The Effects of Heterogeneity in Price Setting on Price and Inflation Inertia

Carlos Viana de Carvalho*
Department of Economics
Princeton University

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Abstract

This paper analyzes the implications of heterogeneity in price setting for both price and inflation inertia. Standard models based on Taylor- or Calvo-style price setting usually assume ex-ante identical firms, while Calvo's approach implies only ex-post heterogeneity. While it is known that these models have different implications in terms of the dynamic effects of monetary shocks, the role of heterogeneity has not yet been properly explored. In a simple framework, this paper provides analytical results which suggest that ex-ante heterogeneity should have a larger role in models which attempt to understand price and inflation inertia, particularly in low inflation environments.

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*Financial support from Princeton University, under the Harold Willis Dodds Merit Fellowship in Economics, is gratefully acknowledged. Address for correspondence: Department of Economics, Princeton University, Fisher Hall, Princeton, NJ 08544-1021, USA. E-mail: cvianac@princeton.edu.

1 Introduction

Standard models of nominal price rigidity based on the seminal contributions of either Taylor (1979, 1980) or Calvo (1983) usually do not involve any explicit attempt to model heterogeneity in price-setting rules followed by firms. They are usually assumed to be ex-ante identical, modulo the time at which they adjust prices (and therefore the prices that they choose). This in turn implies that they are also ex-post identical in Taylor's specification and heterogeneous in Calvo's only as a direct result of randomization.[1]

There are many reasons why in reality firms could differ with respect to their pricing policies. Menu-costs could be more relevant for some firms, while for others information-gathering costs might dominate. Also, the nature of shocks to the firms' optimal prices as well as the degree of competition across sectors might be quite different. We know from the literature on optimal pricing rules that these differences should be translated into heterogeneity in terms of price setting behavior. In addition to the theoretical arguments, there is also ample evidence that firms do in fact differ substantially in terms of the frequency of price adjustments (see, for instance, Blinder et al., 1998 and Bils and Klenow, 2003). So, apart from analytical convenience, the only reason not to take heterogeneity explicitly into account would be if it did not matter. In this paper I argue that this is not the case.[2]

Recently, a lot of effort has been devoted to understanding the implications of different price setting assumptions for the dynamics of monetary economies.[3] To some extent, this has been

[1] Such randomization was introduced for tractability reasons (more specifically to reduce the number of state variables in the model), not to generate (ex-post) heterogeneity in the duration of price rigidity. This is why I do not consider Calvo's paper to be explicitly concerned with heterogeneity in price setting.

[2] I focus on heterogeneity with respect to the (average) frequency of price adjustments or, alternatively, to the (average) duration of price rigidity. Heterogeneity could also play an important role in other dimensions. For instance, Bonomo (1992) studies the effects of the interaction between firms which follow state-dependent rules and firms which adjust in a time-dependent fashion.

[3] A common departure from the basic settings has been to change Calvo's pricing to account for some sort of indexation (Yun (1996), Gali and Gertler (1999), Woodford (2003)). More recently, some papers which use different assumptions are Mankiw and Reis (2002) who postulate Calvo-style randomization with respect to the arrival of information, rather than the opportunity to adjust prices, and Calvo, Celasun and Kumhof (2003), who assume that firms choose (linear) price paths rather than levels. In a related paper, Devereux and Yetman (2003) compare the

motivated by two facts: i) a widespread perception that the basic Taylor and Calvo models have similar dynamic implications; ii) the failure of these models to generate realistic results in some important dimensions, most notably in terms of inflation inertia.[4]

Taking an important step back, Kiley (2002) argues that there are both quantitative and qualitative differences between Taylor and Calvo models which the literature has overlooked. He shows that, for the same average duration of price rigidity[5], Calvo's specification tends to generate much higher welfare costs of inflation and more persistent real effects of monetary shocks than Taylor's. These results stem from the differences in the distribution of relative prices implied by the two models, which is a direct result of ex-post heterogeneity in the duration of price rigidity in Calvo's model.

While our understanding of the differences between these two leading specifications has improved, the role of heterogeneity in these models has not yet been properly explored. To what extent can heterogeneity be a source of persistence by itself? Are there important differences between ex-ante and ex-post heterogeneity? Once there is heterogeneity, does it matter for the comparison between Taylor and Calvo pricing that we match the average duration of price rigidity rather than the average frequency of price adjustments (its inverse)? Are the implications different in terms of price and inflation inertia?

Some recent papers which involve heterogeneity in terms of price setting are Ohanian et al. (1995), Bils and Klenow (2003), Bils et al. (2003) and Barsky et al. (2003). However, none of them focuses on isolating its effects. This requires comparing models with ex-ante heterogeneity with otherwise identical models in which all firms are ex-ante identical, keeping constant the average frequency of price adjustment/duration of price rigidity. One exception is Woodford (2003), who explores this comparison to argue that ex-ante heterogeneity in the context of Calvo-pricing does have important implications in terms of optimal monetary policy.

Perhaps the work that is most related to this paper is that of Caballero and Engel (1991, 1993). In a different framework in which firms follow state- rather than time-dependent pricing rules,

dynamic implications of fixed-prices versus predetermined price paths in a Calvo-style framework.

[4] While Taylor's model does generate some inflation inertia, Calvo-pricing generates none.

[5] The expression "contract length" will be used occasionally as a substitute for "duration of price rigidity".

they show that ex-ante ("structural") heterogeneity has important effects in terms of the dynamic response of the economy to shocks. In particular, they show that under some circumstances it may slow down the economy's adjustment process. Heterogeneity also plays a role in terms of guaranteeing existence and uniqueness of equilibrium.

In this paper I use a simple continuous time model to try to answer the questions raised above in the context of time-dependent pricing rules. First, assuming heterogeneity in price setting to be exogenous, I provide analytical results which show that it has important implications for the degree of price and inflation stickiness.[6] Then, building on the model of endogenous time-dependent rules developed by Bonomo and Carvalho (2003), I argue that heterogeneity tends to be higher in low inflation economies. This conclusion is also supported by evidence presented in Lach and Tsiddon (1992).

I analyze shocks to the level of the money supply and to its growth rate, and study the behavior of the economy through impulse response functions and functionals of them. The first result is that in the presence of heterogeneity in price setting, the average duration of price rigidity and the average frequency of price adjustment are no longer equivalent in terms of summarizing the degree of nominal rigidity in the economy. While this is the case in identical-firms economies, with heterogeneous firms it turns out that for a given average frequency of price adjustments, the average duration of price rigidity is increasing in the degree of heterogeneity. This raises the question of what is the appropriate way to model the representative firm for a benchmark identical firms economy, given an economy with arbitrary degree of heterogeneity. I show that, in general, this is an empirical question: for a given heterogeneous economy, the duration of price rigidity for the identical firms economy which minimizes the distance between the impulse response functions depends on the whole distribution of contract lengths. Nevertheless, there is a sensible measure of overall inertia which, surprisingly, does not. More precisely, I show that the (normalized) cumulative output effects only depend on the mean of such distribution (in the case of shocks to the money supply)

[6] In order to obtain analytical results, comparison of ex-ante and ex-post heterogeneity is done in the context of Calvo pricing, while comparison of ex-ante and "no heterogeneity" is based on Taylor's approach. I also need to abstract from strategic complementarities in price setting, and illustrate their effects in a separate section with numerical simulations.

or on the mean and the variance (in the case of shocks to the growth rate of the money supply). Furthermore, in the latter case I show that such measure of inertia turns out to be proportional to the sum of the square of the average and the variance of the distribution of contract lengths.

The rest of the paper is organized as follows. Section 2 presents the basic setup and introduces Calvo pricing and Taylor staggered price setting in separate subsections. Section 3 discusses issues related to the assumption of existence of a "representative firm" in terms of adjustment frequency, as usually postulated in the literature. In subsequent sections I discuss the role of heterogeneity for price and inflation inertia, respectively. Results which are specific to either Calvo's or Taylor's specifications are presented in separate subsections. Section 6 presents some numerical results which indicate that heterogeneity interacts with strategic complementarities in price setting to make the process of adjustment to shocks even more sluggish than in a corresponding identical firms economy. In section 7, I endogeneize the degree of heterogeneity and argue that it tends to be higher the lower the inflation rate. The last section concludes. All proofs are in the appendices.

2 Basic setup

This section presents the basic setup, with the assumptions which are common to the models with Taylor and Calvo pricing which will be developed subsequently. The details of each pricing specification are introduced in separate subsections. The modeling strategy is to introduce ex-ante heterogeneity into a standard dynamic sticky price model, building on the static model results of Blanchard and Kiyotaki (1987), and Ball and Romer (1989). Starting from the specification of preferences, endowments and technology, these models derive individual optimal price equations at each moment as a function of aggregate demand (Ball and Romer) or directly as a function of the money supply and price level (Blanchard and Kiyotaki). Here, for simplicity, I choose to model the demand side of the economy in the simplest possible way and to use reduced form equations which are derived from first principles in those papers. The model is set in continuous time.[7]

In the economy there is a continuum of imperfectly competitive firms divided into groups, which differ with respect to the expected frequency of price adjustment. Firms will be indexed by

[7] For a derivation of the discrete-time counterpart to this model from first principles, see Woodford (2003).

their group, n, and by $i \in [0,1]$. Each group is indexed by its expected duration of price rigidity $n \in [0, n^*]$, which therefore ranges from "continuous adjustment" to "adjustment at intervals of length n^*." The distribution of firms across groups is summarized by a density function $f(\cdot)$ on $[0, n^*]$, with cdf $F(\cdot)$.[8] The degree of heterogeneity can be measured by the dispersion of such distribution.

In the absence of frictions to price adjustment it is assumed that the optimal level of the individual relative price, which is the same for all firms, is given by (all variables are in log):

$$p^*(t) - p(t) = \theta y(t) \tag{1}$$

where p^* is the individual frictionless optimal price, p is the aggregate price level and y is aggregate demand.[9] For simplicity, $p(t)$ is evaluated according to:

$$p(t) = \int_0^{n^*} f(n) \int_0^1 p_{n,i}(t) didn$$

where $p_{n,i}(t)$ is the price charged by firm i from group n at time t. The corresponding "identical firms" benchmark economy in each case will be obtained by assuming that all firms behave as "the representative firm" in a sense to be made precise in the next sections.

For simplicity, nominal aggregate demand is given by the quantity of money:

$$y(t) + p(t) = m(t)$$

Substituting the above equation into equation 1 yields:[10]

$$p^*(t) = \theta m(t) + (1-\theta)p(t) \tag{2}$$

If there were no frictions to price adjustment, each firm would choose $p_{n,i}(t) = p^*(t)$ and the resulting aggregate price level would be $p(t) = m(t)$. Thus, aggregate output and individual prices

[8] All results hold in the case of discrete or mixed distributions.

[9] Equation 1 can be derived from utility maximization in a yeoman farmer economy, as in Ball and Romer (1989).

[10] This equation can also be derived directly from other specifications, such as Blanchard and Kiyotaki (1987), where real balances enter the utility function.

would be given by $y(t) = 0$ and $p_{n,i}(t) = m(t)$, respectively. To obtain analytical results, from here on I assume that $\theta = 1$. This rules out strategic complementarities in price setting, which usually amplify the effects of changes in monetary policy. Section 6 presents some numerical simulations which illustrate the role of strategic complementarities and show that the results are qualitatively similar.[11]

2.1 Calvo pricing

In this subsection frictions are introduced through pricing behavior *à la* Calvo (1983). For each firm, the opportunity to change prices arrives according to a Poisson process, with hazard rate given by the inverse of the expected duration of price rigidity for the firms' group ($\frac{1}{n}$).

For simplicity, based on a second order approximation to the loss incurred from not charging the optimal price, firms are assumed to set prices to minimize expected squared deviations from the optimal price:[12]

$$\begin{aligned} p_{n,i}(t) &= \arg\min_{x(t)} \int_0^\infty e^{-\frac{1}{n}s} E_t \left[x(t) - p^*(t+s)\right]^2 ds \\ &= \int_0^\infty \frac{1}{n} e^{-\frac{1}{n}s} E_t p^*(t+s)\, ds \end{aligned}$$

The aggregate price level is then given by:

$$p(t) = \int_0^{n^*} f(n) \int_{-\infty}^{t} \frac{1}{n} e^{-\frac{1}{n}(t-s)} p_{n,i}(s)\, ds dn$$

2.2 Taylor staggered price setting

This subsection introduces staggered price setting *à la* Taylor (1979, 1980). In this framework, firms are assumed to set prices for a fixed period of time. Firms from group n set prices for a

[11] This is the case both for time- and state-dependent pricing rules. For the former see, for example, Bonomo and Carvalho (2003). For state-dependent rules see Caplin and Leahy (1991,1997) and Caballero and Engel (1991, 1993).

[12] This yields results which correspond to the ones obtained in the now standard DSGE models with sticky prices, after log-linearization of the first order conditions.

period of length n. Adjustments are uniformly staggered across time in terms of both firms and groups.

Assuming the same second order approximation to the loss incurred from not charging the optimal price, firms set prices according to:

$$\begin{aligned} p_{n,i}(t) &= \arg\min_{x(t)} \int_0^n E_t \left[x(t) - p^*(t+s)\right]^2 ds \\ &= \frac{1}{n} \int_0^n E_t p^*(t+s)\, ds \end{aligned}$$

The aggregate price level is then given by:

$$p(t) = \int_0^{n^*} f(n) \left(\frac{1}{n} \int_0^n p_{n,i}(t-s)\ ds \right) dn$$

3 "The Representative Firm"

In order to isolate the effects of heterogeneity, one needs to construct a benchmark economy with identical firms, retaining the same degree of (ex-ante) nominal rigidity in some sense. With identical firms the degree of nominal rigidity can be equivalently summarized by the average duration of price rigidity or by the average frequency of price adjustment (its inverse). With heterogeneous firms this is no longer the case. Matching both economies in one dimension necessarily implies that the heterogeneous economy will display a higher average in the other dimension. This result is formalized below.

Proposition 1 *If the benchmark and the heterogeneous economy have the same average duration of price rigidity, then the heterogeneous economy will display a higher average frequency of price adjustment. Likewise, if both economies match in terms of average frequency of price adjustment, then the heterogeneous economy will have a higher average duration of price rigidity. Furthermore, the difference is increasing in the degree of heterogeneity.*

The implications of this result are not trivial. In particular, it raises the question of what is the appropriate way to construct the representative firm for the benchmark economy, given an economy with arbitrary degree of heterogeneity. Put differently, to construct an "identical firms" economy while keeping the same average degree of nominal rigidity, should one match the average duration of price rigidity or the average frequency of price adjustments? The results in the next sections help clarify this question in the context of specific price setting models.

4 Price inertia

This section analyzes the implications of heterogeneity for price-level inertia. For this purpose, I propose the following experiment: the economy is initially in a zero-inflation steady state with constant money supply, which for convenience is normalized to zero. At $t = 0$ the monetary authority announces an instantaneous, once-and-for-all increase in the money supply to $m(t) = \overline{m}$ for all $t \geq 0$. The announcement is fully believed. The behavior of firms in this experiment is straightforward: they set $p_{n,i}(t) = 0$ for $t < 0$ and then set $p_{n,i}(t) = \overline{m}$ whenever they get to change prices after $t = 0$.

In Calvo's specification the aggregate price level is given by:

$$\begin{aligned} p(t) &= \int_0^{n^*} f(n) \int_{-\infty}^{t} \frac{1}{n} e^{-\frac{1}{n}(t-s)} p_{n,i}(s)\, ds dn \\ &= \int_0^{n^*} f(n) \int_0^{t} \frac{1}{n} e^{-\frac{1}{n}(t-s)} \overline{m} ds dn \\ &= \overline{m} \int_0^{n^*} f(n) \left[1 - e^{-\frac{1}{n}t}\right] dn \\ &= \left(1 - \int_0^{n^*} f(n)\, e^{-\frac{1}{n}t} dn\right) \overline{m}, \end{aligned}$$

In the case of staggering *à la* Taylor, $p(t) = \overline{m}$ for $t \geq \mathbf{n}^*$. For $\mathbf{0} \leq \mathbf{t} < \mathbf{n}^*$, the aggregate price level is given by:

$$
\begin{aligned}
p(t) &= \int_0^t f(n)\left(\frac{1}{n}\int_0^n p_{n,i}(t-s)\ ds\right)dn + \int_t^{n^*} f(n)\left(\frac{1}{n}\int_0^t p_{n,i}(t-s)\ ds\right)dn \\
&= \int_0^t f(n)\,\overline{m}dn + \int_t^{n^*} f(n)\,\frac{t}{n}\overline{m}dn \\
&= \left(1-\int_t^{n^*} f(n)\,dn + \int_t^{n^*} f(n)\,\frac{t}{n}dn\right)\overline{m} \\
&= \left(1-\int_t^{n^*} f(n)\left(1-\frac{t}{n}\right)dn\right)\overline{m},
\end{aligned}
$$

In general, heterogeneity affects the dynamic response of the economy to such a shock to the money supply. For both price setting models, the path of the aggregate price level (or equivalently the impulse-response function) depends on the whole distribution of contract lengths. Figures 1 and 2 show the typical pattern of the aggregate price level under Taylor's and Calvo's specifications, respectively.[13] In each case the behavior of an identical-firms economy with the same average duration of price rigidity is also presented. Initially, adjustment is faster in the heterogeneous economy, because a relatively higher measure of firms with shorter contract lengths gets to adjust earlier. As time passes, the distribution of duration of price rigidity among firms which have not yet adjusted becomes more and more dominated by firms with relatively longer contract lengths. So, the speed of adjustment slows down through time, and, overall, the process takes longer when there is heterogeneity.

Recently, a strand of the literature has estimated some parameters in DSGE models with sticky prices by fitting the model's impulse response functions to those of an estimated VAR.[14] In the context of heterogeneity, this could also be a sensible way to pick the contract length for the benchmark identical firms economy.[15] The following result shows that, in general, the whole distribution of contract lengths will influence the outcome. I present it in the context of Calvo pricing, but a similar result holds for Taylor's specification. Also, using least absolute deviations instead of least squares yields qualitatively similar results.

[13] In both cases a uniform distribution over $[0, 2]$ was used. The pattern is the same, irrespective of the distribution. This can be show analytically, although it requires some tedious algebra.

[14] See, for example, Rotemberg and Woodford (1997) and Christiano et al. (2003).

[15] For a quantitative exploration of this idea, see the NBER Working Paper version of Bils and Klenow (2003).

Proposition 2 *Under Calvo pricing, the contract length for the identical firms economy which best matches the behavior of the heterogeneous economy (in the sense of minimizing the integral of the square of the differences between the aggregate price paths) depends on the distribution of contract lengths.*

This result shows that finding the benchmark economy that best matches the impulse response function of the heterogeneous economy (in the least squares sense) is an empirical question. In particular, constructing an identical firms economy simply by assuming the average contract length might not be the best alternative. Altough shown formally in the absence of strategic complementarities, the qualitative results should also hold in the general case (see section 6).

Surprisingly, there is a sensible measure of overall price inertia which does not depend on the distribution of contract lengths: the (normalized) cumulative output effects, given by $\frac{1}{\overline{m}} \int_0^\infty \overline{m} - p(t)\, dt$. It takes into account both the intensity and the persistence of the effects of the shock. The following proposition summarizes this result.

Proposition 3 *For both price setting models, although the path of the aggregate price level does depend on the distribution of pricing rules, price-level inertia as measured by* $\frac{1}{\overline{m}} \int_0^\infty \overline{m} - p(t)\, dt$ *is equal to the average duration of price rigidity in the economy (modulo a normalization).*

Corollary 1 *An economy with an arbitrary degree of heterogeneity will display more price-level inertia (as measured by the normalized cumulative output effects) than a benchmark identical-firms economy with the same average frequency of price adjustments. The difference is increasing in the degree of heterogeneity.*

These results refer to the (normalized) cumulative real effects of shocks to the money supply and, as such, do not take into account the differences in the shape of the impulse response functions. Nevertheless, they do suggest that matching the average duration of price rigidity is the right way to construct a benchmark identical firms economy if one is interested in such cumulative effects.

5 Inflation inertia

In this section I analyze the implications of heterogeneity for inflation inertia with the following experiment: the economy is initially in a constant inflation steady state with a constant rate of money growth (μ). At $t = 0$ the monetary authority announces a permanent change in the rate of money growth. After a normalization, this is equivalent to assuming that at $t = 0$ the monetary authority halts money growth and announces that from then on the money supply will be kept constant at $m(t) = 0$. The announcement is fully believed.

Although the quantitative equation does not take into account explicitly the effect of inflation reduction on money demand, one can interpret m as nominal aggregate demand, and assume that the monetary authority sets the trajectory of the money supply that corresponds to such path for nominal aggregate demand.[16] Under this interpretation, for example if the variable m is halted, money supply is increased by an amount just enough to satisfy the higher money demand due to lower inflation expectations and maintain the nominal aggregate demand constant.

In the inflationary steady state prior to $t = 0$, money supply is given by $m(t) = \mu t$. In the next subsections I derive the behavior of firms for each price setting model and evaluate the resulting inflation inertia. In Taylor's specification (but not in Calvo's), it is still the case that the time paths of aggregate variables depend on the whole distribution of contract lengths. I focus on the normalized cumulative effect on output ($\frac{1}{\mu}\int_0^\infty p(t)\,dt$), which again turns out to depend only on some moments of such distribution.

5.1 Calvo pricing

In the inflationary steady state, firm i from group n sets its price as

$$\begin{aligned} p_{n,i}(t) &= \int_0^\infty \frac{1}{n} e^{-\frac{1}{n}s} \mu(t+s)\,ds \\ &= \mu(t+n) \end{aligned}$$

[16] This strategy for modelling monetary disinflations is also followed by Mankiw and Reis (2002). See also Woodford (2003).

The aggregate price level is given by

$$\begin{aligned}
p(t) &= \int_0^{n^*} f(n) \int_{-\infty}^{t} \frac{1}{n} e^{-\frac{1}{n}(t-s)} p_{n,i}(s)\, ds dn \\
&= \int_0^{n^*} f(n) \int_{-\infty}^{t} \frac{1}{n} e^{-\frac{1}{n}(t-s)} \mu(s+n)\, ds dn \\
&= \mu t,
\end{aligned}$$

and output is constant at the natural rate $y(t) = 0$.

After $t = 0$, every firm that has the opportunity to adjust its price sets it equal to zero. So, after the announcement the aggregate price level is given by:

$$\begin{aligned}
p(t) &= \int_0^{n^*} f(n) \int_{-\infty}^{0} \frac{1}{n} e^{-\frac{1}{n}(t-s)} p_{n,i}(s)\, ds dn \\
&= \int_0^{n^*} f(n) \int_{-\infty}^{0} \frac{1}{n} e^{-\frac{1}{n}(t-s)} \mu(s+n)\, ds dn \\
&= 0
\end{aligned}$$

That is, ex-ante heterogeneity does not change the fact that Calvo pricing is incapable of generating inflation inertia.

5.2 Taylor staggered price setting

In Taylor's model, prior to $t = 0$ firm i from group n sets its price as

$$\begin{aligned}
p_{n,i}(t) &= \frac{1}{n} \int_0^{n} \mu(t+s)\, ds \\
&= \mu\left(t + \frac{n}{2}\right)
\end{aligned}$$

and the aggregate price level is therefore given by

$$
\begin{aligned}
p(t) &= \int_0^{n^*} f(n) \left(\frac{1}{n} \int_0^n p_{n,i}(t-s) \; ds \right) dn \\
&= \int_0^{n^*} f(n) \left(\frac{1}{n} \int_0^n \mu \left(t - s + \frac{n}{2} \right) ds \right) dn \\
&= \mu t
\end{aligned}
$$

After $t = 0$, every firm that adjusts sets its price equal to zero. So, after the announcement the aggregate price level is given by (for $\mathbf{0} \leq \mathbf{t} \leq \mathbf{n}^*$):

$$
\begin{aligned}
p(t) &= \int_0^{n^*} f(n) \left(\frac{1}{n} \int_0^n p_{n,i}(t-s) \; ds \right) dn \\
&= \int_0^t \frac{f(n)}{n} \left(\int_0^n p_{n,i}(t-s) ds \right) dn + \int_t^{n^*} \frac{f(n)}{n} \left(\int_0^t p_{n,i}(t-s) ds + \int_t^n p_{n,i}(t-s) ds \right) dn \\
&= \int_t^{n^*} f(n) \left(\frac{1}{n} \left(\int_t^n \mu \left(t - s + \frac{n}{2} \right) ds \right) \right) dn \\
&= \frac{\mu}{2} t \int_t^{n^*} f(n) \left(1 - \frac{t}{n} \right) dn
\end{aligned}
$$

After $t = \mathbf{n}^*$ every firm has had the opportunity to adjust its price, and so $p(t) = 0$.

The following result summarizes the implications of heterogeneity in terms of inflation inertia in the context of Taylor pricing. It provides a nice decomposition of the cumulative output effects which shows that, given the average duration of price rigidity, inflation inertia increases one-to-one with the degree of heterogeneity in price setting (measured by the variance of the duration of price rigidity in the economy).

Proposition 4 *Define $\overline{n} = \int_0^{n^*} f(n) n dn$ (i.e., the average duration of price rigidity) and $\sigma_n^2 = \int_0^{n^*} f(n) (n - \overline{n})^2 dn$ (i.e., the variance of the duration of price rigidity). In the context of Taylor staggered price setting, although the path of the aggregate price level does depend on the distribution of pricing rules, inflation inertia as measured by $\frac{1}{\mu} \int_0^{\infty} p(t) dt$, is proportional to $\left(\overline{n}^2 + \sigma_n^2 \right)$.*

So, in a sense the effect of heterogeneity is of the same order as that of the average duration of price rigidity, and might be quantitatively very important. In the sample analyzed in Bils and Klenow (2003), for instance, the magnitudes are $\overline{n} = 6.6$ months, with $\sigma = 7.1$.

6 Strategic complementarities

When there are strategic complementarities ($\theta < 1$), analytical results are no longer available. Therefore, I provide some numerical simulations which illustrate that the results are qualitatively similar, and that, in quantitative terms, the effects of heterogeneity are amplified.

I compute the dynamic response of the economies in the context of the experiment described in section 5. In the identical firms benchmark economy, firms set prices for periods of length one. To use a particularly simple form of heterogeneity, I assume that in the heterogeneous economy half of the firms sets prices for periods of length $1-\alpha$, while the other half sets prices for $1+\alpha$ periods. The average duration of price rigidity is, therefore, the same in both economies.

Figure 3 displays the effects on output when $\alpha = 0.8$ and $\theta = 0.1$. The results for the benchmark economy are also depicted ($\alpha = 0$). In both cases, I also included the output paths when there are no strategic complementarities ($\theta = 1$).[17] With strategic complementarities adjustment is still more sluggish in the heterogeneous economy: the recession trough is delayed and output is lower than in the benchmark economy essentially during the whole process.[18] The cumulative effects on output can be obtained through numerical integration. With $\theta = 1$, the results are -0.08 for $\alpha = 0$, and -0.13 for $\alpha = 0.8$. Note that the ratio of this measure in the heterogeneous economy to the benchmark economy is 1.63, quite close to the theoretical value implied by proposition 5, which equals $\left(1+\alpha^2\right) = 1.64$. With $\theta = 0.1$, the results are -0.26 when $\alpha = 0$, and -0.56 for $\alpha = 0.8$. The ratio is now 2.15. Looking at the same result from another angle, note that the cumulative effects in the benchmark economy increase by a factor of 3.3 with strategic complementarities, while they increase by a factor of 4.27 in the heterogeneous economy. These results suggest that strategic compementarities interact with heterogeneity to make the adjustment process even more sluggish.

[17] In the simulations, I set $\mu = 1$ in the initial inflationary steady state.

[18] Formally, with strategic complementarities the adjustment process in the benchmark economy takes as long as in the heterogeneous economy, because output only returns to the natural level asymptotically.

7 Endogenous heterogeneity and inflation

Up to now I have assumed an exogenous degree of heterogeneity. In this section, based on the model of endogenous time-dependent rules proposed by Bonomo and Carvalho (2003), I argue that heterogeneity in price setting tends to be higher in low inflation economies. This conclusion is also supported by evidence presented in Lach and Tsiddon (1992).[19]

The modeling strategy is to introduce heterogeneity by assuming that firms face different adjustment/information gathering costs.[20] A single lump-sum cost is assumed to allow the firm to gather information about its optimal price and make price adjustments. Such assumption is appealing because it rationalizes pricing behavior *à la* Taylor.

In the economy there is a continuum of imperfectly competitive firms facing different adjustment/information gathering (lump-sum) costs (k). Each firm is indexed by (i,k), where i is distributed uniformly over $[0,1]$ and k is distributed over $[0,\overline{k}]$. The distribution of firms across groups is summarized by a density function $g(\cdot)$ on $\left[0,\overline{k}\right]$.[21] Therefore, k has the role of grouping firms according to the lump-sum cost they face.

The equation which determines the frictionless optimal price is (1), with the addition of a firm-specific shock:

$$p_{i,k}^{*}(t) - p(t) = \theta y(t) + e_{i,k}(t)$$

For each firm (i,k), $e_{i,k}$ follows a driftless Brownian motion with coefficient of diffusion σ, $de_{i,k} = \sigma dw_{i,k}$. Those individual processes $e_{i,k}'s$ are assumed to be independent of each other.

The corresponding aggregate price level is now given by

$$p(t) = \int_0^{\overline{k}} g(k) \int_0^1 p_{i,k}(t) di dk$$

The above equations, combined with the equation for nominal aggregate demand, yield

[19] See their Table 4 in page 366 and Figure 4 in page 382.

[20] Differences in menu-costs are the source of heterogeneity in the state-dependent pricing policies in Caballero and Engel (1991, 1993). Dotsey et al. (1999) also assume that firms face different adjustment costs, but in their case there is only ex-post heterogeneity.

[21] As will be seen below, this will induce a distribution of optimal contract lengths.

$$p^*_{i,k}(t) = \theta m(t) + (1-\theta)p(t) + e_{i,k}(t) \tag{3}$$

If there were no costs to adjust prices and/or obtain information about the frictionless optimal price level, each firm would choose $p_{i,k}(t) = p^*_{i,k}(t)$ and the resulting aggregate price level would be $p(t) = m(t)$. Thus, aggregate output and individual prices would be given by $y(t) = 0$ and $p_{i,k}(t) = m(t) + e_{i,k}(t)$, respectively.

Following Bonomo and Carvalho (2003), firms can neither observe the stochastic components of $p^*_{i,k}$ nor adjust their prices based on the known components of $p^*_{i,k}$ without paying a lump-sum cost k (which varies with the firm's group). On the other hand, to let the price drift away from the optimal level entails expected profit losses, which flow at rate $E_{t_0}(p_{i,k}(t) - p^*_{i,k}(t))^2$, where t_0 is the last time of observation and adjustment. Time is discounted at a constant rate ρ.

I focus on inflationary steady states. Again, the money supply follows $m(t) = \mu t$. Given the stochastic process for the optimal price, each price setter solves for the optimal pricing rule. Appendix B shows that in steady state the aggregate price level grows at the same rate μ. As a consequence, the frictionless optimal price follows a Brownian motion with a drift given by the rate of the money supply growth:

$$dp^*_{i,k} = \mu dt + \sigma dw_{i,k} \tag{4}$$

So, each firm faces the optimization problem solved in Bonomo and Carvalho (2003), given its group-specific lump-sum cost. For a generic firm i from group k such problem can be formalized through the following Bellman equation:

$$V_\mu(k) = \min_{p_{i,k}(t),n} E_t \int_0^n [p_{i,k}(t) - p^*_{i,k}(t+s)]^2 e^{-\rho s} ds + ke^{-\rho n} + V_\mu(k)\ e^{-\rho n} \tag{5}$$

where $V_\mu(k)$ represents the value function for the steady state problem with money growth rate μ for a firm from group k.[22] The first order conditions are:

[22] The value function in steady state will be the same for all firms within each group, because it depends on the parameters of the stochastic process for $p^*_{i,k}$ and not on its realizations.

$$p_{i,k}(t) = \frac{\rho}{1-e^{-\rho n}} \int_0^n E_t \left[p_i^*(t+s)\right] e^{-\rho s} ds \tag{6}$$

$$\rho \left(V_\mu(k) + k\right) = E_t[p_{i,k}(t) - p_{i,k}^*(t+n)]^2 \tag{7}$$

Bonomo and Carvalho (2003) show that the resulting time-dependent pricing rule involves setting prices for intervals of length $n = \tau(k, \mu, \sigma, \rho)$ satisfying, in particular:[23]

$$\frac{\partial \tau(k,\mu)}{\partial \mu} \mu \quad < \quad 0 \tag{8}$$

$$\frac{\partial \tau(k,\mu)}{\partial k} \quad > \quad 0 \tag{9}$$

That is, the optimal duration of price rigidity decreases with the absolute value of the inflation rate, and increases with the lump-sum adjustment/information gathering cost. Thus, $g(\cdot)$ induces a distribution of optimal contract lengths over $[0, \overline{n}(\mu)]$, where $\overline{n}(\mu) = \tau\left(\overline{k}, \mu\right)$. It has density given by:[24]

$$f(n,\mu) = \frac{g\left(\tau^{-1}(n,\mu)\right)}{\frac{\partial \tau\left(\tau^{-1}(n,\mu),\mu\right)}{\partial k}}$$

where $\tau^{-1}(n,\mu)$ is the inverse with respect to the coordinate k, i.e., it satisfies $n = \tau\left(\tau^{-1}(n,\mu), \mu\right)$.

The fact that there is no closed form solution for $\tau(k, \mu, \sigma, \rho)$ makes it hard to derive analytical results for the effects of inflation on the degree of heterogeneity in price setting. Therefore, I provide numerical results which indicate that heterogeneity decreases with inflation.

The intuition is clear from Figure 4. An increase in the inflation rate from μ to μ' decreases the optimal duration of price rigidity for all groups. But the effect tends to be smaller the lower the adjustment/information gathering cost. To gain intuition on why this is so, notice that when $k = 0$ the optimal contract length is zero (continuous adjustment), irrespective of the inflation rate. Thus, higher inflation tends to generate a distribution of optimal contract lengths that is more concentrated on a smaller support. In the limit, as inflation increases without bounds, all

[23] The arguments σ and ρ will be omitted for notational simplicity.

[24] Just apply the Jacobian transformation.

firms tend to adjust very frequently, because the losses from price erosion induced by high inflation become large relative to all values of the lump-sum cost.[25]

Figures 5, 6 and 7 display distributions of optimal contract lengths for some parameter values. The support of the underlying distribution of lump-sum costs in all cases is $[0, \overline{k}] = [0.000001, 0.003]$. The distributions underlying $g(\cdot)$ are truncated normal for Figure 4, uniform in the case of Figure 5 and triangular for Figure 6.[26] The distributions of optimal contract lengths are displayed for inflation rates (μ) of 0.03, 0.1 and 1. The values of the other parameters are $\rho = 0.025$ and $\sigma = 0.03$ per year.

Table 1 reports some descriptive statistics which confirm that heterogeneity decreases with the inflation rate. For $\mu = 0.03$, optimal contract lengths range from two weeks to two years. With $\mu = 0.1$ the maximum contract length drops to 13 months. Finally, with $\mu = 1$ the optimal duration of price rigidity ranges from 6 days to 3 months. The standard deviation of the distribution of optimal contract lengths decreases monotonically with inflation for all distributions of lump-sum costs.

8 Concluding Remarks

In this paper, I argued that heterogeneity should have a larger role in models which attempt to understand price and inflation inertia, particularly in low inflation environments. Standard models of nominal price rigidity usually assume that all firms are identical in terms of price-setting behavior. This would be a good approximation either if empirically the degree of heterogeneity were small or if, despite significant in the real world, heterogeneity turned out not to matter much. Available empirical evidence points to the existence of a high degree of heterogeneity in price-setting behavior, and I provided results which show that it does matter significantly for the dynamic response of economies to monetary shocks.

[25] This result would still hold if heterogeneity was, instead, generated by differences in other parameters.

[26] Always centered around $\frac{\overline{k}}{2}$.

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Appendix A

Proposition 1 *If the benchmark and the heterogeneous economy have the same average duration of price rigidity, then the heterogeneous economy will display a higher average frequency of price adjustment. Likewise, if both economies match in terms of average frequency of price adjustment, then the heterogeneous economy will have a higher average duration of price rigidity. Furthermore, the difference is increasing in the degree of heterogeneity.*

Proof. The results follow directly from Jensen's inequality. Pick the average duration of price rigidity in the benchmark economy, $\overline{n}$, so that it matches that of the heterogeneous economy: $\overline{n} = \int_0^{n^*} f(n)\, n \, dn$. Then, the average frequency of price adjustments in the heterogeneous economy equals $\int_0^{n^*} f(n) \frac{1}{n} \, dn > \frac{1}{\int_0^{n^*} f(n) n dn} = \frac{1}{\overline{n}}$, which is the average frequency of price adjustments in the benchmark economy. The slack in the inequality is increasing in the dispersion of the distribution summarized by $f(\cdot)$. The proof for the other case is analogous. ■

Proposition 2 *Under Calvo pricing, the contract length for the identical firms economy which best matches the behavior of the heterogeneous economy (in the sense of minimizing the integral of the square of the differences between the aggregate price paths) depends on the distribution of contract lengths.*

Proof. In the context of the shock to the money supply described above, let $p_x(t)$ denote the aggregate price level in an identical firms economy with average duration of price rigidity equal to x. Then, for $t \geq 0$, $p_x(t) = \overline{m}\left(1 - e^{-\frac{1}{x}t}\right)$. The price level in the heterogeneous economy evolves according to $p(t) = \overline{m}\left(1 - \int_0^{n^*} f(n)\, e^{-\frac{1}{n}t} dn\right)$. The contract length x that minimizes the integral of the square of the differences between both price paths solves:

$$\min_x \int_0^{\infty} (p(t) - p_x(t))^2 \, dt \Longleftrightarrow \min_x \int_0^{\infty} \left(e^{-\frac{1}{x}t} - \int_0^{n^*} f(n)\, e^{-\frac{1}{n}t} dn\right)^2 dt$$

The first order condition is

$$\int_0^\infty \left(e^{-\frac{1}{x}t} - \int_0^{n^*} f(n)\, e^{-\frac{1}{n}t} dn\right) te^{-\frac{1}{x}t} dt = 0$$

$$\Longleftrightarrow \int_0^\infty \left(te^{-\frac{2}{x}t} - \int_0^{n^*} tf(n)\, e^{-\left(\frac{1}{n}+\frac{1}{x}\right)t} dn\right) dt = 0$$

$$\Longleftrightarrow \frac{1}{4} - \int_0^{n^*} f(n)\, \frac{n^2}{(n+x)^2} dn = 0$$

which depends on the whole distribution of contract lengths. ■

Proposition 3 *For both price setting models, although the path of the aggregate price level does depend on the distribution of pricing rules, price-level inertia as measured by* $\frac{1}{\overline{m}} \int_0^\infty \overline{m} - p(t)\, dt$ *is equal to the average duration of price rigidity in the economy (modulo a normalization).*

Proof. i) Calvo Pricing. The proposed measure of price inertia is given by

$$\begin{aligned}
\frac{1}{\overline{m}} \int_0^\infty \overline{m} - p(t)\, dt &= \int_0^\infty 1 - \left(1 - \int_0^{n^*} f(n)\, e^{-\frac{1}{n}t} dn\right) dt \\
&= \int_0^\infty \int_0^{n^*} f(n)\, e^{-\frac{1}{n}t} dndt \\
&= \int_0^{n^*} \int_0^\infty f(n)\, e^{-\frac{1}{n}t} dtdn \\
&= \int_0^{n^*} f(n) \int_0^\infty e^{-\frac{1}{n}t} dtdn \\
&= \int_0^{n^*} f(n)\, ndn,
\end{aligned}$$

which is exactly the average duration of price rigidity in the economy.ii) Taylor pricing.

$$\begin{aligned}
\frac{1}{\overline{m}} \int_0^\infty \overline{m} - p(t)\, dt &= \int_0^{n^*} 1 - \left(1 - \int_t^{n^*} f(n) \left(1 - \frac{t}{n}\right) dn\right) dt \\
&= \int_0^{n^*} \int_t^{n^*} f(n) \left(1 - \frac{t}{n}\right) dndt \\
&= \int_0^{n^*} 1 - F(n)\, dn - \frac{1}{2} \int_0^{n^*} nf(n)\, dn \\
&= \frac{1}{2} \int_0^{n^*} f(n)\, ndn
\end{aligned}$$

■

Corollary 1 *An economy with an arbitrary degree of heterogeneity will display more price-level inertia (as measured by the normalized cumulative output effects) than a benchmark identical-firms economy with the same average frequency of price adjustments. The difference is increasing in the degree of heterogeneity.*

Proof. The result follows directly from Propositions 1 and 2. ■

Proposition 4 *Define* $\overline{n} = \int_0^{n^*} f(n)\, ndn$ *(i.e., the average duration of price rigidity) and* $\sigma_n^2 = \int_0^{n^*} f(n)\,(n-\overline{n})^2\, dn$ *(i.e., the variance of the duration of price rigidity). In the context of Taylor staggered price setting, although the path of the aggregate price level does depend on the distribution of pricing rules, inflation inertia as measured by* $\frac{1}{\mu}\int_0^{\infty} p(t)\, dt$*, is proportional to* $\left(\overline{n}^2 + \sigma_n^2\right)$.

Proof. As measured by the normalized cumulative real effects on output, inflation inertia is given by

$$
\begin{aligned}
\frac{1}{\mu}\int_0^{\infty} p(t)\, dt &= \frac{1}{\mu}\int_0^{\infty} \frac{\mu}{2} t \int_t^{n^*} f(n)\left(1-\frac{t}{n}\right) dndt \\
&= \frac{1}{2}\left(\int_0^{n^*} t\,(1-F(t))\, dt - \int_0^{n^*}\left(t^2\int_t^{n^*} f(n)\,\frac{1}{n} dn\right) dt\right) \\
&= \frac{1}{2}\left(\frac{1}{2}\int_0^{n^*} f(t)\, t^2 dt - \int_0^{n^*}\left(\int_t^{n^*} f(n)\,\frac{1}{n} dn\right) t^2 dt\right) \\
&= \frac{1}{2}\left(\frac{1}{2}\int_0^{n^*} f(n)\, n^2 dn - \frac{1}{3}\int_0^{n^*} f(n)\, n^2 dn\right) \\
&= \frac{1}{12}\int_0^{n^*} f(n)\, n^2 dn \\
&= \frac{1}{12}\left(\overline{n}^2 + \sigma_n^2\right)
\end{aligned}
$$

■

Appendix B

Here I show that in steady state the aggregate price level does, in fact, grow at rate μ. Using the method of undetermined coefficients, assume that the price level evolves according to $p(t) = a + bt$. Plug this expression into (6) and aggregate according to:

$$p(t) = \int_0^{\overline{k}} g(k) \frac{1}{k} \int_0^{\tau(k,\mu)} p_k(t-s)\, dsdk$$

where $p_k(s)$ is the average price set by firms from group k which adjust at time s. Since within each group the idiosyncratic shock is the only component specific to firm i and vanishes with the averaging,

$$p_k(s) = p_{i,k}(s) - e_{i,k}(s)$$

The next step is to find the expressions for a and b that are consistent with the resulting equation for $p(t)$. This yields:

$$b = \mu$$

and

$$a = \int_0^{\overline{k}} g(k) \left[-\frac{1}{2}\mu \frac{\tau(k,\mu)\rho e^{\rho\tau(k,\mu)} + \rho\tau(k,\mu) - 2e^{\rho\tau(k,\mu)} + 2}{\theta\rho\left(-1 + e^{\rho\tau(k,\mu)}\right)} \right] dk$$

The resulting expression for $p(t)$ depends on the distribution of optimal contract lengths.

Finally, substitute this expression in (3) to arrive at the following expression for $p^*_{i,k}(t)$:

$$\begin{aligned} p^*_{i,k}(t) &= \theta m(t) + (1-\theta)p(t) + e_{i,k} \\ &= \mu t - \int_0^{\overline{k}} g(k) \left[-\frac{1}{2}\mu \frac{\tau(k,\mu)\rho e^{\rho\tau(k,\mu)} + \rho\tau(k,\mu) - 2e^{\rho\tau(k,\mu)} + 2}{\theta\rho\left(-1 + e^{\rho\tau(k,\mu)}\right)} \right] dk + e_{i,k} \end{aligned}$$

Thus, in steady state, both $p(t)$ and $p^*_{i,k}(t)$ grow at the same constant rate μ.

Impulse Response Functions: Taylor's Model

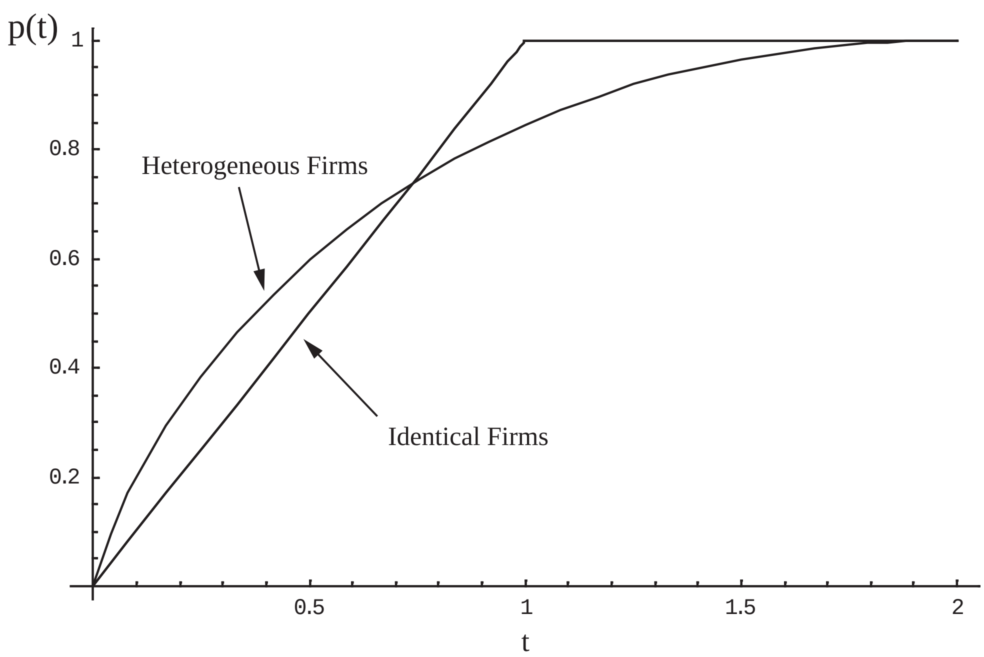

Figure 1

Impulse Response Functions: Calvo's Model

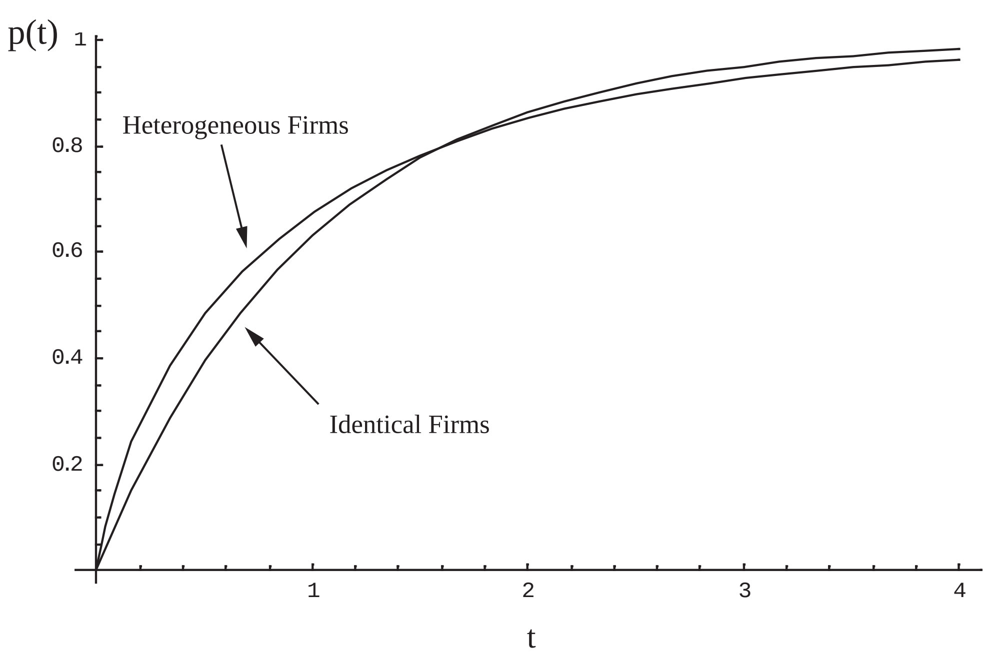

Figure 2

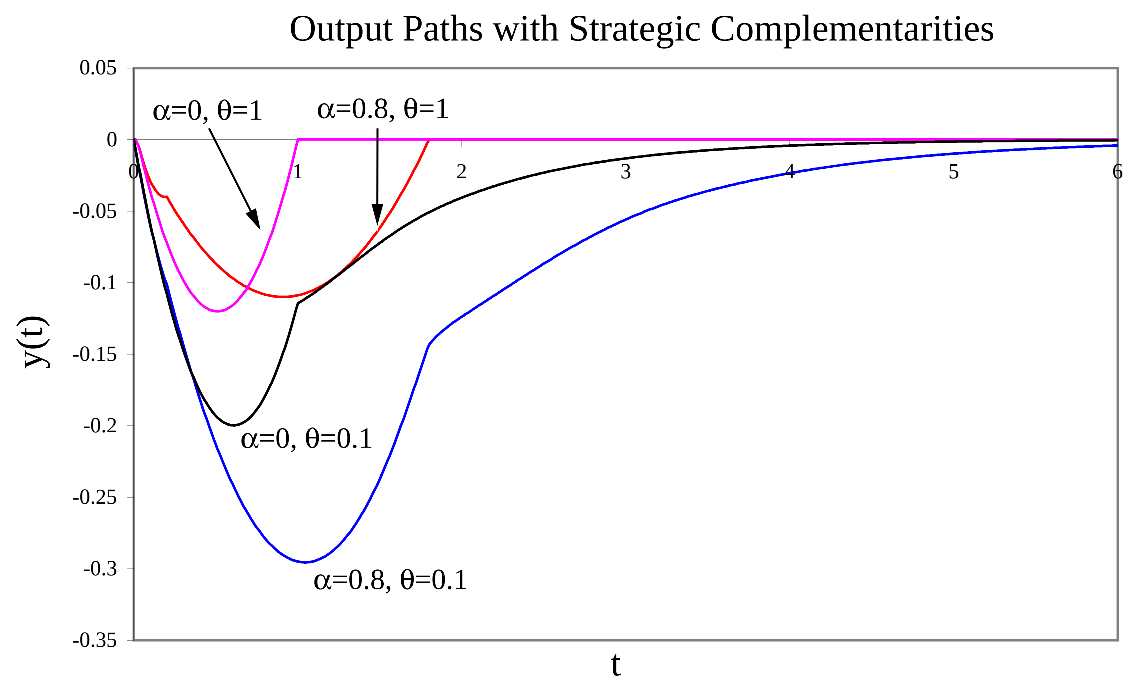

Figure 3

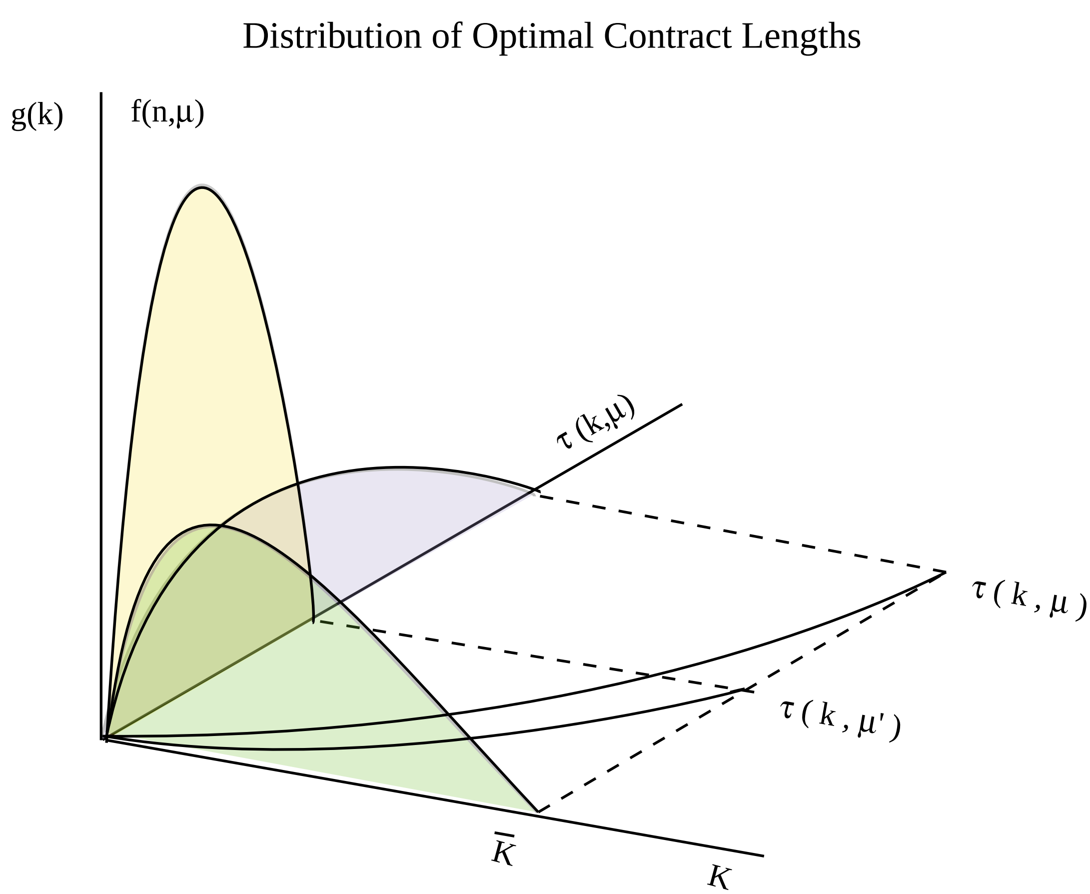

Figure 4

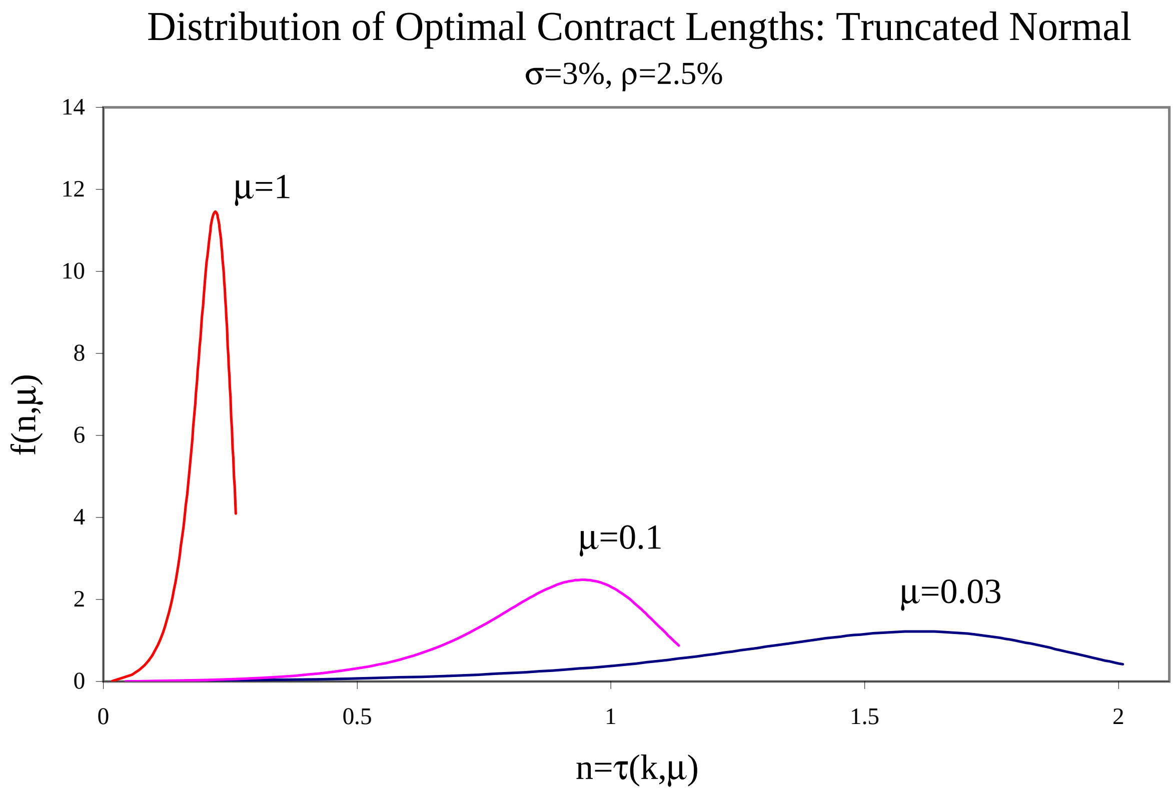

Figure 5

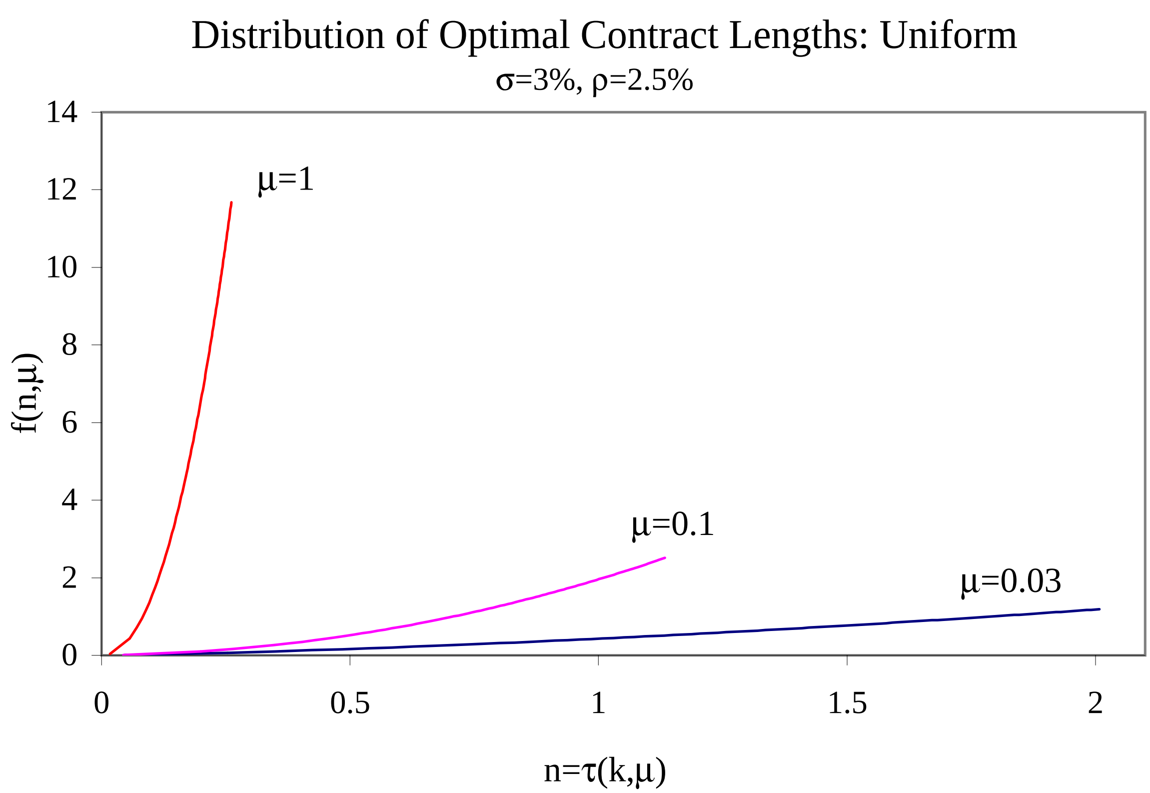

Figure 6

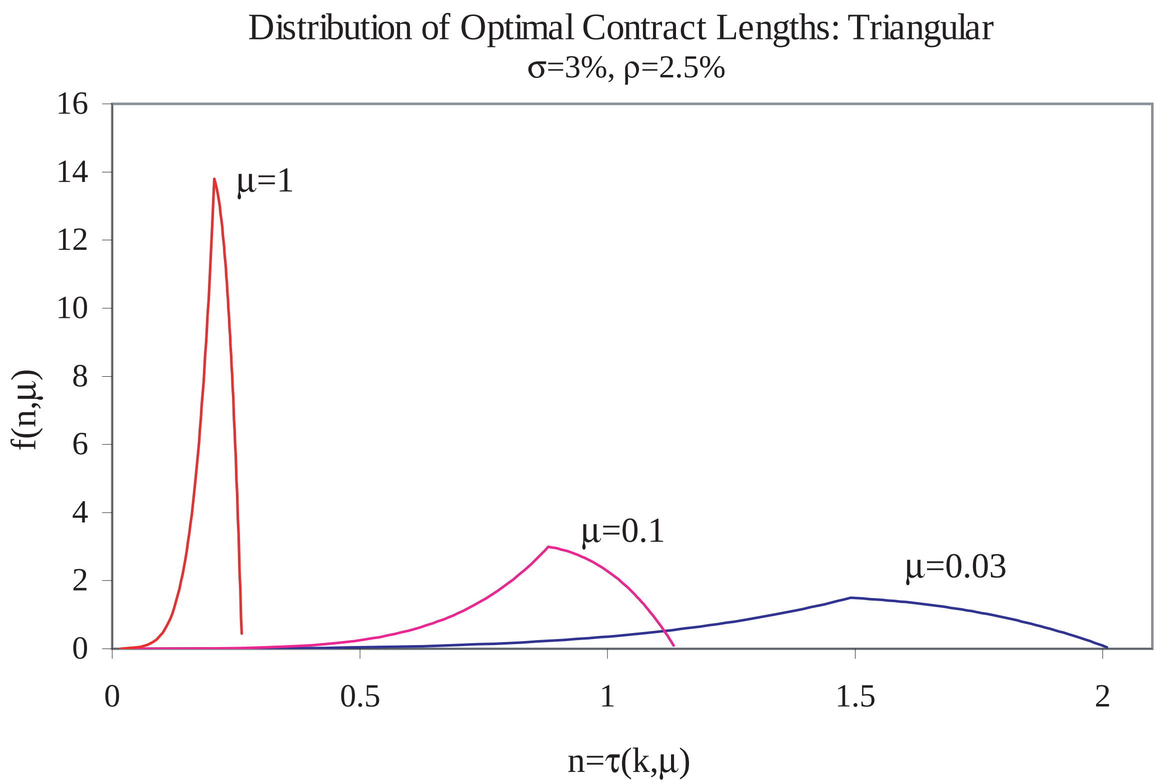

Figure 7

Truncated Normal			
$\mu=$	0.03	0.1	1
Mean	1.44	0.85	0.20
Standard deviation	0.36	0.19	0.04
Max	2.01	1.13	0.26
Min	0.05	0.04	0.02
Uniform			
$\mu=$	0.03	0.1	1
Mean	1.41	0.83	0.20
Standard deviation	0.47	0.25	0.06
Max	2.01	1.13	0.26
Min	0.05	0.04	0.02
Triangular			
$\mu=$	0.03	0.1	1
Mean	1.45	0.86	0.20
Standard deviation	0.31	0.16	0.03
Max	2.01	1.13	0.26
Min	0.05	0.04	0.02

Obs: σ=3%, ρ=2.5%.

Table1