PROGNOSES FOR A NON-PREDICTABLE DISCOUNTED COMMODITY PRICE PROCESS

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Abstract

In this paper we consider the behavior of the price of a continuously stored commodity, for which discounted price is a non-constant martingale, and thus not-predictable. Given any arbitrary probability less than 1 and given any arbitrary neighborhood of zero, we prove that for any date beyond a finite, state-independent horizon, the discounted price realization is, with at least the given probability, within the given neighborhood of zero. Furthermore, with probability 1, the path of discounted price realizations will lie within a given neighborhood of zero beyond a finite state-dependent horizon. The martingale property implies that for a sufficiently long series of initial dates, the average of returns over a given horizon approximates the opportunity cost of capital arbitrarily exactly. But the average of returns for the same initial dates, over a sufficiently extended horizon, reflects the eventual and permanent divergence of price realizations from the profile of conditional expectations at any date.

1. INTRODUCTION

In a model in the tradition of Gustafson (1958), Wright and Williams (1982), Scheinkman and Schechtman (1983), and Deaton and Laroque (1992), Bobenrieth, Bobenrieth and Wright (2002) characterize a commodity price model in which expected price, conditional on any arbitrary price realization, increases without bound, price has a unique invariant distribution with no atoms, and Corollary 2 (p.1218) implies that given any initial price realization p_m, at any time m, the sequence $\{\delta^t p_{m+t}\}_{t\geq 0}$ of discounted prices converges in probability to zero.

Here we consider a modification of this model,with zero storage cost (apart from interest), for which Corollary 2 continues to hold. The sequence of discounted prices in this model is a martingale that is non-constant, and hence non-predictable. Using the fact that the Markov operator of the price process is quasicompact we prove that the sequence of discounted prices converges in probability to zero, uniformly in p_m.

To complement the above state-independent probability result, we have the following state-dependent almost sure result: conditional on any date, beyond a far enough finite horizon the discounted price lies within any given neighborhood of

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zero. Equivalently, beyond some finite horizon that depends upon the path of price realizations, that path is bounded strictly below a given arbitrary fraction of the initial profile of conditional price expectations, almost surely. Realized price is of smaller order of magnitude than its expectation conditional on price at any given date.

Among the empirical implications of the model we note the following: a strong law of large numbers implies that, with probability one, for a large enough sample of initial prices, the average of the realized rates of increase in price from each initial date, calculated over a given horizon, approximates the opportunity cost of capital, arbitrarily exactly. This implication can distinguish this model from any given alternative specification in which the invariant measure of states with zero stocks is positive, and price has a finite upper bound (as, for example, in the empirical implementation of the model of Deaton and Laroque 1992, 1996).

However we show that from each element of a finite sequence of initial price realizations of any given finite sample size, the discounted gross relative price change is within any given neighborhood of -100% with at least any given joint probability less than one, beyond some finite horizon independent of the finite sequence of initial price realizations. Given the same finite sequence of initial price realizations, there exists a finite state-dependent horizon beyond which the result holds with probability 1.

2. THE MODEL

We consider a competitive market for a single storable consumption commodity. Time is discrete. All agents have rational expectations.

Production is subject to a common exogenous i.i.d. disturbance $\omega \in [0, \bar{\omega}],\ 0 < \bar{\omega} < \infty$, and ω has a mixed discrete-continuous distribution with a countable set of atoms, one of which is at zero. More precisely, the distribution of ω is of the form $\alpha L_d + (1-\alpha)L_c$, where $\alpha \in (0,1),\ L_d$ is a discrete distribution that has an atom at 0, and L_c is an absolutely continuous distribution, with continuous derivative when restricted to its support $[0, \bar{\omega}]$.

Assume that there is a continuum of identical producers, a continuum of identical storers, and a continuum of identical consumers; each of the three has total measure one. There is a one-period lag between the producers' choice of effort $\lambda \geq 0$ and output $\omega'\lambda$, where ω' is next period's productivity shock. Cost of effort is given by a function $g : \mathbb{R}_+ \to \mathbb{R}_+$, with $g(0) = 0,\ g'(0) = 0$, and $g'(\lambda) > 0,\ g''(\lambda) > 0$ for all $\lambda > 0$. Storers can hold output from one period to the next, and the sole cost of storage is the cost of capital invested. Given storage x and effort λ, the next period's total available supply is $z' \equiv x + w'\lambda$. Producers and storers are risk neutral and have a common constant discount factor $\delta \equiv 1/(1+r)$, where $r > 0$ is the discount rate.

The utility function of the representative consumer $U : \mathbb{R}_+ \to \mathbb{R}_+$ is continuous, once continuously differentiable, strictly increasing and strictly concave. It satisfies $U(0) = 0,\ U'(0) = \infty$. The inverse consumption demand curve is then $f = U'$. We assume U has a finite upper bound, and thus total revenue $cf(c)$ is also bounded.

The perfectly competitive market yields the same solution as the surplus maximization problem. The Bellman equation for the surplus problem is:

$$\nu(z) = \max_{x,\lambda}\{U(z-x) - g(\lambda) + \delta E[\nu(z')]\}, \quad \text{subject to}$$
$$z' = x + \omega'\lambda,$$
$$x \geq 0, \quad z - x \geq 0, \quad \lambda \geq 0,$$

where $E[.]$ denotes the expectation with respect to next period's productivity shock ω'.

By standard results (see for example Stokey and Lucas with Prescott, 1989), ν is continuous, strictly increasing, strictly concave, and the optimal policy functions $x(z)$ and $\lambda(z)$ are single valued and continuous.

Consumption and price are given by the functions $c(z) \equiv z - x(z)$, $p(z) \equiv f(z - x(z))$.

The policy functions x and λ satisfy the Euler conditions:

$$(1) \qquad f(z - x(z)) \geq \delta E[\nu'(x(z) + \omega'\lambda(z))], \qquad \text{with equality if } x(z) > 0,$$

$$(2) \qquad g'(\lambda(z)) \geq \delta E[\omega'\nu'(x(z) + \omega'\lambda(z))], \qquad \text{with equality if } \lambda(z) > 0,$$

and the envelope condition $\nu'(z) = f(z - x(z))$.

Given initial available supply $z > 0$, condition (1) implies that $z' > 0$ and $x(z') > 0$, and this arbitrage condition holds with equality in the current period and for the indefinite future. Storage $x(z)$ is strictly increasing with z, and effort $\lambda(z)$ is decreasing with z. Note that $p(0) = f(0) = \infty$.

Define available supply at time t as z_t. Given arbitrary fixed $z_0 > 0$, the function that yields the supremum of the support of z_{t+1} is $\hat{z}(z_t) \equiv x(z_t) + \lambda(z_t)\bar{\omega}$. From the facts that there exists a unique fixed point z^* of $\hat{z}$ and that $\hat{z}(z) < z$ for all $z > z^*$, we conclude that $z_t \leq \bar{z} \equiv \max\{z_0, \max\{\hat{z}(z) : 0 \leq z \leq z^*\}\}$, for all $t \geq 0$. Then a suitable state space is $S \equiv [0, \bar{z}]$. Storage takes values in the set $[0, \bar{x}]$, where $\bar{x} \equiv x(\bar{z})$.

Bobenrieth, Bobenrieth and Wright (2002, section 3) prove that the sequence of probability measures of z_t, $\{\gamma_t\}_{t=0}^{\infty}$, converges in the total variation norm to a unique invariant probability γ_*, regardless of the value of z_0. It follows immediately that the sequence of probability measures of prices $\{\gamma_t c^{-1} f^{-1}\}_{t=0}^{\infty}$ converges in the total variation norm to the unique invariant probability measure $\gamma_* c^{-1} f^{-1}$. Note that $\text{Prob}[p_t \geq y] = (\gamma_t c^{-1} f^{-1})([y, \infty])$, where $p_t = f(c(z_t))$ is the price at time t. $H_t(y) \equiv \text{Prob}[p_t \geq y]$ converges uniformly to a unique invariant upper c.d.f. H_*, with $\lim_{p \to \infty} H_*(p) = 0$.

The support of the invariant distribution of prices is an interval $[\underline{p}, \infty]$ with $0 < \underline{p} < \infty$. Without loss of generality, we take a finite initial price p_0 in this support.

3. THE BEHAVIOR OF DISCOUNTED PRICE

Corollary 2 of Bobenrieth, Bobenrieth and Wright (2002, p.1218) implies that given any price realization p_m, the sequence of conditional probability measures of prices is tight. That is, given $\varepsilon > 0$ and a subsample of size $N \in \mathbb{N}$, there exists a finite price bound B such that:

$$\text{Prob}[p_{m+t} < B, p_{m+t+1} < B, \cdots, p_{m+t+N-1} < B \mid p_m] \geq 1 - \varepsilon, \quad \forall\ t \geq 0.$$

Thus price realizations tend to cluster below a bound that does not depend upon the initial date of the subsample, even though storage is strictly positive and the Euler condition for storage arbitrage (1) ensures that the price expectation for next period always exceeds current price, by a fixed proportion.

Bobenrieth, Bobenrieth and Wright (2002, p. 1216) proved that the sequence of probability measures of prices conditional on any initial price p_m converges strongly to a unique invariant measure, which has no atoms, at a geometric rate. In this paper we prove that this convergence is uniform in p_m, and consequently the sequence of discounted prices converges in probability to zero, uniformly in p_m. In the proof we use the facts that the Markov operator is stable and quasicompact, and that given any initial price, any neighborhood of infinity, and any integer k, the price process visits that neighborhood in a time that is some multiple of k, with positive probability.

More precisely, we prove:

THEOREM 1: *Given $\beta > 0$ and $\varepsilon > 0$, there exists $T \in \mathbb{N}$ such that for any price realization p_m,*

$$\textit{Prob}\left[\delta^t p_{m+t} < \beta \mid p_m\right] \ \geq\ 1 - \varepsilon, \quad \forall\ t \geq T.$$

Theorem 1 implies that for any sample size $N \in \mathbb{N}$, given any finite sequence of realized initial prices $\{p_m, p_{m+1}, \cdots, p_{m+N-1}\}$, we have the following bound on the joint probability of the gross discounted relative price changes from each initial price in the sample, beyond a finite T', where T' is independent of the finite sequence of initial price realizations:

$$\text{Prob}\left[\frac{\delta^t p_{m+t}}{p_m} < \beta,\ \frac{\delta^t p_{m+1+t}}{p_{m+1}} < \beta,\ \cdots,\ \frac{\delta^t p_{m+N-1+t}}{p_{m+N-1}} < \beta \ \mid\ p_{m+N-1}\right] \ \geq\ 1 - \varepsilon,$$

for all $t \geq T'$.

PROOF OF THEOREM 1: Consider the probability of the complement,

$$\text{Prob}\left[\delta^t p_{m+t} \geq \beta \mid p_m\right] = \text{Prob}\left[p_{m+t} \geq \frac{\beta}{\delta^t} \mid p_m\right] = \mu_t\left(\left[\frac{\beta}{\delta^t}, \infty\right]\right),$$

where μ_t is the probability measure of the price at time $m+t$, conditional on p_m. Furthermore,

$$\mu_t\left(\left[\frac{\beta}{\delta^t},\infty\right]\right) \;\leq\; |\mu_t - \mu_*| \;+\; \mu_*\left(\left[\frac{\beta}{\delta^t},\infty\right]\right),$$

where μ_* is the invariant probability measure of the price process and $|\cdot|$ denotes the total variation norm.

The transition probability of the price process satisfies, with respect to the point ∞, what is called in Futia a Generalized Uniqueness Criterion (Futia, 1982, p.390). In addition, the corresponding Markov operator L is stable and quasicompact (Theorems 4.6 and 4.10 in Futia, 1982, p.394 and p. 397). Using Theorem 3.6 in Futia (1982, p.390), and Theorem 4 in Yosida and Kakutani (1941, p.200), we obtain the following conclusion : independent of p_m, there exist constants $M > 0, \eta > 0$, such that :

$$||(L^*)^t \;-\; L_1^*|| \;\leq\; \frac{M}{(1+\eta)^t} \qquad \forall\; t \in \mathbb{N},$$

where L^* is the adjoint of the Markov operator L, L_1^* is a continuous linear operator, the image of which consists precisely of the fixed points of L^*, and $||\cdot||$ is the operator norm. Therefore, if δ_{p_m} denotes the unit point mass at p_m, then :

$$|\mu_t - \mu_*| \;=\; |(L^*)^t(\delta_{p_m}) - L_1^*(\delta_{p_m})| \;\leq\; ||(L^*)^t \;-\; L_1^*|| \;\leq\; \frac{M}{(1+\eta)^t} \quad \forall\; t \in \mathbb{N}.$$

Finally, since μ_* has no atom at infinity, we have that $\lim_{t\to\infty} \mu_*\left(\left[\frac{\beta}{\delta^t},\infty\right]\right) \;=\; 0.$

Q.E.D.

The existence of a unique invariant distribution which is a global attractor implies for this price process that, with probability one, the sequence of price realizations is dense on the support $[\underline{p},\infty]$ of the invariant distribution. The infinite sequence of price realizations visits every neighborhood of every price in the support, no matter how high, infinitely often, almost surely. Given this fact, the following proposition regarding discounted prices might not be surprising:

PROPOSITION 1: *For any given price realization p_m, for arbitrary positive real number D, there exists a horizon $d \in \mathbb{N}$, such that:*

$$Prob[\delta^t p_{m+t} > D \mid p_m] > 0, \quad \forall\; t \geq d.$$

Thus the maximum of the support of the conditional distribution of discounted price goes to infinity as the horizon increases, in contrast to the case for the standard model with bounded price, where the maximum goes to zero. But this proposition is, nevertheless, consistent with the following fact: For any finite sequence of discounted price realizations generated by this model there is a second sequence of discounted prices that lie, pointwise, within any given arbitrary neighborhood of the original sequence, where the second sequence of realizations of discounted prices is generated by a standard model with bounded price. To prove Proposition 1, we need Proposition 2, which might be counter-intuitive given Proposition 1.

For the discussion that follows, given a price realization p_m, let $E_m[.]$ denote the expectation conditional on p_m.

PROPOSITION 2: *Given any price realization p_m, the sequence of discounted prices, $\{\delta^t p_{m+t}\}_{t\geq 0}$, goes to zero, almost surely (as $t \to \infty$).*

PROOF OF PROPOSITION 2: The Euler condition for storage arbitrage (1) implies that $\{\delta^t p_{m+t}\}_{t\geq 0}$ is a martingale and that $\sup\{\, E_m[\delta^t p_{m+t}] \;:\; t \geq 0\,\} = p_m < \infty$. By the Martingale Convergence Theorem (due to Doob) we conclude that $\delta^t p_{m+t} \to Y$ a.s. (as $t \to \infty$), where Y is a real random variable. By Theorem 1, $\delta^t p_{m+t} \to 0$ in probability (as $t \to \infty$), and hence $Y = 0$ almost surely. *Q.E.D.*

PROOF OF PROPOSITION 1: If not, there exist a price realization p_m, a real number $D > 0$ and a sequence of natural numbers $\{t_k\}_{k\in\mathbb{N}} \uparrow \infty$ with $\text{Prob}[\delta^{t_k} p_{m+t_k} > D \mid p_m] = 0$, for all t_k. Therefore $\delta^{t_k} p_{m+t_k} \leq D$ a.s., for all t_k. Then the Lebesgue dominated convergence theorem and the fact that $\lim_{t_k\to\infty} \delta^{t_k} p_{m+t_k} = 0$ a.s. imply that $\lim_{t_k\to\infty} E_m[\delta^{t_k} p_{m+t_k}] = 0$, a contradiction to $E_m[\delta^{t_k} p_{m+t_k}] = p_m > 0$, for all t_k. *Q.E.D.*

We have that $E_m[\delta^t p_{m+t}] = p_m, \ \forall\ t \geq 0$. Nevertheless, Proposition 2 states $\{\delta^t p_{m+t}\}_{t\geq 0}$, converges to zero almost surely, implying that $\{E_m[\delta^t p_{m+t}]\}_{t\geq 0}$ does not converge to the expectation of the almost sure limit of $\{\delta^t p_{m+t}\}_{t\geq 0}$. As a consequence, the sequence of discounted prices is not uniformly integrable.

Proposition 2 is easy to understand in a model with bounded price, but how can the discounted price be going to zero, almost surely, if there is positive probability that discounted price exceeds D at any sufficiently far horizon? The explanation hinges on the distinction between a profile of expectations conditional on a price realization and the path of realizations. By Proposition 2, with probability 1, for any given path of discounted price realizations there is a time beyond which that path lies permanently below D. But by Proposition 1, there is no finite horizon beyond which all paths possible from time m would have done so. In fact, at any finite horizon, there is with positive probability a price bubble, with price rising at a rate greater than the discount rate r, continuously within that horizon. Although a path of discounted price realizations eventually remains permanently below D, before it does so, it can exceed any given arbitrary high finite bound. It is recognition of such a possibility that keeps $E_m[\delta^t p_{m+t}]$ equal to p_m as the horizon, and the probability that the discounted price will be below D at that horizon, both increase.

Proposition 2 implies that, given a price realization p_m, the sample mean and sample variance of a discounted price sequence go to zero almost surely, that is:

$$N^{-1} \sum_{t=0}^{N-1} \delta^t p_{m+t} \to 0 \quad \text{a.s.} \qquad (\text{as} \ \ N \to \infty)\ , \qquad \text{and}$$

$$N^{-1} \sum_{t=0}^{N-1} \left[\delta^t p_{m+t} - N^{-1} \sum_{j=0}^{N-1} \delta^j p_{m+j} \right]^2 \to 0 \quad \text{a.s.} \qquad (\text{as} \ \ N \to \infty).$$

Thus the estimators are consistent with respect to the first two moments of the limiting distribution of discounted price. The sample average of discounted price realizations starting at any price realization p_m, is eventually permanently below any arbitrary positive fraction of the profile of expectations, conditional on p_m, of discounted price. Nevertheless the variance of the distribution of discounted price, conditional on p_m, goes to infinity as $t \to \infty$.

4. THE BEHAVIOR OF PRICE

The behavior of the price path is related to the profile of conditional expectations at time m by the following theorem:

THEOREM 2: *Given any price realization p_m, with probability one, for any $1 \leq l < \infty$, there exists a finite time $\tau(l)$, which depends on the sequence of price realizations, such that:*

$$\frac{E_m[p_{m+t}]}{l} > p_{m+t}, \quad \forall\, t \geq \tau(l),$$

implying that

$$p_{m+t} = o(E_m[p_{m+t}]), \quad a.s.$$

PROOF OF THEOREM 2: By Proposition 2, $\delta^t p_{m+t} \to 0$ (as $t \to \infty$), with probability one. Therefore, given any l, $1 < l < \infty$, there exists a time $\tau(l)$ that satisfies $\delta^t p_{m+t} \cdot l < p_m = \delta^t E_m[p_{m+t}]$, $\forall\, t \geq \tau(l)$. *Q.E.D.*

Theorem 2 defines a sequence of upper bounds on the path of price realizations. Note that the profile of conditional expectations $E_m[p_{m+t}]$ is itself an upper bound beyond some date $\tau(1)$. Any given fraction of the profile of expectations conditional on initial price is an upper bound on any price realized beyond some fixed horizon, with probability one.

The behavior of price expectations and realizations in the model is elucidated in figure 1. The profile of conditional expectations, $E_m[p_{m+t}]$, rises to infinity at the discount rate. (In any given standard model of commodity storage with positive probability of stockouts there is a finite uniform upper bound, possibly well above the range shown in this illustration, on any conditional expectations profile.) The horizontal line at price B is an upper probability bound on a sequence of N consecutive price realizations, with probability at least $1 - \varepsilon$.

A possible sequence of price realizations is illustrated as a series of dots beginning at p_m. After $\tau(1)$, all the realizations of price lie below $E_m[p_{m+t}]$. The curve $E_m[p_{m+t}]/2$ shows another bound effective beginning at date $\tau(2)$. Further bounds generated by successively higher values of l imply that the long-run rate of increase of realized price is strictly lower than the discount rate, even though the storage arbitrage condition (1) holds, with equality, each period, and that price bubbles of any finite length, understood as sequences of prices rising faster than the discount rate, recur infinitely often along the path of realizations, almost surely.

5. SOME EMPIRICAL IMPLICATIONS

Implications of the model for the empirical behavior of sample averages of returns on the stocks, that are held in the optimal solution over specific intervals, are summarized in the following theorem:

THEOREM 3: *With probability one, for any given path of price realizations* $\{p_t\}_{t\geq 0}$, *for any* $n \in \mathbb{N}$ *and for any* $\beta > 0$, *there exist* $J = J(\{p_t\}_{t\geq 0}, n, \beta) \in \mathbb{N}$, $k = k(\{p_t\}_{t\geq 0}, J, \beta) \in \mathbb{N}$, $k > n$, *and* $K = K(\{p_t\}_{t\geq 0}, k, \beta) \in \mathbb{N}$, $K > J$, *such that:*

(i) $J^{-1} \sum_{t=0}^{J-1} \left[\dfrac{\delta^n p_{t+n} - p_t}{p_t} \right] \in (-\beta,\, \beta),$

(ii) $J^{-1} \sum_{t=0}^{J-1} \left[\dfrac{\delta^k p_{t+k} - p_t}{p_t} \right] \in (-1,\, -1+\beta),$ *and*

(iii) $K^{-1} \sum_{t=0}^{K-1} \left[\dfrac{\delta^k p_{t+k} - p_t}{p_t} \right] \in (-\beta,\, \beta).$

PROOF OF THEOREM 3: For $j \in \mathbb{N}$ and for $t \in \mathbb{N} \cup \{0\}$, let $Y_{t+j} \equiv \dfrac{\delta^j p_{t+j} - p_t}{p_t}$. The arbitrage equation for storage (1) implies that there exists $\bar{p} \geq p(x^j(z_t))$, $\bar{p}$ depends on z_t, such that $\delta^j \alpha_1^j \bar{p} = p_t$, where α_1 is the size of the atom at zero of the distribution of ω, and $x^j \equiv x \circ x \circ \cdots \circ x$ (j times). Therefore,

$$-1 \leq Y_{t+j} \leq \delta^j \frac{\bar{p}}{p_t} = \frac{1}{\alpha_1^j}.$$

The same arbitrage equation (1) implies $E_t[Y_{t+j}] = 0$. Hence the sequence $\{X_t\}_{t\geq 0}$, where $X_t \equiv Y_{t+j}$, is uniformly bounded, and $\sum_{i=1}^{\infty} \sup_t |\mathrm{Cov}(X_t, X_{t-i})| < \infty$. A strong law of large numbers (see Davidson 1994, p.297) implies that

$$\lim_{N\to\infty} N^{-1} \sum_{t=0}^{N-1} \left[\frac{\delta^j p_{t+j} - p_t}{p_t} \right] = 0, \quad \text{a.s.} \tag{2}$$

Evaluating (2) for $j = n$ we conclude that there exists $J \in \mathbb{N}$ such that (i) holds. For this J, by proposition 2,

$$\lim_{k\to\infty} J^{-1} \sum_{t=0}^{J-1} \left[\frac{\delta^k p_{t+k} - p_t}{p_t} \right] = -1, \quad \text{a.s.,}$$

establishing (ii) for large enough k. Finally, evaluating (2) for $j = k$ we obtain $K,\ K > J$, satisfying (iii). *Q.E.D.*

Expression (i) of Theorem 3 shows that the average excess rate of return on stocks held over n periods is greater than a given, arbitrary $-\beta$, for a sufficiently large sample size J, as implied by a strong law of large numbers. Expression (ii) states that, with the same sample of initial holding dates, if we increase the holding interval sufficiently, to k periods, (and increase the sample size by $k - n$ periods to accommodate the extended lead), the average gross discounted return is within an arbitrary β of a total loss. At this sample size, the sample average (ii) could be considered a downward-biased estimator of the expected $k-$period rate of increase in price, which in this model is constant. Expression (iii) reflects the fact that the sample average for the longer holding period approaches the conditional expectation for that horizon, when the sample size is sufficiently increased.

Comparison of results (i) through (iii) has another interpretation, more relevant for estimation of the long-run return on storage from any given time zero. As the horizon is increased, the discounted present value of price realizations conditional on any price p_t in the sample of size J in (i), eventually converges, along the path of realizations, to a neighborhood of zero in finite time, as stated in Proposition 2. ¿From this point of view, comparison of (ii) with (i) reflects the convergence of the gross discounted value to its almost sure limit of a one hundred percent loss over the holding period, as the latter goes to infinity. But (iii) shows an increase in the average excess rate of return back to an arbitrary neighborhood of the conditional expectation of zero when sufficient observations are added to include some that have high rates of price increase through the fixed horizon. Note that (iii) does not imply that an initial investment at time 0 can be restored to profitability if held for a sufficiently long time.

6. CONCLUSIONS

The stationary price process that we have examined reveals the importance of distinguishing any given profile of conditional price expectations from the path of price realizations. The rate of increase of any profile of conditional price expectations is constant at the discount rate, while the realized price at a sufficiently far horizon is bounded below any given positive fraction of the profile of expectations conditional on the current price. Furthermore, there is a state-independent horizon beyond which a discounted price realization lies within a given neighborhood of zero with at least any given probability less than 1.

For processes of the type we consider, the fact that the average rate of price increase declines eventually, as the lead over which the increase is measured is extended in a given sample, must be interpreted with care. The average realized rate of price increase constitutes an underestimate of the expected rate of return at a sufficiently long horizon. This estimate converges on the expected rate of return at that horizon, if the sample size is increased sufficiently. Emergence of a negative bias in the sample rate of return as an estimator of the conditional expectation of the rate of return, as the horizon is extended for a given finite sequence of initial price realizations, reflects the fact that the discounted value of a unit of the commodity converges to its long-run limit of zero almost surely, as the horizon goes to infinity, even though discounted price is a martingale. Indeed the price of the commodity is of smaller order of magnitude than its own expectation, conditional on initial price.

These results for a commodity price process raise questions regarding the appropriate criteria for forecasting long-horizon returns, and the interpretation of such forecasts.

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