Endogenous Collateral

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Summary

We study an economy where there are two types of assets. Consumers' promises are the primitive defaultable assets secured by collateral chosen by the consumers themselves. The purchase of these personalized assets by financial intermediaries is financed by selling back derivatives to consumers. We show that nonarbitrage prices of primitive assets are strict submartingales, whereas nonarbitrage prices of derivatives are supermartingales. Next we establish existence of equilibrium, without imposing bounds on short sales. The nonconvexity of the budget set is overcome by considering a continuum of agents.

Keywords: Endogenous Collateral; Non Arbitrage.

1 Introduction

1.1 Motivation

Housing mortgages stand out as the most clear and most common case of collateralized loans. In the past, these mortgages were entirely financed by commercial banks who had to face a serious adverse selection problem in addition of the risks associated with concentrating investments in the housing sector. More recently, banks have managed to pass these risks to other investors. The collateralized mortgage obligations (C.M.O.) developped in the eighties and nineties are an example of a mechanism of spreading risks of investing in the housing market. These obligations are derivatives backed by a big pool of mortgages which was split into different contingent flows.

Collateralized loans were first addressed in a general equilibrium setting by Dubey, Geanakoplos and Zame [9]. Collateral was modelled by these authors as a bundle of durable goods, purchased by a borrower at the time assets are sold and surrendered to the creditor in case of default. Clearly, in the absence of other default penalties, in each state of nature, a debtor will honor this commitments only when the debt does not exceed the value of the collateral. Similarly, each creditor should expect to receive the minimum between his claim and the value of the collateral. This pionnering work studied a two-period incomplete markets model with default and exogenous collateral coefficients and discussed also the endogenization of these coefficients, allowing for some coefficients to prevail in equilibrium, out of a possible finite set of strictly positive values, but for a fixed composition in terms of durable goods. Araujo, Páscoa and Torres-Martinez [5] extended the exogenous collateral model to infinite horizon economies with one-period assets and showed that Ponzi schemes can be avoided without imposing transversality or debt constraints.

Araujo, Orrillo and Páscoa [3] studied existence of equilibria in an economy where borrowers may choose collateral bundles under the restriction that the value of the collateral, per unit of asset and at the time when it is constituted, must exceed the asset price by some arbitrarily small amount exogenously fixed. Under this requirement the loan can only finance up to some certain fraction of the value of the house. Lenders were assumed not to trade directly with individual borrowers, but rather to buy obligations backed by a weighted average of the collaterals chosen by individual borrowers, with the individual sales serving as weights. Borrowers sell at different prices depending on the collateral choice, as there is a spread which is a discounted expectation of default given in the future. Hence, borrowers choose the composition of the collateral in terms of durable goods and the collateral margin (which is not necessarily equal to the exogenous lower bound as more collateral reduces the spread).

However, the model suffered from three important drawbacks that we try to overcome in the current paper. First, short sales were bounded due to the above exogenous lower bound on the difference between the value of the collateral and the asset price (in fact, first period budget feasibility implies that short sales must be bounded by the upper bound on endowments divided by the exogenously fixed lower bound on the difference between the value of the collateral and the asset price). It is hard to accept the existence of an exogenous uniform upper bound on the fraction of the value the house that can be financed by a loan.

Secondly, the payoffs of the derivative were constructed in a way that implied that in equilibrium, in each state of nature, either all borrowers would honor their debts or all borrowers would default (even though the collateral bundle might vary across borrower). In fact, derivative's payoffs were assumed to be the minimum between the debt and the value of the depreciated weighted average of all collateral bundles. If we require, as we do in this paper, that the derivative's payoff in each state is just the weighted average of borrowers' repayments (which may be the full repayment of the debt for some borrowers or the value of the depreciated personalized collateral for others), then, in equilibrium, some borrowers may default while others will pay back their loans.

Third, derivative aggregate purchases were required to match, in units, aggregate short-sales of primitive assets, but this equality should only be required in value. That is, each financial intermediary should be financing the purchase of the consumers' promises on a certain primitive asset by issuing the respective derivative, thereby making zero profit at the initial date (and also at any future state of nature due to the above requirement that the derivative's endogenous payoff should be the weighted average of consumers' effective repayments).

1.2 Results and Methodology

It is well known that in incomplete markets with real assets equilibrium might not exist without the presence of a bounded short sales condition (see Hart [14] for a counter-example and Duffie and Shafer [10] on generic existence). In a model with exogenous collateral this bounded short sales condition does not need to be imposed arbitrarily but it follows from the fact that collateral must be constituted at the exogenously given coefficients. An important question is whether existence of equilibria may dispense any bounded short sales conditions in a model with endogenous collateral. The fact that the borrower holds and consumes the collateral discourages him from choosing the collateral so low that default would become a sure event. To be more precise, the borrower should always provide enough collateral so that its depreciated value matches the promised payment in at least one state. Otherwise, the borrower's utility could be increased by raising the collateral without changing net returns and by making the reduced default spread cancel out the increased collateral costs. If equilibrium levels of the collateral coefficients are bounded away from zero, then equilibrium aggregate short sales are bounded.

Allowing borrowers to choose their collateral bundles introduces a non-convexity in the budget set, which is overcome by considering a continuum of agents. This large agents set is actually a nice set up both for the huge pooling of individual mortgages and for the spreading of risks across many investors. However, for a continuum of agents, having established that aggregate short sales are endogenously bounded does not imply that the short sales allocation is uniformly bounded. But if it were not, short sale prices, net of collateral costs, would not be uniformly bounded away from zero and all agents would respond by choosing low margins and arbitrarily large short sales, contradicting the already established bounded aggregated short sales. Then, short sales allocations are endogenously uniformly bounded, as desired to prove existence using a multi-dimensional version of Fatou's lemma applied to a sequence of equilibria of truncated auxiliary economies whose bundles and portfolios are bounded.

1.3 Arbitrage and Pricing

The existence argument uses a pricing formula suggested by a study of the nonarbitrage conditions for asset pricing in the context of a model where purchases of the collateralized derivatives and sales of individual assets yield different returns. This nonarbitrage analysis was absent in the earlier work by Araujo, Orrillo and Páscoa [3], where budget feasible short sales were bounded.

Our analysis of the nonarbitrage conditions is close to the study made by Jouini and Kallal [16] in the presence of short sales constraints. In fact, the individual promises of homeowners are assets that can not be bought by these agents and the collateralized derivatives bought by investors is an asset that can not be short sold by these agents. These sign constraints determine that purchase prices of the the collateralized derivatives follow supermartingales, whereas sale prices of homeowners promises follow submartingales. Actually, the latter must be *strict* submartingale when collateral is consumed by borrowers, since short sales generate utility returns also, and in this respect, our analysis differs from Jouini and Kallal [16].

The nonarbitrage conditions identify several components in the price of a consumer's promise: a base price common to all consumers, a spread that depends on the future default, a positive term reflecting the difference between current and future collateral values, a nonnegative tail due to the sign constraints and a negative tail on the sale price due to utility returns from consumption of the collateral. We also show that the price of the minimal cost superhedging strategy is the supremum over all discounted expectations of the claim, with respect to every underlying probability measure (and similarly, the price of a maximal revenue subhedging strategy is instead the infimum over those expectations, in the spirit of the Cvitanic and Karatzas [7] and El Karoui and Quenez [12] approaches to pricing in incomplete markets).

In equilibrium agents will face price functions, as in Araujo, Orrillo and Páscoa [3], rather than price vectors. More precisely, we propose price formulas both for the primitive assets and the derivatives which are suggested by our arbitrage analysis. The state prices entering in these equilibrium price functions and the negative tail of the primitive asset prices are both taken as given and common to all agents. That is, equilibrium prices of derivative or primitive assets are given by super or sub martingales, respectively, with respect to a common measure, but can also be written as super or sub martingales for consumer specific measures implied by the personal choice of collateral and effective returns (namely using the Kuhn-Tucker multipliers as deflators).

1.4 Relation to Other Equilibrium Concepts

We close the paper with a discussion of the efficiency properties of equilibria. We show that an equilibrium allocation is undominated by allocations that are feasible and provide income across states through the same given equilibrium spot prices, although may be financed in the first period in any other way (possibly through transfers across individuals). This results extends usual constrained efficiency results to the case of default and endogenous collateral. An implication is that the no-default equilibrium, the exogenous collateral equilibrium or even the endogenous collateral equilibrium with bounded short sales are concepts imposing further restrictions on the welfare problem and should be expected to be dominated by the proposed equilibrium concept.

In this paper we simplify the mixing of individual promises by assuming that each collateralized derivative mixes the promises of all sellers of a certain primitive asset. Since the collateral choice personalizes the asset the resulting derivative represents already a significative mixing across assets with rather different default profiles. Further work should address the composition of derivatives from different primitive assets and certain chosen subsets of debtors. We do not deal also with the case of default penalties entering the utility function and the resulting adverse selection problems. The penalty model was extensively studied by Dubey, Geanakoplos and Shubik [8], extended to a continuum of states and infinite horizon by Araujo, Monteiro and Páscoa [1, 2] and combined with the collateral model by Dubey, Geanakoplos and Zame [9]. Our default model differs also from the bankruptcy models where agents do not honor their debts only when they have no means to pay them, or more precisely, when the entire financial debt exceeds the value of the endowments that creditors are entitled to confiscate (see Araujo and Páscoa [4]).

The paper is organized as follows. Section 2 presents the basic model of default and collateral choice. Sections 3 and 4 address arbitrage and pricing. Section 5 presents the definition of equilibrium and the existence result. Section 6 contains the existence proof and Section 7 discusses the efficiency properties. A mathematical appendix contains some results used in the existence proof.

2 Model of Default and Collateral Choice

We consider an economy with two periods and a finite number S of states of nature in the second period. There are L physical durable commodities traded in the market and J real assets that are traded in the initial period and yield returns in the second period. These returns are represented by a random variable $R : S \mapsto I\!R_+^{JL}$ such that the returns from each asset are not trivially zero, that is , $R_{sj} > neq0 \;\; \forall s, j$. In this economy each sale of asset j (promise) must be backed by collateral. This collateral will consist of goods that depreciate at some rate Y_s depending on the state of nature $s \in S$ that occurs in the second period.

Each seller of assets chooses also the collateral coefficient for the different assets that he sells and we suppose that the mean collateral coefficients can be known by consumers. For each asset j denote by $M_j \in I\!R_+^L$ the choice of collateral coefficients. The mean collateral coefficients will be denoted by $C \in I\!R_+^{JL}$. Each agent in the economy is a small investor whose portfolio is $(\theta, \varphi) \in I\!R_+^J \times I\!R_+^J$, where the first and second components are the purchase of the derivative and sale of the primitive assets, respectively. The collateral bundle choosen by borrower will be $M\varphi$ and his whole first period consumption bundle is $x_o + M\varphi$.

Denote by $x_s \in I\!R_+^l$ the consumption vector in state of nature s. Agent´s endowments are denoted by $\omega \in I\!R_{++}^{(S+1)L}$. Let π_1 and π_2 be the vectors of purchase prices of the derivatives and of sale prices of primitive assets, respectively. Then, the budget constraints of each agent will be the following

$$p_o x_o + p_o M\varphi + \pi_1\theta \leq p_o\omega_o + \pi_2\varphi \tag{1}$$

$$p_s x_s + \sum_{j=1}^{J} D_{sj}\varphi_j \leq p_s\omega_s + \sum_{j=1}^{J} N_{sj}\theta_j + \sum_{j=1}^{J} p_s Y_s M_j \varphi_j + p_s Y_s x_o, \ \ \forall s \in S \quad (2)$$

Here $D_{sj} \equiv \min\{p_s R_s^j, p_s Y_s M_j\}$ and N_{sj} are what he will paid and received with the sale and purchase of one unit of the primitive asset j and one unit of its derivative, respectively. Now we will represent equations (1) and (2) in matrix form:

$$p \,\square(\tilde{x} - \omega) \leq A(x_o, \theta, \varphi) \quad (3)$$

where $\tilde{x} = (0, x_1, \ldots, x_S)$, $\omega = (\omega_o, \omega_1, \ldots, \omega_S)$, $p \,\square(\tilde{x} - \omega)$ is the column vector whose components are $p_s \cdot (\tilde{x}_s - \omega_s)$ for $s = 0, 1, \ldots, S$ and

$$A = \begin{bmatrix} -p_o & -\pi_1 & \pi_2 - p_o M \\ p_1 Y_1 & N_1 & p_1 Y_1 M - D_1 \\ p_2 Y_2 & N_2 & p_2 Y_2 M - D_2 \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ p_S Y_S & N_S & p_S Y_S M - D_S \end{bmatrix}$$

3 Arbitrage and Collateral

Now we will define arbitrage in our context where both sales of collateralized assets and additional purchases of durable goods have utility returns that have to be taken into account together with pecuniary returns. Moreover, agents' preferences are assumed to be monotonic.

Definition 1 *We say that there exist arbitrage opportunities if* $\exists\, M_j > 0,\ j = 1, .., J,\ \theta \geq 0$ *and* (x_o, φ) *such that*

$$T(x_o, \theta, \varphi) > 0 \quad (4)$$

where

$$T = \begin{bmatrix} & A & \\ I & 0 & 0 \\ 0 & 0 & I \end{bmatrix}$$

Notice that even when there are no pecuniary net returns and zero net cost the agent may still gain from the utility returns of consuming durable goods, serving or not as collateral, that is, through a collateralized short sale ($\varphi_j > 0$) or a non financed purchase ($x_o > 0$).

Theorem 1 *There are no arbitrage opportunities if and only if there exists* $\beta \in I\!R^S_{++}$ *such that for each* $j = 1, 2, .., J$

$$\pi^j_1 \geq \sum_{s=1}^{S} \beta_s p_s N^j_s \tag{5}$$

$$\pi^j_2 < \sum_{s=1}^{S} \beta_s p_s R^j_s - \sum_{s=1}^{S} \beta_s (p_s R^j_s - p_s Y_s M_j)^+ + (p_o M_j - \sum_{s=1}^{S} \beta_s p_s Y_s M_j) \tag{6}$$

and

$$p_o > \sum_{s=1}^{S} \beta_s p_s Y_s \tag{7}$$

Proof:
Let $B = \{T(x_o, \theta, \varphi) : \theta \geq 0\}$ and $\tilde{B} = \{T(x_o, \theta, \varphi) : \theta = 0\}$, which are a convex cone and a linear subspace, respectively. Let $K = I\!R_+ \times I\!R^S_+ \times I\!R^{L+J}_+$.

Absence arbitrage is equivalent to $K \cap B = \{0\}$. By the theorem of separation of convex cones, we have that $K \cap B = \{0\}$ if and only if $\exists f \neq 0$ linear: $f(z) < f(y)$, $\forall z \in B, y \in K \backslash \{0\}$.

Now $f(z) = 0$, $\forall z \in \tilde{B}$, since $\tilde{B}$ is a linear subspace. Then $f(y) > 0$, $\forall y \in K \backslash \{0\}$ and it follows that $f(z) \leq 0$ $\forall z \in B$. Hence $\exists$ $(\tilde{\alpha}, \tilde{\beta}, \tilde{\mu}, \tilde{\eta}) >> 0$: $f(v, c, x_o, \varphi) = \tilde{\alpha} + \tilde{\beta} c + \tilde{\mu} x_o + \tilde{\eta} \varphi \leq 0$, $\forall (v, c, x_o, \varphi) \in B$. Take $\beta = \tilde{\beta}/\alpha$, $\mu = \tilde{\mu}/\alpha$ and $\eta = \tilde{\eta}/\alpha$, and we have (5) when $(x_o, \varphi) = 0$. To obtain (6) and (7) let $\theta = 0$ and recall that $f(z) = 0$, $\forall z \in \tilde{B}$, implying

$$p_o M_j - \pi^j_2 = \sum_s \beta_s (p_s Y_s M_j - D_{sj}) + \eta_j$$

and

$$p_o = \sum_s \beta_s p_s Y_s + \mu. \blacksquare$$

Comment

Durable goods prices (p_0) and net prices ($p_0 M^j - \pi^j_2$) of the joint operation of constituting collateral and short-selling a primitive asset are both superlinear functions of pecuniary returns, by the Theorem above, due to the additional utility returns from consumption (of x_0 and of $M^j \varphi_j$, respectively).

Corollary 1

$$p_o M_j - \pi_2^j > 0 \quad when \quad M_j \neq 0, \ \forall j$$

Since short-sales lead to nonnegative net yields in the second period (once we add the depreciated collateral to returns) and also to consumption of the collateral bundle in the first period, nonarbitrage requires the net coefficient of short-sales in the first period budget constraint to be positive.

If we had considered the collateral as being exogenous, we would have the following result:

Corollary 2 *There are no arbitrage opportunities if and only if there exists* $\beta \in I\!R^{(S)}_{++}$ *such that*

$$\sum_{s=1}^{S} \beta_s D_{sj} \leq \pi^j < \sum_{s=1}^{S} \beta_s D_{sj} + (p_o - \sum_{s=1}^{S} \beta_s p_s Y_s) C_j$$

, which implies

$$(p_o - \sum_{s=1}^{S} \beta_s p_s Y_s) C_j > 0, \quad and \ \ p_o C_j - \pi^j > 0, \ \forall j \in J.$$

For more details on the implications of the absence of arbitrage in the exogenous collateral model see Fajardo [13].

In contrast with the fundamental theorem of asset pricing in frictionless financial markets, we can obtain an alternative result for the default model with collateral where discounted nonarbitrage asset prices are no longer martingales with respect to some equivalent probability measure. This result is presented in the next section.

4 Pricing

4.1 A Pricing Theorem

Let $I\!R$ be the real line and $\overline{I\!R} = I\!R \cup \{-\infty, +\infty\}$ the extended real line. Let $\Omega = \{1, 2, .., S\}$, $(\Omega, \mathcal{F}, P)$ be a probability space and $X = I\!R^S$. We say that $f : X \mapsto I\!R$ is a positive linear functional if $\forall\, x \in X^+,\ \ f(x) > 0$, where $X^+ = \{x \in X / P(x \geq 0) = 1 \ \text{ and } \ P(x > 0) > 0\}$. The next result follows in spirit of the result in Jouini and Kallal [16].

Let $\overline{\pi}_2^j = \pi_2^j - p_o M_j < 0,\ \forall j$ which will be refered to as the net sell price and let $\overline{D}_{sj} = D_{sj} - p_s Y_s M_j,\ \forall j$ and $\forall s$.

Denote by $\iota(x)$ the smallest amount necessary to get at least the payoff x for sure by trading in the underlying defaultable assets. Then no investor is willing to pay more than $\iota(x)$ for the contingent claim x. The specific expression for ι is given by

$$\iota(x) = \inf_{(\theta,\varphi)\in\Theta} \{\pi_1\theta - \overline{\pi}_2\varphi > 0 \Big/ G(\theta,\varphi) \geq x \ \ a.s.\}$$

where

$$G(\theta,\varphi) = \sum_{j=1}^{J} [N_j\theta^j - \overline{D}_j\varphi^j]$$

Theorem 2 *i) There are no arbitrage opportunities if and only if there exist probabilities $\beta_s^*,\ s = 1,..,S$ equivalent to P and a positive γ such that the normalized (by γ) purchase prices of the derivatives are supermartingales and the normalized (by γ) net sale prices of the primitive assets are submartingales under this probability. when the collateral is consumed by the borrower, the net sale price is a strict submartingale*

ii) Let $\mathcal{Q}^*$ be the set of β^* obtained in (i) and Γ be the set of positive linear functionals ξ such that $\xi|_{\mathcal{M}} \leq \iota$, where $\mathcal{M}$ is a convex cone representing the set of marketed claims. Then there is a one-to-one correspondence between these functionals and the equivalent probability measures β^* given by:

$$\beta^*(B) = \sum_{s=1}^{S} \beta_s^* 1_B(s) = \xi(1_B) \text{ and } \xi(x) = E^*(\frac{x}{\gamma})$$

where E^* is the expectation taken with respect to β^*

iii) For all $x \in \mathcal{M}$ we have

$$[-\iota(-x), \iota(x)] = cl\{E^*(\frac{x}{\gamma}) : \beta^* \in \mathcal{Q}^*\}$$

Proof:

(i) Let $\beta_o = \sum_{s=1}^{S} \beta_s$ and $\beta_s^* = \frac{\beta_s}{\beta_o}$ in theorem 1, we obtain:

$$\pi_1^j/\beta_o \geq \sum_{s=1}^{S} \beta_s^* N_{sj}$$

and

$$\pi_2^j/\beta_o \leq \sum_{s=1}^{S} \beta_s^* p_s R_s^j - \sum_{s=1}^{S} \beta_s^* (p_s R_s^j - p_s Y_s M_j)^+$$
$$+(p_o M_j/\beta_o - \sum_{s=1}^{S} \beta_s^* p_s Y_s M_j)$$

Take $\gamma = 1/\beta_o$. From the above equations it follows that π_1^j and $\pi_2^j - p_0 M_j$ are super and sub martingales, respectively.
Now, if there is a probability measure and a process γ such the normalized prices are sub and supermartingales, we have

$$E^* \left(\sum_{j=1}^{J} [N^j \theta^j - \overline{D}^j \varphi^j] \right) \leq \gamma[\pi_1 \theta - \overline{\pi}_2 \varphi]$$

Then there can not exists arbitrage opportunities.

(ii) Given $\beta^* \in \mathcal{Q}^*$ define $\xi(x) = E^*(\frac{x}{\gamma})$, then

$$\xi(x) = \sum_s \left(\frac{x_s}{\gamma} \frac{\beta_s^*}{P_s} \right)$$

it is a continuous linear functional. Since β^* is equivalent to P and taking the infimum over all supereplicating strategies :

$$E^*(x) \leq E^* \left(\sum_{j=1}^{J} [N^j \theta^j - \overline{D}^j \varphi^j] \right) \leq \gamma[\pi_1 \theta - \overline{\pi}_2 \varphi]$$

we have $\xi \in \Gamma$.
Now take $\xi \in \Gamma$ and define $\beta^*(B) = \sum_{s=1}^{S} \beta_s^* 1_B(s) = \xi(1_B)$. Since S is

finite, β^* is equivalent to P.

Now since $\xi(1_S) = 1$, we have $\beta^*(S) = 1 = \sum_{s=1}^{S} \beta^*_s$, so β^* is a probability.

(iii) By part (ii) take a $\xi \in \Gamma$ then $\forall x \in \mathcal{M}$

$$\xi(x) \leq \iota(x) \Rightarrow -\xi(-x) \leq \iota(x)$$

then replacing x by $-x$ we have

$$\xi(x) \geq -\iota(-x)$$

Hence

$$cl\{\xi(x)/\xi \in \Gamma\} \subset [-\iota(-x), \iota(x)]$$

For the converse, $-\iota(-x) = \iota(x)$ the proof is trivial. Then we suppose that $-\iota(-x) < \iota(x)$. Now it is easy to see that ι is l.s.c. and sublinear. Then the set $K = \{(x, \lambda) \in \mathcal{M} \times I\!R : \lambda \geq \iota(x)\}$ is a closed convex cone. Hence $\forall \epsilon > 0$ we have that $(x, \iota(x) - \epsilon) \notin K$. Applying the strict separation theorem we obtain that there exist a vector ϕ and there exists real number α such that$\phi \cdot (x, \iota(x) - \epsilon) < \alpha$ and $\phi \cdot (x, \lambda) > \alpha \ \ \forall (x, \lambda) \in K$. Then we can rewrite these inequalities as:

$$\phi_o \cdot x + \phi_{S+1}(\iota(x) - \epsilon) < \alpha$$

$$\phi_o \cdot x + \phi_{S+1}\lambda > \alpha \quad \forall (x, \lambda) \in K$$

where $\phi_o = (\phi_1, ..., \phi_S)$ and, since K is a convex cone, we must have $\alpha < 0$. This implies $\phi_o \cdot x + \phi_{S+1}(\iota(x) - \epsilon) < 0$ and $\phi_o \cdot x + \phi_{S+1}\lambda \geq 0 \ \ \forall (x, \lambda) \in K$. Hence $\phi_{S+1} > 0$ and we can define $\nu(x) = -\frac{\phi_o}{\phi_{S+1}} \cdot x$. It is easy to see that ν is a continuous linear functional and $\nu(x) \leq \iota(x)$, $\forall x \in \mathcal{M}$, since $(x, \iota(x)) \in K$. Also $\nu(x) > \iota(x) - \epsilon$. Now for all $x \in X_+$, we have $\nu(-x) \leq \iota(-x) \leq 0$, so $\nu(x) \geq 0$. With an analoguous argument, we obtain $\nu'(x) \in \Gamma$ such that $\nu'|_{\mathcal{M}} \leq \iota$ and

$$-\iota(-x) \leq \nu'(x) \leq -\iota(-x) + \epsilon$$

Since $\{\nu \in \Xi/\nu|_{\mathcal{M}} \leq \iota\}$ is a convex set and $\{\nu(x)/\nu|_{\mathcal{M}} \leq \iota \ , \ \nu \in \Gamma\}$ is an interval we obtain the inclusion.$\blacksquare$

Remark

- Our definition of maximal willingness to pay $\iota(x)$ is in the spirit of the super replication approach of El Karoui and Quenez [12] and Cvitanić and Karatzas [7] to pricing in incomplete markets. We consider as superhedging strategies the defaultable assets.

  Theorem 2, (ii) establishes a one to one correspondence between linear pricing rules, bounded from above by $\iota(x)$, and measures β^*, considered in the sub and supermartingale pricing formulas
  Our result (iii) implies

$$[\inf_{\beta^* \in \mathcal{Q}^*} E^*(\frac{x}{\gamma}), \sup_{\beta^* \in \mathcal{Q}^*} E^*(\frac{x}{\gamma})] = [-\iota(-x), \iota(x)]$$

5 Equilibria

In this section borrowers (sellers of assets) will choose the collateral coefficients. We assume that there is a continuum of agents $H = [0, 1]$ modeled by the Lebesgue probability space $(H, \mathcal{B}, \lambda)$. Each agent h is characterized by his endowments ω_h and his utility U^h. Each agent sells in the initial period J assets that will be backed by a chosen collateral bundle and purchases also the derivatives; in the second period will receive the respective returns.

The allocation of the commodities is an integrable map $x : H \to I\!R_+^{(S+1)L}$. The derivative purchase and primitive assets short sale allocations are represented by two integral maps; $\theta : H \to I\!R_+^J$ and $\varphi : H \to I\!R_+^J$, respectively. Each borrower h will choose the collateral coefficients for each portfolio sold .The allocation of collateral coefficients chosen by borrowers is described by the function $M : H \to I\!R_+^J$.

Consumers short-sell and collateralize the primitive assets but can only buy a derivative issued by a financial intermediary that buys the primitive assets. The value of the derivative's aggregate purchases must match the value of the primitive asset's aggregate short-sales (and the value of the aggregate respective returns should also be equal in any state of nature in the future). Each buyer of assets (lender) will take as given the derivatives' payoffs N_{sj} and a mean collateral coefficients vector $C \in I\!R_+^{JL}$ as given. Let $x_{-o}^h = (x_1^h, \ldots, x_S^h)$ be the commodity consumption in the several states of the world in the second period.

Sale prices of primitive assets are assumed to consist of a base price minus a discounted expected value of future default plus a term reflecting the collateral requirements (which entail a cost but yield a depreciated collateral bundle) and an addicional negative tail $\delta_j \equiv -(p_0 - \sum_s \gamma_s p_s Y_s) C_j$ which is independent of the collateral choice. More specifically we assume

$$\pi_{2j} = q_j - \sum_s \gamma_s (p_s R_{sj} - p_s Y_s M_j)^+ + (p_o - \sum_s \gamma_s p_s Y_s)(M_j - C_j) \qquad (8)$$

The state prices γ_s are common to all agents and taken as given together with the base price q. The vector of prices for the collateralized derivatives, whose returns are given by N_s, is π_{1j}. We will show that for an asset j which is traded we have $q_j = \sum_s p_s R_{sj}$ and the price of the respective derivative $\pi_{1j} = \sum_s \gamma_s N_{sj}$.

Then the individual problem is

$$\max_{(x^h, \theta^h, \varphi^h, M^h) \in B^h} U^h(x_o^h + M^h \varphi^h, x_{-o}^h) \qquad (9)$$

where B^h is the budget set of each agent $h \in H$ given by:

$$B^h(p, \pi_1, q, \gamma, C, N) = \big\{(x, \theta, \varphi, M) \in I\!R^{L(S+1)+2J+JL} : \text{(1) and (2) hold for } \pi_2 \text{ given by (8)}\big\}$$

Definition 2 *An equilibrium is a vector* $((p, \pi_1, \pi_2, C, N), (x^h, \theta^h, \varphi^h, M^h)_{h \in H})$ *such that:*

- $$(x^h, \theta^h, \varphi^h, M^h)$$
  solves problem (9)

- $$\int_H \left(x_o^h + \sum_{j \in J} M_j^h \varphi_j^h \right) dh = \int_H \omega_o^h dh \tag{10}$$

$$\int_H x^h(s) dh = \int_H \left(\omega^h(s) + \sum_{j \in J} (Y_s M_j^h \varphi_j^h + Y_s x_o^h) \right) dh \tag{11}$$

- $$\int_H M_j^h \varphi_j^h dh = C_j \int_H \varphi_j^h dh \ \ \forall\, j \in J \tag{12}$$

- $$N_{sj} \int_H \theta_j^h dh = \int_H D_{sj} \varphi_j^h dh,\ \forall j \in J,\ \forall s \in S \tag{13}$$

- $$\pi_1^j \int_H \theta_j^h dh = \int_H \pi_2^{jh} \varphi_j^h dh. \tag{14}$$

Some Remarks

- Equations (10) and (11) are the usual market clearing conditions. Equation (12) says that in equilibrium the anonymous collateral coefficient C_j is anticipated as the weighted average of the collateral coefficients allocation M_j.

- Equation (13) says that aggregate yields of each derivative must be equal to aggregate actual payments of the underlying primitive assets. This implies that aggregate default suffered must be equal to aggregate default given, for each state and each promise:

  $$\int_{h \in \mathcal{S}_s^j} (p_s R_s^j - N_{sj})^+ \theta_j^h dh = \int_{h \in \mathcal{G}_s^j} (p_s R_s^j - p_s Y_s M_j^h)^+ \varphi_j^h dh \ \ \forall s \in S,\ \forall j \in J$$

  Where $\mathcal{S}_s^j = \{h \in H : p_s R_s^j > N_{sj}\}$ is the set of agents that suffered default in state of nature s on asset j and $\mathcal{G}_s^j = \{h \in H : p_s R_s^j > p_s Y_s M_j^h\}$ is the set of agents that give default in state of nature s on asset j. Note that $\mathcal{S}_s^j$ is equal to H or ϕ, since $p_s R_s^j$ and N_{sj} do not depend on h.

- The above equilibrium concept portraits equilibria in housing mortgages markets where individual mortgages are backed by houses and then huge pools of mortgages are split into derivatives.

  In our anonymous and abstract setting, any agent in the economy may be simultaneously a homeowner and an investor buying a derivative. The above equilibrium concept assumes the existence of J financial institutions, each one buying the pool of mortgages, written on primitive asset j, from consumers at prices π^h_{2j} and issuing the respective derivative, which is sold to consumers at prices π_{1j}. These financial institutions make zero profits in equilibrium both at the initial date and at any future state of nature.

  To simplify, we mix promises of different sellers of a same asset but do not mix different assets into derivatives. This simplification is not too strong, since different sellers of a same asset end up selling personalized assets due to different choices of collateral. A more elaborate model should allow for the mix of different primitive assets and for the strategic choice of the mix of assets and debtors by the issuer of the derivative. Putting together in a same model the price-taking consumers and investments banks composing the derivatives strategically may be a difficult task, since the latter would have to anticipate the Walrasian response of the former.

We will now fix our assumptions on preferences.

Assumption (P) : preferences are time and state separable, monotonic, representable by smooth strictly concave utility functions u^h

Theorem 3 *If consumers's preferences satisfy assumption (P) and the endowments allocation ω belongs to $L^\infty(H, \mathbb{R}^{(S+1)L}_{++})$, then, there exist equilibria where borrowers choose their respective collateral coefficients.*

6 Proof of the Existence Theorem

Let us first address the case where bundles and portfolios are bounded from above. More precisely, nonfinanced consumption bundles x^h, portfolios

(θ^h, φ^h) and collateral coefficients M_j^h are bounded by n in each coordinate. Then we will let n go to ∞.

Truncated Economy

Define a sequence of truncated economies $(\mathcal{E}_n)_n$ such that the budget set of each agent h is

$$B_n^h(p, \pi_1, q, \gamma, C, N) := \{(x_n^h, \theta_n^h, \varphi_n^h, M_n^h) \in [0, n]^{L(S+1)+2J+JL} : x_{0n}^h + M_n^h \varphi_n^h \leq n\mathbf{1},$$

$$(1) \text{ and } (2) \text{ hold } \}$$

We assume that $C \in [0, n]^{LJ}$. We denote by $\mathbf{1}$ the vector (1,...,1).

Generalized Game

For each $n \in \mathbb{N}$ we define the following generalized game played by the continuum of consumers and some additional atomic players. Denote this game by $\mathcal{J}_n$ which is described as follows:

- Each consumer $h \in H$ maximizes U^h in the constrained strategy set $B_n^h(p, q, \pi_1, C, \gamma)$.
- The auctioneer of the first period chooses $p_o \in \triangle^{L-1}$ in order to maximize
$$p_o \int_H (x_o^h + \sum_j M_j^h \varphi_j^h - \omega_o^h) dh$$
- The auctioneer of state s of the second period chooses $p_s \in \triangle^{L-1}$ in order to maximize
$$p_s \int_H (x_s^h - Y_s(\sum_j M_j^h \varphi_j^h + x_o^h) - \omega_s^h) dh.$$
- The first JL fictitious agents chooses $C_{jl} \in [0, n]$ in order to minimize
$$\left(C_{jl} \int_H \varphi_j^h dh - \int_H M_{jl}^h \varphi_j^h dh \right)^2.$$

- Another fictitious agent chooses $\pi_{1j} \in [0,1]$, $q_j \in [0, \overline{\gamma} \max_{s,k} R_{sjk}]$, $N_{sj} \in [0,n]$ and $\gamma_s \in [0,\overline{\gamma}]$ for every j and s in order to minimize

$$\sum_j \Bigg((\pi_{1j} \int_H \theta_j^h dh - \int_H \pi_{2j}^h \varphi_j^h dh)^2 + (q_j - \sum_s \gamma_s p_s R_{sj})^2 \int_H \theta_j^h dh$$

$$+ \sum_s (N_{sj} \int_H \theta_j^h dh - \int_H \min \{ p_s R_{sj}, p_s Y_s M_j^h \} \varphi_j^h dh)^2 \Bigg)$$

This game has an equilibrium in mixed strategies (see lemma 8) and, by Liapunov's Theorem (see lemma 9), there exists a pure strategies equilibrium.

Now let us define a free disposal equilibrium for the truncated economy as a pair consisting of a price vector (p, π_1, γ, C, N) and an allocation $(x, \theta, \varphi, M)^H)$ such that $(x, \theta, \varphi, M)(h)$ maximizes consumer h's utility U^h on the constrained budget set of the truncated economy given the price vector and

$\int_H (x_0^h + M^h \varphi^h - \omega_0^h) dh = 0$

$\int_H (x_s^h - \omega_s^h - Y_s x_0^h - Y_s M^h \varphi^h) dh \leq 0$

$N_{sj} \int_H \theta_j^h dh \leq \int_H D_{sj}^h \varphi_j^h dh$

$\pi_1^j \int_H \theta_j^h dh = \int_H \pi_2^{jh} \varphi_j^h dh$

$C_j \int_H \varphi_j^h dh = \int_H M_j^h \varphi_j^h dh$

$\int_H \varphi_j^h dh = 0 \Longleftrightarrow \int_H \theta_j^h dh = 0$

Lemma 1 *For n large enough, there exists a free-disposal equilibrium for the truncated economy.*

Proof:

Let $z = (x^h, \theta^h, \varphi^h, M^h) : H \to [0, n]^{L(S+1)+2J+LJ}$, $(p_o, q, \pi_1, \gamma,\ p_s, C)$ be an equilibrium in pure strategies for $\mathcal{J}_n$. Now, $C_j \int_H \varphi_j^h dh = \int_H M_j^h \varphi_j^h dh$. In fact, the equality holds trivially when $\int_H \varphi^h dh = 0$ and, otherwise, notice that $\int_H M_j^h \varphi_j^h dh / \int_H \varphi_j^h dh \leq n$ and therefore C_j can be chosen in $[0, n]$ to make this equality hold.

Claim: (1) $\pi_1 \int_H \theta^h dh = \int_H \pi_2^h \varphi^h dh$, (2) $q_j = \sum_s \gamma_s p_s R_{sj}$ when $\int_H \theta_j^h dh \neq 0$, (3) $\int_H \theta^h dh = 0$ if and only if $\int_H \varphi^h dh = 0$.

In fact, if $\int_H \theta_j^h dh \neq 0$ the financial intermediary sets $q_j = \sum_s \gamma_s p_s R_{sj}$ and chooses $\pi_{1j} \in [0, 1]$ and $\gamma \in [0, \overline{\gamma}]^S$ so that

$$\begin{aligned} \pi_{1j} \int_H \theta_j^h dh &= \int_H \pi_{2j}^h \varphi_j^h dh \\ &= \sum_s \gamma_s \int_H D_j^h \varphi_j^h dh \end{aligned}$$

If $\varphi_j^h = 0$ for a.e. h but $\int_H \theta_j^h dh \neq 0$, then N_j and $\pi_{1j}$are set equal to zero, implying that θ_j^h could be instead set equal to zero, for a.e. h, without affecting any of the strategic equilibrium conditions.

If $\int_H \theta_j^h dh = 0$ the financial intermediary sets γ so that $p_0 \geq \sum_s \gamma_s p_s Y_s$ and makes $q_j = \sum_s \gamma_s (p_s R_{sj} - p_s Y_s M_j)^+$ implying

$$\pi_{2j}^h = (p_0 - \sum_s \gamma_s p_s Y_s)(M_j - C_j).$$

Notice that this agent may not be able to make $p_0 = \sum_s \gamma_s p_s Y_s$ as γ may have to be low enough so that $\pi_{1k} \int_H \theta_k^h dh = \sum_s \gamma_s \int_H D_k^h \varphi_k^h dh$ holds with $\pi_{1k} \leq 1$ for other assets with $\int_H \theta_k^h dh > 0$.

When $p_0 = \sum_s \gamma_s p_s Y_s$ all borrowers choose $\varphi_j^h = 0$ and when $p_0 > \sum_s \gamma_s p_s Y_s$ all borrowers choose $M_j^h \geq C_j$ and $C_j \int_H \varphi^h dh = \int_H M^h \varphi_j^h dh$ implies $M_j^h = C_j$. Then, $\pi_{2j}^h = 0$ and $\varphi_j^h = 0, \forall h. \Box$.

Claim: (1) $N_{sj} \int_H \theta_j^h dh \leq \int_H \min \left\{ p_s R_{sj}, p_s Y_s M_j^h \right\} \varphi_j^h dh, \forall s$ and (2) $\pi_{1j} \geq \sum_s \gamma_s N_{sj}$

In fact, these inequalities hold as equalities when $\int_H \theta_j^h dh = 0$ (as seen above) or when $\int_H D_{sj}^h \varphi_j^h dh / \int_H \theta_j^h dh$ does not exceed n, for every s. Otherwise, the strict inequalities hold in (1) for some s and in (2).□

Now, the optimality conditions of the auctioneers' problems imply that

$$\int_H (x_o^h - \omega_o^h + M^h \varphi^h) dh \leq 0 \tag{15}$$

$$\int_H (x_s^h - \omega_s^h - Y_s M^h \varphi^h - Y_s x_o^h) dh \leq 0 \tag{16}$$

After integrating the budget constraint of the second period, we obtain

$$p_s \int_H (x_s^h - \omega_s^h - Y_s M^h \varphi^h - Y_s x_o^h) dh \leq 0, \forall s \in S \tag{17}$$

For n larger enough, we must have $p_{ol} > 0, \forall l \in L$. Otherwise, every consumer would choose $x_{ol}^h = n$ and we would have contradicted (15) But when $p_{ol} > 0$ we must have

$$\int_H (x_{ol}^h - \omega_{ol}^h + (M^h \varphi^h)_l) dh = 0 \ \forall l \in L \tag{18}$$

since the aggregate budget constraint of the first period is a null sum of non positive terms and therefore a sum of null terms. ■

Asymptotics of truncated free-disposal equilibria

Now let $\{(x_n^h, \theta_n^h, \varphi_n^h, (M_n^h)_{\{h \in H\}}), p_n, \pi_{1n}, q_n, \gamma_n, C_n, N_n\}$ be the sequence of free-disposal equilibria corresponding to $\mathcal{E}_n$. Let $n \to \infty$ and examine the asymptotic properties of the sequence.

Lemma 2 $p_{sl}^n \nrightarrow 0 \ \forall_{s,l}$

Proof:

Suppose $p_{sl}^n \to 0$ for some (s, l). Since $\int x_{sl}^{hn} dh$ is bounded it follows, by Fatou's lemma, that for h in a full measure set $F_0$the sequence $\{x_{sl}^{hn}\}$has a cluster point. Moreover for each n there is a full measure set F_n of consumers that are optimizing in the equilibrium of the economy $\mathcal{E}_n$. Let $F = \bigcap_{n=0}^{\infty} F_n$

which is a full measure set and take $h \in F$. Passing to a subsequence if necessary, we can take $x_{sl}^{hn} < n - v$ for every n, where $v = \min_l \varpi_{sl}^h$, and let $\widehat{x}^n \in \Re_+^{L(S+1)}$ be such that $\widehat{x}_{sl}^n = x_{sl}^{hn} + v < n$ and $\widehat{x}_{ki}^n = x_{ki}^{hn}$ for $(k,i) \neq (s,l)$. Now, by monotonicity of preferences

$$u^h((1-p_{sl}^n)\widehat{x}+ve_{sl}, M^{hn}(1-p_{sl}^n)\varphi^{hn}) > u^h(x^{hn}, M^{hn}\varphi^{hn}) \quad \text{for } n \quad \text{large enough.}$$

Moreover, the vector $(1- p_{sl}^n)(\widehat{x}, \theta^{hn}, \varphi^{hn})$ together with M^{hn} is budget feasible since $(1- p_{sl}^n)p_s^n\varpi_s^h + p_{sl}^n v \leq p_s^n\varpi_s^h$. This contradicts the optimality of $(x^{hn}, \theta^{hn}, \varphi^{hn}, M^{hn})$.■

The sequences $\{M_{jl}^{hn}\}_n$ and $\{C_{jl}^{hn}\}_n$admit $(\max_{s,k,j} R_{sk}^j)/(\min_s p_{sl}Y_{sl})$ as an upper bound. In fact, any choice of collateral coefficients beyond this bound determines sure repayment and would be equivalent to constituting collateral just up to this bound and consuming the remaining in the form of a bundle not serving as collateral (that is, as part of x_0^n).

Lemma 3 $C_j^n \not\to 0$ *as* $n \to \infty$. *Actually, there exist uniform positive lower bounds, across consumers, for the sequence* M^{hn} *of equilibrium collateral coefficients*

Proof:

Let $\mathfrak{S}_j^{hn} = \{s \in S : p_s^n R_s^j > p_s^n Y_s M_j^{hn}\}$ be the set of states where agent h gives default in promise j and let $\left(\mathfrak{S}_j^{hn}\right)'$ be it's complement. Now, $\left(\mathfrak{S}_j^{hn}\right)' \neq \emptyset \;\; \forall\, h, j$ for n large enough, when asset j is traded. Otherwise, the Kuhn-Tucker first order necessary condition in M_{jl}(see section 9.4 in the appendix where the necessity is established) would become $u'_{ol}\varphi_j \leq 0$, which is impossible.

Now let $T_{sj}^n = \left\{z \in R_+^l : p_s^n Y_s z \geq p_s^n R_{sj}\right\}$ and $T_j^n = \bigcup_{s=1}^{S} T_{sj}^n$. Then, for each n, $\forall h$, $M_j^{hn} \in T_j^n$ and $C_j^n \in \overline{conT_j^n}$. Notice that for n large enough $p_{sl}^n \neq 0$ and therefore $0 \notin \overline{conT_j^n}$. Define the corresponding sets at the cluster point $(p_s)_{s=1}^S \gg 0 : T_{sj} = \left\{z \in I\!R_+^l : p_s Y_s z \geq p_s R_{sj}\right\}$ and $T_j = \bigcup_{s=1}^{S} T_{sj}$. We must have the cluster point C_j of the sequence C_j^n belonging to $\overline{con}T_j$ which does not contain the origin, hence $C_j \neq 0$. This completes the proof of lemma 3.■

The intuition behind the claim in the above proof that consumers never choose collateral coefficients so low that they end up defaulting in every state lies in the fact that, by increasing the collateral coefficients up to the point where the depreciated value of the collateral exactly matches the promised payment in some state, consumers still have a zero net return, from the joint operation of short selling and constituting collateral (as they keep on surrending the value of the depreciated collateral), and manage to mantain also the same net price of this joint operation as the short sale price increase (due to reduced default spread) equals the increase in collateral costs, but utility has meanwhile gone up as more collateral is being consumed. In fact, let the collateral coefficients rise from $M_j^{h1}$to M_j^{h2}. The short sale price π_{2j}^h is given by $q_j - \sum_s \gamma_s def_{sj} + (p_0 - \sum_s \gamma_s p_s Y_s)(M_j^h - C_j)$, where def_{sj} is the default on asset j in state s. Then, the increase in π_{2j}^h is equal to $p_0 \Delta M_j^h$ since $\Delta def_{sj} = p_s R_{sj} - p_s Y_s M_j^{2h} - (p_s R_{sj} - p_s Y_s M_j^{1h}) = -p_s Y_s \Delta M_j^h$.

Lemma 4 $\{\int_H (x_n^h, \varphi_n^h, M_n^h \varphi_n^h) dh\}$ *is a bounded sequence.*

Proof:

By definition of equilibrium,

$\int_H x_{no}^h dh \leq \int_H \omega_o^h dh$ and $\int_H M_n^h \varphi_n^h dh \leq \int_H \omega_o^h dh$.

So

$$\int_H x_{ns}^h dh < \int_H (\omega_s^h + 2Y_s \omega_o^h) dh, \forall s \in S. \tag{19}$$

For each $l \in L$ the following holds

$$\int_H M_{lnj}^h \varphi_n^h dh = C_{lnj} \int_H \varphi_{jn}^h dh \tag{20}$$

and therefore

$$C_{jnl} \int_H \varphi_{nj}^h dh \leq \int_H \omega_{ol}^h dh, \forall l \in L \tag{21}$$

Then, by lemma 3, $\int_H \varphi_{nj}^h dh$ is bounded. ■

Lemma 5 *The aggregate purchase of the derivative can also be taken as bounded, along the sequence of equilibria for the truncated economies.*

Proof:

Let $N(n) = \max_s N^n_{sj}$ and use the homogeneity of degree -1 of demand for the derivative with respect to (N_j, π_{1j}) to replace N^n_j, π^n_{1j} and θ^{nh}_j by $\widetilde{N}^n_{sj} = N^n_{sj}/N(n), \widetilde{\pi}^n_{1j} = \pi^n_{1j}/N(n)$ and $\widetilde{\theta}^{hn}_j = \theta^{hn}_j N(n)$. Then, $\widetilde{N}^n_{sj}$ has a cluster point also,$\forall_s$ and actually, passing to a subsequence if necessary, $\widetilde{N}^n_{sj}$ is equal to one for some s and every n. Now,

$$\widetilde{N}_{sj} \int_H \widetilde{\theta}^h_j dh \leq \int_H \min\left\{p_s R_{sj}, p_s Y_s M^h_j\right\} \varphi^h_j dh \leq p_s R_{sj} \int_H \varphi^h_j dh$$

and therefore $\int_H \widetilde{\theta}^{hn}_j dh \nrightarrow \infty$.

In the rest of the proof, to simplify the notation, let us take θ^n to be actually the allocation $\widetilde{\theta}^n$.■

Lemma 6 *the sequence of allocations* $\{x_n, \theta_n, \varphi_n, M_n\}$ *is uniformly bounded.*

Proof:

By the two preceding lemmas, the sequence $z^h_n \equiv (x^h_n, \theta^h_n, \varphi^h_n, M^h_n)$ satisfies the hypothesis of the weak version of Fatou's Lemma. Therefore $\exists z$ integrable such that

$$z^h \in cl\{z_n(h)\} \text{ for a.e } h$$

Notice also that $p_n, \pi_{1n}, q_n, \gamma_n$ have cluster points.

Claim $z^h \equiv (x^h, \theta^h, \varphi^h, M^h)$ maximizes U^h at the cluster point of $(p^n, q^n, \gamma^n, C^n, N^n)$, for almost every h.

In fact, z^h is budget feasible at $(p, q, \pi_1, \gamma, C, N) = \lim_{n\to\infty}(p^n, q^n, \pi^n_1, \gamma^n, C^n, N^n)$, passing to a subsequence if necessary and consumers' optimal choice correspondences are closed (see appendix).□

Individual optimality at the cluster points implies that $p_{0l}^n \nrightarrow 0$ $(l = 1, ..., L)l$ and $\pi_{1j}^n \nrightarrow 0$ $(j = 1, ..., J)$. It follows immediately that

$$x_{ol}^{hn}, \theta_j^{hn} \leq (ess \sup_{h,l} \omega_{ol}^h)/(\min_{l,j} \left\{ \lim_{n\to\infty} p_{ol}^n, \lim_{n\to\infty} \pi_{1j}^n \right\}). \quad \blacksquare$$

Lemma 7 *The short sales allocation is also uniformly bounded*

Proof:

Suppose not, then there is a sequence $h(n)$ of agents such that $\varphi_n^{h(n)} \to \infty$, even though $\varphi_n^h \nrightarrow \infty$ for almost every $h \in \{h(n)\}_n$. Now:

$$p_o^n M_j^{h(n)} - \pi_{2j}^{h(n)} = -q_j^n + \sum_s \gamma_s^n (p_s^n R_{sj} - p_s^n Y_s M_j^{h(n)})^+ + \sum_s \gamma_s^n p_s^n Y_s M_j^{h(n)}$$

$$+(p_o^n - \sum_s \gamma_s^n p_s^n Y_s) C_j^n$$

equivalently

$$p_o^n M^h(n)_j - \pi_{2j}^{h(n)} = a_j^n + b_j^{h(n)}$$

where

$$a_j^n \equiv -q_j^n + (p_o^n - \sum_s \gamma_s^n p_s^n Y_s) C_j^n + \sum_s \gamma_s^n p_s^n R_{sj}$$

$$b_j^{h(n)} \equiv \sum_s \gamma_s^n p_s^n Y_s M_j^{h(n)} - \sum_s \gamma_s^n D_j^{h(n)}$$

and

$$D_j^{h(n)} \equiv \min \left\{ p_s^n R_{sj}, p_s^n Y_s M_j^{h(n)} \right\}$$

Claim $q_k < \sum_s \gamma_s p_s R_{sk} + (p_o - \sum_s \gamma_s p_s Y_s) C_k$, for any asset k

Otherwise any agent could make $p_o M_k - \pi_k^{2h} = \lim_{n\to\infty} a_k^n \leq 0$ for M_k^h sufficient small but different from zero so that default occurs in every state (or the payment exactly matches the value of the depreciated collateral).

Such a choice for M_k^h would be accompanied by choosing φ_k arbitrary large, which can not occur since there is a finite optimal choice z^{nh} for almost every h (see the claim in the proof of lemma 6). $\square$

Now, for any asset k, $b_k^{h(n)} \geq 0$ and, for n large enough, the above claim implies that $a_k^n > 0$, so that $p_o^n(M_k^{h(n)} - \pi_k^{2h(n)}) > 0$ for any asset k and n large enough. Then, the first period budget constraint implies that $p_o^n(M_j^{h(n)} - \pi_j^{2h(n)})\varphi_j^{h(n)}$ is bounded by $\sup_{h,l} \varpi_{0l}^h$. Hence $\varphi_n^{h(n)} \to \infty$ implies $(p_o^n M_j^{h(n)} - \pi_j^{2h(n)}) \to 0$ and therefore $a_j^n \to 0$, contradicting the above claim. $\blacksquare$

Then, $\{M_{jln}^h \varphi_{jn}^h\}$ is uniformly bounded and, from (2), $\{x_{sln}^h\}$ is also uniformly bounded. All these facts imply that the sequence $(x_n, \theta_n, \varphi_n, M_n\varphi_n)$ is uniformly bounded.

We can now continue the proof of existence of equilibria for the economy $\mathcal{E}$ using the strong version of Fatou's lemma (see Appendix):

$\int_H x^h dh = \lim_{n\to\infty} \int_H x_n^h dh$, $\int_H \theta^h dh = \lim_{n\to\infty} \int_H \theta_n^h dh$,

$\int_H \varphi^h dh = \lim_{n\to\infty} \int_H \varphi_n^h dh$ and

$$\int_H M^h \varphi^h dh = \lim_{n\to\infty} \int_H M_n^h \varphi_n^h dh$$

Thus all markets clear in the $\mathcal{E}$. We also have $C_{jl} \int_H \varphi_j^h dh = \int_H M_{jl}^h \varphi_j^h dh$.

Moreover, $N_{sj} \int_H \theta_j^h dh = \int_H \min\left\{p_s R_{sj}, p_s Y_s M_j^h\right\} \varphi_j^h dh$. Suppose not, then, using the notation in the proof of lemma 5, we would have for all n large enough $\int_H D_j^{hn} \varphi_j^{hn} dh / (N(n) \int_H \theta_j^{hn} dh) > n$, for some (s, j), implying $\int_H D_j^{hn} \varphi_j^{hn} dh / \int_H \theta_j^{hn} dh > nN(n)$. If $N(n) < n$, then the inequality would hold as equality. If $N(n) = n$, then $n^2 \int_H \theta_j^{hn} dh$ would be bounded, implying that $\int_H \widetilde{\theta}_j^{hn} dh = n \int_H \theta_j^{hn} dh \to 0$. Now, $\widetilde{\pi}_{1j}^n \int_H \widetilde{\theta}_j^{hn} dh = \sum_s \gamma_s^n \int_H D_{sj}^{hn} \varphi_j^{hn} dh$ where $\widetilde{\pi}_{1j}^n = \pi_{1j}^n / N(n)$ is bounded. Hence $\gamma^n \to 0$ or $\int_H D_j^{hn} \varphi_j^{hn} dh \to 0$, but the former implies the latter, since we would have $\pi^h_{2j} = p_0(M_j - C_j)$ which would lead every agent to choose $M^h_j \geq C_j, \forall_h$, and, therefore, $M^h_j = C_j$, implying that $\pi^h_{2j} = 0$ and $\varphi^h_j = 0, \forall_h$, contradicting the supposed strict inequality.

Moreover, $\pi_{1j} = \sum_s \gamma_s N_{sj}$ when asset j is traded, since

$$\pi_{1j} \int_H \theta^h_j dh = \int_H \pi^h_{2j} \varphi^h_j dh = \sum_s \gamma_s \int_H D^h_{sj} \varphi^h_j dh = \sum_s \gamma_s N_{sj} \int_H \theta^h_j dh.$$

7 Efficency

In this section we prove that an equilibrium allocation is constrained efficient among all feasible allocations that provide income across states through the same spot prices (the given equilibrium prices). In comparison with the equilibrium obtained by Araujo, Orrillo and Páscoa [3], we can say that our equilibrium is Pareto superior, since we are not impossing any kind of bounded short sale.

As in the work of Magill and Shafer [18], we compare the equilibrium allocation with one feasible allocation whose portfolios do not necessarily result from trading competitively in asset markets. That is, in alternative allocations agents pay participation fees which may differ from the market portfolio cost. Equivalently, we allow for transfers across agents which are being added to the usual market portfolio cost.

Proposition 1 *Let* $((\overline{x}, \overline{\theta}, \overline{\varphi}, \overline{M}), \overline{p}, \overline{\pi}_1, \overline{\pi}_2, \overline{C}, \overline{N})$ *be an equilibrium. The allocation* $(\overline{x}, \overline{\theta}, \overline{\varphi}, \overline{M})$ *is efficient among all allocations* (x, θ, φ, M) *for which there are transfers* $T^h \in I\!R$ *across agents and a vector* $C \in I\!R^{JL}_+$*, such that*

(i) $\int_H (x^h_o + M^h \varphi^h) dh = \int_H \omega^h_o dh$, $\int_H x^h_s = \int_H (\omega^h_s + Y_s M^h \varphi^h + Y_s x^h_o) dh$, $\overline{\pi}_1 \int_H \theta^h dh = \int_H \overline{\pi}_2 \varphi^h dh$

(ii)

$$\overline{p}_s(x^h_s - \omega^h_s - Y_s x^h_o) + \sum_{j \in J} \min\{\overline{p}_s R^j_s, \overline{p}_s Y_s M^h_j\} \varphi^h_j$$

$$= \sum_{j \in J} \overline{N}^j_s \theta^h_j + \sum_{j \in J} \overline{p}_s Y_s M^h_j \varphi^h_j, \ \ \forall s, \ \ a.e. \ h$$

(iii) $\overline{p}_o(x_o^h + M^h\varphi^h - \omega_o^h) + \overline{\pi}_1\theta^h - \overline{\pi}_2\varphi^h + T^h = 0$

(iv) $\int_H T^h dh = 0$

(v) $C_j \int_H \varphi_j^h dh = \int_H M_j^h \varphi_j^h, \ \ \forall j$

where the equilibrium prices are given by

$$\overline{\pi}_1 = \overline{q} - \sum_s \overline{\gamma}_s \overline{g}_{1s}$$

and

$$\overline{\pi}_2 = \overline{q} - \sum_s \overline{\gamma}_s \overline{g}_{2s} + (\overline{p}_o - \sum_s \overline{\gamma}_s \overline{p}_s Y_s)\overline{M}_j - (\overline{p}_o - \sum_s \overline{\gamma}_s \overline{p}_s Y_s)\overline{C}_j$$

Proof:

Suppose not, say $(x, \theta, \varphi, M, C)$ together with some transfer fraction T satisfies (i) through (v); $u^h(x_o^h + M^h\varphi^h, x_{-o}^h) \geq u^h(\overline{x}_o^h + \overline{M}^h\overline{\varphi}^h, \overline{x}_{-o})$ for $a.e$ h and $u^h(x_o^h + M^h\varphi^h, x_{-o}^h) > u^h(\overline{x}_o^h + \overline{M}^h\overline{\varphi}^h, \overline{x}_{-o})$ for h in some positive measure set G of agents. Then, for $h \in G$, the first period constraint must be violated, that is,

$$\overline{p}_o(x_o^h + M^h\varphi^h - \omega_o^h) + \overline{\pi}_1\theta^h - \overline{\pi}_2\varphi^h > 0 \tag{22}$$

Now remember that

$$g_s^h = (\overline{p}_s R_s - \overline{p}_s Y_s M^h)^+ \varphi^h - (\overline{p}_s R_s - \overline{N_s})^+ \theta^h$$

$$= (\overline{p}_s R_s - D_s^h)\varphi^h - (\overline{p}_s R_s - \overline{N}_s)\theta^h$$

By continuity of preferences and monotonicity we can take $G = H$, without loss of generality. Then $\int_H g_s^h dh > 0$ for some s, by (22) and (i), implying $\int_H N_s\theta^h dh > \int_H D_s^h \varphi^h dh$. Now, by (ii),

$$\overline{p}_s \cdot \int_H (x_s^h - \omega_s^h - Y_s(M^h\varphi^h + x_o^h))dh = \int_H N_s\theta^h dh - \int_H D_s^h\varphi^h dh$$

where the right hand side is strictly positive, contradicting $\int_H (x_s^h - \omega_s^h - Y_s(M^h\varphi^h + x_o^h))dh = 0$.■

The above weak constrained efficiency property is in the same spirit as properties found in the incomplete markets model without default (see Magill and Shafer [18]) and also in the exogenous collateral model (without utility penalties) of Dubey, Geanakoplos and Zame [9]. As in these models, it does not seem to be possible to show that equilibrium allocations are undominated when prices are no longer assumed to be constant at the equilibrium levels. However equilibria with default and endogenous collateral, as proposed in this paper, is Pareto superior to the no-default equilibria, to the exogenous collateral equilibria and even to the bounded short-sales endogenous collateral equilibria of Araujo, Orrillo and Páscoa [3], since our equilibria is free of any of the constraints which are used in the definition of these equilibrium concepts (that is, absence of default, exogeneity of collateral and bounded short-sales).

8 Conclusions

In this paper we have obtained a no arbitrage characterization of the prices of collateralized promises, where the collateral coefficients are choosen by borrowers as in Araújo, Orrillo, Páscoa [3]. We also obtained a pricing result consistent with the observation made by Jouini and Kallal [16] for the case of short sale constraints, more precisely we have shown that our buy and net sell prices are supermartingale and submartingales, respectively, under some probability measures. For these probabilities we have found lower and upper bounds for the prices of derivatives written on the primitive defaultable assets. Finally using the nonarbitrage characterization of asset prices we proposed an equilibrium pricing formula and showed the existence of equilibrium in the model where borrowers choose the collateral coefficients, without imposing uniform bounds on short-sales (thus avoiding a drawback of the work by Araújo, Orrillo and Páscoa [3]) and showed that this equilibrium is constrained efficient.

9 Appendix

9.1 Mathematical Preliminarities

- Let $C(K)$ the Banach space of continuous functions on the compact metric space K. Let $L^1(H, C(K))$ be the Banach space of Bochner integrable functions whose values belong to $C(K)$. For $z \in L^1(H, C(K))$,

$$||z||_1 := \int_H \sup_K |Z^h| dh < \infty$$

Let $\mathcal{B}(K)$ denotes the set of regular measures on the Borelians of K. The dual space of $L^1(H, C(K))$ is $L^\infty_\omega(H, \mathcal{B}(K))$, the Banach space of essentially strong bounded weak $*$ measurable functions from H into $\mathcal{B}(K)$. We say that $\{\mu_n\} \subset L^\infty(H, \mathcal{B}(K))$ converges to $\mu \in L^\infty_\omega(H, \mathcal{B}(K))$ with respect to the weak * topology on the dual $L^1(H, C(K))$, if

$$\int_H \int_K z^h d\mu_n^h dh \rightarrow \int_H \int_K z^h d\mu^h dh \,, \forall f \in L^1(H, C(K))$$

- We will use in this work the following lemmas (in m-dimension).
Fatou's lemma (Weak Version)
Let $\{f_n\}$ be a sequence of integrable functions of a measure space $(\Omega, \mathcal{A}, \nu)$ into $\mathbb{R}^m_+$. Suppose that $\lim_{n\to\infty} \int_\Omega f_n d\nu$ exists. Then there exists an integrable function $f : \Omega \mapsto \mathbb{R}^m_+$ such that:

  1. $f(w) \in cl\{f_n(w)\}$ for a.e w, and
  2. $\int_\Omega f d\nu \leq \lim_{n\to\infty} \int_\Omega f_n d\nu$

  Fatou's lemma (Strong version)
If in addition the sequence $\{f_n\}$ above is uniformly integrable, then the inequality in 2. holds as an equality.

9.2 Extended Game

We extend the generalized game by allowing for mixed strategies both in portfolios and collateral bundles. Remember that, for each player a mixed strategy is a probability distribution on his set of pure strategies. In this case the set of measures on the Borelians of $K_n = [0,n]^J \times [0,n]^J \times [0,n]^{LJ}$. We denote by $\mathcal{B}$ the set of mixed strategies of each consumer. Since we are not interested in a mixed strategies equilibrium, per se, we will extend the previous game to a game $\overline{\mathcal{J}}_n$ over mixed strategies (that we call extended game) whose equilibria: 1) exist 2) can be purified and 3) a pure version is an equilibrium for the original game. First, before extending the game to mixed strategies, let us rewrite the payoffs of the fictitious agents replacing consumption bundles by the following function of portfolios and collateral:

$$d^h(\theta^h, \varphi^h, M^h) = \text{ arg } \max\{u^h : x_s^h \in [0,n]^L, s = 1, ..., S, x_0^h + M^h\varphi^h \leq n\mathbf{1},$$

$$\text{satisfying (1) and (2)}\}$$

That is, function d^h solves the utility maximization problem for a given portfolio (θ^h, φ^h) and given collateral coefficients M_j^h. By the maximum theorem and the fact that consumers' choice correspondences are closed (see Proposition below), d^h is continuous. Secondly, we extend the payoffs to mixed strategies.

(i) Each consumer $h \in H$ chooses $(x^h, \mu^h) \in [0,n]^{L(S+1)} \times \mathcal{B}$ in order to maximize $\int_{K_n} U^h(x_o^h + M^h\varphi^h, x_{-o}^h)d\mu^h$ subject to $x_0^h + \int_{K_n}(M^h\varphi^h)d\mu^h \leq n\mathbf{1}$ and the following extended budget constraints:

$$p_o(x_o^h - \omega_o^h) + \int_{K_n}[\pi_1\theta^h + p_oM^h\varphi^h - \pi_2^h\varphi^h]d\mu^h \leq 0$$

$$p_s(x_s^h - \omega_s^h - Y_sx_o^h) \leq \int_{K_n}\sum_j(N_s^j\theta_j^h - D_s^j\varphi_j^h + p_sY_sM_j^h\varphi_j^h)d\mu^h \text{ for } s \in S$$

(ii) The auctioneer of the first period chooses $p_o \in \triangle^{L-1}$ in order to maximize

$$p_o\int_H\int_{K_n}[d_o^h(\theta^h, \varphi^h, M^h) + \sum_j M_j^h\varphi_j^h - \omega_o^h]d\mu^h dh$$

(iii) The auctioneer of state s in the second period chooses $p_s \in \triangle^{L-1}$ in order to maximize

$$p_s\int_H\int_{K_n}[d_s^h(\theta^h, \varphi^h, M^h) - \sum_j Y_sM_j^h\varphi_j^h - \omega_s^h - Y_sd_o^h(\theta^h, \varphi^h, M^h)]d\mu^h dh$$

(iv) The first JL fictitious agents chooses $C_{jl} \in [0, n]$ in order to minimize

$$\left(\int_H \int_{K_n} [C_{jl}\theta_j^h - M_{jl}^h \varphi_{jl}^h] d\mu^h dh\right)^2$$

- Another fictitious agent chooses $\pi_{1j} \in [0,1]\,, q_j \in [0, \max_{s,k} R_{sjk}]\,, N_{sj} \in [0, n]$ and $\gamma_s \in [0, \overline{\gamma}]$ for every j and s in order to minimize

$$\sum_j \Bigg((\pi_{1j} \int_H \int_{K_n} \theta_j^h d\mu^h dh - \int_H \int_{K_n} \pi_{2j}^h \varphi_j^h d\mu^h dh)^2$$

$$+(q_j - \sum_s \gamma_s p_s R_{sj})^2 \int_H \int_{K_n} \theta_j^h d\mu^h dh$$

$$+ \sum_s (N_{sj} \int_H \int_{K_n} \theta_j^h d\mu^h dh - \int_H \int_{K_n} \min\left\{p_s R_{sj}, p_s Y_s M_j^h\right\} d\mu^h dh)^2 \Bigg)$$

Lemma 8 *$\overline{\mathcal{J}}_n$ has an equilibrium, possibly in mixed strategies over portfolio and collateral together.*

Proof:

The existence argument in Ali Khan [17] can be modified to allow for some atomic players. First, by the Proposition below, consumers' pure strategies choice correspondences are closed, and therefore, upper semicontinuous in the truncated economy. Now, mixed strategies choice correspondences are the closed convex hull of the pure strategies choice correspondences and, therefore, will be also upper semicontinuous.

Now, define the correspondence:

$$\alpha(p, \pi, C) = \{f \equiv (x, \mu) \in ([0, n]^{L(S+1)} \times \mathcal{B})^H : f(h) \in \nu^h(p, \pi, C)\}$$

Which is also convex valued and upper semicontinuous . The best response correspondences $\mathcal{R}^i$ of the $r = S + 2 + JL$ fictitious agents are convex valued and upper semicontinuous on the profile of consumers' probability measures on K_n (with respect to the weak * topology on the dual of $L^1(H, C(K_n)$). The profiles set is compact for the same topology and Fan - Glicksberg fixed point theorem applies to $\alpha \times \prod_{i=1}^r \mathcal{R}^i$. ■

Lemma 9 $\overline{\mathcal{J}}_n$ *has an equilibrium in pure strategies.*

Proof:

In this part Liapunov's theorem will be fundamental. First, notice that the payoffs of the atomic players in $\overline{\mathcal{J}}_n$ depend on the profile of mixed strategies $(\mu^h)_h$ only through finitely many e indicators of the form ($e = L + S + SL + 2JL$).

$$\int_H \int_{K_n} Z_e^h(\theta^h, \varphi^h, M^h) d\mu^h dh \text{ where } Z_e \in L(H, C(K_n))$$

Secondly, let $E^h(p, \pi, C) = \prod_2 \nu^h(p, \pi, C)$ and $Z = (Z_1, \ldots, Z_e)$. Now,

$$\int_{K_n} Z^h(\theta^h, \varphi^h, M^h) dE^h(p, \pi, C) = conv \int_{K_n} Z^h(\theta^h, \varphi^h, M^h) d(extE^h(p, \pi, C))$$

where the integral on the left hand side is the set in $\mathbb{R}^e$ of the all integrals of the form $\int_{K_n} Z^h(\theta^h, \varphi^h, M^h) d\mu^h$, for $\mu^h \in E^h(p, \pi, C)$. The integral on the right hand side is defined endogenously. The equality above follows by linearity of the map

$$\mu^h \mapsto \int_{K_n} Z^h(\theta^h, \varphi^h, M^h) d\mu^h$$

Then, Theorem I.D.4 in Hildenbrand [15] implies

$$\int_H \int_{K_n} Z^h(\cdot) dE^h(p, \pi, C) dh = \int_H \int_{K_n} Z^h(\cdot) d(extE^h(p, \pi, C)) dh$$

Then, given a mixed strategies equilibrium profile $(\mu^h)_h$, there exists $(\theta^h, \varphi^h, M^h)$ such that the Dirac measure at $(\theta^h, \varphi^h, M^h)$ is an extreme point of E^h (evaluated at the equilibrium levels of the variables chosen by the atomic players) and $(\theta^h, \varphi^h, M^h)_h$ can replace $(\mu^h)_h$ and keep all equilibrium conditions satisfied, without changing the equilibrium levels of the variables chosen by the atomic players but replacing the former equilibrium bundles by $d^h(\theta^h, \varphi^h, M^h)$.■

9.3 Closedness of consumers' choice correspondences

Since $p_0 \in \triangle^{L-1}$ consumers' budget correspondence always has the origin as an interior point of its values, implying that the interior of the budget correspondence is lower-semicontinuous and, therefore, the budget correspondence itself is also lower semi-continuous.

Lemma 10 *The budget correspondence is lower semicontinuous .*

Proof:
Let $B^h(p,q,\gamma,C)$ be the budget set (of the untruncated economy) and let $B_0^h(p,q,\gamma,C)$ be its subset where all $S+1$ budget constraints hold as strict inequalities.

Claim 1: $B^h(p,q,\gamma,C)$ is the closure of $B_0^h(p,q,\gamma,C)$

To prove this claim, let $z=(x,\theta,\varphi,M) \in B^h(p,q,\gamma,C)$. We want to find a sequence (z_n) such that $z_n \in B_0^h(p,q,\gamma,C)$ and $z_n \to z$.

Let $z_n = (k_n x, k_n\theta, k_n\varphi, M)$ where $k_n = 1-1/n$. Let

$$h_0(x,\theta,\varphi,M) = p_0x_0 + p_0M\varphi - \pi_2\varphi + \pi_1\theta$$

and $h_s(x,\theta,\varphi,M) = p_sx_s - N_s\theta - p_sY_sM\varphi + D_s\varphi$ for $s \geq 1$, where $D_{sj} = \min\{p_sY_sM, p_sR_{sj}\}$. Now $h_s(x,\theta,\varphi,M) = p_s\varpi_s > 0$ for $s \geq 0$ and therefore $h_s(z_n) = k_nh_s(z) < h_s(z)$ for $s \geq 0$, that is, $z_n \in B_0^h(p,q,\gamma,C)$, as desired.□

Claim 2: $B_0^h(p,q,\gamma,C) \neq \phi$.

To see this let $x^h = 0, \theta^h = 0, \varphi^h = 0$ and $M^h = 0$. The values thus chosen for these variables satisfy the budget constraint of agent h with strict inequality, as desired.□

Claim 3: B_0^h is lower semicontinuous.

To prove this claim let $\lim_{k\to\infty}(p^k,q^k,\gamma^k,C^k) = (p,q,\gamma,C)$ and $(x^h,\theta^h,\varphi^h,M^h) \in B_0^h(p,q,\gamma,M)$. Then for every $\{(x_k^h,\theta_k^h,\varphi_k^h,M_k^h)\}$ such that

$$\lim_{n\to\infty}(x_k^h, \theta_k^h, \varphi_k^h, M_k^h) = (x^h, \theta^h, \varphi^h, M^h)$$

and for n large enough, the strict budget inequalities hold. Thus

$$(x_k^h, \theta_k^h, \varphi_k^h, m_k^h) \in B_0^h(p^k, q^k, \gamma^k, C^k)$$

for k large enough, as desired.$\square$.

Then lemma follows from Hildenbrand [15], pag. 26, fact 4.$\blacksquare$

It can also be verified that budget correspondences of truncated economies enjoy also the same property. Let us see that choice correspondences of truncated economies are closed. Consumers' optimal choice correspondences are closed at any (p, q, γ, C) satisfying the assumptions of the previous lemma: if $(p^k, q^k, \gamma^k, C^k) \to (p, q, \gamma, C)$, $\overline{z}^k$ is an optimal choice of consumer h at $(p^k, q^k, \gamma^k, C^k)$ and $\overline{z}^k \to \overline{z}$, given any $z \in B^h(p, q, \gamma, C)$, $\exists (z^k) \to z$ such that $z^k \in B^h(p^k, q^k, \gamma^k, C^k)$ and z^k is not prefered to $\overline{z}^k$ by consumer h, implying, by continuity of u^h that $\overline{z}$ is an optimal choice at (p, q, γ, C).

Comment

Consider an economy where derivative and primitive asset aggregates are also required to match in value (but not in quantity) but collateral margin requirements are bounded from below, say $p_0 M_j^h - \pi_{2j} \geq \epsilon$ (or that $p_0 M_j^h - q_j \geq \epsilon$, as in Araujo, Orrillo and Pascoa (2000)), when $\varphi_j^h > 0$. Then, the lower semi-continuity of the budget correspondence holds. In fact, taking $p_0 \in \triangle^{L-1}$ and $\pi_{2j}^h \equiv q_j - \sum_s \gamma_s (p_s R_{sj} - p_s Y_s M_j)^+$ (that is, setting the negative tail in the equilibrium sale price of primitive assets to be $\delta_j \equiv (p_0 - \sum_s \gamma_s p_s Y_s) M_j^h)$, the constraint $p_0 M_j^h - q_j \geq \epsilon$ is always well-defined and admits an interior solution (with M_j^h large enough) which is compatible with the interior solution $(x, \theta, \varphi) = 0$ of the other budget constraints.

9.4 On Necessary Conditions for Utility Maximization

We will examine in this section a first order necessary condition for utility maximization in the budget set of the truncated economy. We will address only the case when the consumer defaults on an asset in every state as desired for the proof of Lemma 3 above (to establish that actually it is never optimal to default in every state, in a large enough truncated economy). In this case, a constraint qualification holds and a Kuhn-Tucker necessary condition can be derived.

Lemma 11 *For truncated economies $\mathcal{E}_n$, with n large enough, if a consumer's optimal choice would lead to default on asset j in every state, then the following Kuhn-Tucker condition on M_j should hold: $u'_{ol}\varphi_j \leq 0$ and $u'_{ol}\varphi_j M_{jl} = 0$, for every commodity l.*

Proof:

Let us modify the problem of maximizing the utility of consumer h in the budget set $B^h_n(p^n, \pi^n_1, q^n, \gamma^n, C^n, N^n)$ of the truncated economy by fixing the values of the variables φ_k and M_k, $k \neq j$, to be equal to the respective truncated equilibrium values φ^{hn}_k and M^{hn}_k. We will now examine a constraint qualification that ensures necessity of the Kuhn-Tucker conditions of this modified maximization problem, with respect to all remaining choice variables, if the set $(\mathfrak{S}^{hn}_j)'$ were empty at an optimal solution. Notice that in this case all functions entering the budget constraints become differentiable, with respect to these remaining choice variables, at the optimal choice vector. We suppose that n is large enough so that $p^n_s \gg 0$ for every $s \geq 1$, which is possible due to Lemma 2.

Denote by $h(z) = 0$ the system of $S+1$ budget constraints in the variables $z = (x, \theta, \varphi_j, M_j)$ and denote by $g(z) \leq 0$ the system of inequaties given, in this order, by $x_s \geq 0$, $x_s \leq n\mathbf{1}$ $(s \geq 1)$, $\theta \geq 0$, $\theta \leq n\mathbf{1}$, $\varphi_j \geq 0$, $\varphi_j \leq n$, $M_j \geq 0$, $M_j \leq n\mathbf{1}$, $x_0 \geq 0$ and $x_0 + M\varphi \leq n\mathbf{1}$ (where φ_k and M_k for $k \neq j$ are set equal to φ^{hn}_k and M^{hn}_k, respectively.).

Let us check that the Mangasarian-Fromovitz constraint qualification (also known as the modified Arrow-Hurwicz-Uzawa condition, see Mangasarian (1994) 11.3.5), holds at a vector $\overline{z}$ for which $(\mathfrak{S}^{hn}_j)' = \emptyset$. We have to show that $\nabla h(\overline{z})$ has full row rank and that the system constituted by $\nabla g_I(\overline{z})z > 0$ together with $\nabla h(\overline{z})z = 0$ has a solution $z \in \Re^a$, where $a = L(S+1)+J+1+L$ and $I = \{i : g_i(\overline{z}) = 0\}$.

First, notice that $\nabla_x h(\overline{z})$ has full row rank since $(\nabla_{x_s} h_s(\overline{z}))_{s=1}^S$ is block diagonal and nonsingular (by lemma 3).

Secondly, $\nabla g_I(\overline{z})z > 0$ is equivalent to (i) $-z_i > 0$ if $\overline{z_i} = 0$, (ii) $z_i > 0$ if $\overline{z_i} = n$ (both (i) and (ii) for z_i equal to x_{sl},θ_k,φ_j or M_j), (iii) $-x_{0l} > 0$ if $\overline{x}_{0l} = 0$ and (iv) $x_{0l} + \overline{M_j}\varphi_j + M_j\overline{\varphi_j} > 0$ if $\overline{x}_{0l} + \overline{M_l}\overline{\varphi} = n$. Let us start by trying to make $\nabla h_s(\overline{z})z = 0$ for $s \geq 1$ and let y be such that $z = (x_s, y)$.

(a) If $\nabla_y h_s(\overline{z})y \leq 0$ let x_s be such that $p_s x_s = -\nabla_y h_s(\overline{z})y \geq 0$ which is possible since x_s does not need to have negative components, as $\overline{x}_s > 0$.

(b) If $\nabla_y h_s(\overline{z})y > 0$, that is, if $p_s Y_s x_0 + N_s\theta < 0$, then $\overline{\theta}_k < n$ for some k or $\overline{x}_{0l} < n$ for some l.

(b1) When $\overline{\theta} = 0$ and $\overline{x}_0 = 0$, let x_s be such that $p_s x_s = -\nabla_y h_s(\overline{z})y < 0$ which is possible since for some commodity l we can make x_{sl} negative because $\overline{x}_{s_l}$ can not be equal to n for every l. Otherwise, $h_s(\overline{z}) = 0$ implies $n \leq p_s\varpi_s + \sum_{k\neq j} p_s Y_s \overline{M}_k \overline{\varphi}_k \leq \max_l \varpi_{sl} + n p_s Y_s$, that is, $p_s Y_s \geq 1 - \max_l \varpi_{sl}/n$ which can not hold for n large due to lemma 2 and since $Y_s < 1$.

(b2) When $\overline{\theta}_k > 0$ for some k or $\overline{x}_{0l} > 0$ for some l, then the respective variable θ_k or x_{0l} can be chosen positive and large so that $p_s Y_s x_0 + N_s\theta \geq 0$ and (b) does not occur, in any state s.

Finally, let us make $\nabla h_0(\overline{z})z = 0$, or equivalently, $p_0 x_0 + \widehat{\pi}_j\varphi_j + \pi_1\theta = 0$, where $\varphi_j \geq 0$ and $\widehat{\pi}_j \equiv -q_j + (p_0 - \sum_s \gamma_s p_s Y_s)C_j + \sum_s \gamma_s p_s R_{sj}$.

If $\widehat{\pi}_j\varphi_j + \pi_1\theta \geq 0$ let x_0 be such that $p_0 x_0 = -(\widehat{\pi}_j\varphi_j + \pi_1\theta) \leq 0$, by choosing $x_{0l} < 0$ for at least one commodity l. Even when $\overline{x}_{0l} + \overline{M_l}\overline{\varphi} = n$, the requirement $x_{0l} + \overline{M_j}\varphi_j + M_j\overline{\varphi_j} > 0$ can be fulfilled by making M_{jl} positive and large enough.

The case $\widehat{\pi}_j\varphi_j + \pi_1\theta < 0$ can be avoided when $\overline{\theta}_k > 0$ for some k or $\widehat{\pi}_j > 0$ (by making θ_k or φ_j positive and large enough) and can be dealt with easily when $\overline{x}_{0l} > 0$ for some l (by making x_{0l} positive and large enough). Suppose $\overline{x}_0 = 0$, $\overline{\theta} = 0$ and $\widehat{\pi}_j \leq 0$, which implies $\theta \ll 0$, $x_0 \ll 0$ and $\overline{M}_{jl}\overline{\varphi}_j = n$ for every l. Actually, every agent would respond to $\widehat{\pi}_j \leq 0$ by choosing $M_{jl}^{hn}\varphi_j^{hn} = n$ and M_{jl}^{hn} small enough in order to default on asset j in every state (so that $p_0^n M_j^{hn} - \pi_{2j}^{hn} = \widehat{\pi}_j \leq 0$), implying that $\int M_{jl}^{hn}\varphi_j^{hn} dh = n > \int \varpi_{0l}^h dh$ for n large enough, contradicting a feasibility condition that must hold at an equilibrium for the truncated economy.

We have shown that the Mangasarian-Fromovitz constraint qualification of the utility maximization problem with φ_k and M_k, $k \neq j$, fixed at the respective truncated equilibrium values φ_k^{hn} and M_k^{hn} is satisfied when $(\mathfrak{S}_j^{hn})' = \emptyset$ at an optimal solution. The necessity of the Kuhn-Tucker condition follows from Proposition 11.3.6 in Mangasarian (1994).

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