Noisy Evolution in Normal Form Games

Christoph Kuzmics

J. L. Kellogg School of Management
Northwestern University
2001 Sheridan Road
Evanston, IL 60208-2009
email: C.A.Kuzmics.99@cantab.net
phone: (847) 491 2978
fax: (847) 467 1220

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Abstract

This paper analyzes a stochastic model of evolution in normal form games. The long-run behavior of individuals in this model is investigated in the limit where mutation rates tend to zero, while the expected number of mutations, and hence population sizes, tend to infinity. It is shown that weakly dominated strategies do not survive evolution. Also strategies which are not rationalizable in the game obtained from the original game by the deletion of all weakly dominated strategies disappear in the long-run. Furthermore it is shown that if evolution leads to a unique prediction this prediction must be equivalent to a trembling-hand perfect equilibrium.

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1 Introduction

Evolution in evolutionary models of behavior in normal form games does not necessarily eliminate weakly dominated strategies (see Samuelson, 1993). This is true for dynamic models such as the replicator dynamics of Taylor and Jonker (1978), and for the stochastic model of Kandori, Mailath, and Rob (1993).

In deterministic models weakly dominated strategies can survive evolution when all opponents' strategies, against which the weakly dominated strategy performs poorly, diminish much faster than the weakly dominated strategy and then vanish before the weakly dominated strategy does (see e.g. Example 3.4 in Weibull, 1995).

In a stochastic finite-population model a la Kandori, Mailath, and Rob (1993), weakly dominated strategies may feature in the support of the limiting invariant distribution of play because of the possibility of "evolutionary drift". Suppose play is currently in a state in which two conditions are satisfied. First, a given weakly dominated strategy is not played by anyone in the relevant player population. Second, opponents' strategies, against which the weakly dominated strategy performs worse than the strategy it is dominated by, are not present either. But then the weakly dominated strategy is an alternative best reply in the given state, and if employed by one individual in the relevant population by mutation, there is no evolutionary pressure to remove it. In fact one could have a series of single mutations in this population toward more and more individuals playing the weakly dominated strategy. If nothing else changes, i.e. no other individual in any other population changes strategy, evolutionary pressure does not bear on individuals using the weakly dominated strategy, as it continues to be an alternative best reply in these circumstances. For a concrete example of a game in which this can happen see Samuelson (1993).

The fact that we can assume that (at least with positive probability) nothing else changes in the play of the game seems to depend a lot on the fact that each population in the game is of finite size, which is a feature of the stochastic model employed both in Kandori, Mailath, and Rob (1993) and Samuelson (1993).

In this paper I employ the model of Nöldeke and Samuelson (1993) for normal form games, which gives rise to a sticky myopic best-reply dynamics. As in Kuzmics (2003) I investigate the limiting behavior of individuals in this stochastic evolutionary model when mutation rates tend to zero and when simultaneously the expected number of mutations per period, hence also the population sizes, tend to infinity. Under these conditions I can show that weakly dominated strategies will not be played to a significant extent in the long-run. I then show that in fact any strategy which is not **strongly rationalizable** will not be played by a significant proportion of individuals in the long-run. A strategy is called strongly rationalizable if it is rationalizable (Bernheim, 1984, and Pearce, 1984) in the game obtained from the original game by deletion of all weakly dominated strategies. Hence, this paper provides an evolutionary justification for the use of what has been termed the W^1S^∞-procedure, which stands for the deletion first of all weakly dominated strategies and then iteratively of all strictly dominated strategies. While epistemic conditions for the use of this procedure have been identified by Dekel and Fudenberg (1990), Brandenburger (1992), Börgers (1994), Gul (1996), and Ben Porath (1997), I believe that this paper is the first to identify conditions on the evolutionary process under which non-W^1S^∞ compatible strategies do not survive evolution.

I move on to prove that if evolution, as modelled here, leads to a unique prediction, to be made precise in the paper, then this unique prediction must be equivalent to a trembling-hand perfect equilibrium (Selten, 1975).

2 Model

The following model is a straightforward adaptation of the model of Nöldeke and Samuelson (1993) to normal form games. Let Γ be a normal form game. Let there be n players $i \in N = \{1, ..., n\}$ and let each player i be replaced by a population of individuals $M(i)$ with population size $m_i = |M(i)|$. Let S_i denote player i's set of pure strategies. Individuals are characterized by the pure strategy they are playing*. A state is a characterization for each individual in each population. Let the state space be denoted by Ω.

Individuals in every period t play against every possible configuration of opponents. Between times t and $t+1$ each individual in each population first receives a draw from a Bernoulli random variable either to learn with probability σ or not to learn, and then receives a second draw from an independent Bernoulli variable either to experiment with probability μ or not to experiment.

If an agent learns, the agent chooses a best reply to the aggregate behavior of individuals at time t. If there are multiple best replies the agent chooses one according to a fixed probability distribution with full support over all best replies. If the agent already plays a best reply she is assumed to continue playing it. If she does not learn, the agent continues to play her old strategy.

If the agent receives an experimentation-draw she chooses an arbitrary strategy according to a probability distribution with full support over all

*In the model of Nöldeke and Samuelson (1993), which is designed for extensive form games, individuals are characterized by a strategy as well as a conjecture about every other individual's strategy. By omitting the latter I here implicitly assume that when individuals play the game they realize which strategy every player is using as in Kandori, Mailath, and Rob (1993). This assumption, while not very reasonable for extensive form games, seems fairly innocuous for normal form games.

strategies available to this agent (including the one she is playing at the moment). In the other case she does not change her strategy.

The above mutation-selection mechanism constitutes a Markov chain on the state space Ω with transition probability matrix denoted by Q_μ^m, indicating that it is different for different population sizes and different experimentation rates. The transition probabilities also vary with different learning probabilities σ. In this paper, however, σ is assumed to be fixed at a value strictly between 0 and 1.

The Markov chain induced by the above selection-mutation dynamics is aperiodic and irreducible. Hence, it has a unique stationary distribution, which shall be denoted by π_μ^m, and satisfies

$$\pi_\mu^m Q_\mu^m = \pi_\mu^m. \tag{1}$$

Let $\Theta = \times_{i \in N} \Delta(S_i)$ be the space of all independent mixed strategy profiles, where $\Delta(D)$ is the set of all probability distributions over D.

Definition 1 *For a fixed vector of population sizes, m, a state $\omega \in \Omega$ is said to be* **equivalent** *to a (mixed) strategy profile $x \in \Theta$ if for all player populations $M(i)$ the proportion of individuals playing action $s_i \in S_i$ in state ω is identical to the probability attached by player i to strategy s_i in strategy profile x.*

Note that a mixed strategy profile is often equivalent to a specific state only for a particular vector of population sizes m. Note also that irrational mixed strategy profiles, in the sense that at least one entry in its distribution is an irrational number, cannot be equivalent to any state for any vector of population sizes m. Throughout this essay, though, I will not be interested in any irrational mixed strategy. I will, therefore, ignore this problem.

Compare Kandori, Mailath, and Rob (1993) and Young (1993) for the following definition of a stochastically stable state.

Definition 2 *A (possibly mixed) strategy profile $x \in \Theta$ is* **stochastically stable** *if for some finite m there is a state $\omega \in \Omega$, which is equivalent to x and satisfies*

$$\lim_{\mu \to 0} \pi_{\mu}^{m}(\omega) > 0. \tag{2}$$

For $x \in \Theta$ let Ψ_{ϵ}^{x} denote the set of states in Ω which are equivalent to a mixed strategy profile $x' \in \Theta$ such that $\|x' - x\|_{\infty} \leq \epsilon$, where $\|\cdot\|_{\infty}$ is the maximum norm. I.e. Ψ_{ϵ}^{x} is the set of states for which for each player population $M(i)$ the difference between the proportion of individuals playing strategy $s_i \in S_i$ and the probability attached to action s_i in the mixed strategy x is not greater than ϵ in absolute values. Let $\rho_{\mu}^{m} = \left(\mu, \frac{1}{m_1 \mu}, ..., \frac{1}{m_n \mu}\right)$ and let $\rho_{\mu}^{m} \to 0$ mean that each component of ρ_{μ}^{m} tends to zero.

Definition 3 *A (possibly mixed) strategy profile $x \in \Theta$ is* **stable under**

noise (SUN) *if for all* $\epsilon \in (0,1)$

$$\lim_{\rho_\mu^m \to 0} \pi_\mu^m \left(\Psi_\epsilon^x\right) > 0. \tag{3}$$

The strategy profile $x \in \Theta$ *is* **strongly stable under noise (strongly SUN)** *if for all* $\epsilon \in (0,1)$

$$\lim_{\rho_\mu^m \to 0} \pi_\mu^m \left(\Psi_\epsilon^x\right) = 1. \tag{4}$$

3 Results

Let $i \in N$ be an arbitrary player and let $x \in S_i$ be an arbitrary strategy available to individuals at population $M(i)$. Let $\Lambda_k^{i,x}$ denote the set of states in which the proportion of individuals at population $M(i)$ is playing strategy x is $\frac{k}{m_i}$.

Lemma 1 *Let* $i \in \{1, ..., n\}$ *be an arbitrary player and* $x \in S_i$ *an arbitrary strategy available to individuals at population* $M(i)$*.*

$$\lim_{\mu \to 0,\ m_i \mu \to \infty} \pi_\mu^m \left(\Lambda_0^{i,x}\right) = 0. \tag{5}$$

The proof of this lemma is identical, modulo notation, to the combined proof of Lemma 1 and Corollary 1 in Kuzmics (2003). It is nevertheless given in the appendix for convenience. The following corollary is immediate from Lemma 1.

Corollary 1 *Denote by* Ψ *the set of states, in which there is a population such that at least one strategy is not played by any individual at this population, i.e.*

$$\Psi = \bigcup_{i=1}^{n} \bigcup_{x \in S_i} \Lambda_0^{i,x}. \tag{6}$$

Then

$$\lim_{\rho_\mu^m \to 0} \pi_\mu^m (\Psi) = 0. \tag{7}$$

For $i \in N$ and $x \in S_i$ let $\Phi_\tau^{i,x} = \bigcup_{k \leq \tau m_i} \Lambda_k^{i,x}$ denote the set of states in which not more than a proportion of τ individuals play x at player population $M(i)$.

Lemma 2 *Let* $i \in N$ *and* $x \in S_i$ *be a pure strategy available to individuals in population* $M(i)$*. Let* $B^x \subset \Omega$ *denote the set of states in which* x *is a best reply (given conjectures after learning) for individuals at population* $M(i)$*. Suppose* $\lim_{\rho_\mu^m \to 0} \pi_\mu^m(B^x) = 0$*. Then for all* $\epsilon \in (0,1)$*:*

$$\lim_{\rho_\mu^m \to 0} \pi_\mu^m \left(\Phi_\epsilon^{i,x} \right) = 1. \tag{8}$$

The proof of this lemma is virtually identical, modulo notation and taking set complements, to the proof of Lemma 4 in Kuzmics (2003). It is, however, given in the appendix for convenience. Lemma 2 enables me to prove a series of results.

Theorem 1 *Let $w_i \in S_i$, $i \in N$, be a weakly dominated strategy. Then for any $\epsilon \in (0,1)$:*

$$\lim_{\rho_\mu^m \to 0} \pi_\mu^m \left(\Phi_\epsilon^{i,w_i} \right) = 1. \tag{9}$$

Proof: Strategy w_i can be a best reply only when there is some strategy available to players in some other population $M(j)$ which is not played by any individual at $M(j)$, i.e. w_i can be a best reply only for states in Ψ as defined in Corollary 1. The set Ψ, however, has zero limiting probability. Hence Lemma 2 applies. QED

The fact that all weakly dominated strategies "disappear" in the long run is more than can be said when evolution is modelled by either monotonic deterministic dynamics or stochastic dynamics with fixed population sizes (see Samuelson, 1993). Hence, it is necessary to take population sizes to infinity in the limit, to be able generally to exclude weakly dominated strategies. It is not clear, however, whether it is necessary also to have the expected number of mutations per period tend to infinity. In this essay I will not investigate the question of necessary and sufficient conditions for deleting weakly dominated strategies. I will, rather, try to analyze whether more can be said about the behavior of the invariant distribution under the given limiting conditions.

Let Γ^1 denote the game which remains when all such weakly dominated strategies are eliminated. I.e. Γ^1 is derived from Γ by reducing each player's pure strategy set by all weakly dominated strategies, while the payoff function is the same (with restricted domain). Let S_i^1 denote the restricted strategy set for player i. The next theorem states that strategies which are never a best reply in Γ^1 must essentially disappear in the limit I consider.

Theorem 2 *For $i \in N$ let $d_i \in S_i^1$ be a strategy which is never a best reply in Γ^1. Then for any $\epsilon \in (0,1)$:*

$$\lim_{\rho_\mu^m \to 0} \pi_\mu^m \left(\Phi_\epsilon^{i,d_i} \right) = 1. \tag{10}$$

Proof: A strategy $d_i \in S_i^1$ is never a best reply in Γ^1 if for any (pure or mixed) strategy combination $\alpha_{-i} \in \times_{i' \neq i} \Delta \left(S_{i'}^1 \right)$ there is another strategy $s_i(\alpha_{-i}) \in S_i^1$ such that $s_i(\alpha_{-i})$ gives a higher payoff against α_{-i} than d_i does by at least some $\delta > 0$. For $\eta > 0$ let W_η denote the set of states such that at all populations the proportion of individuals who play a weakly dominated strategy (in Γ) is below η. By continuity of the payoff functions and the fact that $\delta > 0$ we have that, provided η is small enough, d_i can only be a best reply for states in W_η^c, which, by Theorem 1 and the finiteness of N and the finiteness of all S_i for all $i \in N$, has zero limiting probability. Hence, Lemma 2 applies. QED

The above argument can be iterated any finite number of times. A strategy which survives the iterated deletion of never best replies is called rationalizable (Bernheim, 1984, and Pearce, 1984). Let a strategy which is rationalizable in the game obtained from the original by deletion of all weakly dominated strategies be termed **strongly rationalizable**. We then have the following

Theorem 3 *For $i \in N$, let $d_i \in S_i$, be a strategy which is not strongly rationalizable. Then for any $\epsilon \in (0,1)$:*

$$\lim_{\rho_\mu^m \to 0} \pi_\mu^m \left(\Phi_\epsilon^{i,d_i} \right) = 1. \tag{11}$$

While epistemic conditions for the use of what has been termed the $W^1 S^\infty$-procedure, which stands for the deletion of first all weakly dominated strategies and then iteratively all strictly dominated strategies, have been identified by Dekel and Fudenberg (1990), Brandenburger (1992), Börgers (1994), Gul (1996), and Ben Porath (1997), the above theorem provides an evolutionary justification for its use.

A corollary is immediate from Theorem 1.

Corollary 2 *Let $\Gamma = (I, S, u)$ be a 2-player normal form game, i.e. $I = \{1, 2\}$. Let $x = (x_1, x_2) \in S = S_1 \times S_2$ be a pure strategy profile. If x is strongly SUN then x must be a (trembling-hand) perfect (Nash) equilibrium.*

This is because in 2-player games every undominated Nash equilibrium is perfect (see Theorem 3.2.2 in van Damme, 1991).

The game in Table i, taken from Exercise 6.10 in Ritzberger (2002), suggests that a similar statement to Corollary 2 could hold also for normal

	A	B
A	1,1,1	1,0,1
B	1,1,1	0,0,1

A

	A	B
A	1,1,0	0,0,0
B	0,1,0	1,0,0

B

Table i: A 3-player game, taken from Exercise 6.10 in Ritzberger (2002), with an undominated Nash equilibrium which is not perfect.

form games with more than 2 players. In this game there are two admissible (undominated) equilibria, (A,A,A) and (B,A,A). Only the first one is perfect. Player 2's as well as player 3's strictly dominated strategy B disappear in the long-run in our evolutionary dynamics in the sense of Theorem 1. Hence, strategy B, in player populations 2 and 3, will be played by less than any tiny proportion, ϵ_2 and ϵ_3, of individuals in the limit. For player 1 A is best if and only if $1-\epsilon_2\epsilon_3 > (1-\epsilon_2)(1-\epsilon_3)+\epsilon_2\epsilon_3$, i.e. if and only if $\epsilon_2+\epsilon_3 > 3\epsilon_2\epsilon_3$. But this is true for any combination of ϵ_2 and ϵ_3 provided they are small enough. In fact this means that (A,A,A) is strictly perfect (Okada, 1981). Hence, for player population 1 the states in which B is a best-reply has zero limiting probability. Then, by Lemma 2, player 1's strategy B must vanish in the long-run. In this game, then, only the perfect Nash equilibrium (A,A,A) survives evolution in the long run. The second, also undominated, Nash equilibrium (B,A,A) is not SUN.

In fact, it can be shown for any normal form game (with any number of players) that if evolution, as modelled here, leads to a unique long-run prediction, a strongly SUN pure strategy profile, then this strategy profile must be a perfect Nash equilibrium. I need two lemmas before I can prove this claim.

Selten (1975) introduced the concept of a (trembling-hand) perfect (Nash) equilibrium. One possible characterization of a perfect equilibrium is given in the following lemma, which is also due to Selten (1975) (see also e.g. Proposition 6.1 in Ritzberger, 2002, for a textbook treatment).

Lemma 3 *A (possibly mixed) strategy profile* $x \in \Theta$ *is a (trembling-hand) perfect (Nash) equilibrium if there is a sequence* $\{x_t\}_{t=1}^{\infty}$ *of completely mixed strategy profiles (i.e. each* $x_t \in int(\Theta)$*) such that* x_t *converges to* x *and* x *is a best reply to* x_t *for all* t.

In the following lemma I establish what is essentially the inverse of this condition. For $x \in \Theta$ and for $\epsilon > 0$ let $U_\epsilon^x \subset \Theta$ denote an ϵ-ball around x.

Lemma 4 *Let* $x \in \Theta$ *be a Nash equilibrium which is not (trembling-hand) perfect. Then there is an* $\bar{\epsilon} > 0$ *such that for all* $\epsilon \in (0, \bar{\epsilon})$ *it is true that for all* $y \in U_\epsilon^x \cap int(\Theta)$ x *is not a best reply to* y.

Proof: Suppose not. Then for all $\bar{\epsilon}$ there is an $\epsilon \in (0, \bar{\epsilon})$ such that there is a $y = y_{\bar{\epsilon}} \in U_\epsilon^x \cap \text{int}(\Theta)$ such that x is a best reply to $y_{\bar{\epsilon}}$. But then

$\{y_{\bar{\epsilon}}\}_{\bar{\epsilon}>0}$ constitutes a sequence of completely mixed strategy profiles which converges to x (for $\bar{\epsilon}$ tending to zero) and is such that x is a best reply to each member in the sequence. Hence, by Lemma 3, x is perfect. Thus, we arrive at a contradiction. QED

Theorem 4 *Let $\Gamma = (I, S, u)$ be an n-player normal form game, i.e. $I = \{1, 2, ..., n\}$. Let $x = (x_1, x_2, ..., x_n) \in S = S_1 \times S_2 \times ... \times S_n$ be a pure strategy profile. If x is strongly SUN then x must be a perfect Nash equilibrium.*

Proof: Suppose x is strongly SUN but not perfect. Then by lemma 4 there is a $\bar{\epsilon} > 0$ such that for all $\epsilon \in (0, \bar{\epsilon})$ it is true that for all $y \in U^x_\epsilon \cap \text{int}(\Theta)$, x is not a best reply to y. Let $C^{j,x_j} \subset \Omega$ denote the set of states in which x_j, player j's strategy in $x \in S$, is not a best reply (after learning). Fix $\epsilon \in (0, \bar{\epsilon})$. There is a player $j \in I$ such that $\lim_{\rho^m_\mu \to 0} \pi^m_\mu \left(C^{j,x_j} \cap \Psi^x_\epsilon\right) = \delta > 0$, where Ψ^x_ϵ is the set of states in which for all $i \in N$ the proportion of individuals playing a pure strategy $s_i \in S_i$ and the probability attached to s_i in x do not differ by more than ϵ. Suppose this were not true. Then for all $j \in I$ and for all $s_j \in S_j$ $\pi^m_\mu \left(C^{j,x_j} \cap \Psi^x_\epsilon\right) \to 0$. However, it must be that $\Psi^x_\epsilon \subset \bigcup_{j \in I} \Lambda^{j,x_j}_{m_j} \cup \left(\bigcup_{j \in I} \left(C^{j,x_j} \cap \Psi^x_\epsilon\right)\right)$. This is due to the fact that if a state ω is in Ψ_ϵ and strictly interior (not in $\Lambda^{j,x_j}_{m_j}$ for any $j \in N$), then it must be in one of the sets C^{j,x_j} for some $j \in N$ by force of Lemma 4, given that

$\epsilon \in (0, \bar{\epsilon})$. But then by the fact that the game is finite we have that

$$\pi_{\mu}^{m}\left(\Psi_{\epsilon}^{x}\right) \leq \sum_{j \in I} \pi_{\mu}^{m}\left(\Lambda_{m_j}^{j,x_j}\right) + \sum_{j \in I} \pi_{\mu}^{m}\left(C^{j,x_j} \cap \Psi_{\epsilon}^{x}\right). \tag{12}$$

But the right hand side tends to zero under the supposition, while the left hand side tends to one. Hence, we arrive at a contradiction. Therefore, x strongly SUN and yet not perfect implies that there is a player $j \in I$ such that $\lim_{\rho_{\mu}^{m} \to 0} \pi_{\mu}^{m}\left(C^{j,x_j} \cap \Psi_{\epsilon}^{x}\right) = \delta > 0$.

For this player $j \in I$ I will investigate the limiting probability of $V = \left(\Phi_{(1-\epsilon)}^{j,x_j}\right)^{c}$, the set of states in which more than a proportion of $(1-\epsilon)$ individuals play x_j at population $M(j)$. By the fact that x is strongly SUN we know that $\pi_{\mu}^{m}(V)$ tends to one in the limit. By equation (1) we also know that

$$\pi_{\mu}^{m}(V) = \sum_{\omega' \in \Omega} \pi_{\mu}^{m}(\omega') \left(Q_{\mu}^{m}\right)_{\omega' V}, \tag{13}$$

where $\left(Q_{\mu}^{m}\right)_{\omega' V} = \sum_{\omega \in V} \left(Q_{\mu}^{m}\right)_{\omega' \omega}$. Hence,

$$\pi_{\mu}^{m}(V) = \sum_{\omega' \in C^{j,x_j} \cap \Psi_{\epsilon}^{x}} \pi_{\mu}^{m}(\omega') \left(Q_{\mu}^{m}\right)_{\omega' V} + \tag{14}$$

$$+ \sum_{\omega' \notin C^{j,x_j} \cap \Psi_{\epsilon}^{x}} \pi_{\mu}^{m}(\omega') \left(Q_{\mu}^{m}\right)_{\omega' V}. \tag{15}$$

In the limit then

$$\lim_{\mu \to 0,\ m_{j'}\mu \to \infty\ \forall j' \in I} \pi_{\mu}^{m}(V) \leq 1 - \delta + \tag{16}$$

$$\lim_{\mu \to 0,\ m_{j'}\mu \to \infty\ \forall j' \in I} \pi_{\mu}^{m}\left(C^{j,x_j} \cap \Psi_{\epsilon}^{x}\right) \max_{\omega' \in C^{j,x_j} \cap \Psi_{\epsilon}^{x}} \left(Q_{\mu}^{m}\right)_{\omega' V}. \tag{17}$$

But for $\omega' \in C^{j,x_j} \cap \Psi_\epsilon^x$, $\left(Q_\mu^m\right)_{\omega' V}$ is bounded away from 1, provided m_j is large enough.

To see this let $\{\Omega \times \Omega, P\}$ denote a probability space where P is such that $P(\omega', \omega) = \pi_\mu^m(\omega') \left(Q_\mu^m\right)_{\omega', \omega}$ for all $(\omega', \omega) \in \Omega \times \Omega$. Define $U : \Omega \times \Omega \to \{0, 1, ..., m_i\}$ such that $U(\omega', \omega)$ is the number of individuals at population $M(j)$ who play x_j in state ω. Similarly let $V : \Omega \times \Omega \to \{0, 1, ..., m_i\}$ be a random variable such that $V(\omega', \omega)$ is the number of individuals at population $M(j)$ who play x_j in state ω. Note that

$$\{\omega \in \Omega | U(\omega', \omega) = k\} = \{\omega \in \Omega | W(\omega', \omega) = k\} = \Lambda_k^{j,x_j}.$$

Let $Z : \Omega \times \Omega \to \{-m_i, -m_i + 1, ..., -1, 0, 1, ..., m_i\}$ denote a third random variable such that $Z(\omega', \omega)$ is the "loss" of x_j-players at population $M(j)$ in the transition from state ω' to ω. Obviously we must have that $Z(\omega', \omega) = -\left(V(\omega', \omega) - U(\omega', \omega)\right)$, or $Z = U - V$. Note that $\max_{\omega' \in C^{j,x_j} \cap \Psi_\epsilon^x} \left(Q_\mu^m\right)_{\omega' V}$ is achieved by an $\omega' \in \Lambda_m^{j,x_j}$. Hence,

$$\max_{\omega' \in C^{j,x_j} \cap \Psi_\epsilon^x} \left(Q_\mu^m\right)_{\omega' V} = \max_{\omega' \in C^{j,x_j} \cap \Lambda_m^{j,x_j}} \left(Q_\mu^m\right)_{\omega' V}.$$

However,

$$\max_{\omega' \in C^{j,x_j} \cap \Lambda_m^{j,x_j}} \left(Q_\mu^m\right)_{\omega' V} = P(Z \le \epsilon m_j | U = m_j).$$

Given $U(\omega', \omega) = m_j$, however, Z is binomially distributed with parameters σ and m_j. This implies that the expectation satisfies $E(Z|U = m_j) = \sigma m_j$

while the variance satisfies $V(Z|U = m_j) = \sigma(1-\sigma)m_j$. Then

$$\begin{aligned} P(Z \leq \epsilon m_j | U = m_j) &= P\left(Z - \sigma m_j \leq (\epsilon - \sigma) m_j | U = m_j\right) && (18) \\ &= P\left(\sigma m_j - Z \geq (\sigma - \epsilon) m_j | U = m_j\right) && (19) \\ &\leq P\left(|\sigma m_j - Z| \geq (\sigma - \epsilon) m_j | U = m_j\right) && (20) \\ &\leq P\left(|Z - \sigma m_j| \geq (\sigma - \epsilon) m_j | U = m_j\right) && (21) \\ &\leq \frac{\sigma(1-\sigma)m_j}{(\sigma - \epsilon)^2 m_j^2}, && (22) \end{aligned}$$

where the last inequality is an application of Chebyshev's inequality. Given $\sigma > 0$ is fixed, we can of course choose $\bar{\epsilon}$ and hence ϵ small enough such that $\epsilon < \sigma$, which is needed in the equations above. But then for m_j large enough we have $\frac{\sigma(1-\sigma)m_j}{(\sigma-\epsilon)^2 m_j^2} < 1$.

Hence, we have

$$\lim_{\rho_\mu^m \to 0} \pi_\mu^m (V) < 1. \qquad (23)$$

Thus we arrive at a contradiction. QED

4 Discussion

I this paper I show that evolution, as modelled here, eliminates weakly dominated strategies. Also strategies which are not rationalizable in reduced game, the game obtained from the original by deletion of all weakly dominated strategies disappear in the long-run. A natural question to ask is whether evolution, as modelled here, also eliminates strategies which are only weakly dominated in the reduced game, i.e. whether evolution allows the iterated deletion of weakly dominated strategies. This is not true, however. In Example 2 I show that a second order weakly dominated strategy may well survive evolution, as modelled here.

In this paper I call a pure strategy profile, which is the unique long-run prediction of the evolutionary process, in the sense of Definition 3, strongly stable under noise (strongly SUN). Of course, in some games evolution, as modelled here, does not give rise to a unique, hence strongly SUN, prediction. The game discussed in Example 3 illustrates this. But when there is a pure strategy profile which strongly SUN then it must constitute a trembling-hand perfect equilibrium (Selten, 1975). In Example 2 I demonstrate that a strategy profile which is strongly SUN does not necessarily constitute a proper equilibrium (Myerson, 1978). The coordination game discussed in Example 1 illustrates furthermore that neither a mixed nor pure proper (and hence perfect) equilibrium is necessarily stable under noise (SUN), notwithstanding strongly SUN. To summarize: Let $x \in \Theta$. Then

$$x \text{ is strongly SUN} \quad \Rightarrow \quad x \text{ is perfect} \tag{24}$$

$$x \text{ is strongly SUN} \quad \not\Rightarrow \quad x \text{ is proper} \tag{25}$$

$$x \text{ is proper} \quad \not\Rightarrow \quad x \text{ is SUN} \tag{26}$$

Example 1 *This example is to demonstrate that perfect or even proper* *equilibria (Myerson, 1978) may not necessarily be SUN. Consider the coordination game with payoffs given in Table ii. First, note that no small*

	A	B
A	2,2	0,0
B	0,0	1,1

Table ii: A coordination game.

ϵ*-ball around the mixed Nash equilibrium* $((1/3, 2/3); (1/3, 2/3))$*, which, being strictly interior, is proper and, hence, perfect, can carry positive limiting probability. But likewise the proper equilibrium in pure strategies (B,B) fails to be SUN. As Kandori, Mailath, and Rob (1993) showed, (B,B) is not stochastically stable. Note that taking the population size to infinity cannot alter that fact that all limiting probability will center on (A,A) in the long-run. This is due to the fact that the transition probability from the risk-dominant (A,A) to the risk-dominated equilibrium (B,B) will still be orders smaller.*

Example 2 *The game given in Table iii taken from Samuelson (1992) illustrates that evolution, as modelled here, does not generally eliminate iteratively weakly dominated strategies (see also Example 5.5 of Ritzberger, 2002). It also demonstrates that a strategy profile which is strongly SUN does not need to constitute a proper equilibrium. In this game strategies C*

	D	E	F
A	1,1	1,1	2,1
B	1,1	0,0	3,1
C	1,2	1,3	1,1

Table iii: A game from Samuelson (1992).

and F are both weakly dominated and by Theorem 1 must essentially vanish in the limit. This leaves us with a 2×2 *game where strategies B and E are weakly but not strictly dominated. The now weakly dominated strategies are a best reply in some states which may carry positive limiting probability. B is a best reply if the number of individuals in player population* $M(2)$ *who play E is lower than the number of people playing F. Similarly E is a best reply if the number of individuals in player population* $M(1)$ *who play B is lower than the number of people playing C. Which strategy profile is SUN, unfortunately, depends on the conditional mutation probabilities given by the vector* λ*. Suppose first that* $1 - \lambda_E > 1 - \lambda_F$*, i.e. when individuals at* $M(2)$ *mutate they are more likely to mutate to* E*. Suppose similarly that* $1 - \lambda_B > 1 - \lambda_C$*. Then (A,D) is strongly SUN. This is because when the mutation rate becomes smaller the probability of more individuals playing F than E, or more C then B, tends to zero. Hence the set of states in which either B or E is a best reply tends to zero. Consider now the case where*

$1-\lambda_E > 1-\lambda_F$ *and* $1-\lambda_B < 1-\lambda_C$. *Then for player population 1 the argument above still applies. However, players from population 2 now mostly face a situation in which there are more C-players than B-players at* $M(1)$. *Hence, with a probability tending to one, E is the unique best reply. Hence in this case (A,E) is strongly SUN. Similarly one can find circumstances under which (B,D) is strongly SUN. It is even possible, when all* λ*'s are identical, that only the set of all states in which payoffs are 1 to each player and players do not play C or F, is strongly SUN, if we define strongly SUN for sets in the obvious way.*

	H	T
H	1,-1	-1,1
T	-1,1	1,-1

Table iv: Matching Pennies.

Example 3 *The game in Table iv, Matching Pennies, illustrates that for fixed* σ *even strategy profiles which are not an equilibrium in any reasonable definition can be SUN. In this game there is only one Nash equilibrium, which is completely mixed at* $((1/2,1/2);(1/2,1/2))$. *It is easy to see that with a fixed* $\sigma > 0$ *we cannot expect the evolutionary system to stay too close to this Nash equilibrium. To see this suppose that at the moment the state is such that just a little less than* $1/2$ *of the player population 1 plays* H, *and* *that the same is true for player population 2. Now for player population 1 T is the unique best reply in this state. Of all H-players at population 1, then, we expect a proportion of σ individuals to learn and switch to T. But then next period we expect the state to be such that only approximately $1/2 - \sigma/2$ individuals play H, which is obviously less than $1/2 - \epsilon$ for any small ϵ. Similarly for player population 2, H is the unique best reply in the given state. Hence, we expect to see approximately $1/2 + \sigma/2$ individuals playing H next period. This, again, is quite far off $1/2 + \epsilon$ for any small ϵ. Similar statements can be made for almost any state close to the Nash equilibrium state. Hence, we cannot expect that in the long-run the system will be close to the Nash equilibrium state. Yet it seems that the smaller σ the closer states will generally be to the equilibrium in the long-run. Investigating this might provide an interesting line of research from here.*

A Proof of Lemma 1

Define D_x^i as the set of states such that $x \in S_i$ is not a best reply for any agent in population $M(i)$. Let λ_x denote the conditional probability that if an agent mutates she does not mutate to playing strategy x.

Given the property of the invariant distribution (1) any probability

$\pi_\mu^m(\omega)$ can be expressed as

$$\pi_\mu^m(\omega) = \sum_{\omega' \in \Omega} \pi_\mu^m(\omega') \left(Q_\mu^m\right)_{\omega'\omega}, \tag{27}$$

where $\left(Q_\mu^m\right)_{\omega'\omega}$ is the transition probability that the system moves from ω' to ω.

Equivalently for any set of states, Λ,

$$\pi_\mu^m(\Lambda) = \sum_{\omega \in \Lambda} \sum_{\omega' \in \Omega} \pi_\mu^m(\omega') \left(Q_\mu^m\right)_{\omega'\omega}. \tag{28}$$

Changing the order of summation yields

$$\pi_\mu^m(\Lambda) = \sum_{\omega' \in \Omega} \pi_\mu^m(\omega') \left(Q_\mu^m\right)_{\omega'\Lambda}, \tag{29}$$

where $\left(Q_\mu^m\right)_{\omega'\Lambda} = \sum_{\omega \in \Lambda} \left(Q_\mu^m\right)_{\omega'\omega}$.

We are interested in the set $\Lambda = \Lambda_0^{i,x}$. It is easy to show that for any $\omega' \in \Lambda_k^{i,x}$,

$$\left(Q_\mu^m\right)_{\omega'\Lambda_0^{i,x}} \begin{cases} = & p_{k0} \quad \forall\, \omega' \in D_x^i \\ \leq & p_{k0} \quad \text{otherwise} \end{cases}, \tag{30}$$

where

$$p_{k0} = \sum_{j=0}^{k} \sigma^j (1-\sigma)^{k-j} \binom{k}{j} (\mu\lambda_x)^{k-j} \left(1 - \mu(1-\lambda_x)\right)^{m_i - k + j}. \tag{31}$$

This is because there are many ways to move from a state where k out of m_i individuals at population $M(i)$ play x to a state where none do. Suppose the current state ω is in D_x^i. A possible transition is that any $j \leq k$ individuals who are currently playing x learn and change their strategy and the remaining $k-j$ agents mutate to play anything other than x, while everyone else does not change their strategy to x. p_{k0} is then just the sum of all the probabilities of these various possible transitions.

Careful inspection of equation (31) reveals that

$$\begin{aligned} p_{k0} &= (1-\mu(1-\lambda_x))^{m_i-k}\sum_{j=0}^{k}\binom{k}{j}(\sigma(1-\mu(1-\lambda_x)))^j((1-\sigma)(\mu\lambda_x))^{k-j} \\ &= (1-\mu(1-\lambda_x))^{m_i-k}(\mu\lambda_x+\sigma(1-\mu))^k . \end{aligned} \tag{32}$$

Hence, for all $k < m_i$,

$$\frac{p_{k+1,0}}{p_{k0}} = \frac{\mu\lambda_x+\sigma(1-\mu)}{1-\mu(1-\lambda_x)}, \tag{33}$$

which is less than 1 for small μ.

Using equations (29) and (30) yields

$$\pi_\mu^m\left(\Lambda_0^{i,x}\right) \le \sum_{k=0}^{m_i}\pi_\mu^m\left(\Lambda_k^{i,x}\right)p_{k0}. \tag{34}$$

Rearranging leads to

$$\pi_\mu^m\left(\Lambda_0^{i,x}\right) \le \frac{1}{1-p_{00}}\sum_{k=1}^{m_i}\pi_\mu^m\left(\Lambda_k^{i,x}\right)p_{k0} \tag{35}$$

and hence

$$\pi_\mu^m\left(\Lambda_0^{i,x}\right) \le \frac{1-\pi_\mu^m\left(\Lambda_0^{i,x}\right)}{1-p_{00}}\max_{k\ge 1}\{p_{k0}\}. \tag{36}$$

Finally

$$\pi_\mu^m\left(\Lambda_0^{i,x}\right) \le \frac{1}{1+\frac{1-p_{00}}{\max_{k\ge1}\{p_{k0}\}}}. \tag{37}$$

By equation (33) $\max_{k\geq 1}\{p_{k0}\} = p_{10}$ for μ small enough. This confirms the intuition that the easiest way to move to $\Lambda_0^{i,x}$ is coming from $\Lambda_1^{i,x}$.

Now, by equations (32) and (33),

$$p_{10} = (1-\mu(1-\lambda_x))^{m_i} \frac{\mu\lambda_x + \sigma(1-\mu)}{1-\mu(1-\lambda_x)}. \tag{38}$$

Hence,

$$\forall m_i \ \forall \kappa > 1 \ \exists \bar{\mu} : \forall \mu \leq \bar{\mu}$$

$$p_{10} \leq \kappa\sigma \left(1-\mu(1-\lambda_x)\right)^{m_i}. \tag{39}$$

Hence, for all m_i and for all $\kappa > 1$ there is a $\bar{\mu}$ such that for all $\mu < \bar{\mu}$,

$$\pi_\mu^m\left(\Lambda_0^{i,x}\right) \leq \frac{1}{1 + \frac{1-(1-\mu(1-\lambda_x))^{m_i}}{\kappa\sigma(1-\mu(1-\lambda_x))^{m_i}}}, \tag{40}$$

To show that $\pi_\mu^m\left(\Lambda_0^{i,x}\right)$ tends to zero in the case where μ goes to zero while $m_i\mu$ tends to infinity, it is enough to prove that $(1-\mu(1-\lambda_x))^{m_i}$ goes to zero under these circumstances.

$$(1-\mu(1-\lambda_x))^{m_i} = (1-\mu(1-\lambda_x))^{\frac{\mu m_i}{\mu}} \tag{41}$$

$$= \left[(1-\mu(1-\lambda_x))^{\frac{1}{\mu}}\right]^{\mu m_i} \tag{42}$$

and the fact that $(1-\mu(1-\lambda_x))^{\frac{1}{\mu}}$ tends to $e^{-(1-\lambda_x)} < 1$ as μ tends to zero. QED

B Proof of Lemma 2

Let $\{\Omega \times \Omega, P\}$[†] denote a probability space, where P is such that[‡] $P(\omega, \omega') = \pi_\mu^m(\omega)\left(Q_\mu^m\right)_{\omega,\omega'}$ for all $(\omega, \omega') \in \Omega \times \Omega$. Define $U : \Omega \times \Omega \to \{0, 1, ..., m_i\}$ such that $U(\omega, \omega')$ is the number of individuals at population $M(i)$ who play x in state ω. Similarly let $V : \Omega \times \Omega \to \{0, 1, ..., m_i\}$ be a random variable such that $V(\omega, \omega')$ is the number of individuals at population $M(i)$ who play x in state ω'. Note that

$$\{\omega \in \Omega | U(\omega, \omega') = k\} = \{\omega' \in \Omega | W(\omega, \omega') = k\} = \Lambda_k^{i,x}.$$

Let $Z : \Omega \times \Omega \to \{-m_i, -m_i + 1, ..., -1, 0, 1, ..., m_i\}$ denote a third random variable such that $Z(\omega, \omega')$ is the "loss" of x-players at population $M(i)$ in the transition from state ω to ω'. Obviously $Z(\omega, \omega') = U(\omega, \omega') - V(\omega, \omega')$.

Note that $P(U = k) = \pi_\mu^m\left(\Lambda_k^{i,x}\right)$ by definition. Also $P(V = k) = \pi_\mu^m\left(\Lambda_k^{i,x}\right)$. To see this let $(\cdot, \omega') = \{(\omega, \omega') | \omega \in \Omega\}$. Then $P(\cdot, \omega') = \sum_{\omega \in \Omega} P(\omega, \omega') = \sum_{\omega \in \Omega} \pi_\mu^m(\omega)\left(Q_\mu^m\right)_{\omega,\omega'} = \pi_\mu^m(\omega')$ by definition of the invariant distribution. But the set of states where $V = k$ is just $\bigcup_{\omega' \in \Lambda_k^{i,x}}(\cdot, \omega')$. Hence, $P(V = k) = \pi_\mu^m\left(\Lambda_k^{i,x}\right)$.

Given this we have $E(U) = E(V)$ and, hence, $E(Z) = 0$. The expectation of Z can be written as $E\left(E(Z|U)\right)$ by the law of iterated expectations.

[†] As the state space is finite I omit the sigma-algebra, which can be taken as the set of all subsets of $\Omega \times \Omega$, in the description of the probability space.

[‡] Given the axioms of a probability measure this is sufficient to uniquely define P.

Let $B = \{(\omega, \omega')|\omega \in B^x\}$. Obviously $P(B) = \pi_\mu^m(B^x)$. Then

$$
\begin{aligned}
E\left(\frac{Z}{m_i}\right) &= \pi_\mu^m\left(B^x\right) E\left(\frac{Z}{m_i}\middle| B\right) + &(43)\\
&+ \sum_{k=0}^{m_i} \pi_\mu^m\left(\Lambda_k^{i,x} \setminus B^x\right) E\left(\frac{Z}{m_i}\middle| U = k \wedge B^c\right). \\
&= \pi_\mu^m\left(B^x\right) E\left(\frac{Z}{m_i}\middle| B\right) + &(44)\\
&+ \sum_{k=0}^{m_i} \pi_\mu^m\left(\Lambda_k^{i,x}\right) E\left(\frac{Z}{m_i}\middle| U = k \wedge B^c\right) - \\
&- \sum_{k=0}^{m_i} \pi_\mu^m\left(\Lambda_k^{i,x} \cap B^x\right) E\left(\frac{Z}{m_i}\middle| U = k \wedge B^c\right).
\end{aligned}
$$

Now, $E\left(\frac{Z}{m_i}\middle| U = k \wedge B\right) \geq -1$ as the greatest change in x-players can never exceed the total number of individuals at $M(i)$. Similarly $E\left(\frac{Z}{m_i}\middle| U = k \wedge B^c\right) \leq 1$. We then have

$$
0 = E\left(\frac{Z}{m_i}\right) \geq -2\pi_\mu^m\left(B^x\right) + \sum_{k=0}^{m_i} \pi_\mu^m\left(\Lambda_k^{i,x}\right) E\left(\frac{Z}{m_i}\middle| U = k \wedge B^c\right). \quad (45)
$$

Let $\alpha_k = E\left(\frac{Z}{m_i}\middle| U = k \wedge B^c\right)$. Rearranging then yields

$$
2\pi_\mu^m\left(B^x\right) \geq \sum_{k=0}^{m_i} \pi_\mu^m\left(\Lambda_k^{i,x}\right) \alpha_k. \quad (46)
$$

Let $k^* = \lceil \epsilon m_i \rceil$, where $\lceil r \rceil$ denotes the smallest integer greater than r. By Lemma 5 below there is an $\bar{\alpha} > 0$ such that for all $k \geq k^*$ $\alpha_k \geq \bar{\alpha}$ provided μ is small enough. Also $\alpha_k \geq \alpha_0 = -\mu(1 - \lambda_x)$ for all k, in particular also for all $k < k^*$.

Hence,

$$
\sum_{k=0}^{m_i} \alpha_k \pi_\mu^m\left(\Lambda_k^{i,x}\right) \quad \geq \quad \sum_{k=0}^{k^*-1} \bar{\alpha} \pi_\mu^m\left(\Lambda_k^{i,x}\right) - \sum_{k=k^*}^{m_i} \mu(1 - \lambda_x) \pi_\mu^m\left(\Lambda_k^{i,x}\right) \quad (47)
$$

$$\geq \quad \bar{\alpha}\pi_{\mu}^{m}\left(\Phi_{\epsilon}^{i,x,c}\right) - \mu(1-\lambda_x)\pi_{\mu}^{m}\left(\Phi_{\epsilon}^{i,x}\right) \tag{48}$$

$$\geq \quad \bar{\alpha}\pi_{\mu}^{m}\left(\Phi_{\epsilon}^{i,x,c}\right) - \mu(1-\lambda_x)\left(1-\pi_{\mu}^{m}\left(\Phi_{\epsilon}^{i,x,c}\right)\right) \tag{49}$$

$$\geq \quad \left(\bar{\alpha} + \mu(1-\lambda_x)\right)\pi_{\mu}^{m}\left(\Phi_{\epsilon}^{i,x,c}\right) - \mu(1-\lambda_x) \tag{50}$$

Combining inequalities (46) and (50), we obtain

$$\left(\bar{\alpha} + \mu(1-\lambda_x)\right)\pi_{\mu}^{m}\left(\Phi_{\epsilon}^{i,x,c}\right) - \mu(1-\lambda_x) \leq \sum_{k=0}^{m_i} \alpha_k \pi_{\mu}^{m}\left(\Lambda_k^{i,x}\right) \leq 2\pi_{\mu}^{m}(B^x). \tag{51}$$

Taking $\mu \to 0$ while $m_i\mu$ for all $i \in N$ tends to infinity in inequality (51), we obtain

$$\bar{\alpha} \lim_{\rho_{\mu}^{m} \to 0} \pi_{\mu}^{m}\left(\Phi_{\epsilon}^{i,x,c}\right) \leq 0 \tag{52}$$

Hence, $\pi_{\mu}^{m}\left(\Phi_{\epsilon}^{i,x,c}\right) \to 0$. QED

Lemma 5 *Let $k^* = \lceil \epsilon m_i \rceil$. There is an $\bar{\alpha} > 0$ and a $\bar{\mu} > 0$ such that for all $k \geq k^*$ $\alpha_k \geq \bar{\alpha}$ provided $\mu < \bar{\mu}$. Also $\alpha_k \geq \alpha_0 = -\mu(1-\lambda_x)$ for all k.*

Proof: By definition $\alpha_k = \frac{1}{m_i} E\left(Z | U = k \wedge B^c\right)$. To calculate the term $E\left(Z|U = k \wedge B^c\right)$ note that Z can be written as the difference of two random variables Y and X (different from V and U), where $X(\omega, \omega')$ is the number of individuals at $M(i)$ who, in the transition from ω to ω', switch strategy from something other than x to x, and $Y(\omega, \omega')$ is the number of individuals at $M(i)$ who, in the transition from ω to ω', switch strategy from x to anything other than x. Conditional on $U(\omega, \omega') = k$ and $(\omega, \omega') \in B^c$

both X and Y are binomially distributed, i.e. $X \sim \text{Bin}\,(m_i - k, \mu(1-\lambda_x))$ and $Y \sim \text{Bin}\,(k, \sigma(1-\mu) + \mu\lambda_x)$. Hence, the term $E\,(Z|U = k \wedge B^c)$ is the difference between the expectation of these two binomial variables and given by

$$E\,(Z|U = k \wedge B^c) = k\,(\sigma(1-\mu) + \mu\lambda_x) - (m_i - k)\mu(1-\lambda_x).$$

The term α_k is then negative if and only if

$$\frac{k}{m_i} > \frac{\mu(1-\lambda_x)}{\sigma(1-\mu)+\mu}. \tag{53}$$

In particular if $k = \epsilon m_i$, $\alpha_k > 0$ if

$$\epsilon > \frac{\mu(1-\lambda_x)}{\sigma(1-\mu)+\mu}. \tag{54}$$

It is easy to see that $\alpha_0 < 0$. However, for an arbitrary $\epsilon > 0$, $\alpha_k > 0$ for all $k > \epsilon m_i$, provided μ is small enough. Indeed there is a $\bar{\mu} > 0$ and an $\bar{\alpha} > 0$ such that for all $\mu \leq \bar{\mu}$ we have that $\alpha_k \geq \bar{\alpha}$ for all $k > k^*$. Suppose, for the sake of simplicity, that ϵm_i is an integer. Then $\alpha_{k^*} = \epsilon\,(\sigma(1-\mu) + \mu\lambda_x) - (1-\epsilon)\mu(1-\lambda_x)$. One might, for instance, set $\bar{\alpha} = \frac{\epsilon\sigma}{2}$. Also observe that for all k we have that $\alpha_k \geq \alpha_0 = -\mu(1-\lambda_x)$. QED

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