R} \\ v_{T,R} \end{bmatrix} - \gamma_R \begin{bmatrix} p_0(1-\beta) \\ \hat{x}_0 - \hat{x}_R \end{bmatrix} \right].$$

The weights $(\hat{\lambda}_1, \hat{\lambda}_2, \hat{\gamma}_1)$ represent a 1-flexible rule with delays (and $\hat{\lambda}_1 + \hat{\lambda}_2 = 1$). If $\gamma_1 > 0$ (so $\hat{\gamma}_1 > 0$), then by Claim 3 there exists another 1-flexible rule with weights $(\tilde{\lambda}_1, \tilde{\lambda}_2, 0)$ that is better. Let $\lambda'_k = h\tilde{\lambda}_k$ for $k = 1, 2$, $\gamma'_1 = 0$, $\lambda'_k = \lambda_k$ for $k \geq 3$, and $\gamma'_k = \gamma_k$ for $k \geq 2$. The policy (λ', γ') is better than the policy (λ, γ) and has $\gamma'_1 = 0$. Now, let $h = \lambda'_2 + \lambda'_3$, $\hat{\lambda}_k = \lambda'_k/h$ for $k = 2, 3$, and $\hat{\gamma}_2 = \gamma'_2/h$. The weights $(\hat{\lambda}_2, \hat{\lambda}_3, \hat{\gamma}_2)$ represent a 2-flexible rule with delays. Again, if $\hat{\gamma}_2 > 0$, Claim 3 implies that there exists a better 2-flexible rule without delays that can be used to modify (λ', γ') and construct a new policy (λ'', γ'') that is better, uses the same budget, and has $\gamma''_1 = \gamma''_2 = 0$. Continuing this way, after T steps, we will have constructed a policy (λ^*, γ^*) with $\gamma^* = 0$, that uses the same budget and has a better value than (λ, γ).

Finally, by Claim 2 (or Theorem 1), for any weights λ^*, there exist R and an R-flexible rule that uses the same budget $b = \sum_k \lambda_k b_{T,k}$ and delivers a (weakly) better value. Therefore, the optimal value function V_T for the continuous-time economy coincides with that for the discrete-time economy (as defined in Theorem 1). ■

Proof of Theorem 3: Recall that we defined $c_0 = \bar{w}(1-\beta) - p_0$ and $c_{T+1} = \underline{w}(1-\beta)$, so that $w_{1,1}(c_0) = \bar{w}$ and $w_{T,T+1}(c_{T+1}) = \underline{w}$.

Let $B(w, c) = w - c/(1-\beta)$ be the budget left for durables when the total wealth is w and the agent consumes a constant per period budget c on non-durables. For fixed α and w, the function $\varphi(c) = \hat{u}(c)/(1-\beta) + V_\alpha(B(w, c))$ is concave. Thus $\hat{c}$ maximizes $\varphi(c)$ if and only if $0 \in \partial\varphi(\hat{c})$ (that is, 0 is a subdifferntial of φ at $\hat{c}$) or equivalently, if and only if $\hat{u}'(\hat{c}) \in \partial V_\alpha(B(w, \hat{c}))$. There are two cases corresponding to the situations where (1) V_α is differentiable at $B(w, \hat{c})$; and (2) V_α has a kink at $B(w, \hat{c})$.

Case 1: Observe that $B(w, c_R) \in (b_{\alpha,R+1}, b_{\alpha,R})$ if and only if $w \in (w_{\alpha,R+1}(c_R), w_{\alpha,R}(c_R))$. Now, if $B(w, c_R) \in (b_{\alpha,R+1}, b_{\alpha,R})$ for some R, then $\hat{u}'(c_R) = A_R = V'_\alpha(B(w, c_R))$, and c_R is the optimal solution of problem (1). That is, when $w \in (w_{\alpha,R+1}(c_R), w_{\alpha,R}(c_R))$, it is optimal to consume a constant flow c_R of non-durables and follow an R-flexible purchasing rule for the durable good. One can check that $B(w, c_\alpha) = b_{\alpha+1,\alpha+1} \Leftrightarrow w = w_{\alpha+1,\alpha+1}(c_\alpha)$ and $B(w, c_\alpha) = b_{\alpha-1,\alpha} \Leftrightarrow w = w_{\alpha-1,\alpha}(c_\alpha)$, and $w_{\alpha,\alpha+1}(c_\alpha) < w_{\alpha+1,\alpha+1}(c_\alpha) < w_{\alpha-1,\alpha}(c_\alpha) < w_{\alpha,\alpha}(c_\alpha)$. Therefore, $\delta^*(w)$ is given by (4).

Case 2: Observe that $A_{R-1} \leq \hat{u}'(c_{\alpha,R}(w)) \leq A_R$ if and only if $c_R \leq c_{\alpha,R}(w) \leq c_{R-1}$, or alternatively, if and only if $w \in [w_{\alpha,R}(c_R), w_{\alpha,R}(c_{R-1})]$. Since $B(w, c_{\alpha,R}(w)) = b_{\alpha,R}$ and $\partial V_\alpha(b_{\alpha,R}) = [A_{R-1}, A_R]$, if $\hat{u}'(c_{\alpha,R}(w)) \in [A_{R-1}, A_R]$ for some R, then $c_{\alpha,R}(w)$ is the optimal solution of problem (1). That is, it is optimal to consume a constant flow $c_{\alpha,R}(w)$ of non-durables and follow the R-fixed purchasing rule for the durable good. In particular, $\delta^*(w) = 1$ if $R \leq \alpha$ (or equivalently, if $w \geq w_{\alpha,\alpha}(c_\alpha)$) and $\delta^*(w) = 0$ if $R > \alpha$, as stated in (4).

For a fixed α, the intervals corresponding to case 1 alternate with those corresponding to case 2. Moreover, collectively, they are mutually exclusive and cover the whole wealth range. ■

Proof of Proposition 1:

(i) Since $w_{\alpha,R}(c_{R-1}) - w_{\alpha,R}(c_R) = (c_{R-1} - c_R)/(1-\beta)$, the total size of the fixed-rule classes is

$$\varphi_T = \frac{1}{1-\beta}\sum_{R=1}^{T} R[c_{R-1} - c_R] + \frac{c_T}{1-\beta} - \underline{w} = \frac{1}{1-\beta}\sum_{R=1}^{T-1}[c_R - c_T] + \bar{w} - \underline{w} - \frac{p_0}{1-\beta}.$$

(Recall that $c_0 = (1-\beta)\bar{w} - p_0$ and $c_{T+1} = (1-\beta)\underline{w}$.) Similarly, since $w_{\alpha,R}(c_R) - w_{\alpha,R+1}(c_R) = b_{\alpha,R} - b_{\alpha,R+1}$, we have that $\mu(C_{R,R+1}) = p_0/(1-\beta^R)$ for $1 \leq R \leq T-1$, and $\mu(C_{T,T+1}) = p_0/(1-\beta)$. Therefore, the total size of the flexible-rule classes is

$$\psi_T = \sum_{R=1}^{T-1} \frac{p_0}{1-\beta^R} + \frac{p_0}{1-\beta}.$$

Note that $\theta = \psi_T/[\psi_T + \varphi_T]$ and that ψ_T does not depend on g. Therefore,

$$\frac{\partial\theta}{\partial g} = \frac{-\psi_T}{(\psi_T+\varphi_T)^2}\frac{\partial\varphi_T}{\partial g} \qquad \text{and} \qquad \frac{\partial\varphi_T}{\partial g} = \frac{1}{1-\beta}\sum_{R=1}^{T-1}\left(\frac{\partial c_R}{\partial g} - \frac{\partial c_T}{\partial g}\right).$$

A simple computation shows that

$$A_R = \frac{1}{p_0}\sum_{\alpha=0}^{R-1}\beta^\alpha(\hat{x}_\alpha - \hat{x}_R) = g\left[\frac{1-\beta}{\rho p_0}\right]\sum_{\alpha=0}^{R-1}\beta^\alpha(R-\alpha).$$

Since $\hat{u}'(c_R) = A_R$,

$$\frac{\partial c_R}{\partial g} = \frac{1}{\hat{u}''(c_R)}\frac{\partial A_R}{\partial g} = \frac{1}{\hat{u}''(c_R)}\frac{A_R}{g} = \frac{1}{g}\frac{\hat{u}'(c_R)}{\hat{u}''(c_R)}.$$

If u has decreasing (increasing) absolute risk aversion then so does $\hat{u}$, and since $c_1 > c_2 > \cdots > c_T$, for every $R = 1, \ldots, T-1$,

$$\frac{\partial c_R}{\partial g} - \frac{\partial c_T}{\partial g} = \frac{1}{g}\left[\frac{\hat{u}'(c_R)}{\hat{u}''(c_R)} - \frac{\hat{u}'(c_T)}{\hat{u}''(c_T)}\right] < 0 \ (> 0).$$

(ii) Let $x_\alpha = g(T-\alpha)$ for $\alpha = 1, \ldots, T$ and $x'_\alpha = g(T'-\alpha)$ for $\alpha = 1, \ldots, T'$, where $T < T'$. Then, for all $R = 1, \ldots, T$,

$$A_R = \frac{1}{p_0}\sum_{\alpha=0}^{R-1}\beta^\alpha(\hat{x}_\alpha - \hat{x}_R) = \frac{1}{p_0}\sum_{\alpha=0}^{R-1}\beta^\alpha(\hat{x}'_\alpha - \hat{x}'_R) = A'_R.$$

This implies that the economy where durables last T periods and the economy where durables last $T' > T$ periods have identical consumption levels $c_1, \ldots, c_T$.

Clearly, θ is increasing in T if and only if

$$\frac{\varphi_{T+1}}{\varphi_T} < \frac{\psi_{T+1}}{\psi_T}. \tag{6}$$

For all consumption classes to be non-empty, the interval $[\underline{w}, \bar{w}]$ must be such that $\bar{w}(1-\beta) > c_1 + p_0$ and $\underline{w}(1-\beta) < c_T$. We now set $\underline{w} = 0$, in order to guarantee that class C_{T+1} is non-empty for all T. Inequality (6) is harder to satisfy for smaller values of $\bar{w}$. Therefore, we set $\bar{w} = (c_1 + p_0)/(1-\beta)$. If inequality (6) is satisfied for this $\bar{w}$ then it must also hold for any larger $\bar{w}$. Since $\hat{u}(c) = [(1-\beta)/\rho]^\gamma [c^{1-\gamma}/(1-\gamma)]$, $c_R = (1-\beta)A_R^{-1/\gamma}/\rho$. The left hand side of (6) is then

$$\frac{\varphi_{T+1}}{\varphi_T} = \frac{A_1^{-1/\gamma} + \sum_{R=1}^{T}\left(A_R^{-1/\gamma} - A_T^{-1/\gamma}\right)}{A_1^{-1/\gamma} + \sum_{R=1}^{T-1}\left(A_R^{-1/\gamma} - A_{T-1}^{-1/\gamma}\right)}.$$

Noticed that since A_R is proportional to g, the above expression does not depend on g, and is only a function of β and γ. Similarly, ψ_{T+1}/ψ_T depends only on β. Numerical computations shows that there exists $\hat{\gamma}(\beta) > 0$ such that (6) holds for all $\gamma \in (0, \hat{\gamma}(\beta)]$. If we let $\gamma^* = \min_\beta \hat{\gamma}(\beta) = 1.36$, then, independent of β, θ is increasing in T. ∎

Proof of Proposition 2: Let S_k denote the size of the k-th purchased durable. When the endowed durable is of size S_0 and age α, the value and the cost of a durable purchasing policy (τ, S) are respectively

$$v = S_0^\eta X_{\alpha,r_0} + \sum_{k\geq 1} \beta^{\tau_k} S_k^\eta X_{0,r_k} \qquad \text{and} \qquad b = p_0 \sum_{k\geq 1} \beta^{\tau_k} S_k.$$

Given a budget b, the consumer wants to maximize the value v. The first-order condition for S_k is $\eta S_k^{\eta-1} X_{0,r_k} = \mu p_0$, where $\mu > 0$ is a Lagrange multiplier. This condition implies that the optimal sizes depend on the holding time only. That is, for each R for which $K_R \neq \emptyset$, there exists a common size S_R such that $S_k = S_R$ for all $k \in K_R$. Then, following the notation in the proof of Theorem 1, the policy (τ, S) can alternatively be represented as a policy $(\lambda, \hat{S})$, and its corresponding value and budget can be expressed as

$$v = \sum_{R=1}^{T+1} \lambda_R \hat{S}_R^\eta v_{T,R} = \sum_{R=1}^{T+1} \lambda_R \left[\frac{v_{T,R}}{b_{T,R}^\eta}\right] B_R^\eta \qquad \text{and} \qquad b = \sum_{R=1}^{T+1} \lambda_R \hat{S}_R b_{T,R} = \sum_{R=1}^{T+1} \lambda_R B_R,$$

where $B_R = b_{T,R}\hat{S}_R$ for $1 \leq R \leq T$ and $B_{T+1} = 0$. In this format, the consumer chooses the weights λ_R so as to maximize v while satisfying the budget constraint. By definition, $[v_{T,R^*}/b_{T,R^*}^\eta] > [v_{T,R}/b_{T,R}^\eta]$ for all $R \neq R^*$. Therefore, it is optimal to set $\lambda_{R^*} = 1$ and $\lambda_R = 0$ for all $R \neq R^*$. That is, for any budget b, the optimal policy is an R^*-fixed rule with a constant size, where the size $S^*(b)$ is adjusted to spend the budget b exactly: $S^*(b) = b/b_{T,R^*}$. ∎

Proof of Proposition 3: As argued in the proof of Proposition 2, in an optimal policy, for each R there exists an optimal size $\hat{S}_R$ such that whenever a durable of age R is replaced,

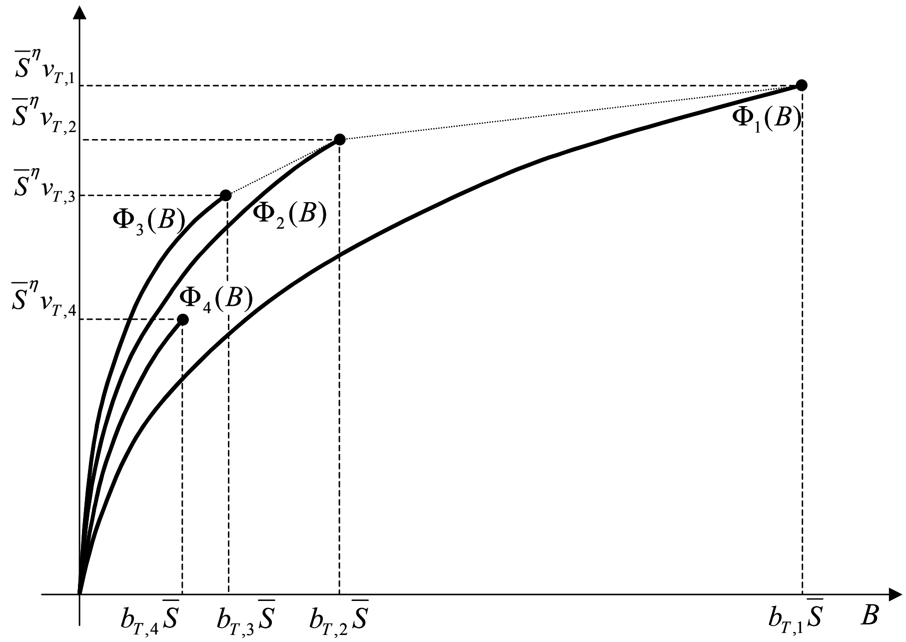

Figure 7: Functions $\Phi_R(B)$ for $T = 4$ and $R^* = 3$.

the new durable is of size $\hat{S}_R$. Let $\Phi_R(B) = [v_{T,R}/b_{T,R}^{\eta}]B^{\eta}$ for $B \in [0, \bar{S}b_{T,R}]$. Then, the optimal durable value function is

$$
\begin{aligned}
J(b) = \max \ & \sum_{R=1}^{T} \lambda_R \Phi_R(B_R) \\
\text{s.t.} \ & \sum_{R=1}^{T} \lambda_R B_R = b, \quad \sum_{R=1}^{T} \lambda_R = 1 \\
& B_R \in [0, \bar{S}b_{T,R}], \quad \lambda_R \geq 0, \quad 1 \leq R \leq T.
\end{aligned}
$$

That is, $J(b)$ is the convex envelope of the functions Φ_R on $[0, \bar{S}b_{T,R}]$, $1 \leq R \leq T$. Since $v_{T,R}/b_{T,R}^{\eta}$ is single-peaked in R and $\bar{S}b_{T,R}$ is monotonically decreasing in R, the graphs of the functions Φ_R are ordered in the following fashion. For each $R < R^*$, the graph of Φ_{R+1} lies above (or "dominates") the graph of Φ_R on the interval $[0, \bar{S}b_{T,R+1}]$, but while the graph of Φ_{R+1} stops at $\bar{S}b_{T,R+1}$, the graph of Φ_R extends to the right until $\bar{S}b_{T,R} > \bar{S}b_{T,R+1}$. In the other direction, for each $R > R^*$, the graph of Φ_{R^*} dominates the graph of every Φ_R and the domain of the former $[0, \bar{S}b_{T,R^*}]$ includes the domain of the latter $[0, \bar{S}b_{T,R}]$. Therefore, $J(b) = \Phi_{R^*}(b)$ for all $b \in [0, \bar{S}b_{T,R^*}]$. To the right of $\bar{S}b_{T,R^*}$, we show that the assumption on η implies that J is the piecewise linear function obtained by joining for each $R < R^*$ the vertices $(\bar{S}b_{T,R+1}, \Phi_{R+1}(\bar{S}b_{T,R+1}))$ and $(\bar{S}b_{T,R}, \Phi_R(\bar{S}b_{T,R}))$. Note that $\Phi_R(\bar{S}b_{T,R}) = \bar{S}^{\eta}v_{T,R}$, so that the piecewise linear function coincides with $\bar{S}^{\eta}V_T(b/\bar{S})$ for $b \in [\bar{S}b_{T,R^*}, \bar{S}b_{T,1}]$.

To verify our last claim, all we need to do is to check that for each $R < R^*$, the straight line segment from $(\bar{S}b_{T,R+1}, \Phi_{R+1}(\bar{S}b_{T,R+1}))$ to $(\bar{S}b_{T,R}, \Phi_R(\bar{S}b_{T,R}))$ lies above the graph of Φ_R in the interval $[\bar{S}b_{T,R+1}, \bar{S}b_{T,R}]$. Since Φ_R is concave and the slope of the line segment is

$\bar{S}^{\eta-1}A_R$, this is the case if and only if

$$\Phi_R'(\bar{S}b_{T,R}) = \eta \left[\frac{v_{T,R}}{b_{T,R}^{\eta}}\right][\bar{S}b_{T,R}]^{\eta-1} \geq \bar{S}^{\eta-1}A_R.$$

That is, if and only if $\eta \geq A_R b_{T,R}/v_{T,R}$ for all $R < R^*$. Our assumption on η guarantees the last inequality independent of the value of R^*. ■

9 Appendix 2: Alternative models of durables

The frictionless PIH model: The stock of durable good, K_t, evolves according to

$$K_{t+1} = (1-\delta)\,K_t + E_t,$$

where $\delta < 1$ is the rate of economic depreciation and E_t is the current expenditure on durable goods. Service flow from the durable is proportional to K_t. Assume the interest rate r is constant and satisfies $\beta = 1/(1+r)$, where β is the discount factor. A consumer's problem is[14]

$$\max_{(c_t,K_t)} \sum_{t\geq 0}\beta^t\left[\ln c_t + \ln K_t\right] \quad \text{subject to} \quad \sum_{t\geq 0}\frac{1}{(1+r)^t}\left(c_t + K_{t+1} - (1-\delta)K_t\right) = w.$$

The optimal solution is $K_t = Aw$ and $c_t = Bw$ for all t, where $A = (1-\beta)/[r+2\delta]$ and $B = (r+\delta)A$. Therefore, the optimal durable expenditure every period is $E = \delta Aw$. Suppose that a shock changes the wealth from w to $(1+\epsilon)w$. Then, the non-durable consumption level changes from Bw to $(1+\epsilon)Bw$ and desired durable stock changes from Aw to $(1+\epsilon)Aw$. Therefore, the current period durable expenditure is $(1+\epsilon)Aw - (1-\delta)Aw = (1+\epsilon/\delta)\delta Aw$. That is, the *short-run* wealth elasticity of demand is 1 for the non-durable good and $1/\delta$ for the durable good.

The (s,S) replacement model: Consider a model in continuous time with continuous technological obsolescence, where a durable good of age $t \geq 0$ and size S provides a service flow $S^\eta x(t)$. Assume that $\eta \in (0,1)$, and that $x(t) > 0$ and $x'(t) < 0$ for all $t \geq 0$. The discount factor is ρ, the interest rate is $r = \rho$, and the price of a durable of size S is $p_0 S$. A consumer with wealth w chooses non-durable consumption flow c, durable purchase size S and holding time R to maximize lifetime utility subject to a wealth constraint. That is, his problem is

$$\max_{c,S,R}\frac{u(c)}{\rho} + S^\eta\frac{\int_0^R x(t)e^{-\rho t}dt}{1-e^{-\rho R}} \quad \text{subject to} \quad \frac{c}{\rho} + \frac{p_0 S}{1-e^{-\rho R}} = w.$$

The first order conditions for this problem are

$$u'(c) = \lambda, \qquad \eta S^{\eta-1}\int_0^R x(t)e^{-\rho t}dt = \lambda p_0, \quad \text{and}$$

$$S^\eta\left[\frac{\int_0^R x(t)e^{-\rho t}dt}{(1-e^{-\rho R})^2}\rho e^{-\rho R} - \frac{x(R)e^{-\rho R}}{1-e^{-\rho R}}\right] = \frac{\lambda p_0 S}{(1-e^{-\rho R})^2}\rho e^{-\rho R}.$$

[14] The results are similar for a more general class of preferences and other values of r. See, for example, Carroll and Dunn (1997).

The last condition can be rewritten as

$$S^{\eta} \int_0^R [x(t) - x(R)] e^{-\rho t} dt = \lambda p_0 S.$$

Substituting the FOC for S into FOC for R, we get

$$\int_0^R [x(t) - x(R)] e^{-\rho t} dt = \eta \int_0^R x(t) e^{-\rho t} dt.$$

Therefore, the optimal holding time does not depend on wealth. Given that every consumer replaces durables with the same frequency, c and S are strictly increasing in w. Therefore, R will not respond to an aggregate wealth shock, but both c and S will.