Network Formation and Cooperation with an Application to Social Capital

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Abstract

In this paper we adopt Granovetter's view expressed in his famous article "Economic Action and Social Structure: The Problem of Embeddedness" , where he argues that the concept of man in economics is extremely undersocialized because it ignores the importance of social networks. In so doing the incentives to mutual cooperation in social matching games in which the social network is endogenously determined are studied. The main result shows that in atomized societies where there is no information flows between different pairs of individuals and the rest of the society, individuals choose either to form the maximal number of links possible or to form no links. Whereas in embedded societies where information transmission is allowed, the type of social networks that arise take different architectures some of them symmetric and some of the asymmetric.

This allows us to improve our understanding of a wide variety of phenomena as occupational mobility, informal credit markets in rural areas, cooperative formation, social capital, segmented labor markets, international trade and so on. In partricular, the model results are used to explain the concept of social capital , its benefits and costs.

1 Introduction

Grannovetter (1985) in his famous article "Economic Action and Social Structure: The Problem of Embeddedness" argue that the concept of man in economics is extremely undersocialized because it ignores the importance of personal contacts and networks of relations–what he calls the embeddedness of economic transactions in social relations– used for many people to achieve their goals. In fact he argues that "the behavior and institutions to be analyzed are so constrained by ongoing relationships that to construe them as independent is a grievous misunderstanding" (p. 481). Granovetter, however, does not postulate that the alternative to an undersocialized man is a oversocialized one that fully internalizes the consequences of his actions for his network of relations or social contacts. In fact he argues that "Despite the apparent contrast between under- and oversocialized views, we should note an irony of great theoretical importance: both have in common a conception of action and decision carried out by atomized actors. In the undersocialized account, atomization results from narrow pursuit of self-interest; in the oversocialized one, from the fact that behavioral patterns have an internalized and ongoing social relations thus have only peripheral effects on behavior" (p. 485). What Grannovetter proposes is not to abandon the assumption of fully rational agents pursuing their own self-interest, but embed them in social structure were they are involved and make decisions. This view is well-captured by the following quote: "My claim here is that however naive the psychology (of rational choice) may be, this is not where the main difficulty lies–it is rather in the neglect of social structure" (p. 506).

In this paper Granovetter's view is adopted by considering a model in which the society is conformed by infinitely-lived individuals that during the first period choose to form links–that is they choose the social structure in which future transactions will take place. A link between any two individuals is formed by mutual consent, links are assumed to be costly to form and they can be of two types: good and bad links. After links are formed, individuals play a repeated Prisoner's Dilemma with changing partners a la Kandori (1992), where the payoffs of the game depend on whether a link is good or bad. Good links yield a larger payoff than bad links. In each period pair-wise matching occurs only between linked individuals, unlinked individuals remain unmatched for the rest of their lifes.

A key aspect of the model is that information on how players have behaved in the past diffuses through the social network only gradually. Specifically, it is assumed that, player i is discovered by player j right after a deviation with positive probability if and only if they are linked. When i is not discovered right after he deviates, his defection will be forgotten in all subsequent periods. Thus, information can range from atomized (people know only what has happened in their own interactions) to comprehensive (people know what has happened in all previous interactions, and everything about the game is common knowledge). The main implication of the assumption that the probability of becoming informed is network dependent is that, in general, the architecture of the social network has bearing on the extent of cooperation that the network can support in a self-sustainable fashion. In addition, the information transmission mechanism assumed encompasses the two most emblematic repeated game models: when the probability of becoming informed is equal to one for each ij link, the model corresponds to the one in which everyone observes everyone else play in each period, while when that probability is zero, the model corresponds to the one where there is no information transmission between any given ij link and the rest of the society.

In this context the existence of pair-wise equilibrium networks–that is networks that are Nash equilibrium and pair-wise stable– is established, and a characterization of pair-wise equilibrium networks in terms of a particular set of architectures is provided. In general pair-wise equilibrium networks have the exclusive groups architecture; that is, networks that have distinctive groups of completely connected players and a group of isolated players, yet in some cases for the same set of parameters more than one pair-wise equilibrium network exist. A typical example of this, is a parametrization in which the empty, the complete and an asymmetric network in which there is a complete component and the rest are isolated players are pair-wise equilibirum networks.

There are certain feature of pair-wise equilibirum networks that are wortwhile to highlight here. First, in some networks individuals are segregated in two distinctive groups: one group conformed by good links but in which cooperation does not take place and another group formed by good and bad links, but in which its members cooperate in each encounter. In addition, the size of the group of cooperators is non-increasing in the probability that a given player's partners become informed about that player's action in his last encounter. The reason being that for any number of links an improved information transmission implies a larger expected punishment, and therefore less links are needed to induce cooperation.

Second, in some networks individuals are segregated in two groups: one group of individuals that cooperate among themselves and the rest are isolated individuals, and the group of cooperators is in general composed of both, good and bad links.

Third, in some networks some individuals form links that are not profitable. They do so because in that way they increase the expected punishment, and therefore cooperation with some of their links become self-sustainable. That is, unprofitable links serve as a kind of commitment device that makes cooperation self-sustainable.

The results of the paper are used to gain a better understanding of the concept of social capital. Social capital, in general, has entered economics as being an under-appreciated and independent factor of production, which in general is considered to be complementary with human capital. The reason being that human capital will amount to little unless the person having it can be in contact to others to inform, correct, assist with and disseminate his work and/or ideas. Much of the interest in social capital by economists has been fueled by definitions that include not only the structure of social networks and social relations, but also behavioral dispositions such as trust, reciprocity, and honesty and institutional quality measures such as rule of law, contract enforceability, and civil liberties and personal characteristic as social skills, charisma, intellectual skills. Examples of behavioral dispositions are LaProta et al (1997) and Knack and Keefer (1997). The former reports a positive correlation of some measure of trust and judicial efficiency and the latter between the same measure of trust and growth rate[1]. An example of personal characteristic is Glaser et al. (2002) who define social capital as a person's social characteristics–including social skills, charisma, and of his Rodolex– which enables him to reap market and non-market returns from interactions with others[2]. Thus, economists studying social capital have more often than none adopted definitions of social capital that make no difference between the resources available per-se and the ability to obtain them by virtue of membership in different social networks and that mix the consequences and determinants of social capital. Those studies, empirically or theoretically, that have either or both problems are bound to end up finding that successful people (nations) succed, as it is usually the case. In view of this critique, we believe that a definition of social capital must be able: (i) first, to distinguish social capital from other forms of capital so that social capital can have a distinct meaning, and (ii)

[1] See Durlauf (2002) for a critique of the econometric methods used to identify the positive effects of social capital.

[2] This definition however does not suffer from Portes and Landolt's critique that social capital cannot be thought of as a community based attribute.

second, to allow to implement empirical strategies capable of identifying the empirical benefits and "costs" of social capital. The model analyzed in this paper allows us to propose a definition of social capital that satisfies these two premises. Indeed, social capital is defined as follows: *An individual social capital refers to the number and type of costly links formed by mutual consent with the goal of achieving certain personal ends that cannot be achieved in the absence of them.*

We do not intend to claim that these is "the" definition of socical capital but among the many possibles one at least is capable of achieving the two goals proposed above. In addition to this, the definition proposed is at individual level and not at the aggregate level. This avoids the problem of aggregation that arises in definition based on collectives as Putman's definition, who by analogy with notions of physical capital and human capital–tools and training that enhance individual productivity–defines social capital as features of social organization, such as networks, norms, and trust, that facilitate coordination and cooperation for mutual benefit.

The are two strands of the literature that are related to this paper. The first one is related to the literature on repeated game models and the second one is the literature on strategic network or link formation.

Repeated game models usually ignore the role of social network formation and focus on random pairwise matching of identical individuals, who may have different histories of play (Ellison (1994), Kandori (1992), Okuno-Fujiwara and Postlewaite (1995)). This approach has resulted on the well-known Folk Theorem, which establishes that any payoff larger than the minmax payoff is supported as a subgame perfect equilibrium by mean of strategies that either punish deviators or reward conformers or use a mix of both.[3] This result hinges on the existence of information flows from each pairwise match concerning past actions to the rest of the society. For instance, if someone does not cooperate in a pairwise play of the prisoner's dilemma game, there is someone else in the society with whom there is a positive probability to be matched in the future who bcomes informed about that player not cooperating in the past. This kind of modeling while interesting in its own right, it ignores two crucial dimensions of social network which are (i) individual heterogeneity and (ii) that social networks are endogenously formed, and therefore, so they are the interaction probabilities. Other related papers are, Fujiwara-Greve (2001), Ghosh and Rey (1996), Datta (1996) and Kranton (1996). Ghosh and Ray (1994) show that the presence of heterogeneity among agents coupled with asymmetric information may induce players to cooperate even in the absence of information flows. Deviators face the potential cost of being matched with myopic players that never cooperate and thereby non-myopic players may choose to cooperate after a testing period. In this paper, contrary to Kandori (1992) and Ellison (1994), matched players may choose to continue a relationship instead of being forced to separate. This combined with the potential cost of being matched with myopic players, provide non-myopic players with incentives to cooperate. Kranton (1996) and Datta (1996) show the existence of cooperative equilibria that are characterized by buildup of cooperation over time when agents have incomplete information in the former and complete information in the latter.

The strategic network formation literature is concerned with obtaining the stable network formation where individuals' decisions are to either form or not a link. Watts (2001) analyzes the process of network formation in a dynamic framework, where self-interested individuals can form and sever links. She determines which network structures the formation process will converge to. Jackson and Wolinsky 1996, Bala and Goyal (2000), and Jackson and Watts (1999) are also concerned with link formation. Jackson and Wolinsky (1996) examine a static model in which self-interested individu-

[3]The minmax payoff is the minimum payoff that a player may hold someone else when he plays his best response to that strategy.

als can form and sever links. They determine which networks are stable and which networks are efficient. Thus, they leave open the question of which stable networks will form. Bala and Goyal (2000) simultaneously examine network formation in a dynamic setting. However, their approach differs from Watts (2001) in that Bala and Goyal restrict attention to models where links are formed unilaterally; *i.e.,* one player does not need another players permission to form a link in a noncooperative game and focus on learning as a way to identify equilibria. Jackson and Watts (1999) also analyze the formation of networks in a dynamic framework. Jackson and Watts extend Watts' model to a general network setting where players occasionally form or delete links by mistake; thus, stochastic stability is used as a way to identify limiting networks. However, the paper closest to this one is Goyal and Joshi (2003), who study a model where costly links are formed by mutual consent and positve as well as negative spillovers between each link and the rest of the network are allowed. By imposing assumptions on the marginal benefits from links they derive the pairwise equilibrium networks. In particular the impose in one version of the model that the gross marginal benefit of each link depends only on the number of links possesed by the individuals forming the link and on the other version the marginal benefit depends only on the number of own links and the total number of links formed by the rest of the society.

The remainder of the paper proceeds as follows. In the next section the model is presented. In the next section **??**, the general model is analyzed. In section **??**, the equilibrium is derived in steady-state network. In the next section, section **??**, the equilibrium outcomes are analyzed for three evolving networks. In section **??**, three applications are discussed at length and in the final section, concluding remarks are presented.

2 The Matching Game

2.1 The Static Game

The society consists on $N+1$ infinitely lived players who may interact through a collection of infinitely repeated games. At $t=0$, before repeated interactions start, the $N+1$ individuals choose to form links with each other in a way that will be explained below and after that, at each period $t \geq 1$, player i knows $n_i(g)$ individuals and he is matched with one of them, where $n_i(g)$ is the cardinality of the set $N_i(g)$ that is the set of individuals known by i, and g is the social network to which player i belongs. The probability of being matched with player $j \in N_i(g)$ is time independent and given by $p_{ij}(g)$. For each pair of players who actually interact, $i, j \in g$, the stage game they play is an idionsyncratic prisoner's dilemma (PD, hereafter) with a payoff matrix given by:

	D	C
D	$d+I_{ij}\theta, d+I_{ij}\theta$	$b+I_{ij}\theta, 0$
C	$0, b+I_{ij}\theta$	$c+I_{ij}\theta, c+I_{ij}\theta$

where, as customary, D stands for defection and C for cooperation and I_{ij} is an indicator function that takes the value 1 when i and j are a good match to each other and takes the value 0 otherwise. Furthermore, the payoffs satisfy the following: $b > c > d \geq 0$.

It is assumed that each of the PD games are choice independent, in the sense that players' decisions in the past do not restrict the feasible behavior in the future. They need not be, however, strategically independent since the behavior in the future may be made contigent on the information of what has occured in the past.

A key aspect of the approach adopted here is that information on how players have behaved in the past diffuses through the social network only gradually. Specifically, it is assumed that, player i is discovered by player j right after a deviation with probability $\pi_{ji}(g)$. When he is not discovered right after he deviates, his defection will be forgotten in all subsequent periods. Thus, information can range from atomized (people know only what has happened in their own interactions) to comprehensive (people know what has happened in all previous interactions, and everything about the game is common knowledge). The main implication of the assumption that $\pi_{ij}(g)$ is network dependent is that, in general, the architecture of the social network has bearing on the extent of cooperation that the network can support in a self-sustainable fashion.[4]

Notice that when $\pi_{ij}(g) = 1$ for all ij pairs, the model corresponds to the one in which everyone observes everyone else play in each period, while when $\pi_{ij}(g) = 0$, the model corresponds to the other extreme when there is no information transmission between any given pair ij and the rest of the society. So the model encompasses the two most emblematic repeated game models.

Finally, each individual discounts the future with a discount factor equal to δ.

2.2 Social Networks

By a social network it is understood a set of people or groups of people with some pattern of interaction or "ties" between them. Friendship among a group of individuals, business relationships between companies, church participation, marriages among individuals of the same race or religion are all examples of social networks that have been studied in the past.

The social network or network of connections among individuals is described by a graph $g \in G \equiv \{g \mid g \subseteq g^{N+1}\}$ which is an $N+1 * N+1$ matrix, where G is the set of graphs of $N+1$ and g^{N+1} is the complete network. Each element of g is denoted by ij and g is a symmetric matrix, *i.e.,* $ij = ji$. The set of i's direct contacts is $N_i(g) \equiv \{j \neq i : ij \in g\}$ which is of size $n_i(g)$. Thus, the size of g is $n(g) = \sum_{i \in N_t} n_i(g)/2$ and if $n_i(g) = v$ for all $i \in N+1$, then g is a symmetric social network of degree v, denoted by $g(v)$. In addition, $g + ij$ (resp. $g - ij$) denotes the network obtained by adding (resp. substracting) the link formed by player i and j to (resp. from) g.

A network is connected if there exists a path between any pair $ij \in N+1$. A network $g' \subset g$ is a component of g if for all $i, j \in g'$, $i \neq j$, there exists a path in g' connecting i and j, and for all $i \in g'$ and $k \in g$, $g_{ik} = 1$ implies that $k \in g'$. A component is complete if $g_{ij} = 1$ for all $i, j \in g'$. The complete network is a symmetric network of degree N for all $i \in g(N)$, while the empty network is a symmetric network of degree 0 for all $i \in g(0)$.

A network is asymmetric when at least one pair of players have a different number of links–that is $n_i(g) \neq n_j(g)$.

Let $N_1(g)$, $N_2(g)$,...,$N_m(g)$ be a partition of players corresponding to the number of links that players have, i.e., $i, j \in N_k(g)$, $k = 1, 2, .., m$ if and only if $n_i(g) = n_j(g)$. Note that k refers to the order in the partition and not the exact number of links that players have. An inter-linked stars architecture has at least two members in the above partition, and the maximally and minimally linked groups, respectively, satisfy the following two conditions: (i) $n_i(g) = N - 1$ for $i \in N_m(g)$ and (ii) $N_i(g) = N_m(g)$ for $i \in N_1(g)$. The star network is an special case of such an architecture with $|N_m(g)| = 1$ and $|N_1(g)| = N - 1$. An exclusive groups architecture is characterized by $m+1$ groups, a group of isolated players $A_1(g)$ and m distinct groups of completely connected players,

[4] There are many different ways in which information transmission can be introduced in the model, yet the one chosen is the simplest one capable of creating network dependant strategic effects.

$A_2(g), ..., A_{m+1}(g)$. Thus, $n_i(g) = 0$, for $i \in A_1(g)$, while $n_j(g) = |A_x(g)| - 1$, for $j \in A_x(g)$ for $x \in \{2, ..., m+1\}$. A special case of this architecture is the dominant group network in which there is only one complete component and the rest of the players are isolated–that is $m = 1$.

2.3 Information Transmission and Matching Probabilities.

In order to focus on the role of social networks it is assumed that individual i's pure actions are observed by the current partner and with probability $\pi_{ij}(g)$, i learns j's history of play before the stage game is played. If $\pi_{ij}(g) = 0$ for all ij pairs, then there is no information transmission regarding past actions between two partners and the rest of the society while when $\pi_{ij}(g) > 0$ for at least some ij pairs, then there is information transmission from partners to the rest of the society. When $\pi_{ij}(g) = 1$ the interactions are perfectly embedded since i becomes complete informed about the history of play of j. The information transmission probability from j to i satisfies the following: $\pi_{ij}(g) : G \longrightarrow [0,1]$ and $\pi_{ij}(g) \geq 0$ if and only if $g_{ij} = 1$ and $\pi_{ij}(g) = 0$ otherwise. That is, individual i learns j's history of play only if they form a link or know each other.

The matching probability between i and j satisfies the following: $p_{ij}(g) : G \longrightarrow [0,1]$ and $p_{ij}(g) \geq 0$ if and only if $g_{ij} = 1$ and $p_{ij}(g) = 0$ otherwise. That is, individual i and j are matched to each other with positive probability in any given period if and only if they form a link or know each other. Thus, while whether i and j are matched in any given period is random, i and j are matched in any given period if and only if they are linked.

2.4 Network Formation

Before the matching game starts all individual announce all the links that they want to form. For all $i, j \in N$, $s_{ij} = 1$ if i wants to form a link with j, and $s_{ij} = 0$ otherwise. By convention $s_{ii} = 0$. A link is created if and only if $s_{ij} * s_{ji} = 1$. Links are thus created by mutual consent. Forming links is not costless. Individual i's cost of forming a link with individual j is $r_i(n_i(g))$. The cost of each link dependens on the network structure.

2.5 The Equilibrium Concept

The equilibrium concept used is pairwise-equilibrium networks. That is the strategy profile $(s_i, a_i)(g) \equiv \left(s_{i1}, s_{i2}, ..., s_{iN}, a_{ij}^0, a_{ij}^1, ..., a_{ij}^t\right)(g)$ is sub-game perfect and no pair of players gains by altering the current configuration of links. Thus, g is a pairwise-equilibrium network (PWSE, here thereafter) if and only if there is a sub-game perfect equilibrium strategy profile $(s_i, a_i)(g)$ which supports g and is pairwise-stable.[5]

Let $g \in G$. Individual i's expected payoff is given by:

$$U_i(s_i, a_i; g) \equiv (1-\delta) \sum_{t=0}^{\infty} \delta^t \sum_{j \in N_i(g)} p_{ij}(g)\, u\left(a_{ij}^t, a_{ji}^t\right) - r_i(n_i(g)),$$

where the strategy profile (s_i, a_i) has been ommited to save on notation, a_{ij}^t is player i's action when playing the stage game with player j in period t.

[5] The concept of sub-game perfection by itself is too weak a concept. In fact the empty network is always sub-game perfect. More generally, for any pair i and j, it is always mutual best response for the players to offer to form no link.

Pairwise stability implies that for all $ij \notin g$, if $U_i(g+ij) > U_i(g)$ then $U_j(g+ij) < U_j(g)$. Thus, strategy profile (s_i, a_i) is a PWSE network if $U_i\left(g\left(s_i^*, a_i^*, \left(s_{-i}^*, a_{-i}^*\right)\right)\right) \geq U_i\left(g\left(s_i, a_i, \left(s_{-i}^*, a_{-i}^*\right)\right)\right)$ for all $(s_i, a_i) \in \{0,1\}^N \times \{C, D\}$ and if $U_i(g+ij) > U_i(g)$ then $U_j(g+ij) < U_j(g)$.

In words, PWSE networks are such that no player gains by altering the current configuration of links, neither by adding a new link nor by eliminating any subset of existing links.

To simplify the analysis it is assumed that players rely only on trigger strategies. Player i cooperates against j if he has no information that j ever played D before, be it against i or against some third party h. On the other hand, if player i gets information that j has defected (against i or h), then i chooses defection himself against j in all interactions with j after receiving the information on j's defection. Mutul defections do not evoke sanctions, when it is a part of the prescribed pattern of behavior. That is, if k defects against i in order to punish the latter for cheating j, then k is not cheating but rather carrying out a prescribed punishment, so others observing the defection would not punish k in turn. It is also assumed that if k observes i cheating j then this is common knowledge between k and i.

3 Preliminaries

Given any network g, player i's expected payoff from following the trigger strategy given that each of his links follows the trigger strategy is given by:

$$V_i(C) \equiv (1-\delta)\, c_{ik} + \delta \sum_{j \in N_i(g)} p_{ij}(g)\, c_{ij}.$$

Player i's expected payoff from deviating during the simultaneous move game against partner k and then conforming to the trigger strategy forever thereafter is given by:

$$V_i(D) \equiv (1-\delta)\, b_{ik} + \\ \delta\left(p_{ik}(g)\, d_{ik} + \textstyle\sum_{j \in N_i(g)/k} p_{ij}(g)\left[\pi_{ji}(g)\, d_{ij} + (1-\pi_{ji}(g))\, c_{ij}\right]\right).$$

The first term is self-explanatory. The term in parenthesis includes three terms. The first one is the probability of being matched with player k and being punished by her forever thereafter. The second one is the expected payoff from being matched with someone who learned that i deviated in the last period, which is equal to mutual defection, and the third term is the expected payoff from being matched with someone who did not learn that i deviated in the last period, which is equal to mutual cooperation. In the last two terms the expected payoff is taken over all i's partners that choose cooperation from the first period onwards.

Player i cooperates in each encounter from time t onwards if and only if $V_i(C) \geq V_i(D)$–that is,

$$\delta \geq \delta_{ik}(g) \equiv \frac{b-c}{b-c+(c-d)\sum_{j \in N_i(g)} \pi_{ji}(g)\, p_{ij}(g)}, \tag{1}$$

where $\pi_{ki} = 1$.

If there is no information transmission between any pair of individuals and the rest of the society–that is the society is atomized–$\pi_{ji}(g) = 0$ for all $j \in N_i(g)$ and for all $i \in g$, cooperation between i and k is self-sustained if and only if $\delta \geq \overline{\delta}_{ik}(g) \equiv \frac{b-c}{b-c+(c-d)p_{ik}(g)} > 0$ while if the society is fully embedded–that is $\pi_{ji}(g) = 1$ for all $j \in N_i(g)$ and for all $i \in g$–cooperation between i and k is self-sustained if and only if $\delta \geq \underline{\delta}_{ik}(g) \equiv \frac{b-c}{b-c+(c-d)\sum_{j\in N_i(g)} p_{ij}(g)}$, with $\underline{\delta}_{ik}(g) < 1$. Furthermore, notice that $\underline{\delta}_{ik}(g) < \delta_{ik}(g) < \overline{\delta}_{ik}(g)$ as long as $\pi_{ji}(g) \in (0,1)$ for some $ji \in g$. That is, network-based effects resulting from information transmission can only help in supporting cooperation.

In general the strength of those effects depends positively on the number of neighbors a player has, how valuable these are, and his information gathering efficiency. That is third-party sanctions are more effective as $\pi_{ji}(g)$ increases.

This leads to the following result.

Proposition 1 *(i) Player i and k cooperate in each encounter if and only if $\delta \geq \max\{\delta_{ik}(g), \delta_{ki}(g)\}$; and (ii) $\delta_{ik}(g) \in \left[\underline{\delta}_{ik}(g), \overline{\delta}_{ik}(g)\right]$ for all $\pi_{ji}(g)$ and $p_{ij}(g)$ and $\delta_{ik}(g)$ is decreasing in $\pi_{ji}(g)$ and $p_{ij}(g)$.*

Given the condition in equation (1), the set of i's direct contacts with whom cooperating is self-sustainable can be defined as $C_i(g) \equiv \{j \in N_i(g) : \delta \geq \max\{\delta_{ij}(g), \delta_{ji}(g)\}\}$ and the set of i's direct contacts with whom cooperation is not self-sustainable is given by $D_i(g) = N_i(g)/C_i(g)$. Notice that $n_i(g) = c_i(g) + d_i(g)$.

In equilibrium individual i's expected payoff can be written as:

$$U_i(g) = \sum_{j\in C_i(g)} p_{ij}(g)\, c_{ij} + \sum_{j\in D_i(g)} p_{ij}(g)\, d_{ij} - r_i(n_i(g)).$$

Given the expected payoff, individual i's marginal gross return from establishing a link with individual h–that is $U_i(g+ih) - U_i(g)$–is given by:

$$\triangle U_i(g+ih, g) \equiv \begin{cases} p_{ih}(g+ih)(u + I_{ih}\theta) + \\ \sum_{j\in C_i(g)\cap C_i(g+ih)} [p_{ij}(g+ih) - p_{ij}(g)](c + I_{ij}\theta) + \\ \sum_{j\in D_i(g)\cap D_i(g+ih)} [p_{ij}(g+ih) - p_{ij}(g)](d + I_{ij}\theta) + \\ \sum_{j\in D_i(g)\cap C_i(g+ih)} [p_{ij}(g+ih)(c + I_{ij}\theta) - p_{ij}(g)(d + I_{ij}\theta)] + \\ \sum_{j\in C_i(g)\cap D_i(g+ih)} [p_{ij}(g+ih)(d + I_{ij}\theta) - p_{ij}(g)(c + I_{ij}\theta)], \end{cases} \tag{2}$$

where $u = c$ if $ih \in C_i(g+ih)$ and $u = d$ if $ih \in D_i(g+ih)$.

The first term is the direct benefit from adding one more link: $p_{ih}(g+ih)(u + I_{ih}\theta)$ is i's payoff from forming a link and entering in a long-run relationship with player h. The second term is the decrease in i's expected payoff from those relationships in which cooperation is self-sustainable before and after the link ih is added and the third term is the decrease in i's expected payoff from those relationships in which defecting is optimal before and after the link ih is added. The fourth term is the change in i's payoff from those relationships in which cooperation was optimal before adding the link ih and is no longer optimal after the link is added. The fifth term is the change in i's payoff from those relationships in which cooperation was not optimal before adding the link ih and is optimal after the link is added. These last two terms arise when the threshold for the discount factor $\delta_{ik}(g)$ changes as the social network changes. Thus, adding one more link has a straightforward direct effect and two different kind of indirect effects: the first one that is the change in the matching probability between any pair of known players, and the second one that is the change in the optimal strategy from defection to cooperation or vice-versa in some of i's relationships–that is changes in the severity of the punsihment as a result of changes in $p_{ji}(g)$ and $\pi_{ji}(g)$.

It is clear from equation 2 that how fast the information flows through the network changes only the severity of the punishment while a change in the matching probability changes both the severity of the punishment as the direct expected benefit of an encounter.

At the level of generality considered adding one more link then can either favor or instill cooperation. That is, link formation may have either positive or negative spillovers on the welfare of other individuals. In particular, on those who are connected to the individuals that form a link.

4 Two Benchmarks

4.1 Atomized Societies

In this section it is assumed that $\pi_{ij}(g) = 0$, $p_{ij}(g) = p \leq \frac{1}{N}$ for all $ij \in g$ and $r_i(n_i(g)) = n_i(g)r$ for all i. That is the marginal cost of adding an extra links is constant. and individual i is informed on the history of his own past interactions but he has no information on his partner's past behavior. In particular, he has no information on his partners' behavior in interactions with third parties. Thus, the way that player i treats his potential partner j when they meet depends only on their past interactions. This case is then the analogue of the standard repeatead prisoner's dilemma with changing partners.

Notice that while we called this case atomized is not atomized in the way Granovetter suggests; that is anonimous interactions in perfectly competitive markets. It is true here that third parties do not affect the actions taken by any given player, but individuls may interact over and over again with the same group of individuals and have specific information on their behavior on past encounters, which means that interactions are not anonymous.

In this case $\delta_{ik}(g) = \delta(p) \equiv \frac{b-c}{b-c+p(c-d)}$ for all $ik \in g$ and for all g. Notice that $\delta(p)$ decreases in p, which in what follows it is called the intensity of a relationship. That is the more frequent i and k interact the more likely that they cooperate.

In this case the marginal gross return becomes $\triangle U_i(g + ih, g) \equiv pu$, where $u = c$ when $\delta \geq \delta(p)$ and $u = d$ when $\delta < \delta(p)$. Furthermore, $\triangle U_i(g + ih, g)$ is independent of the number of own and third-party links–that is there is no network or spillovers effects. In this case the following proposition can be easily shown.

Proposition 2 *(i) Suppose that $\delta \geq \delta(p)$, if $pc \geq r$, then the complete network is the unique PWSE networks, while if $pc < r$, then the empty network is the unique PWSE network; and*
(ii) suppose that $\delta < \delta(p)$, if $pd \geq r$, then the complete network is the unique PWSE network, while if $pd < r$, then the empty network is the unique PWSE network.

The intuition is simple. By adding links player i neither alters his incentives to cooperate nor he changes other individuals' incentives–that is if $\delta \geq \delta(p)$ cooperation takes place irrespective of the number of links that any player may form. This implies that a link is created if and only if the marginal benefit, which is pu is larger than the marginal cost, which is r.

The intensity of a relationship also increases the incentive to form links. This implies that as p increases the network becomes denser. This however is in stark form since the network may go from the empty network to the complete network as p increases.

Finally , notice that total welfare created by the PWSE network is given by

$$W(g) = \sum_{i \in g} \left(\sum_{j \in C_i(g)} pc + \sum_{j \in D_i(g)} pd - n_i(g) r \right). \tag{3}$$

This implies that the complete PWSE network is the only efficient network when $pu \geq r$, where $u = c$ if $\delta \geq \delta(p)$ and $u = d$ otherwise, and the empty network is the unique efficient network when $pd < r$.

4.2 Embedded Societies

In this section it is assumed that $\pi_{ij}(g) = \pi$ and $p_{ij}(g) = p \leq \frac{1}{N}$ for all $ij \in g$ and that $r_i(n_i(g)) = n_i(g) r$ for all i. That is the marginal cost of adding an extra link is constant, and there is a positive probability that each of player i's partners learns the actions taken by him against a third party in the last period . Notice that when $\pi = 1$ this is the analogue of the standard repeatead prisoner's dilemma with changing partners and perfect information flows.

The assumptions $\pi_{ij}(g) = \pi$ and $p_{ij}(g) = p$ may seem extreme assumptions, yet they are less extreme than in many repeated game models since they $\pi_{ij}(g) > 0$ and $p_{ij}(g) > 0$ if and only if i and j form a link. Thus, while $\pi_{ij}(g)$ and $p_{ij}(g)$ do not depend on the number of links, they are endogeneous in the sense that they are positive if and only if i and j choose to form a link.

An example that fits well with this case when p and π are large is the one cited by Coleman about Jewish diamond merchants in New York. According to Coleman they save a great deal in lawyers' fees by conducting their transactions informally. Sacks of jewels worth thousands of dollars are lent for examination overnight without any contract signed. What makes associates not to shirk their obligations is that anyone found guilty of malfesance can kiss good-bye his future chances of being part of such a profitable business. This occurs because merchants in this market belong to the same tight social circle and information flows well in that circle.

For all $i \in g$, player i's, relationships are all symmetric, and equation 1 becomes

$$\delta \geq \delta_i(\pi, p) \equiv \frac{b - c}{b - c + (c - d) p [1 + (n_i(g) - 1) \pi]}, \tag{4}$$

with $n_i(g) \geq 1$.

Notice that contrary to the atomized case, here, the number of own links that are willing to cooperate changes the incentives to cooperate. In fact, adding links makes cooperation more likely to be self-sustainable. The intuition is simple. The larger player i's number of links, the larger the expected punishment from deviating, since a defection by any player is punished by defection forever thereafter by all i's partners that become informed about his past actions against a third party.

Because $\delta_i(\pi, p)$ decreases as $n_i(g)$ increases, when the number of own links exceeds the threshold $\tilde{n}(\pi, p)$ cooperation becomes self-sustainable. In what follows it is assumed that $\tilde{n}(\pi, p) \geq 2$; that is, a necessary condition for player i to be willing to cooperate is that he has at least two partners.[6]

Under the assumptions made, player i's set of cooperators $C_i(g)$ is given by:

$$C_i(g) \equiv \begin{cases} \{j \neq i : j \in N_i(g),\ n_j(g) \geq \tilde{n}(\pi, p)\} & \text{if } n_i(g) \geq \tilde{n}(\pi, p) \text{ and} \\ \phi \text{ if } n_i(g) < \tilde{n}(\pi, p), \end{cases}$$

[6] If $\tilde{n}(\pi, p) = 1$, then if $pc \geq r$, the only PWSE network is the complte network.

and his marginal gross return becomes

$$\triangle U_i(g+ih,g)=\begin{cases} pd & \text{for } n_i(g+ih)<\tilde{n}(\pi,p), \\ pd+p(c-d)c_i(g+ih) & \text{for } n_i(g+ih)=\tilde{n}(\pi,p), \\ pu & \text{for } n_i(g+ih)>\tilde{n}(\pi,p), \end{cases} \tag{5}$$

where $c_i(g+ih)$ denotes the cardinality of the set $C_i(g+ih)$, $u=c$ if $n_h(g+ih)\geq\tilde{n}(\pi,p)$ and $u=d$ otherwise.

Lemma 3 *Take any PWSE network g. If $pu\geq r$, where $u=c$ if $\min\{n_i(g),n_h(g)\}\geq\tilde{n}(\pi,p)$ and $u=d$ otherwise, then $g_{ih}=1$.*

Proof. Suppose that g is PWSE equilibrium network and $g_{ih}=0$. Because $\triangle U_i(g+ih,g)\geq u$ and $\triangle U_h(g+ih,g)\geq u$ since $p(c-d)c_i(g+ih)\geq 0$, $\triangle U_i(g+ih,g)\geq r$ and $\triangle U_h(g+ih,g)\geq r$. Thus, by pairwise stability i and h choose to form a link with each other. ■

This simple result carries important consequences for the network architecture. First, it implies that in any PWSE network g all $ih\in N$ pairs that yield a direct expected payoff, pu, larger than the cost of forming that link, r, must be linked. This implies that if $pd\geq r$, everyone must be mutually linked irrespective of whether cooperation is feasible or not.

It also implies that in any PWSE network g, if $pc\geq r$ and $\min\{n_i(g),n_h(g)\}\geq\tilde{n}(\pi,p)$, then i and h must be linked. Thus in a PWSE network, all individuals willing to cooperate must be mutually linked. That is cooperators form a complete component of g.

The analysis is split in two cases: one in which $N<\tilde{n}(\pi,p)$ and the other in which $N\geq\tilde{n}(\pi,p)$. The former corresponds to the case in which the number of potential links is so that cooperation is not feasible and the latter assumes the opposite.

In the next proposition the case in which cooperation is not possible between any pair of players is analyzed.

Proposition 4 *Suppose that $N<\tilde{n}(\pi,p)$.*
(i) If $pd\geq r$, then the complete network is the unique PWSE network; and
(ii) if $pd<r$, then the empty network is the unique PWSE network.

Given that the maximum number of possible links is lower than the minimum requiered for cooperation between any pair of players to be self-sustainable, there is no network g that induces cooperation. Lemma 3 ensures that when $pd\geq r$, a link is formed.

Consider next the case in which $N\geq\tilde{n}(\pi,p)$; that is, for a sufficiently dense network cooperation is feasible.

Proposition 5 *Suppose that $N\geq\tilde{n}(\pi,p)$.*
(i) If $pd\geq r$, then the complete network is the unique PWSE network;
(ii) if $pc\geq r>pd$, then the complete and the empty network are the unique symmetric PWSE networks and any asymmetric PWSE network has the dominant group architecture in which the complete component has a degree of at least $\tilde{n}(\pi,p)$; and[7]
(iii) if $pc<r$, then the empty network is the unique PWSE network.

[7] There is a multiplicity of equilibrium in the network formation process. We could have used other cooperative refinements as Strong Nash Equilibrium. In fact, under strong Nash the unique stron nash equilibrium network is the complete network. Yet, strong Nash is harder to justify than pairwise equilibrium in a non-cooperative network formation game.

This proposition establishes that the number of own and third party links have positive spillover effects in the sense that the more links players have, the larger the incentive to form links and to cooperate. Thus, large networks are more likely to induce cooperation and the larger the benefits from cooperation the larger the network. The intuition is simple. Given that cooperation is induced by a social norm in which a deviating player is punished not only by the one being cheated, but also by all those who become informed about a player's defection, inducing cooperation requires a minimum number of potential partners that can punish a deviator. Knowing few people in this case is bad since by cheating a partner one risks to be punished by few people only and so the expected punishment is small. Thus, in order to benefit from cooperation, the network must have a minimum density, as given by the average degree of links, and minimum span, as embodied by the relative size of the largest component.

This result is partially driven by the fact that p and π are both constant in the number of own and third-party links. If they were decreasing on the number of either own or third-party links or both, the results may be reversed. This is discussed at length in the next section.

Notice also that the quality of information, which is assumed exogenous in the model, also affects the arquitecture of the PWSE network. Indeed, the more likely to learn a partner's play in the last period, the more likely is that individuals cooperate. As a consequence of this, each individual needs to have a lower number of own links to be willing to cooperate. Thus, a smaller network is needed to induce cooperation.

Proposition 6 *As p and π increase, in a dominant group architecture the lowest degree of the complete component decreases.*

Proof. This follows from that $\tilde{n}(\pi, p)$ is decreasing in (π, p) and that the complete component in the dominant group architecture has a degree of at least $\tilde{n}(\pi, p)$. ■

In this case total welfare in network g is given by

$$W(g) = \sum_{i \in g} \left(\sum_{j \in C_i(g)} pc + \sum_{j \in D_i(g)} pd - rn_i(g) \right). \tag{6}$$

It is easy to see from equation 6 that the complete network is the unique efficient network when $pu \geq r$, where $u = c$ if $\delta \geq \delta_i(\pi, p)$ and $u = d$ otherwise, and the empty network is the unique efficient network when $pd < r$. Thus, an asymmetric PWSE network is never efficient. Ths reason being that in an asymmetric network there are isolated players whose lack of interaction preclude them from the benefits of cooperation.

5 Generalizations

The two cases studied so far were intended to capture the main trade-off in extremely simple setting. In particular, all links were assumed to be equally costly, all matches were assumed equally productive and the interaction probability as the quality of information transmission were assumed to be independent of the number of links. In this section, the model in sub-section 4.2 is used as plataform for the analysis and each of the following sub-sections one of the assumptions is relaxed in a ceteris-paribus fashion.

5.1 Increasing Marginal Costs

In this section we keep the assumptions that $\pi_{ij}(g) = \pi$, $p_{ij}(g) = p$ and $I_{ij} = 0$ for all $ij \in g$, but it is assumed that the marginal cost of adding a link is strictly increasing in the number of own links. In particular, it is assumed that $r_i(n_i(g)) = r(n_i(g))^2$. Thus, the marginal cost of going from $n_i(g) - 1$ to $n_i(g)$ links is given by $(2n_i(g) - 1)r$ and is increasing in the number of own links.

In this case each relationship is symmetric, and as before cooperation between i and any player who is willing to cooperate is self-sustainable if and only if $\delta \geq \delta_i(\pi, p)$, where the latter is given by equation 4.

Given the assumptions made the marginal gross return is given by equation 5, and therefore player i and h are willing to add a link if and only if

$$\triangle U_i(g + ih, g) \geq (2n_i(g) + 1)r \text{ and } \triangle U_h(g + ih, g) \geq (2n_i(g) + 1)r.$$

In the next proposition the case in which cooperation is not possible for any given number of links between any pair of players is analyzed. Let define n_L as the largest number of own links that makes $pd \geq (2n_i(g) - 1)r$.

Proposition 7 *Suppose that* $N < \tilde{n}(\pi, p)$.
(i) If $pd \geq r$, *then the symmetric network* $g(n_L)$ *is the unique PWSE network; and*
(ii) if $pd < r$, *then the empty network is the unique PWSE network.*

Given that the maximum number of links is lower than the minimum requiered for cooperation between any pair of players to be self-sustainable, there is no network g that induces cooperation. Lemma 3 ensures that when $pd \geq \max\{(2n_i(g) - 1)r, (2n_h(g) - 1)r\}$, a link between i and h is formed, and therefore, links are formed until $pd \geq (2n_i(g) - 1)r$.

Consider next the case in which $N \geq \tilde{n}(\pi, p)$–that is, for a sufficiently dense network cooperation becomes feasible, and define n_H as the largest number of own links that makes $pc \geq (2n_i(g) - 1)r$.

Proposition 8 *Suppose that* $N \geq \tilde{n}(\pi, p)$.
(i) Suppose that $pd \geq r$. *(a) If* $n_L < \tilde{n}(\pi, p)$ *and* $pc \leq (2\tilde{n}(\pi, p) - 1)r$, *the symmetric network* $g(n_L)$ *is the unique PWSE network; (b) if* $n_L < \tilde{n}(\pi, p)$ *and* $pc > (2\tilde{n}(\pi, p) - 1)r$, *the symmetric network* $g(n_H)$ *is a PWSE network, the symmetric network* $g(n_L)$ *and any asymmetric network that has the exclusive groups architecture with two complete components: one with a degree of at least* $\tilde{n}(\pi, p)$ *and at most* n_H *and the other with degree* n_L *are PWSE networks for* $n_L < \tilde{n}(\pi, p) - 1$; *(c) if* $n_L \geq \tilde{n}(\pi, p)$, *the symmetric network* $g(n_H)$ *is the unique PWSE network.*
(ii) Suppose that $pc \geq r > pd$. *(a) If* $n_L < \tilde{n}(\pi, p)$ *and* $pc \leq (2\tilde{n}(\pi, p) - 1)r$, *the empty network is the unique PWSE network; (b) if* $n_L < \tilde{n}(\pi, p)$ *and* $pc > (2\tilde{n}(\pi, p) - 1)r$, *the symmetric network* $g(n_H)$ *is a PWSE network, the empty network and any asymmetric PWSE network that has the dominant group architecture in which the complete component has a degree of at least* $\tilde{n}(\pi, p)$ *and at most* n_H *and the rest are isolated players are PWSE networks for* $n_L < \tilde{n}(\pi, p) - 1$; *and (c) if* $n_L \geq \tilde{n}(\pi, p)$, *the symmetric network* $g(n_H)$ *is the unique PWSE network ; and*
(iii) Suppose that $pc < r$. *Then the empty network is the unique PWSE network.*

As expected when marginal cost of adding one more link is increasing in the number of links, the PWSE networks are less dense and have a lower span relative to the case in which the marginal cost is constant.

What is more interesting is that when $pd \geq r$ two types of symmetric networks may form, one in which there is no cooperation and each individual has n_L links and one in which there is cooperation and each individual has n_H links while when marginal cost is constant a unique PWSE exists in which every is mutually linked and cooperation occurs in all pair-wise matches. When an assymetric network arises the society becomes segregated in two groups: a group of mutually linked players that do not cooperate and another group of mutually linked players that cooperate. Thus, even in societies that are complete homogeneous there are might be equilibria where some people do well relatively to others.

5.2 Link Heterogeneity

In this section we keep the assumptions that $\pi_{ij}(g) = \pi$ and $p_{ij}(g) = p$ for all $ij \in g$ and that $r_i(n_i(g)) = n_i(g)\,r$ for all i, but heterogeneity in links' payoff is introduced. That is, some matches yield an extra payoff equal to θ.. In what follows it is assumed that each player knows which links yield the extra payoff θ before they choose to form links. For the sake of simplicity it is assumed that each individual has the same number of good links available that is assumed to be equal to $N_g + 1$, with $N_g \leq N$ and N even.

Notice that the extra payoff θ reflects the quality of the match and not individual differences, and therefore is not meant to capture differences in ability or in the willingness to cooperate. Indeed, both good and bad links have the same threshold for self-sustainable cooperation and this is still given by equation 4. Thus, under this formulation cooperation between i and j is still self-sustainable if and only if $\min\{n_i(g), n_j(g)\} \geq \tilde{n}(\pi, p)$, where we keep the assumption that $\tilde{n}(\pi, p) \geq 2$,[8] and hence the marginal gross return becomes

$$\triangle U_i(g + ih, g) = \begin{cases} p(d + I_{ih}\theta) & \text{for } n_i(g + ih) < \tilde{n}(\pi, p) \\ p(d + I_{ih}\theta) + p(c - d)\,c_i(g + ih) & \text{for } n_i(g + ih) = \tilde{n}(\pi, p) \\ p(u + I_{ih}\theta) & \text{for } n_i(g + ih) > \tilde{n}(\pi, p), \end{cases} \tag{7}$$

where $u = c$ if $n_h(g + ih) \geq \tilde{n}(\pi, p)$ and $u = d$ otherwise.

This lemma implies that $\triangle U_i(g + ih, g)$ can be defined as a function of $(n_i(g), n_{-i}(g))$.

Lemma 9 *Take any PWSE network g. If $p(u + I_{ih}\theta) \geq r$, where $u = c$ if $\min\{n_i(g), n_h(g)\} \geq \tilde{n}(\pi, p)$ and $u = d$ otherwise, then $g_{ih} = 1$.*

Proof. Suppose that g is PWSE equilibrium network and $g_{ih} = 0$. Then $\triangle U_i(g + ih, g) \geq u + I_{ih}\theta$ and $\triangle U_h(g + ih, g) \geq u + I_{ih}\theta$, and since by hypothesis $u + I_{ih}\theta \geq r$, $\triangle U_i(g + ih, g) \geq r$ and $\triangle U_h(g + ih, g) \geq r$. Thus, by pairwise stability i and h choose to form a link with each other. ■

This simple result has powerful implications for the network structure that may arise in equilibrium. First, it implies that in any PWSE network g all $ih \in N$ pairs that yield a direct expected payoff, $p(u + I_{ih}\theta)$, larger than the cost of forming the link, r, must be linked. This implies that if $p(d + \theta) \geq r > pd$, are all good links are formed irrespective of whether cooperation is self-sustainable while this is not necessarily true for bad links. Thus in any PWSE network g, if $p(d + \theta) \geq r$ the good links must form a complete component of g. That is all must be linked.

[8] If $\tilde{n}(\pi, p) = 1$, then as long as $pc \geq r$, the only PWSE network is the complete network.

Second, it implies that in any PWSE network g, if $p(c + I_{ih}\theta) \geq r$ and $\min\{n_i(g), n_h(g)\} \geq \tilde{n}(\pi, p)$, then i and h must be linked, i.e., $g_{ih} = 1$. Thus in this case in any PWSE network g, the cooperators must form a complete component of g.

Third, it implies that in any PWSE network g if player i forms $n_i(g)$ links and $n_i(g) \leq N_g$, then all his links are best links. This readily follows from the fact that player i knows ex-ante the identity of all the potential links and that all links are equally costly.

The analysis is split in two cases: (i) $N < \tilde{n}(\pi, p)$ and (ii) $N \geq \tilde{n}(\pi, p)$. The latter can also be split in two sub-cases: (i) $c \leq d + \theta$ and (ii) $c > d + \theta$. The former imposes that mutual defection in a good link yields a larger payoff than cooperation in a bad link while the latter imposes the opposite.

In the next proposition the case in which cooperation is not possible between any pair of players is analyzed.

Proposition 10 *Suppose that $N < \tilde{n}(\pi, p)$. Then, (i) If $pd \geq r$, then the complete network is the unique PWSE; (ii) if $p(d + \theta) \geq r > pd$, the dominant group network in which only best matches are formed is the unique PWSE; (iii) $r > p(d + \theta)$, then the empty network is the unique PWSE;*

Given that the maximum number of links available is lower than the minimum number of links requiered to induce cooperation between any pair of players, there is no network g in which at least some pair of players choose cooperation. Lemma 3, however, ensures that good links are formed when $p(d + \theta) \geq r$.

Consider next the case in which $N \geq \tilde{n}(\pi, p)$ and $d + \theta \leq c$. In this case cooperation is always better than defection irrespective of the quality of the link. That, the payoff from cooperation in a bad link is larger than the payoff from defection in a good link.

Proposition 11 *Suppose that $N \geq \tilde{n}(\pi, p)$ and $d + \theta \leq c$.*
(i) If $pd \geq r$, then the complete network is the unique PWSE network;
(ii) if $p(d + \theta) \geq r > pd$, then the complete network is the unique PWSE network if $N_g \geq \tilde{n}(\pi, p) - 1$, otherwise the complete network and any asymmetric network that has the exclusive groups architecture with two complete components: one that has at least $\tilde{n}(\pi, p)$ links per individual and the other that has N_g links per individual are PWSE networks;
(iii) if $pc \geq r > p(d + \theta)$, then the complete network, the empty network and any asymmetric network that has the dominant group architecture with a complete component that has at least $\tilde{n}(\pi, p)$ links per individual is a PWSE network;
(iv) if $p(c + \theta) \geq r > pc$, then the empty network is the unique PWSE network if $N_g < \frac{\tilde{n}(\pi,p)(r-pc)}{p\theta}$, the symmetric $g(\tilde{n}(\pi, p))$, the empty network and the asymmetric network that has a dominant group architecture with a complete component with $\tilde{n}(\pi, p)$ links per individual are PWSE networks if $\frac{\tilde{n}(\pi,p)(r-pc)}{p\theta} \leq N_g < \tilde{n}(\pi, p)$, and the symmetric $g(N_g)$, the empty network, and the asymmetric network that has the dominant group architecture with a complete component with N_g links per individual is a PWSE network if $N_g \geq \tilde{n}(\pi, p)$; and
(v) if $p(c + \theta) < r$, then the empty network is the unique PWSE network.

Proof. see Appendix. ■

With heterogenous links, the PWSE networks are different from those arising under homogeneous links. In the two extreme cases $pd \geq r$ and $p(c + \theta) < r$, the same PWSE networks arise, while when r is between pd and $p(c + \theta)$ PWSE networks may drastically change. There are two types of social networks that have interesting properties. The first one is the exclusive groups architecture with two complete components: one that has at least $\tilde{n}(\pi, p)$ links per individual and the other that has N_g links per individual are PWSE networks. This social network has two distinctive features. First, the society is segregated in two groups: one group conformed by good links but in which cooperation does not take place and another group formed by good and bad links, but in which its members cooperate in each encounter. The members of the second group are better-off since $c > d + \theta$. In addition, the size of the group of cooperators is non-increasing in π. The reason being that for any number of links a larger π implies a larger expected punishment, and therefore less links are needed to induce cooperation.

The second architecture is the dominant group architecture in which the society is segregated in one group of individuals that cooperate among themselves and the rest are isolated individuals. The group of cooperators is in general composed of both, good and bad links. The size of the group of cooperators is non-increasing in π.

Finally, there is one interesting feature in some of the PWSE networks which is that individuals form links that are not profitable. The reason being that by doing so individuals are willing to cooperate. That is, unprofitable links serve as a kind of commitment device that make cooperation self-sustainable. This is the case for instance when $p(c + \theta) \geq r > pc$ and $\frac{\tilde{n}(\pi,p)(r-pc)}{p\theta} \leq N_g < \tilde{n}(\pi, p)$.

Consider next the case in which $N \geq \tilde{n}(\pi, p)$ and $d + \theta > c$. That is defecting in a good link yields a better payoff than cooperating in a bad link.

Proposition 12 *Suppose that $N \geq \tilde{n}(\pi, p)$ and $d + \theta > c$.*
(i) If $pd \geq r$, then the complete network is the unique PWSE network;
(ii) if $pc \geq r > pd$, then the complete network is the unique PWSE network if $N_g \geq \tilde{n}(\pi, p) - 1$, otherwise the complete network and any asymmetric network that has the exclusive groups architecture with two complete components: one with at least $\tilde{n}(\pi, p)$ links per individual and other with N_g links per individual are PWSE networks;
(iii) if $p(d + \theta) \geq r > pc$, then the complete network and any asymmetric network that has the dominant group architecture with a complete component with at least $\tilde{n}(\pi, p)$ links per individual is a PWSE network;
(iv) if $p(c + \theta) \geq r > p(d + \theta)$, then the empty network is the unique PWSE network if $N_g < \frac{\tilde{n}(\pi,p)(r-pc)}{p\theta}$, the symmetric $g(\tilde{n}(\pi, p))$, the empty network and the asymmetric network that has a dominant group architecture with a complete component with $\tilde{n}(\pi, p)$ links per individual are PWSE networks if $\frac{\tilde{n}(\pi,p)(r-pc)}{p\theta} \leq N_g < \tilde{n}(\pi, p)$, and the symmetric $g(N_g)$, the empty network, and the asymmetric network that has the dominant group architecture with a complete component with N_g links per individual is a PWSE network if $N_g \geq \tilde{n}(\pi, p)$; and
(v) if $p(c + \theta) < r$, then the empty network is the unique PWSE network.

Proof. see Appendix. ■

5.3 Negative Network effects on the Interaction Probability

One of the caveat of the model developed so farn and its extensions is that we have assumed that the interaction probabilities as well as the probability of becoming informed are independent of the number of own and third-party links. Assuming that either of these or both change with the number links increase substantially the complexity of model unless some specific assumptions concerning the gross-marginal return are made. In this section, we do so.

In what follows it is assumed that $\pi_{ij}(g) = \pi(N(g), \lambda)$, $p_{ij}(g) = p(N(g))$, $I_{ij} = 0$ for all $ij \in g$ and that $r_i(n_i(g)) = n_i(g) r$ for all i, where $N(g) \equiv \frac{1}{2}\sum_{k\in g} n_k(g)$ and $\pi(N(g), \lambda)$ increases in λ. That is the marginal cost of adding an extra links is constant, the probability that i learns j history of play and the probability that i and j interact depend on the number of own and third-party links and links are homogeneous. Under these assumptions player i is willing to cooperate if and only if

$$\delta \geq \delta(N(g), n_i(g)) \equiv \frac{b-c}{b-c+(c-d)\, p(N(g))\left[1+(n_i(g)-1)\, \pi(N(g), \lambda)\right]}. \quad (8)$$

Furthermore, it is assumed that $p(N(g))[1+(n_i(g)-1)\pi(N(g), \lambda)] > p(N(g)+1)[1+n_i(g)\pi(N(g)+1, \lambda)]$. That is, cooperation become harder to sustain as the number of own and third-party links increase. In words this assumption means that being punished more frequently by less people is worse than being punished less frequent by more people. So, the frequency of punishment is more important than the number of punishers. Thus, it is assumed that there are negative spillovers from the number of links to cooperation.[9]

This assumption results in that the set of cooperators is non-increasing in the number of own and third-party links in any given network. That is, $C_i(g+ih) \subseteq C_i(g)$, $D_i(g) \subseteq D_i(g+ih)$ and $D_i(g) \cap C_i(g+ih) = \phi$.

In this case the marginal gross return $U_i(g+ih) - U_i(g)$ becomes:

$$\triangle U(n_i(g), n_{-i}(g)) \equiv \begin{cases} p(N(g)+1)\, u + \sum_{j\in C_i(g+ih)\cap C_i(g)} \left[p(N(g)+1) - p(N(g))\right] c + \\ \sum_{j\in D_i(g)\cap D_i(g+ih)} \left[p(N(g)+1) - p(N(g))\right] d + \\ \sum_{j\in C_i(g)\cap D_i(g+ih)} \left[p(N(g)+1)\, d - p(N(g))\, c\right], \end{cases} \quad (9)$$

where $u = c$ if $\delta \geq \max\{\delta(N(g), n_i(g)), \delta(N(g), n_h(g))\}$ and $u = d$ otherwise.

Notice that the marginal gross return is at most $p(g+ih)\, u$ since adding more links can only decrease the incentives to cooperate and decrease the probability of interaction.

Following Goyal and Goshi (2003) it is assumed that the gross marginal return $\triangle U(n_i(g), n_{-i}(g))$ is decreasing in own and third-party links. We call the first effect, as they do, negative spillovers on own links (NSOL) and the second effect, negative spillovers on third-party links (NSTP). In addition, to make the analysis interesting it is assumed that for any network g and any player i with $n_i(g) = 0$, $\delta \geq \delta(N(g), 1)$. That is, for player i cooperation is self-sustainable when he has one link only. If this is not the case then under no network cooperation would be self-sustainable.

Suppose a symmetric network of degree v, and lets denote by $v(\pi)$ the degree that satisfies the following, $\triangle U(v, \mathbf{v}) < r \leq \triangle U(v-1, \mathbf{v}-\mathbf{1})$,and by $v(\delta, \pi)$ the degree that solves the following $\delta = \frac{b-c}{b-c+(c-d)p(N(g))[1+(n_i(g)-1)\pi(N(g),\lambda)]}$. Thus, for any symmetric network of degree larger than $v(\delta, \pi)$ cooperation is not self-sustainable. This implies that in symmetric network of degree v the gross marginal return for any $v \leq v(\lambda, \pi) - 1$ is given by:

[9] The case of positive spillovers resembles the one in which $\pi_{ij}(g) = \pi$ and $p_{ij}(g) = p$. Thus, for the sake of brevity the focus is on negative spillovers.

$$\triangle U(v,\mathbf{v}) \equiv \begin{cases} p\left(\frac{v(N+1)}{2}+1\right)c+v\left[p\left(\frac{v(N+1)}{2}+1\right)-p\left(\frac{v(N+1)}{2}\right)\right]c & \text{for } v \leq v(\delta,\pi)-1 \\ p\left(\frac{v(N+1)}{2}+1\right)d+v\left[p\left(\frac{v(N+1)}{2}+1\right)d-p\left(\frac{v(N+1)}{2}\right)c\right] & \text{for } v = v(\delta,\pi) \\ p\left(\frac{v(N+1)}{2}+1\right)d+v\left[p\left(\frac{v(N+1)}{2}+1\right)-p\left(\frac{v(N+1)}{2}\right)\right]d & \text{for } v \geq v(\delta,\pi)+1. \end{cases} \tag{10}$$

The the following is shown in the appendix.

Proposition 13 *Suppose that $\triangle U(n_i(g), n_{-i}(g))$ is decreasing in own and third-party links. If $\triangle U(N, \mathbf{N}) \geq r$, the complete network is the unique PWSE network, if $p(1)c < r$, the empty network is the unique PWSE network and if $p(g(1))c \geq r > \triangle U(N, \mathbf{N})$, then the unique symmetric PWSE network has a degree $v(\pi) \in \{1, 2, ..., N-1\}$.*
(ii) if $p(g(1))c \geq r > p(g(N))d$, then $g(\tilde{n})$ is the unique symmetric PWSE network and any asymmetric PWSE network has the dominant group architecture in which the complete component has a degree of at least $\tilde{n}(\pi, p)$;

This proposition establishes that as long as $p(g(1))c \geq r$, there is a unique symmetric PWSE network and that the degree of this network is non-decreasing in λ. The intuition being simple. As the number of links increases cooperation becomes harder to sustain since it has been assumed that the frequency of punishment is more important than the number of punishers, and an increase in λ increases the frequency of punishment. Since the frequency of punishment increases, more links can be created before cooperation is destroyed.

This is

6 Social Capital

The idea of social capital goes back to Hobbes (1651) who says in Leviathan, "to have friends is power". There, he establishes a distinction between an individual's social and political resources and implies that a person's living standard depends on the resources to his disposal. However, the modern definition of social capital is due to the more recent work of two sociologists, Pierre Bordieu and James Coleman. Bordieu defines social capital as "the aggregate of the actual or potential resources which are linked to possession of a durable network of more or less institutionalized relationships of mutual acquaintance or recognition" (Bordieu 1985, p. 248; 1980). He argues that social networks are not a natural given and must be constructed through strategic investments oriented to the institutionalization of group relations that can be used to obtain other benefits. Thus, Bordieu's definition makes clear that social capital is the result of social relationships that allow individuals to get access to resources possessed by their partners or links and the amount and quality of the resources owned by their partners.

Coleman's definition is less precise. He defines social capital by its functions as "as a variety of entities with two elements in common: They all consist of some aspect of social structures, and they facilitate certain actions of actors–whether persons or corporate actors–within the structure" (Coleman 1988a. p. S98)

The main difference between Bordieu and Coleman's definition is that in the latter's definition there is no difference between the resources available per-se and the ability to obtain them by virtue of membership in different social networks. Equating social capital with the resources acquired through it can easily lead to a tautology. Defining social capital by its a posteriori results can only lead to a empty concept. For instance, saying that successful societies posses social capital and unsuccessful societies do not, is defining social capital by its consequences and not by its determinants, and as such, social capital is bound to have no distinct meaning.

According to Portes and Landolt (1996), Coleman's focus only on the positive effects of social capital has resulted in that social capital has become a property of groups and even complete nations, rather than of individuals. Collective social capital, cannot simply be the sum of individual social capital if the latter is a resource available through social networks since the resources that someone claims must come at the expense of others. A good example of the consequences that equation social capital to its consequences is Putman's (1993) definition of social capital. He, by analogy with notions of physical capital and human capital–tools and training that enhance individual productivity–defines social capital as features of social organization, such as networks, norms, and trust, that facilitate coordination and cooperation for mutual benefit.

Social capital, as defined by Coleman and Putman, is more and more seen as a key ingredient in economic development around the world. For instance, studies of the rapidly growing economies of East Asia almost always emphasize the importance of dense social networks. These networks, often based on the extended family or on close-knit ethnic communities like the overseas Chinese, are seen to be responsible for fostering trust, lowering transaction costs, and facilitating information transmission. In fact, studies show that China's extraordinary economic growth over the last decade has depended less on formal institutions than on personal connections to allocate resources efficiently and make contracts self-enforceable. Social capital has also been important in the development of advanced Western economies. Mark Granovetter has pointed out that economic transactions like contracting or job searches are more efficient when they are embedded in social networks. Studies of highly efficient, highly flexible industrial districts emphasize networks of collaboration among workers and small entrepreneurs; e.g; Silicon Valley. The most complete evidence however in support of social capital comes from micro-data drawing on sophistiacated measures of community networks, the nature and extent of civic participation and exchanges among neighbors. In the OECD countries, the most comprehensive finding is that controlling for other key variables, well-connected people are more likely to be housed, healthy, hired and happy.

Social capital, in general, has entered economics as being an under-appreciated and independent factor of production, which in general is considered to be complementary with human capital. The reason being that human capital will amount to little unless the person having it can be in contact to others to inform, correct, assist with and disseminate his work and/or ideas. Much of the interest in social capital by economists has been fueled by definitions that include not only the structure of social networks and social relations, but also behavioral dispositions such as trust, reciprocity, and nonesty and institutional quality measures such as rule of law, contract enforceability, and civil liberties and personal characteristic as social skills, charisma, intelectual skills. Examples of behavioral dispositions are LaProta et al (1997) and Knack and Keefer (1997). The former reports a positive correlation of some measure of trust and judicial efficiency and the latter between the same measure of trust and growth rate [10]. An example of personal characteristic is Glaser et al.

[10] See Durlauf (2002) for a critique of the econometric methods used to identify the positive effects of social capital.

(2002) who define social capital as a person's social characteristics–including social skills, charisma, and of his Rodolex– which enables him to reap market and non-market returns from interactions with others[11]. As such they claim that social capital can be seen as the social component of human capital. Thus, economists studying the consequences of social capital have adopted definitions of social capital that make no difference between the resources available per-se and the ability to obtain them by virtue of membership in different social networks and that mix its consequences and its determinants.

A causal definition of social capital is needed on two grounds: (i) first to distinguish social capital from other forms of capital so that social capital can have a distinct meaning, and (ii) to implement empirical strategies capable of identifying the empirical benefits and "costs" of social capital.

In our view the results in this paper provide a framework that allows us to proposed a definition capable of providing social capital with a distinct conceptual and empirical meaning. Before doing so is important to emphasize the minimum conditions that a definition of social capital capable of achieving above goals should satisfy.

First, as suggested by Portes and Landolt (1996) already is important that any definition of social capital focuses on its sources rather than its consequences–that is on what social capital is and not in what it does. This immediately eliminates any behavioral disposition as trust or reputation from the definition of social capital. Trust or reputation from an economic point of view is more frequent understood as the outcome of repeated interactions in which individuals follow self-enforcing behavior that creates a reputation of taking certain actions. Second, social capital cannot be defined over an aggregate it has to be defined at the individual level, and this must obtained through personal costly investments oriented to the institutionalization of group relations that can be used to obtain other benefits–that is social capital cannot be thought of as a natural given.

Given these two conditions, social capital is defined as follows: *An individual social capital refers to the number and type of costly links formed by mutual consent with the goal of achieving certain personal ends that cannot be achieved in the absence of them.*

This definition makes clear that social networks are costly to form and maintain. That is, individuals has to spend resources–being those time, effort, physical resources– to form and maintain links. Second, while belonging to a given social network is a necessary condition for having access to the potential resources or benefits that a network may create, it is far from being a sufficient condition. To get large benefits from being part of a social network requires certain recognition or reputation within the network. Those lacking that recognition are unlikely to get the extra benefits from being part of a social network and some network architectures do not provide enough incentives to invest in developing that reputation or recognition. In our specific context, the reputation or recognition of an indiviudual is an attributed or characteristic ascribed to him by his partners, and the empirical basis of an individual's reputation is his observed past behavior, and not the participation on the netwrok per-se. Third, sometimes the same ties that members of a social network have and help them to obtain extra benefits exclude others from these benefits. One example of this is the tight control exercised by descendants of Iatalians, Irish, and Polish immigrants over construction trades and fire and police unions in New York. Another example is Adam Smith's complain that assemblages of merchants inevitably end up as "conspiracies against the public", where the public are all those excluded from the networks and mutual support linking insiders. Fourth, social capital is not defined by its outcome and much less by whether this outcome is pareto-efficient. This avoids

[11] This definition however does not suffer from Portes and Landolt's critique that social capital cannot be thought of as a community based attribute.

the tautology of saying that successful people (nations) succeed, which result from confusing social capital with the benefits of it. Last but not least, this definition also emphasizes that person's social capital is not only given by the number of own links, but also to the type of links since not all links have the same value. This value depends on the structural characteristics of the relations and in fact an indiscriminate increase in the number of friends, links or social contacts may lead to the destruction of social capital. Thus, the common aphorism "It is not what you know, it's who you know" describe quite well what we believe is a good working definition of social capital.

7 Conclusions

In this paper we have shown that Granovetter's inside that economic transactions are embedded in ongoing social relationships has bite on the kind of transactions that can be achieved and the benefits and costs of different transactions. While this conclusion may seem relatively obvious, until recently economics was not much willing to incorporate this kind of analysis to standard economic modeling. Thus, as simple as this conclusion may be it is not irrelevant since it may help to understand real life phenomena that otherwise would not be possible.

Indeed, it allows as to come with what we believe is a definition of social capital capable of distinguishing social capital from other forms of capital so that social capital can have a distinct meaning, and of allowing empirical strategies capable of identifying the empirical benefits and "costs" of social capital.

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A Appendix

Proof. of proposition 5.

Cases (i) and (iii) are straightforward.

(ii) Suppose that $g = g(N)$. Then player i's payoff from this network is $U_i(g(N)) = N(pc - r)$. Adding links is not possible. Suppose then now that player i deletes n links. Doing so it yields a payoff equal to $(N - n)(pu - r)$, where $u = c$ if $n \leq N - \tilde{n}(\pi, p)$ and $u = d$ otherwise. By simple observation it readily follows that this payoff is lower than $U_i(g(N))$ for all $n \leq N - \tilde{n}(\pi, p)$. Thus, deleting $n \leq N - \tilde{n}(\pi, p)$ number of links it is never optimal when the other players choose to form links with everyone else. What about deleting $n > N - \tilde{n}(\pi, p)$ links if possible. Then, the difference between $U_i(g(N))$ and the payoff when n links are deleted is given by:

$$\begin{aligned}&Np(c-d) + n(pd - r) \geq \\ &Np(c-d) + N(pd - r) = \\ &N(pc - r) > 0.\end{aligned}$$

Thus, deleting $n > N - \tilde{n}(\pi, p)$ links is never optimal. This implies that $g(N)$ is a PWSE network since adding more links is impossible and deleting any number of links is never optimal.

Consider an asymmetric network g. Let $N_0(g)$, $N_1(g)$,...,$N_{\tilde{n}}(g)$, $N_{\tilde{n}+1}(g)$,...,$N_N(g)$ be a partition of players corresponding to the number of links that players have, i.e., $i, j \in N_k(g)$, $k = 1, 2, .., N$ if and only if $n_i(g) = n_j(g)$, where k refers to the exact number of links that players in partition k have. Let also denote by $N_{\tilde{n}}(g)$ the partition conformed by all players that have exactly $\tilde{n}(\pi, p)$ links. By lemma 3, $pc \geq r$ implies that all players belonging to a partition with $k \geq \tilde{n}(\pi, p) - 1$ links must be mutually linked. Take now any player $i \in N_k(g)$ with $k < \tilde{n}(\pi, p) - 1$ and $j \in N_{k'}(g)$, $k' = 1, 2, .., N$, and suppose that $g_{ij} = 1$. Then player i has an incentive to delete this link since by doing so it saves r and looses pd and $pd - r < 0$. Thus, the only asymmetric PWSE networks are those in which there are two components, one in which players have no links and one in which all players are mutually linked and therefore each has the same number of links $k \geq \tilde{n}(\pi, p)$. That is, any asymmetric PWSE network has the dominant group architecture with a complete component with a degree of at least $\tilde{n}(\pi, p)$. ■

Proof. of proposition 8.

(i) If $pd \geq r$, there are three different cases to consider: (i) n_L is lower than $\tilde{n}(\pi, p)$ and $pc \leq (2\tilde{n}(\pi, p) - 1)r$; (ii) n_L is lower than $\tilde{n}(\pi, p)$ and $pc > (2\tilde{n}(\pi, p) - 1)r$; and (iii) n_L is larger than or equal to $\tilde{n}(\pi, p)$.

Case (a) follows directly from lemma 3.

Case (b). Notice that in this case the marginal cost is below the marginal gross return up to n_L, is above the marginal gross return between n_L and $\tilde{n}(\pi, p)$, below the marginal gross return between $\tilde{n}(\pi, p)$ and n_H and above again for all $n_i(g) > n_H$. We consider first the symmetric networks and then the asymmetric ones.

Consider the symmetric network $g(n_H)$. Notice that player i's payoff from this network is given by $U_i(g(n_H)) = n_H(pc - rn_H)$ for all i. Adding a link yields $pc - r(2n_H + 1)$ which by definition of n_H is negative. What about deleting links Suppose that player i deletes $n \leq n_H - \tilde{n}(\pi, p)$, then his payoff from this strategy is $(n_H - n)(pc - r(n_H - n))$, which is lower than $U_i(g(n_H))$. Thus, it is never optimal to delete $n \leq n_H - \tilde{n}(\pi, p)$ links. What if he deletes $n_H - n_L \geq n > n_H - \tilde{n}(\pi, p)$. Notice that by lemma 3 it is never optimal to delete more than $n_H - n_L$. In this case player i has two types of links, those with n_H links and those with $n_H - 1$ links. Since his number of links now becomes lower than $\tilde{n}(\pi,p)$, he is not willing to cooperate with any of his partners and therefore he is better-off deleting $n_H - n_L$ links since all links in this range now yield a negative marginal net return. The payoff in this case becomes $n_L(pd - rn_L)$, which is lower than $U_i(g(n_H))$. Thus, it is never optimal to delete $n_H - n_L \geq n > n_H - \tilde{n}(\pi,p)$ number of own links. Thus $g(n_H)$ is a PWSE network.

Consider now the symmetric network $g(n_L)$. By lemma 3 it is never optimal to delete any number of links. Suppose now that i and h choose to form a link. Then the net payoff from this extra link is $pd - r(2n_L + 1)$ if $n_L < \tilde{n}(\pi,p) - 1$ and $pc - r(2n_L + 1)$ if $n_L = \tilde{n}(\pi,p) - 1$. In the former case the marginal net return is negative while in the latter is positive. Thus $g(n_L)$ is a PWSE network if and only if $n_L < \tilde{n}(\pi,p) - 1$. This also implies together with lemma 3 that if $n_L = \tilde{n}(\pi,p) - 1$, the only PWSE network is $g(n_H)$.

Consider now an asymmetric network g. By lemma 3, every player forms at least n_L number of links and at most n_H since adding more links yields a negative net marginal benefit. Let $N_{n_L}(g)$, $N_{n_L+1}(g)$,...,$N_{\tilde{n}}(g)$, $N_{\tilde{n}+1}(g)$,...,$N_{n_H}(g)$ be a partition of players corresponding to the number of links that players have, i.e., $i, j \in N_k(g)$, $k = 1, 2, .., N$ if and only if $n_i(g) = n_j(g)$, where k refers to the exact number of links that players in partition k have. Because $pc \geq \max\{(2n_i(g) - 1)r, (2n_j(g) - 1)r\}$ for every player $i \in N_k(g)$ with $k \geq \tilde{n}(\pi,p) - 1$ and $j \in N_{k'}(g)$ with $k' \geq \tilde{n}(\pi,p) - 1$, lemma 3 implies that i and j must be linked. Thus, all players with a number of own links $k \geq \tilde{n}(\pi,p) - 1$ must be mutually linked. Take now any player $i \in N_k(g)$ with $k < \tilde{n}(\pi,p) - 1$ and $j \in N_{k'}(g)$, $k' = n_L, n_L + 1, .., n_H$, then $g_{ij} = 0$ since the ij link yields $pd - (2n_i(g) - 1)r < 0$ to player i. Thus, the only asymmetric PWSE networks are those in which there are two components, one in which players have n_L links and the other one given by a complete component in which all players have the same number of links $n_H \geq k \geq \tilde{n}(\pi,p)$. That is, any asymmetric PWSE network has the exclusive groups architectures with two complete components: one with a degree of at least $\tilde{n}(\pi,p)$ and at most n_H and the other with degree $n_{L.}$

Case (c) By lemma 3, each player has an incentive to form all those links which yield a positive net marginal return. This implies that each player forms n_H and this is the unique PWSE network.

(ii) Suppose now that $pc \geq r > pd$. The symmetric network $g(n_H)$ is a PWSE network and the proof is the same as the one above. The empty network is also a PWSE since deleting links is not possible and adding a link yields a net marginal return of $pd - r < 0$. Thus the symmetric networks $g(n_H)$ and $g(0)$ are the unique PWSE networks.

Any asymmetric PWSE network has the dominant group architecture with a complete component with a degree of at least $\tilde{n}(\pi,p)$ and the rest are isolated players. The proof is ommited since it is the same as above but now the partition of g allows players with a number of links lower than n_L.

(iii) If $pc < r$, the the marginal gross return is always smaller than the marginal cost of adding a link, and therefore the empty network is the unique PWSE network. ■

Proof. of proposition 11.

Cases (i) and (v) are straightforward.

(ii) Suppose that $p(d + \theta) \geq r > pd$. Because $p(d + \theta) > r$ all good links are formed. Consider the complete network first. Player i's payoff from $g(N)$ is given by $U_i(g(N)) = N(pc - r) + N_g p\theta$. Suppose now that player i deviates and chooses to delete n bad links. A player never deletes a good link since $p(d + \theta) > r$. Deleting $n \leq N - N_g$ bad links yields a payoff equal to $(N - n)(pu - r) + pN_g\theta$, where $u = c$ if $n \leq N - \tilde{n}(\pi,p)$ and $u = d$ otherwise. By simple observation it readily follows that this payoff is lower than $U_i(g(N))$ for all $n \leq N - \tilde{n}(\pi,p)$. Thus, deleting $n \leq N - \tilde{n}(\pi,p)$ number of links it is never optimal when the other players choose to form links with everyone else.

What about deleting $N - N_g \geq n > N - \tilde{n}(\pi, p)$ bad links if possible. Then, the difference between $U_i(g(N))$ and the payoff when n bad links are deleted is given by:

$$\begin{aligned}
&Np(c-d) + n(pd-r) \geq \\
&Np(c-d) + N(pd-r) = \\
&N(pc-r) > 0.
\end{aligned}$$

Thus, deleting $N - N_g \geq n > N - \tilde{n}(\pi, p)$ bad links is never optimal. This implies that $g(N)$ is a PWSE network since adding more links is impossible.

Consider now the symmetric network in which all best links are formed, denoted by $g(N_g)$. If $N_g < \tilde{n}(\pi, p) - 1$, the network $g(N_g)$ is a PWSE equilibrium network. Deleting links is never optimal because of lemma 9. Furthermore, no pair has an incentive to add a link, since adding a link yields a marginal net return of $pd - r < 0$. Whereas if $N_g \geq \tilde{n}(\pi, p) - 1$, any two players i and j with $g_{ij} = 0$, has an incentive to form that link since it yields a marginal net return of $pc - r > 0$. Thus, $g(N_g)$ is a PWSE network if and only if $N_g < \tilde{n}(\pi, p) - 1$.

It follows from 9 that no other symmetric network can be a PWSE network.

Consider now an asymmetric network g. By lemma 9 in a PWSE network g, each player has at least $n_i(g) \geq N_g$ number of links and if $N_g \geq \tilde{n}(\pi, p) - 1$, the only PWSE network is the complete network since any two players i and j has incentive to deviate and form a new link because the new link yields $pc - r > 0$. Suppose then that $N_g < \tilde{n}(\pi, p) - 1$ and let $N_{N_g}(g)$, $N_{N_g+1}(g)$,...,$N_{\tilde{n}}(g)$, $N_{\tilde{n}+1}(g)$,...,$N_N(g)$ be a partition of players corresponding to the number of links that players have, i.e., $i, j \in N_k(g)$, $k = N_g, N_g + 1, .., N$ if and only if $n_i(g) = n_j(g)$, where k refers to the exact number of links that players in partition k have. Because $pc > r$ every player $i \in N_k(g)$ with $k \geq \tilde{n}(\pi, p) - 1$ must be linked to all other players $j \in N_{k'}(g)$ with $k' \geq \tilde{n}(\pi, p) - 1$. Take now any player $i \in N_k(g)$ with $k < \tilde{n}(\pi, p) - 1$ and $j \in N_{k'}(g)$, $k' = N_g, N_g + 1, .., N$, and suppose that $I_{ij} = 0$ and $g_{ij} = 1$. Then player i has an incentive to delete the ij link since it yields $pd - r < 0$. Thus, this implies that in any PWSE asymmetric network every one has at least N_g links and that all players that have a number of links $k \geq \tilde{n}(\pi, p)$ must be mutually linked. Thus, if $N_g < \tilde{n}(\pi, p) - 1$ an asymmetric PWSE network is one in which there are at most two components: a fully connected component in which each player has N_g good links and another fully connected component that has $M \geq \tilde{n}(\pi, p) \geq 2$ players..

(iii) $pc \geq r > p(d + \theta)$. As in case (ii) the complete network is PWSE network and the proof is the same. But now the empty network is also a PWSE network. The proof being trivial. Now however the network in which only all good matches are formed is not a PWSE network. Suppose that $N_g < \tilde{n}(\pi, p) - 1$ and that $g(N_g)$ is a PWSE network. Then any player deleting a link looses $p(d + \theta)$ and saves r. Since $p(d + \theta) < r$, the deviating player is better-off. Suppose now that $N_g \geq \tilde{n}(\pi, p) - 1$ and that $g_{ih} = 0$. Then if players i and h deviate and form a link, each will have $\tilde{n}(\pi, p)$ connections and therefore cooperation becomes self-sustainable for them. Forming the link then yields to each player a net marginal benefit of $pd + p(c - d)c_i(g + ih) - r \geq 0$ and therefore it is worthwhile to form that link. Thus $g(N_g)$ cannot be a PWSE network.

Consider now an asymmetric network g. Let $N_0(g)$, $N_1(g)$,...,$N_{\tilde{n}}(g)$, $N_{\tilde{n}+1}(g)$,...,$N_N(g)$ be a partition of players corresponding to the number of links that players have, i.e., $i, j \in N_k(g)$, $k = 0, 1, .., N$ if and only if $n_i(g) = n_j(g)$, where k refers to the exact number of links that players in partition k have. Because $pc \geq r$ every player $i \in N_k(g)$ with $k \geq \tilde{n}(\pi, p) - 1$ must be linked to every player $j \in N_{k'}(g)$ with $k' \geq \tilde{n}(\pi, p) - 1$. Take now any player $i \in N_k(g)$ with $k < \tilde{n}(\pi, p) - 1$ and $j \in N_{k'}(g)$, $k' = 1, 2, .., N$, and suppose that $g_{ij} = 1$, then player i deletes the link since it yields $p(d+I_{ij}\theta)-r<0$. Thus, this implies that any PWSE asymmetric network has a complete component in which each player has $k\geq\tilde{n}(\pi,p)$ number of own links and the rest are isolated players. Thus, any asymmetric PWSE network has the dominant group architecture.

(iv) $p(c+\theta)\geq r>pc$. If $N_g\geq\tilde{n}(\pi,p)$, then $g(N_g)$ and the empty network are PWSE networks. Player i's payoff from $g(N_g)$ is $U_i(g(N_g))=N_g(p(c+\theta)-r)$. If any two disconnected players form a link, each gets a net payoff of $pc-r<0$, and therefore no pair of players have an incentive to add a link. Furthermore no player i has an incentive to delete links because that eliminates links that are profitable and may, depending of how many links are deleted, destroy cooperation between i and all his partners.

Consider now the case in which $N_g\leq\tilde{n}(\pi,p)-1$ and the symmetric network $g(\tilde{n}(\pi,p))$. Player i's payoff from this network is $U_i(g(\tilde{n}(\pi,p)))=\tilde{n}(\pi,p)(pc-r)+N_gp\theta$. Adding a link is never optimal since it yields $pc-r<0$. If player i deletes a link he destroys cooperation with all his partners since $n_i(g)$ becomes lower than $\tilde{n}(\pi,p)$, then if he is willing to delete a link, he is better-off deleting all links since $p(d+\theta)<0$. His payoff from deleting all his links is 0. So, if $U_i(g(\tilde{n}(\pi,p)))\geq 0$, it is not optimal for player i to delete any number of links. This condition implies that $\tilde{n}(\pi,p)(pc-r)+N_gp\theta\geq 0$ or that $N_g\geq\frac{\tilde{n}(\pi,p)(r-pc)}{p\theta}>0$, where $\frac{(r-pc)}{p\theta}<1$. Thus, $g(\tilde{n}(\pi,p))$ is a PWSE network if $N_g\geq\frac{\tilde{n}(\pi,p)(r-pc)}{p\theta}$.

Consider now any asymmetric network g. Let $N_0(g)$, $N_1(g)$,...,$N_{\tilde{n}}(g)$, $N_{\tilde{n}+1}(g)$,...,$N_N(g)$ be a partition of players corresponding to the number of links that players have. Because $p(c+\theta)\geq r$ every player $i\in N_k(g)$ with $k\geq\tilde{n}(\pi,p)-1$ must be linked to every player $j\in N_{k'}(g)$ with $k'\geq\tilde{n}(\pi,p)-1$ when the ij link is a good link. Take now any player $i\in N_k(g)$ with $k<\tilde{n}(\pi,p)-1$ and $j\in N_{k'}(g)$, $k'=1,2,..,N$, with $g_{ij}=1$. Then player i deletes the ij link since it yields $p(d+I_{ij}\theta)-r<0$. Next, consider player $i\in N_k(g)$ with $k\geq\tilde{n}(\pi,p)$ and player $j\in N_{k'}(g)$ with $k'\geq\tilde{n}(\pi,p)$ with $I_{ij}=0$ and $g_{ij}=1$. If $k>\tilde{n}(\pi,p)$, then i deletes the ij since it yields $pc-r<0$ and does not kill cooperation between i and his other partners. If $k=\tilde{n}(\pi,p)$, deleting the ij link, kills cooperationg with all his partners, and since $p(d+I_{ij}\theta)-r<0$, if he deletes the ij link he is better-off deleting all his links. Thus, it is optimal not to delete any number of own links as long as $N_g\geq\frac{\tilde{n}(\pi,p)(r-pc)}{p\theta}$, otherwise i deletes all his links. So far, this implies that in any asymmetric PWSE network there is one component in which each individuals has at least $k\geq\tilde{n}(\pi,p)$ links.

If $N_g\geq\tilde{n}(\pi,p)$, then any asymmetric network has complete component in which each player has a number of links equal to N_g, and the rest are isolated players is a PWSE network. While if $\frac{\tilde{n}(\pi,p)(r-pc)}{p\theta}\leq N_g\leq\tilde{n}(\pi,p)-1$, an asymmetric network with one component in which each player has $\tilde{n}(\pi,p)$ links and the rest are isolated players is a PWSE network. If $N_g<\frac{\tilde{n}(\pi,p)(r-pc)}{p\theta}$, there is no asymmetric PWSE network. ■

Proof. of proposition 12

Cases (i) and (v) are straightforward.

(ii) $pc\geq r>pd$. The proof is the same as the one in case (iv) in proposition 11..

(iii) if $p(d+\theta)\geq r>pc$. By lemma 9, all good links are formed. If $N_g\geq\tilde{n}(\pi,p)$, then $g(N_g)$ is the unique PWSE network since no one has incentive either to add a link or to delete one. If $N_g\leq\tilde{n}(\pi,p)-1$, then $g(\tilde{n}(\pi,p))$ is a PWSE network if and only if $U_i(g(\tilde{n}(\pi,p)))\geq U_i(g(N_g))$ for all i because adding a link yields $pc-r<0$ and deleting a link kills cooperation and therefore, if it is worthile to delete one bad link it is optimal to delete all the bad links since each yield a negative net marginal payoff. Thus, $g(\tilde{n}(\pi,p))$ is a PWSE network if and only if $\tilde{n}(\pi,p)(pc-r)+N_gp\theta\geq N_g(p(d+\theta)-r)$; that is, $N_g\geq\tilde{n}(\pi,p)\frac{r-pc}{r-pd}$. Otherwise, $g(N_g)$ is a

PWSE network if $N_g < \tilde{n}(\pi,p)-1$ or $N_g = \tilde{n}(\pi,p)-1$, but $N_g < \frac{r-pc}{p(c-d)}$.

Consider now an asymmetric network g. By lemma 9 in a PWSE network g, each player has at least $n_i(g) \geq N_g$ number of links and if $N_g \geq \tilde{n}(\pi,p)$, then $g(N_g)$ is the unique PWSE network. Suppose then that $N_g \leq \tilde{n}(\pi,p)-1$ and let $N_{N_g}(g)$, $N_{N_g+1}(g)$,...,$N_{\tilde{n}}(g)$, $N_{\tilde{n}+1}(g)$,...,$N_N(g)$ be a partition of players corresponding to the number of links that players have. Because $p(c+\theta) \geq r$ every player $i \in N_k(g)$ with $k \geq \tilde{n}(\pi,p)-1$ must be linked to all other players $j \in N_{k'}(g)$ with $k' \geq \tilde{n}(\pi,p)-1$ if the ij link is a good link, while if $I_{ij}=0$ and $g_{ij}=1$, $k=\tilde{n}(\pi,p)-1$ and $k' \geq \tilde{n}(\pi,p)-1$, i deletes the links since it yields $pd-r<0$. Take now any ij link where $i \in N_k(g)$ with $k < \tilde{n}(\pi,p)-1$, $j \in N_{k'}(g)$, $k' = N_g, N_g+1,..,N$, $I_{ij}=0$ and $g_{ij}=1$. Then player i deletes this link since it yields $pd-r<0$. Next, consider player $i \in N_k(g)$ with $k \geq \tilde{n}(\pi,p)$ and player $j \in N_{k'}(g)$ with $k' \geq \tilde{n}(\pi,p)$ with $I_{ij}=0$ and $g_{ij}=0$, then i and j has no incentive to add the link ij since adding that links yields a net marginal benefit equal to $pc-r<0$. So far, this implies that in any asymmetric PWSE network a component given by the good links with the number of links $k \geq \tilde{n}(\pi,p)$ must be formed.

Finally, consider a player $i \in N_k(g)$ with $k = \tilde{n}(\pi,p)-1$ and a player $j \in N_{k'}(g)$ with $k' = \tilde{n}(\pi,p)-1$ with $I_{ij}=0$, then ij has an incentive to add the link when the net marginal benefit is positive. That is, when $pc + N_g p(c-d) + (\tilde{n}(\pi,p)-1-N_g)p(c-d) - r \geq 0$. Thus, the ij link is formed if and only if $\tilde{n}(\pi,p) \geq \frac{r-pd}{p(c-d)}$.

Take now any player $i \in N_k(g)$ with $k < \tilde{n}(\pi,p)-1$ and $j \in N_{k'}(g)$, $k' = 1,2,..,N$, and suppose that $I_{ij}=0$ and $g_{ij}=1$, then player i deletes the link since it yields $pd-r<0$. Thus, this implies that any PWSE asymmetric network has a complete component given by all players that have $k \geq \tilde{n}(\pi,p)$ and all the players in that component has the same number of links. Thus, if $N_g < \tilde{n}(\pi,p)-1$ the only candidate for an asymmetric PWSE network is the one in which there are two components: one in which all players have N_g links and the other is the complete component given by all players with $\tilde{n}(\pi,p)$ good links and the rest $N_g - \tilde{n}(\pi,p)$ are bad links if $N_g \geq \tilde{n}(\pi,p)$.

(iv) $p(c+\theta) \geq r > p(d+\theta)$. The proof is the same as the one in case (iv) in proposition 11. ■

Proof. of proposition 13.

Let g be a symmetric network of degree v. If $r > \triangle U(1,\mathbf{0}) = p(1)c$, then the empty network is clearly a PWSE network. If $\triangle U(N,\mathbf{N}) \geq r$, then the complete network is clearly a PWSE network. No more links can be added and NSOL implies that $\triangle U(v,\mathbf{N}-\mathbf{1}) > \triangle U(N-1,\mathbf{N}-\mathbf{1}) \geq r$ for all $v \leq N-2$. Hence no player has any incentive to delete any number of links. Finally, suppose that $\triangle U(v,\mathbf{v}) < r \leq \triangle U(v-1,\mathbf{v}-\mathbf{1})$ for some $v \in \{1,2,..n-2\}$. Because $\triangle U(v,\mathbf{v}) < r$, no player has an incentive to add a link, while NSOL implies that $\triangle U(l,\mathbf{v}-\mathbf{1}) > \triangle U(v-1,\mathbf{v}-\mathbf{1})$ for all $l \in \{1,2,..v-2\}$, which implies that no player has an incentive to delete links either. Thus a network of degree v is a PWSE network. The uniqueness follows from that $\triangle U(v,\mathbf{v}) < r \leq \triangle U(v-1,\mathbf{v}-\mathbf{1})$ and $\triangle U(v',\mathbf{v}') < r \leq \triangle U(v'-1,\mathbf{v}'-\mathbf{1})$ cannot be simultenously satisfied for v and v', with $v > v'$.

Consider any asymmetric PWSE network g. Let $N_1(g)$, $N_2(g)$,...,$N_N(g)$ be a partition of players corresponding to the number of links that players have, i.e., $i,j \in N_k(g)$, $k=1,2,..,N$ if and only if $n_i(g) = n_j(g)$, where k refers to the exact number of links that players in partition k have. Take any two players $i \in N_k(g)$ and $h \in N_{k'}(g)$ with $\delta \geq \max\{\delta(N(g),n_i(g)),\delta(N(g),n_h(g))\}$ and $g_{ih}=1$. Then all players with degrees smaller than $\min\{k,k'\}$ must be mutually linked. This follows from that if $\min\{\triangle U(n_i(g),n_{-i}(g)),\triangle U(n_h(g),n_{-h}(g))\} \geq r$, then it must be true for any player $j \in N_k(g)$ with $k \leq \min\{k,k'\}$ and player $l \in N_{k'}(g)$ with $k' \leq \min\{k,k'\}$ that $\min\{\triangle U(n_k(g),n_{-k}(g)),\triangle U(n_l(g),n_{-l}(g))\} \geq r$. Likewise if any two players do not have a link then all players with more links must not be linked to each other because of NSOL and NSTP. This rules out inter-linked stars with two or more central players but it allows stars to arise in equilibrium.

Consider now any dominant group architecture with two or more isolated players.

implies that every player $i \in N_k(g)$ with $k \geq \tilde{n}(\pi, p) - 1$ must be linked to all other players $j \in N_k(g)$ with $k \geq \tilde{n}(\pi, p) - 1$. Take now any player $i \in N_k(g)$ with $k < \tilde{n}(\pi, p) - 1$ and $j \in N_k(g)$, $k = 1, 2, .., N$, and suppose that $g_{ij} = 1$. Then player i has an incentive to delete this link since by doing so it saves r and looses pd and $pd - r < 0$. Thus, the only asymmetric PWSE networks are those in which there are two components, one in which players have no links and the other one given by a complete component in which all players have the same number of links $k \geq \tilde{n}(\pi, p)$. That is, any asymmetric PWSE network has the dominant group architecture with a complete component with a degree of at least $\tilde{n}(\pi, p)$ ■