The Formation of Networks with Side-Payments among Players

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Draft: January 26, 2004
Preliminary and Incomplete - please do not circulate

Abstract

We examine the formation of networks among a set of players whose payoffs depend on the structure of the network. We focus on games where players may promise or demand transfer payments when forming links. If players may only make such transfers on the links they are directly involved with, then there are many settings where inefficient networks are the only equilibrium outcomes, and we fully characterize the supportable networks. If externalities are nonpositive and a convexity condition is satisfied, then efficient networks are supportable as equilibria with such direct transfers. If players can also make positive transfers to pay for links they are not involved with, then a convexity condition alone is sufficient for an efficient network to be supportable as an equilibrium. In cases where transfers can be made contingent on the network, then any efficient network is supportable as an equilibrium. We also consider a refinement of equilibrium that allows pairs of players to coordinate their promises and demands on a link. If players can make payments to prevent the formation of a link as well as to form it, then all efficient networks are supportable via the pairwise equilibrium refinement.

JEL Classification Numbers: A14, C71, C72

Keywords: Networks, Network Games, Network Formation, Game Theory, Efficient Networks, Side Payments, Transfers

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1 Introduction

Many social, economic, and political relationships may be thought of as a network of bilateral relationships. This ranges from friendships to trading relationships to political alliances. As the structure of the network of relationships can have a profound impact on the welfare of all the involved parties, it is essential to develop a good understanding of which networks are likely to form and how this depends on the specifics of the circumstances. This paper contributes to a growing literature that models network formation.[1]

Our purpose in what follows is twofold. First, we wish to provide models that will help us to study network formation when the players involved can bargain on possible transfers at the time of forming relationships. For example, when two airlines form a code-sharing agreement, included in that agreement are details on how to split the costs and revenues on cross-booked passengers. Without some transfer payments (in currency or in kind), many such agreements would never exist. This is true in wide variety of network settings, and so we develop models where the formation of links and the agreement over transfers are agreed to at the same time. Second, we wish to understand how the formation of networks depends on the types of transfers that are allowed. For instance, in many settings one player's welfare might depend on how other players are linked. When is it important that a player can make transfers that are contingent on the relationships that another player has? When is it important for players to be able to subsidize relationships between other players? As the formation process and the types of transfers that might be feasible varies with the setting, having an idea of how network formation depends on these aspects of the formation process will help us understand which networks we should expect to see.

This paper fits into a recent literature that examines network formation when players act in their own interest and their payoffs may depend on the structure of the network. In such contexts, Jackson and Wolinsky (1996) showed that the networks that maximize society's overall payoff will often not be stable in an equilibrium sense, regardless of how payoffs are re-allocated (subject to some basic conditions). Their perspective was one where the allocation of payoffs is already set as a function of the network when players decide which links to form. They showed that were simple settings such that, for any allocation of payoffs satisfying an anonymity and component balance condition, the networks that were formed by individuals would fail to maximize overall societal welfare. While they also showed that there are a variety of settings where link formation will lead to efficient networks, their result is still bothersome in that even the ability to reallocate payoffs fairly widely does not overcome difficulties with network externalities (even with complete information).

In recent papers, Currarini and Morelli (2001) and Mutuswami and Winter (2002) show that, at least in some settings, the difficulty of efficient network formation can be overcome if players bargain over the allocation of payoffs at the same time as the network forms. They model network formation as a sequential process where players move in turn and announce the payoff that they demand and the links that they are willing form. The network that forms as a function of the announcements is the largest one such that the demands are

[1]See Jackson (2003b) for a survey of the literature that is most closely related to our work here.

compatible with the value that is generated. They show that the equilibria of such games are efficient networks, assuming that there are no externalities across network components and that some payoff conditions are satisfied. Part of the intuition is that by moving in sequence and making such take it or leave it demands, players can extract their marginal contribution to an efficient network, and this provides correct incentives.

These papers make an important point that the ability to determine payoffs in conjunction with link formation may aid in the emergence of efficient networks. However, these sequential games have special features and are better for illustrating the importance of taking such bargaining seriously, than for providing reasonable models of network formation. In particular, the end-gaming and finite extensive forms drive the results. Moreover, Currarini and Morelli (2001) and Mutuswami and Winter (2002) provide some conditions for the support of efficient networks, but do not give us much of a feel for how generally this might hold, or how this depends on the structure of the process.

Here, we take the important message from these papers that a simultaneous determination of payoffs and link structure can be a critical determinant of which networks form. However, we take a very different approach to modeling these issues, both in the class of games that we consider, and the types of questions that we ask. First, in order to find games that have some robust structure to them, we look at simultaneous move games. While arguably network formation is generally far from simultaneous, these games have a critical feature that is absent from the finite extensive forms. In particular, once the network has formed, no player(s) would benefit from changing their promised or demanded transfers and the links that they form. If one added further stages to the finite extensive forms and allowed players to come back to revisit their actions, the outcomes would change dramatically. As when whatever network formation process reaches some sort of stable point, it should be that no player could gain from some further deviation, the simultaneous move games capture this quite directly. Second, we try to develop a fuller characterization of the networks that can be supported and how this varies with the structure of the externalities and players' payoffs. Third, we tie the type of transfer that is available in the game to the type of networks that are supportable. We compare the networks that emerge when players can only make payments with regards to links that they are involved with, to those networks that emerge when players can subsidize the formation of links that they are not directly involved with. We also study how the ability to make payments contingent on the whole architecture of the network affects network formation. The support of efficient networks ties back to these variations in game structure in intuitive ways relating to the type of externalities present, and the convexity (or lack thereof) of the payoffs. As different applications tend to have different availability of transfers, understanding how outcomes change with transfer structure is vital.

2 Modeling Networks

Players and Networks

The set $N = \{1, \ldots, n\}$ is the set of players.[2]

A network g is a list of which pairs of players are linked to each other. A network is then a list of unordered pairs of players $\{i, j\}$.

For any pair of players i and j, $\{i, j\} \in g$ indicates that i and j are linked under the network g.

For simplicity, we write ij to represent the link $\{i, j\}$, and so $ij \in g$ indicates that i and j are linked under the network g.

For instance, if $N = \{1, 2, 3\}$ then $g = \{12, 23\}$ is the network where there is a link between players 1 and 2, a link between players 2 and 3, but no link between players 1 and 3.

Let g^N be the set of all subsets of N of size 2. The network g^N is referred to as the "complete" network.

$G = \{g \subset g^N\}$ denotes the set of all possible networks on N.

For any network $g \in G$, let $N(g)$ be the set of players who have at least one link in the network g. That is, $N(g) = \{i \mid \exists j \text{ s.t. } ij \in g\}$.

Given a player $i \in N$ and a network $g \in G$, let $L_i(g)$ denote the set of links in g involving player i, $L_i(g) = \{jk \in g | j = i \textit{ or } k = i\}$.

Paths and Components

A *path* in a network $g \in G$ between players i and j is a sequence of players $i_1, \ldots, i_K$ such that $i_k i_{k+1} \in g$ for each $k \in \{1, \ldots, K-1\}$, with $i_1 = i$ and $i_K = j$.

Looking at the path relationships in a network naturally partitions a network into different connected subnetworks that are commonly referred to as components.

A *component* of a network g, is a nonempty subnetwork $g' \subset g$, such that

- if $i \in N(g')$ and $j \in N(g')$ where $j \neq i$, then there exists a path in g' between i and j, and
- if $i \in N(g')$ and $ij \in g$, then $ij \in g'$.

Thus, the components of a network are the distinct connected subnetworks of a network.

Utility Functions

The *utility* of a network to player i is given by a function $u_i : G \to I\!R_+$. Let u denote the vector of functions $u = (u_1, \ldots, u_n)$,

We normalize payoffs so that $u_i(\emptyset) = 0$.

For any network $g \in G$ and subset of links $\ell \subset g$, let

$$mu_i(g, \ell) = u_i(g) - u_i(g \setminus \ell)$$

be the marginal utility for player i from the links ℓ relative to the network g.

Externalities

[2] For background and discussion of the model of networks discussed here, see Jackson (2003b).

The class of utility functions that we consider here is arbitrary, and thus considers very general types of externalities. At times, it is useful to talk about specific forms of externalities. To that end, the following definitions of externalities in payoffs are useful.

The profile of utility functions u satisfies *no externalities* if $u_i(g) = u_i(g + jk)$ for all g, $jk \notin g$, and $i \notin jk$.

The profile of utility functions u satisfies *nonpositive externalities* if $u_i(g) \geq u_i(g + jk)$ for all g, $jk \notin g$, and $i \notin jk$.

The profile of utility functions u satisfies *nonnegative externalities* if $u_i(g) \leq u_i(g + jk)$ for all g, $jk \notin g$, $i \notin jk$.

These definitions of externalities are not exhaustive. There are some settings that do not satisfy any of the above cases, as it may be that some links result in positive externalities and others in negative externalities, or the nature of the externality may differ depending on the players in question and/or the starting network. Nevertheless, these definitions capture many settings of interest and prove useful in talking about some interesting special cases in what follows.

Values and Efficiency

Let v^u denote the total *value* as a function of the network, that is

$$v^u(g) = \sum_i u_i(g).$$

A network $g \in G$ *Pareto dominates* a network $g' \in G$ relative to u if $u_i(g) \geq u_i(g')$ for all $i \in N$, with strict inequality for at least one $i \in N$.

A network $g \in G$ is *Pareto efficient* relative to u if it is not Pareto dominated.

A network $g \in G$ is *efficient* relative to u if it maximizes $v^u(g)$.

In a world where transfers are possible, efficiency and Pareto efficiency (allowing for transfers) are equivalent. Thus, our main focus here is on efficient networks.[3]

3 Network Formation Games

We consider several models of network formation where various types of transfers are available, and examine which networks emerge as equilibria and how that depends on the structure of transfers. There are two basic versions of the game that we consider which differ as to whether players can make transfers on links that do not involve them. Later, we also consider variations of these games when transfers can be contingent on the network that forms. Thus, we consider four main variations of network formation games.

[3]For a detailed discussion of various notions of efficient networks in the presence of transfers, see Jackson (2003a).

The Direct Transfer Network Formation Game

Every player $i \in N$ announces a vector of transfers $t^i \in I\!R^{n-1}$. We denote the entries in this vector by t^i_{ij}, across the $n-1$ potential links that involve i. The announcements are simultaneous.

Link ij is formed if and only if $t^i_{ij} + t^j_{ij} \geq 0$. Formally, the network that forms as a function of the profile of announced vectors of transfers $t = (t^1, \ldots, t^n)$ is

$$g(t) = \{ij \mid t^i_{ij} + t^j_{ij} \geq 0\}$$

In this formation game, player i's payoff is

$$u_i(g(t)) - \sum_{ij \in g(t)} t^i_{ij}.$$

The interpretation is that a player announces a transfer for each possible link that he or she might form. If the transfer is positive, then it represents the amount that the player promises to pay to the other player involved in that link. If the transfer is negative, then it represents the amount that the player must receive in order to be willing to form the given link.

It is conceivable in this game that $t^i_{ij} + t^j_{ij} > 0$. Here, we hold both players to their promises. For instance, if $t^i_{ij} > -t^j_{ij} > 0$, then player i ends up making a bigger payment than player j demanded. Player j only gets his or her demand, and the excess payment is wasted. In equilibrium, this will never arise as either player would benefit from deviating. As will become clear, the game would work equally well if we simply made the transfer in this case either the max or the min of the two promises/demands; as regardless of how this is specified equilibrium will imply that the two will exactly match in equilibrium whenever they are compatible. The same is true of the other games we consider.

The Indirect Transfer Network Formation Game

Every player i announces a vector of transfers $t^i \in I\!R^{(n-1)!}$. We denote the entries in the vector t^i by t^i_{jk}, as varying across all possible links jk.

It is required that if $i \notin jk$, then $t^i_{jk} \geq 0$. Thus, i can make demands on the links that i is involved with (it is permissible to have $t^i_{ij} < 0$), but can only make promises to pay for links that i is not directly involved with.

Link jk is formed if and only if $\sum_{i \in N} t^i_{jk} \geq 0$. Formally, the network that forms as a function of the profile of announced vectors of potential transfers $t = (t^1, \ldots, t^n)$ is

$$g(t) = \{ij \mid \sum_{i \in N} t^i_{jk} \geq 0\}$$

In this formation game, player i's payoff is

$$u_i(g(t)) - \sum_{jk \in g(t)} t^i_{jk}.$$

The variation of this link formation game from the previous one is that players are also allowed to make promises to pay to help support links that they are not directly involved in. That is, player i may promise to contribute a payment t^i_{jk} for the formation of link jk. The restriction that they can only subsidize links that they are not involved with reflects that feature that a player cannot prevent other agents from forming links among themselves, but can subsidize such link formation.

Network Formation Games with Contingent Transfers

In the games we have defined above, players only have a limited ability to condition their actions on the actions of other players. That is, the games do not allow players to say something of the form: "I will pay you to form link ij, but only if link jk is also formed." It turns out that being able to make this kind of contingent promise is very important.

To this end, we consider a variation of each of the above games for the case where a player can make their promises/demands contingent on the network that forms.

Thus, i announces a vector $t^i(g)$ contingent on g forming, for each conceivable nonempty $g \in G$. In the direct transfer game, $t^i(g) \in I\!R^{n-1}$ for each i, while

There are many possible ways to determine which network forms given a set of contingent announcements. We pick one, but it will become very clear that the results are robust to this choice. There is an ordering over G, that is captured by a function σ which maps G onto $\{1, \ldots, \#G\}$. The network that forms is then determined as follows. First look at g^1 such that $\sigma(g^1) = 1$. Look at the profile of transfers $t(g^1)$. Look at $g(t(g^1))$. If $g(t(g^1)) = g^1$ then stop. Otherwise, continue to g^2. Stop at the first k such that $g(t(g^k)) = g^k$. If there is no such k, then the empty network forms.

This defines the network that forms as a function of announced profile of contingent transfers, which we again denote $g(t)$, with the understanding that t is now simply a larger vector that includes payments in all sorts of contingencies. The payoffs to the players are then as before, using the vector of transfers $t(g(t))$.

Equilibrium and Supporting a Network

Given a vector of transfers t for one of the four variations of the game, a players payoff is then

$$\pi_i(t) = u_i(g(t)) - \sum_{jk \in g(t)} t^i_{jk}$$

in the non-contingent game,[4] and

$$\pi_i(t) = u_i(g(t)) - \sum_{jk \in g(t)} t^i_{jk}(g(t))$$

in a contingent game.

[4] This equation includes t^i_{jk}, even when $i \notin jk$, and such transfers are not included in the direct transfer game. Simply set $t^i_{jk} = 0$ when $i \notin jk$ for the direct transfer game.

A vector t forms *an equilibrium* of one of the above games if it is a pure strategy Nash equilibrium of the game. That is, t is an equilibrium if

$$\pi(t) \geq \pi(t_{-i}, \widehat{t}^i),$$

for all i and $\widehat{t}^i$.

We say that a network g is *supported* via a given game relative to a profile of utility functions $u = (u_1, \ldots, u_n)$ if there exists an equilibrium t of the game such that $g(t) = g$.

A Comment on Simultaneous Move Games

A critical advantage of considering a simultaneous version of network formation is that after seeing the resulting network and transfers, players will not wish to make further changes to their transfers and links. This is not true if one instead models network formation sequentially, by having the players move in some order. It could be that the resulting network and transfers would not be stable if players could then come back and make further changes.

Regardless of whether one thinks that network formation is simultaneous, the conditions imposed by equilibrium are necessary conditions for any process to come to a stable position. That is, the equilibrium conditions that are derived here are conditions that capture the idea that we have arrived at a network such that no players would gain from further changes.

A Refinement: Pairwise Equilibrium

There is one issue introduced by the simultaneity of the link formation game. It allows for a multiplicity of equilibrium networks as a result of a coordination failure. This is easily overcome with any of a variety of simple refinements, as we now discuss.

Example **1** *Why refine?*

Consider the following 2 player example, where all of the above transfer games are equivalent.

Note that there are two supported networks. One is the empty network and the other is complete network (one link). For instance, to support the complete network we can set $t^1_{12} = t^2_{12} = 0$. To support the empty network, we can set $t^1_{12} = t^2_{12} = -t$, where $t \geq 1$. The second equilibrium is one in which the link is not formed because both players expect the other to make an unreasonable demand, and so it is a best response for each them to make unreasonable demands.

Note that this second equilibrium supporting the empty network survives an elimination of weakly dominated strategies and is also a trembling hand perfect equilibrium.[5] To eliminate this equilibrium using standard refinements would require the machinery of iterative elimination of strategies, which is cumbersome in games with a continuum of actions.

Alternatively, we should expect players forming a link to be able to coordinate their actions on that formation, as the real-life process that we are modeling would generally already involve some form of direct communication. This suggests a very simple refinement.

Given t, let t_{-ij} indicate the vector of transfers found simply by deleting t_{ij}^i and t_{ij}^j.

A vector t forms is *a pairwise equilibrium* of one of the above games if it is an equilibrium of the game, and there does not exist any $ij \notin g(t)$, and $\widehat{t}$ such that

(1) $\pi_i(t_{-ij}, \widehat{t}_{ij}^i, \widehat{t}_{ij}^j) \geq \pi_i(t)$,

(2) $\pi_j(t_{-ij}, \widehat{t}_{ij}^i, \widehat{t}_{ij}^j) \geq \pi_j(t)$, and

(3) at least one of (1) or (2) holds strictly.[6]

This refinement asks whether or not there are any two agents who have not formed a link, who could benefit from mutually changing their demands/promises to add a link.

Note that the reason the refinement only worries about the addition of links is that players can already unilaterally sever links (simply by increasing their demands) and so equilibrium already captures the essential features of that case.[7]

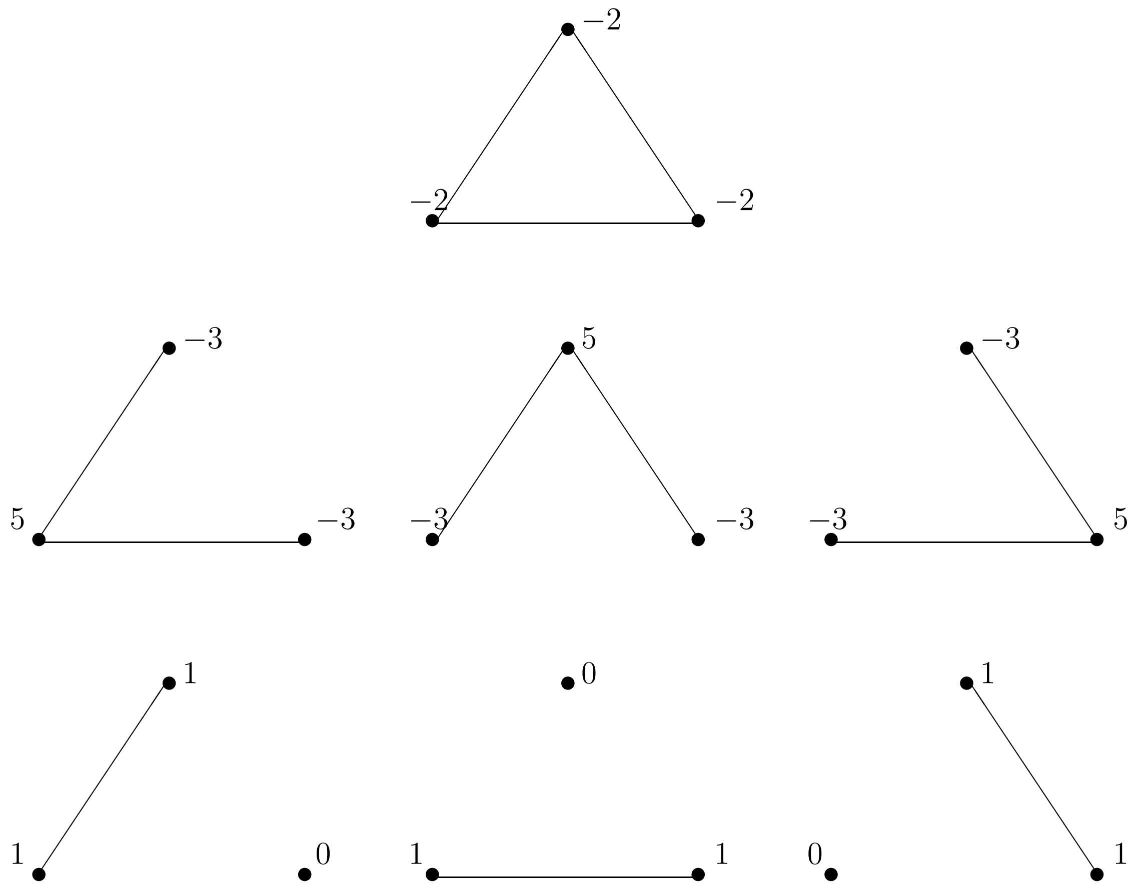
Example **2** *Nonexistence of Pairwise Equilibria*

As opposed to pure strategy Nash equilibria which always exist for all of the games we have discussed, there are some settings in which pairwise equilibria do not exist. Here is such an example. Each player gets 0 in the empty network.

[5]Demanding $-t$ fares well in the case where the other agent happens to offer at least t.

[6]This is easily seen to be equivalent to requiring that both (1) and (2) hold strictly.

[7]There are many other refinements we could also consider - for instance in the indirect transfer game allowing all agents to change their t's on a given link. Once one allows for such group deviations, it makes sense to go all the way to allowing more general groups deviations. At that point one is led to something that is equivalent to the concept of strongly stability with side payments of Jackson and van den Nouweland (2001). Such a refinement is quite stringent, and while it has the nice property of only supporting efficient networks, it only applies in situations where there is substantial room for communication between all individuals.



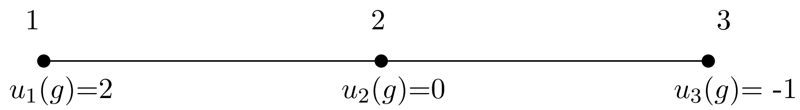
The empty network is a (pure strategy Nash) equilibrium, but not a pairwise equilibrium: two players can set zero demands to form a link and get $1 > 0$. No network that has at least two links can be an equilibrium. Any such network must have some player who gets a negative payoff, who can then get a payoff of at least 0 by setting negative transfers on both links (below -3). Finally, a network with one link cannot be a pairwise equilibrium. For example, if the player who is not linked demanded -3.5 and a player who already has one link offered 3.5, both players would benefit.

4 The Direct Transfer Game

We now provide an analysis of the simplest game, and the one that might best capture the type of direct bargaining that we expect to arise in many applications: the direct transfer network formation game.

We first provide an example that shows that there exist utility functions for which the efficient network is not supported by any equilibrium in the direct transfer game.

Example 3 *Inefficient Network Formation with Direct Transfers and Positive Externalities*



All other networks result in a utilities of 0 for all players.

Consider any equilibrium t and resulting network $g(t)$. It must be that $t^3_{23} \leq -1$ if the network $\{12, 23\} = g(t)$, as otherwise 3 would benefit by lowering t^3. If $t^2_{23} \geq 1 \geq -t^3_{23}$, then 2 will benefit by lowering t^2_{23}, regardless of what other links have formed as u_2 is 0 for all networks. Thus, the network $\{12, 23\}$ cannot be supported in equilibrium.

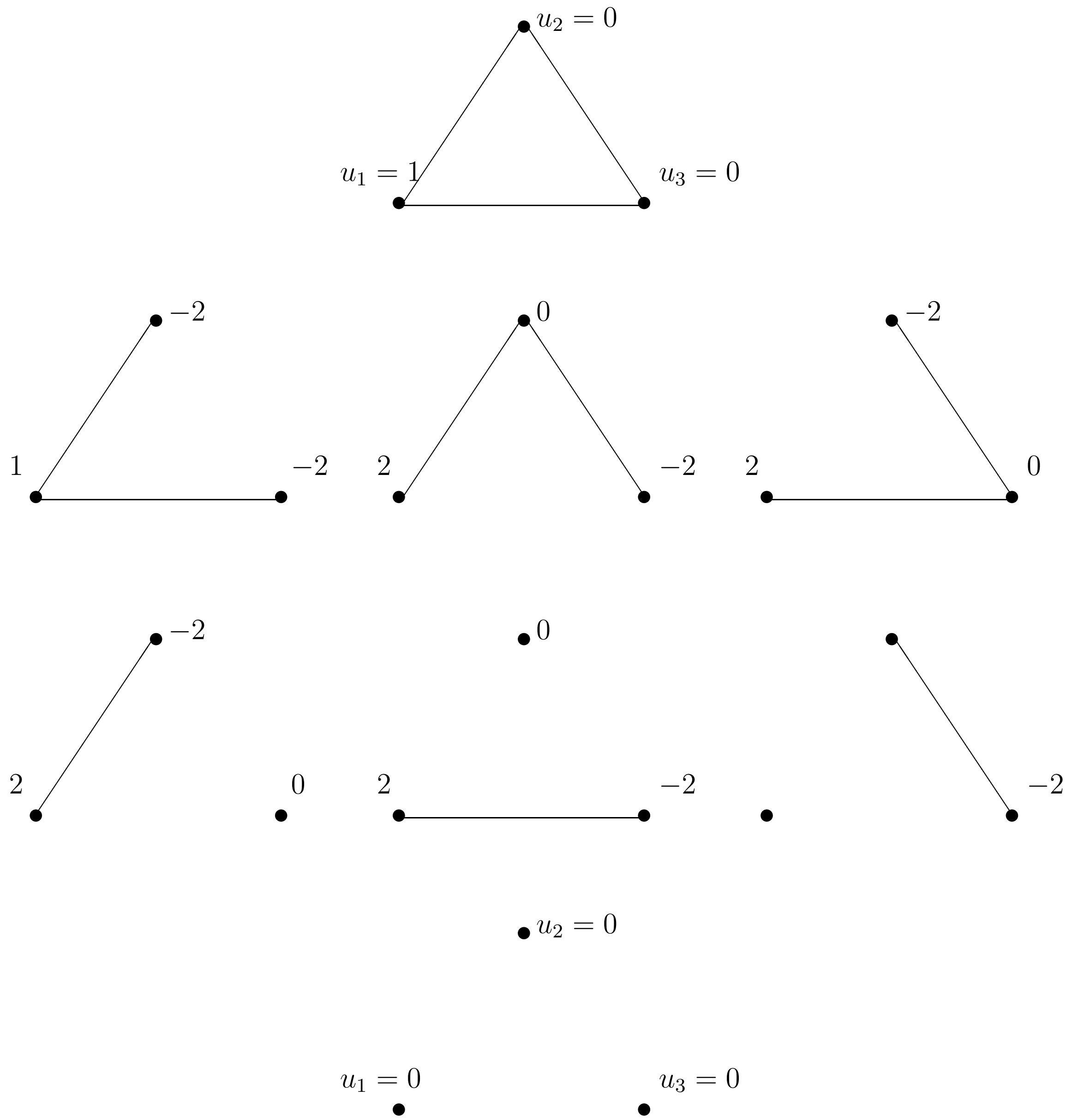
This example shows us that the set of networks that can be supported via the network formation game with direct transfers will have some limitations and that only in some cases will efficient networks be supported. More generally, it points out that we will need indirect transfers in order to support efficient networks in some situations where there are externalities.

Let us get a fuller understanding of what networks can be supported as equilibria under the network formation game with direct transfers. While Example 3 suggests that direct transfers may not always handle externalities, there are hurdles which go even beyond this. Even in cases without externalities, the game must account for payoffs from many different possible deviations in terms of sets of links that might be deleted.

Example 4 *The Efficient Network is Not Supportable in the Complete Absence of Externalities.*

Consider a three-player society and a profile of utility functions described as follows. Any player gets a payoff of 0 if he or she does not have any links. Player 1 gets a payoff of 2 if she has exactly one link, and a payoff of 1 if she has two links. Player 2 gets a payoff of -2 if he has exactly one link, and a payoff of 0 if he has two links. Player 3's payoff function is similar to that of player 2: he gets a payoff of -2 if he has exactly one link, and a payoff of 0 if he has two links.

It is clear from this specification that all players' payoffs depend only on the configuration of their own links and so there are absolutely no externalities in payoffs. This payoff structure is pictured in the network below.



Let us check that there is no equilibrium of the direct transfer game that supports the unique efficient network (the complete network). By setting $t^2_{2i} \leq 0$ for each i, player 2 gets a payoff of at least 0. The same is true for player 3. Thus, players 2 and 3 must have a payoff of at least 0 in any equilibrium. Now, suppose that the complete network were supported in an equilibrium. It follows that $t^1_{1i} \geq 0$ for at least one i, or otherwise one of players 2 and 3 would have a negative payoff. Without loss of generality, say that $t^1_{12} \geq 0$. Player 1's payoff is $1 - t^1_{12} - t^1_{13}$. Suppose that player 1 deviates and changes t^1_{12} so that $t^1_{12} + t^2_{12} < 0$. Then the network that forms will be $12, 23$ and player 1's payoff is then $2 - t^1_{13}$ which is greater than $1 - t^1_{12} - t^1_{13}$ (since $t^1_{12} \geq 0$). This is a contradiction, and so the complete network cannot be supported by any equilibrium.

This example is quite damaging to the hopes of sustaining efficient networks via the direct transfer game. The example is a bit unexpected (by us, at least) in the following sense. If there are no externalities in the network at all, then the value generated can be attributed directly to the links themselves and only affects the players involved in those links. In such a situation, at first glance, it would seem that the players involved in any given link could make transfers that support that link in situations where the total marginal benefit from that link are positive. What is missing from this reasoning is that a given player might have many different combinations of links that they might consider deleting. Each of these combinations might require different transfers to support the links. Indeed, this is the heart of the network balance condition. It might be that some of these combinations are in conflict with each other. In the above example, it is the possibilities that either player 2 or 3 might sever both of his or her links that lies in conflict with what player 1 can get by severing a single link at a time.

This gives us an idea of what characteristics a link formation process must have in order to always support efficient networks. Two things will be needed. First, indirect transfers are needed in order to take care of externalities, as suggested Example 3. Second, Example 4 suggests that we will also need the transfers to be contingent on the network that is formed. In that way the transfers can adjust to the particular combination of links that are formed. We come back to investigate contingencies more fully below.

While Examples 3 and 4 suggest that the direct transfer game will fall short of supporting efficient network in two important regards, there are still many setting with externalities and/or complementarities among links where efficient networks are supported as the equilibria of the direct transfer network formation game.

We now provide a characterization of the networks that are supportable in games with direct transfers, and identify some settings where direct transfers suffice to support efficient networks. First, we offer the complete characterization.

A Complete Characterization of Networks Supported by Direct Transfers: The Network Balance Condition

A set of nonnegative weights $\{\mu^i_\ell\}_{i \in N, \ell \in L_i(g)}$ is *balanced* relative to a network g if

$$\sum_{\ell \subset L_i(g) : ij \in \ell} \mu^i_\ell = \sum_{\ell \subset L_j(g) : ij \in \ell} \mu^j_\ell$$

for each $ij \in g$.

The network g is *balanced* relative to the profile of utility functions u if

$$\sum_i \sum_{\ell \in L_i(g)} \mu_\ell^i mu_i(g, \ell) \geq 0.$$

for every balanced vectors of weights.

We should emphasize that the balance conditions identified here are quite different from the balance conditions used in cooperative game theory. These are not weights that are balanced across coalitions and agents, but equal across particular player-link pairs. These are tailored to the pairwise considerations in network formation, as they should be.

PROPOSITION **1** *A network g is supportable as an equilibrium of the direct transfer network formation game relative to the profile of utility functions u if and only if it is balanced relative to the profile of utility functions u.*

The proof of Proposition 1 appears in the appendix. As one might expect, it uses duality theory from linear programming to turn the equilibrium support conditions into a set of balance conditions.

Proposition 1 only characterizes supportability, and not supportability via pairwise equilibrium. Clearly this provides necessary, but not sufficient conditions for supportability via pairwise equilibrium. The additional constraints imposed by pairwise equilibrium seem to be difficult to capture in a balance sort of condition. Nevertheless, we can identify a sufficient condition, as follows.

PROPOSITION **2** *If a network g is supportable via pairwise equilibrium by the direct transfer network formation, then it is balanced relative to the profile of utility functions u. Conversely, if u satisfies nonnegative externalities, and g is efficient and balanced relative to u, then g is supportable via pairwise equilibrium by the direct transfer network formation game.*

More generally, we show the following lemma.

LEMMA **1** *If g is efficient and supportable via the direct or indirect transfer game, and u satisfies nonnegative externalities, then g is supportable in pairwise equilibrium.*

While the balance conditions are not so obviously interpreted directly, they turn out to be quite useful as is illustrated in the following identification of sufficient conditions for supportability.

Distance-Based Payoffs and Stars

Let $d(i,j)$ denote the distance between i and j in terms of the number of links in the shortest path between them (setting $d(i,j) = \infty$ if there is no path).

Say that a profile of utility functions is *distance-based* if there exist c and f such that

$$u_i(g) = \sum_{j \neq i} f(d(i,j)) - c|L_i(g)|$$

for all i, where $c \geq 0$ is a cost per link, and f is a nonincreasing function.

A distance-based payoff structure is one where players may get benefits from indirect connections, but where those benefits are determined by the shortest paths. Special cases of distance-based payoffs are the connections model and truncated connections models of Jackson and Wolinsky (1996). In such settings, "star" networks play a very central role, as captured in the following proposition.

PROPOSITION **3** *If u is distance-based, then the unique efficient network structure is*

(i) the complete network g^N if $c < f(1) - f(2)$,

(ii) a star encompassing all players if $f(1) - f(2) < c < f(1) + \frac{(n-2)f(2)}{2}$, and

(iii) the empty network if $f(1) + \frac{(n-2)f(2)}{2}$.

In the case where c is equal to $f(1) - f(2)$ or $f(1) + \frac{(n-2)f(2)}{2}$, there are can be a variety of network structures that are efficient. Nevertheless, the star is still efficient in those cases.

The proof of Proposition 3 is an easy extension of the proof of a Proposition in Jackson and Wolinsky (1996), but we include it in the appendix for completeness.

Distance-based settings are ones where efficient networks turn out to be supportable, and via pairwise equilibrium. Even though there are externalities, say in a star network, any player i who enjoys externalities is directly connected to the center who is responsible for providing those externalities. This allows players to pass on their benefits to the center, and helps in supporting the star as an equilibrium. This is captured in the following corollary to Propositions 1 and 2.

COROLLARY **1** *If u is distance-based, then some efficient network is supportable via the direct transfer game and is also supportable via pairwise equilibrium.*

The claim is easy to see directly in cases where either the empty or complete networks are efficient. Consider the remaining case where $f(1)-f(2) \leq c \leq f(1)+\frac{(n-2)f(2)}{2}$, and thus a star involving all players is efficient. Here, we verify the balance conditions. An agent i connected to the center j in a star has only one link, we can simply set $\mu^i_{\{ij\}} = c$ for any $c \geq 0$. Then for the center j, it must be that $\sum_{\ell \subset L_j(g): ij \in \ell} \mu^j_\ell = c$. The fact that a star is balanced then follows from noting that $cmu_i(g,ij) + c\mu_j(g,ij) = 2f(1) + (n-2)f(2) - 2c \geq 0$ in situations where the star is efficient, and noting that the center's payoff is additively separable across links.[8] Proposition 2 implies that we can support an efficient g as a pairwise equilibrium, noting that there are nonnegative externalities in a distance-based u (as adding a link that does not involve i can only increase i's payoff as it may decrease the distance between i and some other agent, but does not impose a cost on i)

Supportability with Nonpositive Externalities and Convexity in Own-Links

In looking for other sufficient conditions for supportability, Examples 3 and 4 are helpful. Examples 3 suggests that we should look at situations where externalities are nonpositive. Example 4 suggests a restriction that marginal payoffs from a given set of links be at least as high as the sum of the marginal payoffs from separate links. This condition is formalized as follows.

A profile of utility functions u are *convex in own-links* if

$$mu_i(g, \ell) \geq \sum_{ij \in \ell} mu_i(g, ij)$$

for all i, g, and $\ell \subset L_i(g)$.

Under these two conditions efficient networks are supportable, as stated in the following proposition.

PROPOSITION 4 *If utility functions are convex in own-links and satisfy nonpositive externalities, then any efficient network g is supportable via the direct transfer game. If utility functions are convex in own links and satisfy no externalities, then g is supportable via a pairwise equilibrium.*

As an example of a setting where we might see convexity in own-links and nonpositive externalities, consider some sort of research partnerships between firms in an oligopoly. For instance, an agreement might lead to the lowering of cost by a firm. If there are diminishing returns to entering into more such relationships, then the convexity in own-links is satisfied. The nonpositive externalities arise as agreements between other firms lowers rivals' costs.

[[Elaborate on this example.]]

A special case of convexity in own links, is the case where payoffs are separable across links.

Link-Separable Payoffs

Let us say that payoffs are *link-separable*, if for each player i there exists a vector $w^i \in \mathbb{R}^{n-1}$, where w^ijk is interpreted as the net utility from jk, and such that

$$u_i(g) = \sum_{jk \in g} w^i jk.$$

[8] This also gives us an idea of which transfers support a star as an equilibrium with agent 1 as the center. Setting $t^i_{1i} = f(1) + (n-2)f(2) - c$, $t^i_{ji} = -(n-1)f(1)$ for $j > 1$, and $t^1_{1i} = -[f(1) + (n-2)f(2) - c]$ for each i. It is easily seen that these form an equilibrium that supports the star.

This condition states that agents view relationships completely separately. A special case is where they only care about the value of their own links.

COROLLARY **2** *If payoffs are link-separable and have nonpositive externalities, then any efficient network g is supportable via the direct transfer game. Furthermore, if payoffs are link-separable and have no externalities, then g is supportable via a pairwise equilibrium if and only if g is efficient.*

The first statement and first part of the second statement follow from Proposition 4. To see the only if claim, suppose to the contrary that g is supportable via a pairwise equilibrium but not efficient. Then there exists g' such that $\sum_i u_i(g') > \sum_i u_i(g)$. As payoffs are link separable and have no externalities, either there exists $ij \in g \backslash g'$ such that $w^i ij + w^j ij < 0$ or there exists $ij \in g' \backslash g$ and $w^i ij + w^j ij > 0$. In the first case, g cannot be supported as an equilibrium, because one of the two players has an incentive to increase her demanded transfer thereby severing the link; in the second case, g cannot be supported as a pairwise equilibrium, since will exist a pair of compatible transfer such that the players have an incentive form the link.

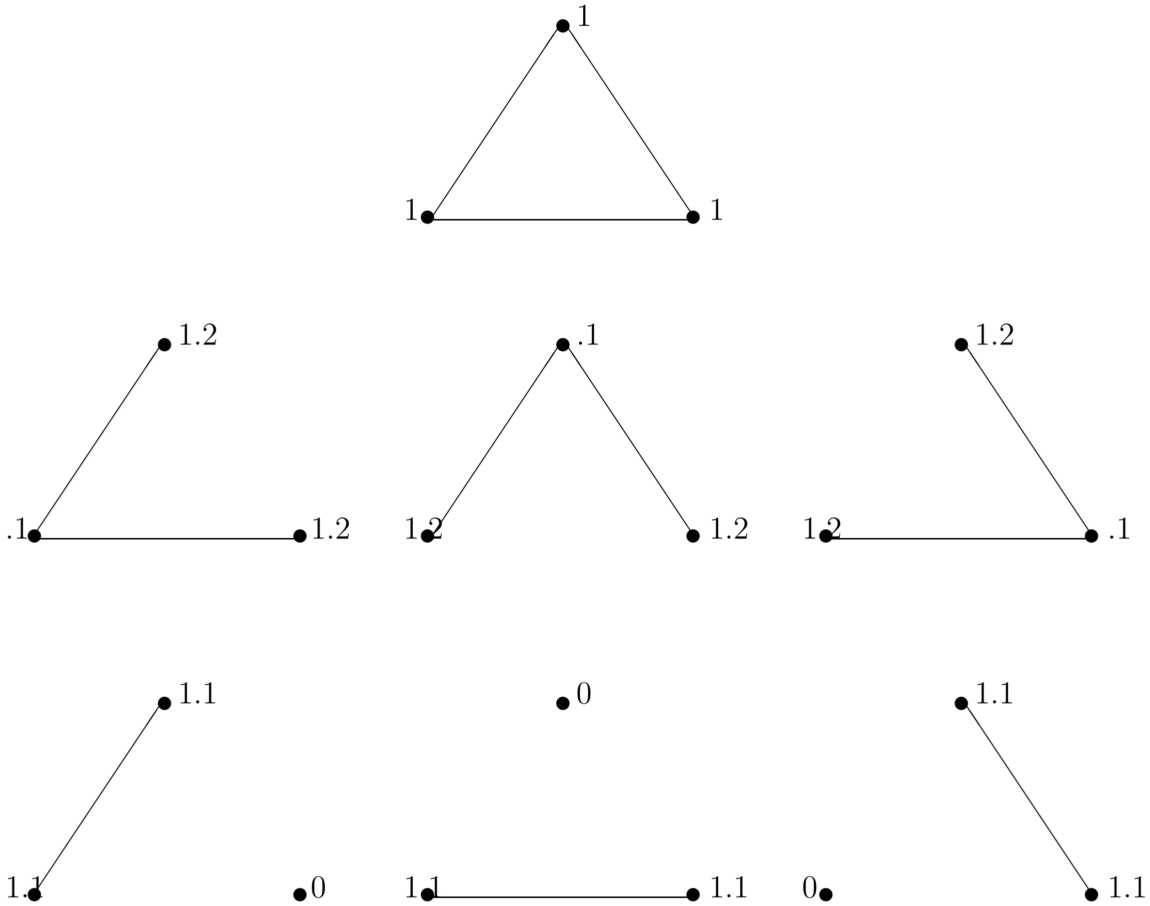
While the above Proposition and Corollary show us that if externalities are nonpositive and payoffs are convex in own-links, then the direct transfer game suffices to support efficient networks; we should emphasize we have also seen that there are also some situations that move beyond this where direct transfers still suffice to support efficient networks. As we have seen in Corollary 1, there are interesting classes where transfers support efficient networks even though both convexity in own-links and nonpositive externalities are violated.

5 Indirect Transfers

As we have seen above, cases where there are positive externalities can make it impossible to support an efficient network via a network formation game with direct transfers. Allowing for indirect transfers will help support efficient networks in situations where there are positive externalities, as players can agree to subsidize links that would indirectly benefit them. While moving to the game with indirect transfers helps us in this way, a different "convexity" problem arises. In the indirect transfer game convexity in own-links is no longer enough to overcome the difficulty faced in terms of affecting several links at once, as a player's deviation (for instance lowering a subsidy) can result in the severance of links not involving that player. This is illustrated in the following example.

EXAMPLE **5** *Efficient Network are not Supportable with Indirect Transfers and Convexity in Own-Links*

Consider a three-player society with payoffs as pictured for networks of one or more links, and payoffs equal to 0 on the empty network.



The complete network is efficient but is not the outcome of any equilibrium of the indirect transfer network formation game. Let us sketch the argument. Consider any player i. Player i must offer to subsidize the link jk by an amount of at least .4, as otherwise at least one of j and k will have an incentive to "sever" the link (set their demand to no more than $-.2$).

Find some player i and a link ij such that $t^i_{ij} \geq 0$. Such a link must exist if the complete network is supported. Consider the following deviation: player i reduces the payment on the link jk and "severs" link ij (setting t^i_{ij} to be low enough so that ij does not form). In that case, the only link formed is link ik, and player i's base payoff is the increased, and transfers have decreased which is strict improvement for player i.

The above network is convex in own-links, as the marginal utility of any second own-link is negative while the marginal utility of any set of two own-links is always positive.

However, note that the convexity in links fails more generally. The marginal utility to player 1 at the complete network of the links 12,23 is negative, while the marginal utility of 23 at the complete network is 1.1, and the marginal utility of 12 is -.2, so the sum of the marginal utilities is positive. Indeed, this is the source of the problem in the example

Convexity in All Links

A profile of utility functions u are **convex in all links** if

$$mu_i(g, \ell) \geq \sum_{jk \in \ell} mu_i(g, jk)$$

for all i, g, and any $\ell \subset g$.

We can now state the following proposition.

PROPOSITION **5** *If payoffs are convex in all links, then any efficient network g is supportable via the indirect transfer game. If payoffs also have nonnegative externalities, then g is supportable via pairwise equilibrium.*

A prominent example where these conditions are satisfied is that of trading networks. Imagine that a group of players (possibly individuals, firms, countries) are involved in bilateral trading or risk-sharing relationships, where gains from trade may pass through the network. This leads to nonnegative externalities. Also, the marginal benefit of adding a new trader decreases in the number already connected, so that payoffs are convex in all links.

[[elaborate on this example.]]

6 Network Contingent Transfers

While the ability of players to make indirect transfers helps in supporting networks, there are still convexity conditions that are necessary to support efficient networks. We now move to exploring contingent transfers to see how that helps. We do this in two parts: first, considering contingent transfers together with direct transfers, and second, considering contingent transfers together with indirect transfers. Let us start with direct transfers.

Let us look back at the reason that the network in Example 3 could not be formed in the direct transfer game. Here, player 1 would like to subsidize the formation of the link 23. However, that is not permitted if transfers can only flow along links. Thus, we saw that the network could not be supported as an equilibrium of the direct transfer game, but could be supported as an equilibrium of the indirect transfer game.

However, we might also consider another possibility. Player 1 might make transfers to player 2, which are then passed on to player 3. The difficulty is that if player 1 makes this transfer to player 2, then player 2 might as well not form the link with player 3 and keep the transfer. This can be rectified if transfers can be made contingent on the network that forms.

As we see now, allowing transfers to be contingent on the network that forms has a big impact on the set of networks that can be supported as equilibrium networks, even when only direct transfers are possible.

PROPOSITION **6** *Consider the contingent version of the direct transfer game and any u. There exists an equilibrium where the network g is formed and the payoffs are $y \in Re^n$ where $y_i \geq 0$ for all $i \in N(g)$ if and only if $\sum_{i \in N(g')} u_i(g) = \sum_{i \in N(g')} y_i$ for all $g' \in C(g)$, and $y_i \neq u_i(g)$ implies $i \in N(g)$.*

COROLLARY **3** *Consider the contingent version of the direct transfer game. Consider any u and network g such that $\sum_{i \in N(g')} u_i(g) \geq 0$ for all components $g' \in C(g)$. There exists an equilibrium supporting g. Moreover, there is an equilibrium corresponding to each allocation $y \in I\!R^n$ such that $\sum_{i \in N(g')} u_i(g) = \sum_{i \in N(g')} y_i$ for each $g' \in C(g)$ and $y_i = u_i(g)$ or $y_i < 0$ implies $i \notin N(g)$.*

While Proposition 6 provides for a very wide set of networks to be supported as equilibria, it is limited by the fact that transfers cannot flow across separate components of a network in the direct transfer game, even if payments are contingent. If we allow for such indirect transfers, then there are additional networks that can be supported.[9]

PROPOSITION **7** *Consider the contingent version of the indirect transfer network formation game. Consider any u, any network g, and any allocation $y \in I\!R^n_+$ such that $\sum_i y_i = v^u(g)$, and $y_i > u_i(g)$ implies $i \in N(g)$. There exists an equilibrium where g is formed and payoffs are y.*

COROLLARY **4** *Consider the contingent version of the indirect transfer network formation game, and any u. Any efficient network such that disconnected players earn zero payoffs is supportable. Moreover, there is an equilibrium supporting each allocation $y \in I\!R^n_+$ such that $\sum_i y_i = v^u(g)$ and $y_i > 0$ implies $i \in N(g)$.*

[9]The y's in Proposition 7 are required to be nonnegative. One can also support the networks from Proposition 6 that are not covered in this Proposition through the construction used there. The difference is that here one sometimes needs a player not in $N(g)$ to subsidize the formation of a component that has a negative value to its members. For this to work, it must be that the disconnected player earns a nonnegative payoff, or they would withdraw their subsidies. Rather than break this into separate cases, we have simply worked with the assumption of nonnegative payoffs.

Pairwise Equilibria with Contingent Transfers

Propositions 6 and 7 have counterparts for pairwise equilibrium,[10] provided the network being supported is efficient and there are nonnegative externalities. This is a simple extension of Lemma 1.

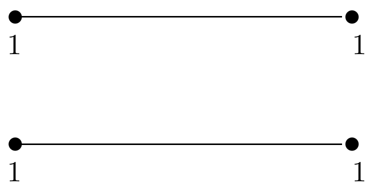
7 Transfers to Prevent Link Formation

We can see from the following example, that in cases with negative externalities; even indirect transfers and contingent payments are not enough to support an efficient network as a pairwise equilibrium.

Example **6** *Negative Externalities and Inefficient Pairwise Equilibria*

Consider a society with four players. If one link forms, the two involved players each get a payoff of 3.

If two (separate) links form, then the four involved players each get a payoff of 1.



All other networks lead to a payoff of 0.

Here, the only pairwise equilibria are inefficient.[11] Two players who are completely disconnected would always benefit from forming a link, and there is no way to give them

[10] In order to define pairwise equilibrium, allow players i and j to vary their announcements $t_{ij}^i(g)$ for all g.

[11] The efficient network is supportable as an equilibrium, where the two disconnected players fail to form a link because each demands too large a transfer. This, again, is a case where pairwise equilibrium is a reasonable refinement.

incentives not to from the efficient network. Here, two players who form a link would like to pay the other players *not* to form a link.

A Game with Payments to Prevent Link Formation

As we saw in the last example, the ability to pay players *not* to form a link can help in supporting efficient networks as pairwise equilibria in the presence of negative externalities. Let us explore such a game.

We first describe the game in the case without contingencies. Consider the indirect link formation game, with the following modification. Each player announces two transfers per link, instead of just one. This pair of announcements by player i relative to link jk is denoted t^{i+}_{jk} and t^{i-}_{jk}. Again, these must be nonnegative if $i \notin jk$, and can be anything otherwise. Player i also announces $m^i_j \in \{+,-\}$ for each $j \neq i$. The interpretation is that i is declaring whether the default decision on link ij is not to add ij or to add ij.

In particular, $g(t,m)$ is determined as follows.

- If $m^i_j \neq m^j_i$, then $ij \notin g$.
- If $m^i_j = m^j_i = +$, then $ij \in g$ if and only if $\sum_k t^{k+}_{ij} \geq 0$.
- If $m^i_j = m^j_i = -$, then $ij \notin g$ if and only if $\sum_k t^{k-}_{ij} \geq 0$.

Payoffs are then

$$u_i(g(t)) - \sum_{jk \in g(t), m^j_k = m^k_j = +} t^i_{jk} - \sum_{jk \notin g(t), m^j_k = m^k_j = -} t^i_{jk}.$$

For this game, equilibrium is again pure strategy Nash equilibrium, and pairwise equilibrium also considers joint deviations by a pair of players ij on their announcements relative to link ij (i and j may change m^i_j, m^j_i, t^{i+}_{ij}, t^{j+}_{ij}, t^{i-}_{ij} and t^{j-}_{ij}).

The contingent version of the game is the version where the t^i and m^i_j's are announced as a function of g.

To see how the game defined above allows payments to prevent link formation, reconsider Example 6.

Example **7** *Negative Externalities with Payments to Prevent Links*

Consider the payoff function of Example 6. Let us find a pairwise equilibrium of the game with payments not to form links that supports an efficient network. Let us support the efficient network $\{12\}$. Have all players set $t^{i+}_{12}(\{12\}) = 0$. Set $t^{1-}_{34}(\{12\}) = t^{2-}_{34}(\{12\}) = 1/2$ and $t^{3-}_{34}(\{12\}) = t^{4-}_{34}(\{12\}) = -1/2$, and $m^3_{34}(g) = m^4_{34}(g) = -$ for all g, and $m^i_{ij}(g) = +$ otherwise. For any other transfers set $t^{i\cdot}_{ij}(g) = -2$, and $t^{i\cdot}_{jk}(g) = 0$ when $i \notin jk$.

Here, players 1 and 2 pay players 3 and 4 if the link 34 is not formed. It is straightforward to check that this is a pairwise equilibrium.

Proposition 8 *In the contingent game with indirect transfers to form or not to form links, any efficient network is supportable via pairwise equilibrium.*

[[Add the proof to the appendix: start with an equilibrium that supports the efficient network with contingent payments and indirect transfers. Alter this to pay to prevent any links that would benefit from forming. Also handle case where payoffs to disconnected players might be ¡0.]]

8 Concluding Remarks

We have shown that ...

9 Relation of Supportable Equilibria to Pairwise Stability

In this section, we compare the networks that are supportable via direct transfers to pairwise stability concepts that identify networks that are supportable without any transfers. This gives some feeling for the differences between transfer-based solutions and ones where payments are fixed.

The following definitions identify networks that are stable when the payoffs are fixed before the formation process.[12]

A network g is *pairwise stable* with respect to a profile of utility functions u if

(i) for all i and $ij \in g$, $u_i(g) \geq u_i(g - ij)$, and

(ii) for all $ij \notin g$, if $u_i(g + ij) > u_i(g)$ then $u_j(g + ij) < u_j(g)$.

This is a self-evident solution concept that requires that no player benefit by severing a link and no two players benefit by adding one.

A network g is *strongly pairwise stable* with respect to a profile of utility functions u if

(i) for all i and $\ell \subset L_i(g)$, $u_i(g) \geq u_i(g \setminus \ell)$, and

(ii) for all $ij \notin g$, if $u_i(g + ij) > u_i(g)$ then $u_j(g + ij) < u_j(g)$.

[12]The first two definitions are from Jackson and Wolinsky (1996). Strong pairwise stability is discussed by Jackson and Wolinsky (1996, section 5), but is not named.

This solution concept is stronger than pairwise stability in that it allows players to sever sets of links rather than just considering one link at a time.

The next definition is a way of incorporating transfers into the study of network formation without actually modeling the bargaining process explicitly.[13]

A network g is *pairwise stable with transfers* with respect to a profile of functions u if

(i) $ij \in g \Rightarrow u_i(g) + u_j(g) \geq u_i(g - ij) + u_j(g - ij)$, and

(ii) $ij \notin g \Rightarrow u_i(g) + u_j(g) \leq u_i(g - ij) + u_j(g - ij)$.

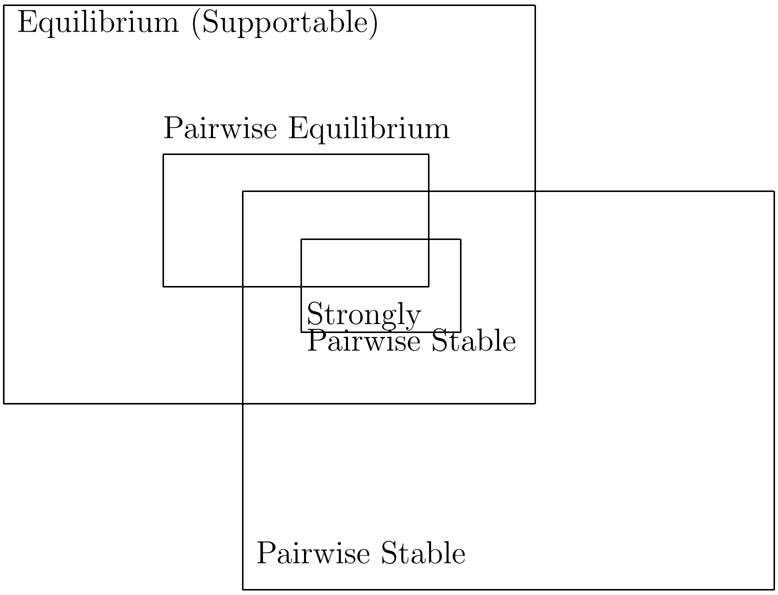
Part (ii) captures the idea that there are no two players who could add a link between them, together with some transfers, and both be better off. Part (i) captures the idea that if a link is in the network, then there must be some transfer (possibly 0) for which both players do not wish to delete the link.

While the notions of pairwise stability and strong pairwise stability can differ quite a bit from the equilibria of the direct transfer game, the notion of pairwise stability with transfers captures some of the spirit of the equilibria of the direct transfer game.

PROPOSITION **9** *The set of networks supportable as pairwise equilibria is exactly the intersection of those networks that are supportable via the direct transfer game and the networks that are pairwise stable with transfers.*

The relationship between supportable networks, pairwise equilibria, and the other pairwise stability concepts is outlined in the following proposition. The relationships between the solution concepts that are outlined in Proposition 10 are captured in the following Venn diagram.

[13]This differs from the concept of pairwise stability allowing for side payments that is discussed by Jackson and Wolinsky (1996). That concept had a stronger requirement in (i), requiring that $u_i(g) \geq u_i(g - ij)$ and $u_j(g) \geq u_j(g - ij)$. If transfers are possible in sustaining a network, and not just in deviations, then arguably the definition here is more appropriate.



PROPOSITION **10**

(i) The set of pairwise equilibria is a subset of the set of equilibria.

(ii) If a network g is strongly pairwise stable relative to a profile of utility functions u, then it is supportable via the direct transfer game and it is pairwise stable.

(iii) There exist u and g for which g is strongly pairwise stable (and thus pairwise stable and supportable), but not supportable via pairwise equilibrium.

(iv) There exist u and g for which g is supported via pairwise equilibrium (and thus supportable) and pairwise stable but not strongly pairwise stable.

(v) There are networks that are supportable and not pairwise stable nor supportable via pairwise equilibrium.

(vi) There are networks that are pairwise stable and not supportable (nor supportable via pairwise equilibrium, nor strongly pairwise stable).

(vii) There are networks that are both supportable and pairwise stable, but not strongly pairwise stable nor supportable via pairwise equilibrium.

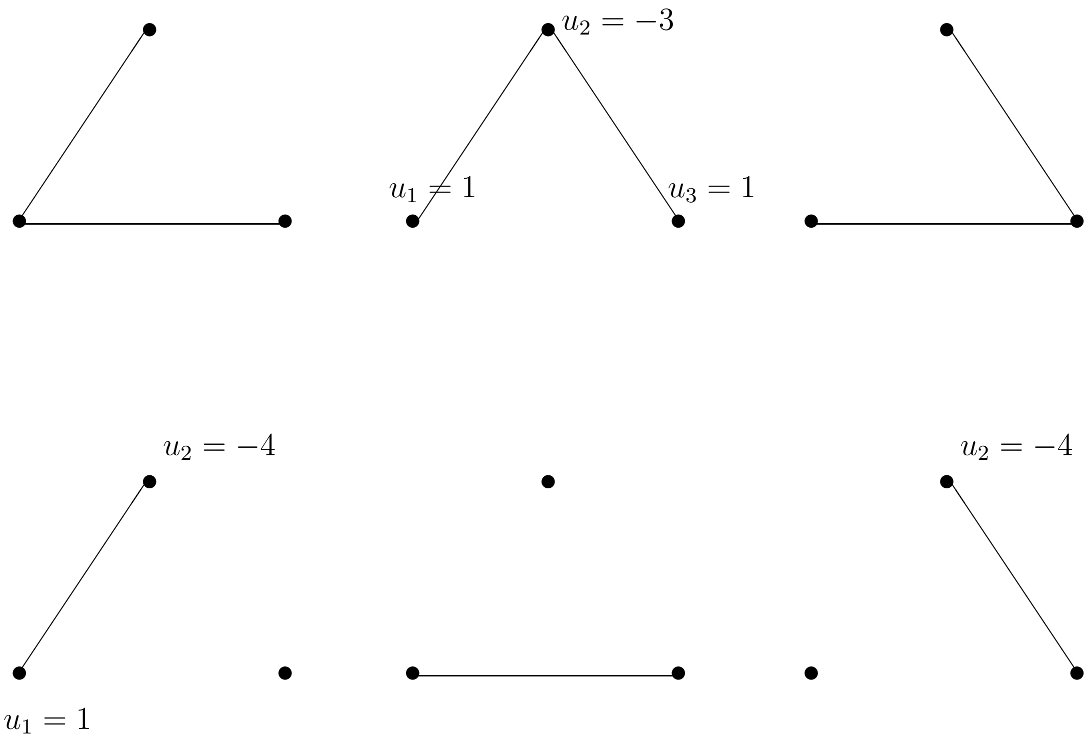
(viii) There are networks that are supportable via pairwise equilibrium and not pairwise stable.

(ix) There exist networks that are strongly pairwise stable (and thus pairwise stable) and at the same time supported via pairwise equilibrium (and thus supportable).

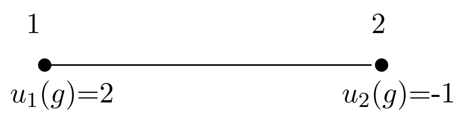
Proof of Proposition 10: (i) follows from the definition of pairwise equilibrium. The pairwise stable part of (ii) is direct. To see the other part of (ii), set $t^i_{ij} = t^j_{ij} = 0$ for each $ij \in g$, and $t^i_{ij} = -X$ for each $ij \notin g$, for some $X > 0$. For large enough X this forms an equilibrium. To see (iii), consider the empty network in Example 9. To see (iv), see Example 10. To see (v), consider the empty network in Example 1. To see (vi), see Example 8. To see (vii), see Example 11. To see (viii), see Example 9. To see (ix), see the complete network in Example 1. ▮

The examples illustrating the claims in Proposition 10 are as follows.

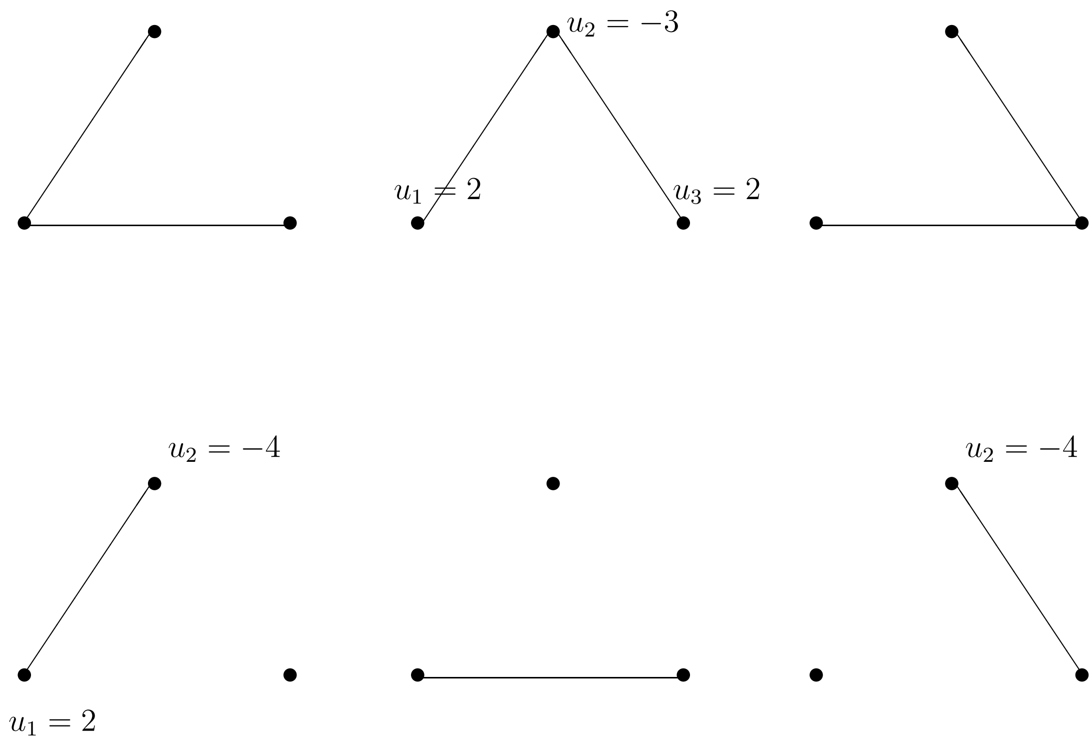
Example 8 *Pairwise stable but not Supportable.*



Example 9 *Supportable via Pairwise Equilibrium but not Pairwise Stable*



Example **10** *Supportable via Pairwise Equilibrium and Pairwise Stable but not Strongly Pairwise Stable*



All other networks have value of 0. The network $\{12, 23\}$ is supportable via pairwise equilibrium and pairwise stable but not strongly pairwise stable.

Example **11** *Supportable and Pairwise Stable but not Strongly Pairwise Stable nor Supportable via Pairwise Equilibrium*

This is the same as Example 10, except that the complete network leads to $u_1 = 6$, $u_2 = -3$, and $u_3 = -1$. The network $\{12, 23\}$ is still supportable and pairwise stable, but no longer supportable via pairwise equilibrium.

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11 Appendix

Proof of Proposition 1: The network g is supported via an equilibrium of the direct transfer network formation game relative to the profile of utility functions u if and only if there exists a vector of transfers t such that:

- $\sum_{ij\in\ell} t^i_{ij} \leq mu_i(\ell)$, for all players i and subsets of their links $\ell \subset L_i(g)$, and
- $t^i_{ij} + t^j_{ij} \geq 0$ for all $ij \in g$.

Furthermore, we know that in equilibrium, we cannot have $t^i_{ij} + t^j_{ij} > 0$ for any ij, as then either one of the players would strictly benefit by lowering their t^i_{ij}.[14]

Therefore, to check whether g is supportable, we can solve the problem
$\min_t \sum_{ij\in g} t^i_{ij} + t^j_{ij}$
subject to:
$-\sum_{ik\in\ell} t^i_{ik} \geq -mu_i(\ell), \forall i \in N, \ell \subset L_i(g)$ and
$t^i_{ij} + t^j_{ij} \geq 0 \forall ij \in g$

and verify that the solution satisfies:
$\min \sum t^i_{ij} + t^j_{ij} = 0.$

The dual of this problem is[15]
$\max_{\{\mu^i_\ell\}_{i\in N,\ell\in L_i(g)},\{\nu_{ij}\}_{ij\in g}} -\sum_i \sum_{\ell\in L_i} \mu^i_\ell mu_i(g,\ell)$ subject to
$\sum_{\ell\subset L_i(g):ij\in\ell} \mu^i_\ell - \nu_{ij} = -1$, for all ordered pairs $i \in N$ and $ij \in g$, and
$\mu^i_\ell \geq 0$ for all $i \in N$ and $\ell \subset L_i(g)$, $\nu_{ij} \geq 0$ for all $ij \in g$.

Since we are free to choose any the ν_{ij}'s do not appear in the objective function, this problem is equivalent to
$\max_{\{\mu^i_\ell\}_{i\in N,\ell\in L_i(g)},\{\nu_{ij}\}_{ij\in g}} -\sum_i \sum_{\ell\in L_i} \mu^i_\ell mu_i(g,\ell)$ subject to
$\sum_{\ell\subset L_i(g):ij\in\ell} \mu^i_\ell - \nu_{ij} = \sum_{\ell\subset L_j(g):ij\in\ell} \mu^j_\ell - \nu_{ij}$ for all ordered pairs $i \in N$ and $ij \in g$, and

[14] We can set $t^i_{ij} = t^j_{ij} = -X$ for some large enough scalar X for any $ij \notin g$, to complete the specification of the equilibrium strategies.

[15] By standard techniques, one can write the $t^i_{ij} = t^{i+}_{ij} - t^{i-}_{ij}$, where t^{i+}_{ij} and t^{i-}_{ij} are both nonnegative. Working across the two inequalities generated by each one of these, we find the equality to -1.

$\mu_\ell^i \geq 0$ for all $i \in N$ and $\ell \subset L_i(g)$.

As the objective can be set to 0 by setting all of the μ_ℓ^i's to 0, we need only verify that $\sum_i \sum_{\ell \in L_i} \mu_\ell^i mu_i(g,\ell)$ is at least 0 for all sets of μ_ℓ^i's that satisfy the constraints. The constraints correspond to the definition of balanced weights, and thus the proposition follows. ∎

Proof of Proposition 2: Given Propositions 10 and 1, the first statement follows directly. Thus, the result follows from Lemma 1. ∎

Proof of Lemma 1: Consider t supporting g in either game. In the indirect transfer game, for any $ij \notin g$ and $k \notin ij$, without loss of generality rearrange transfers so that $t_{ij}^k = 0$. Since g is efficient, and satisfies nonnegative externalities, it must be that $u_i(g+ij) + u_j(g+ij) \leq u_i(g) + u_j(g)$, and so $mu_i(g,ij) + mu_j(g,ij) \leq 0$. Given that $t_{ij}^k = 0$ for all $k \notin ij$, it follows that any joint deviation by i and j on ij that leads to an improvement for one player, must lead to a loss for the other player. ∎

Proof of Proposition 4: Let g be an efficient graph, then for all link ij we must have

$$\sum_k mu_k(g,ij) \geq 0.$$

As the game has nonpositive externalities, this implies that for all links $mu_k(g,ij) \leq 0$ for all $k \neq i,j$. Hence, $mu_i(g,ij) + mu_j(g,ij) \geq 0$. Now by convexity in own-links, $mu_i(g,\ell) \geq \sum_{ij \in \ell} mu_i(g,ij)$ for any $\ell \subset L_i(g)$. Hence

$$\begin{aligned}
\sum_i \sum_{\ell \subset L_i(g)} \mu_\ell^i mu_i(g,\ell) &\geq \sum_i \sum_{\ell \subset L_i(g)} \mu_\ell^i \sum_{ij \in \ell} mu_i(g,ij) \\
&= \sum_i \sum_{ij \in g} mu_i(g,ij) \sum_{\ell \subset L_i(g): ij \in \ell} \mu_\ell^i \\
&= \sum_{ij \in g} (mu_i(g,ij) \sum_{\ell \subset L_i(g): ij \in \ell} \mu_\ell^i + mu_j(g,ij) \sum_{\ell' \subset L_j(g): ij \in \ell'} \mu_{\ell'}^j)
\end{aligned}$$

Now, by balancedness, $\sum_{\ell \subset L_i(g): ij \in \ell} \mu_\ell^i = \sum_{\ell' \subset L_j(g): ij \in \ell'} \mu_{\ell'}^j = \nu_{ij} \geq 0$. Hence,

$$\sum_i \sum_{\ell \subset L_i(g)} \mu_\ell^i mu_i(g,\ell) \geq \sum_{ij \in g} \nu_{ij}(mu_i(g,ij) + mu_j(g,ij)) \geq 0,$$

which is the required balance condition.

The Second statement obtains from Lemma 1. ∎

Proof of Proposition 3:(i) Given that $f(2) < f(1) - c$, any two players who are not directly connected will improve their utilities, and thus the total value, by forming a link.

(ii) and (iii). Consider g', a component of g containing m players. Let $k \geq m-1$ be the number of links in this component. The value of these direct links is $k(2f(1) - 2c)$. This leaves at most $m(m-1)/2 - k$ indirect links. The value of each indirect link is at most $2f(2)$. Therefore, the overall value of the component is at most

$$k(2f(1) - 2c) + (m(m-1) - 2k)f(2). \tag{1}$$

If this component is a star then its value would be

$$(m-1)(2f(1)-2c)+(m-1)(m-2)f(2). \tag{2}$$

Notice that

$$(1)-(2)=(k-(m-1))(2f(1)-2c-2f(2)),$$

, which is at most 0 since $k \geq m-1$ and $c > f(1)-f(2)$, and less than 0 if $k > m-1$. The value of this component can equal the value of the star only when $k = m-1$. Any network with $k = m-1$, which is not a star, must have an indirect connection which has a path longer than 2, getting value at most $2f(2)$. Therefore, the value of the indirect links will be below $(m-1)(m-2)f(2)$, which is what we get with star.

We have shown that if $c > f(1)-f(2)$, then any component of a efficient network must be a star. Note that any component of a efficient network must have nonnegative value. In that case, a direct calculation using (2) shows that a single star of $m+m'$ individuals is greater in value than separate stars of m and m' players. Thus if the efficient graph is nonempty, it must consist of a single star. Again, it follows from (2) that if a star of n players has nonnegative value, then a star of $n+1$ players has higher value. Finally, to complete (ii) and (iii) notice that a star encompassing everyone has positive value only when $f(1)+\frac{(n-2)}{2}f(2) > c$. ∎

Proof of Proposition 5: Let g be an efficient network. If $ij \notin g$, let the transfers be $t^i_{ij} = t^j_{ij} = -X$ and $t^k_{ij} = 0$ for $k \notin ij$, where X is sufficiently large to be exceed the largest marginal utility of any agent for any set of links. If $ij \in g$, by efficiency $\sum_k mu_k(g,ij) \geq 0$. If $mu_k(g,ij) \geq 0$ for all k set all the transfers $t^k_{ij} = 0$. If $mu_i(g,ij) < 0$ and/or $mu_j(g,ij) < 0$ then set the corresponding t^i_{ij} and or t^j_{ij} equal to the marginal utility, and then for each k such that $mu_k(g,ij) > 0$ set $t^k_{ij} \in [0, mu_k(g,ij)]$ so that $\sum_l t^i_{ij} = 0$. This is possible by the efficiency of g.

These t are such that for any $ij \in g$, $mu^l(g,ij) \geq t^l_{ij}$ whenever $l \in ij$ or $l \notin ij$ and $t^l_{ij} > 0$. Let us argue that this forms an equilibrium of the indirect transfer game.

First, note that by the definition of X, if there exists an improving deviation, there will exist one that only changes t's on links in g.

By convexity in all links, if there exists a deviation that is improving for some l on t^l on some set of links, then there exists some deviation that involves at most one link t^l_{ij}, with the possibility that $l \in ij$. For $ij \in g$, increasing transfers is costly and does not change the outcome. Reducing transfers implies that the link will not be formed. Such a deviation cannot be profitable as $mu^l(g,ij) - t^l_{ij} \geq 0$ if $l \in ij$ or if $l \notin ij$ and $t^l_{ij} > 0$. It is not possible to lower t^l_{ij} below 0 if $l \notin ij$.

The last claim in the Proposition follows from Lemma 1. ∎

Proof of Proposition 6: The necessity of $\sum_{i \in N(g')} u_i(g) = \sum_{i \in N(g')} y_i$ for all $g' \in C(g)$, and $y_i \neq u_i(g)$ implies $i \in N(g)$ follow from the balance of transfers across components and the observation that in equilibrium the transfers will sum to 0 on any link that is formed.

To complete the proof, let us show that any such network g and allocation y can be supported as an equilibrium.

Let $Y = 2\max\{\max_i |y_i|; \max_{i,g'} |u_i(g')|\}$.

For $g' \neq g$, set $t^i_{ij}(g') = -Y$ for all i and j.

For g, set transfers as follows. For any $ij \notin g$ set $t^i_{ij} = t^j_{ij} = -Y$.

For $ij \in g$ we set transfers as follows.

Consider a component $g' \in C(g)$.

Find a tree $h \subset g'$ such that $N(h) = N(g')$.[16]

Let player i be a root of the tree.[17] Consider each j who has just one link in the tree. There is a unique path from j to i. Let this path be the network $h' = \{i_1i_2, \ldots, i_{K-1}i_K\}$, where $j = i_1$ and $i = i_K$.

Iteratively, for each $k \in \{1, \ldots, K\}$ set[18]

$$t^{i_k}_{i_{k-1}i_k} = \sum_{k'<k} y_{i_{k'}} - u_{i_{k'}}(g)$$

and

$$t^{i_k}_{i_k i_{k+1}} = \sum_{k' \leq k} -\left(y_{i_{k'}} - u_{i_{k'}}(g)\right)$$

Do this for each path in the tree.

For any link $ij \in g$ but $ij \notin h$, set $t^i_{ij} = t^j_{ij} = 0$.

Under these transfers, g will be the network that forms and y will be the payoff vector. Let us check that there are no improving deviations.

Consider a deviation that leads to another network $g' \neq \emptyset$ being formed. This must involve a net loss for any i as i's payoff must be below $u_i(g') - Y$. Next, consider a deviation that leads to the empty network. It must be that that the deviating player is $i \in N(g)$ in which case the new payoff is 0 for i, which cannot be improving as $y_i \geq 0$. So, consider a deviation by a player i that still leads to g being formed. Player i's promises $t^i_{ij}(g)$ can only have increased, which can only lower i's payoff. ∎

Proof of Proposition 7:

Let $Y = 2\max\{\max_i |y_i|; \max_{i,g'} |u_i(g')|\}$.

For $g' \neq g$, set $t^i_{ij}(g') = -Y$ for all i and j, and set $t^i_{jk}(g') = 0$ for $i \notin jk$.

For g, set transfers as follows. Let $A = \{i | y_i > u_i(g)\}$ and $B = \{i | y_i < u_i(g)\}$.

For $i \in A$ let $\ell_i(g)$ be the number of links that i has in g. Set $t^i_{ij}(g) = \frac{-y_i + u_i(g)}{\ell_i(g)}$ if $ij \in g$ and set $t^i_{ij}(g) = -Y$ if $ij \notin g$, and $t^i_{jk} = 0$ otherwise.

[16] A tree is a network that consists of a single component and has no cycles (paths such that every player with a link in the path has two links in the path).

[17] A root of the tree is a player who lies on any path that connects any two players who each have just one link in the tree.

[18] For $k = 1$ only the second equation applies, and for $k = K$ only the first applies.

For $i \in B$ let

$$\lambda_i = \frac{u_i(g) - y_i}{\sum_{j \in B} u_j(g) - y_j}.$$

Then for $i \in B$ set

$$t^i_{jk}(g)$$

$$= \lambda_i \left(\frac{y_j - u_j(g)}{\ell_j(g)} + \frac{y_k - u_k(g)}{\ell_k(g)} \right) \text{ if } jk \in g, j \in A \text{ and } k \in A,$$

$$= \lambda_i \left(\frac{y_j - u_j(g)}{\ell_j(g)} \right) \text{ if } jk \in g,\ j \in A \text{ and } k \notin A,$$

$$= -Y \text{ if } jk \notin g \text{ and } i \in jk, \text{ and}$$

$$= 0 \text{ otherwise.}$$

For $i \notin A \cup B$, set $t^i_{ij} = -Y$ if $ij \notin g$ and $t^i_{jk} = 0$, otherwise.

Under these transfers, g will be the network that forms and y will be the payoff vector. Let us check that there are no improving deviations.

Consider a deviation that leads to another network $g' \neq \emptyset$ being formed. This must involve a net loss for any i as i's payoff must be below $u_i(g') - Y$. Next, we consider a deviation by a player i that leads to the empty network. This cannot be improving as $y_i \geq 0$. So, consider a deviation by a player i that still leads to g being formed. Player i's promises $t^i_{jk}(g)$ can only have increased, which can only lower i's payoff. ∎

Proof of Proposition 9: It is clear that the set of pairwise equilibria is a subset of the set of equilibria of the direct transfer game. Let us show that any network supportable as a pairwise equilibrium is also pairwise stable with transfers. Consider a pairwise equilibrium $\widehat{t}$. For any link $ij \in g$, player i prefers to announce $\widehat{t^i_{ij}}$ than any transfer X such that $X + \widehat{t^j_{ij}} < 0$. Hence, $u_i(g) - \widehat{t^i_{ij}} \geq u_i(g - ij)$. Similarly, $u_j(g) - \widehat{t^j_{ij}} \geq u_i(g - ij)$. Summing up the two inequalities, $u_i(g) + u_j(g) - (\widehat{t^i_{ij}} + \widehat{t^j_{ij}}) \geq u_i(g - ij) + u_j(g - ij)$ and as $(\widehat{t^i_{ij}} + \widehat{t^j_{ij}}) \geq 0$, $u_i(g) + u_j(g) \geq u_i(g - ij) + u_j(g - ij)$. Conversely, suppose that $ij \notin g$. If $u_i(g) + u_j(g) > u_i(g - ij) + u_j(g - ij)$, define a new transfer vector $\widetilde{t}$ where $\widetilde{t}^h_{kl} = \widehat{t^h_{kl}}$ for all $kl \neq ij$ and $\widetilde{t}^i_{ij} = u_i(g) - u_i(g - ij) - \varepsilon, \widetilde{t}^j_{ij} = u_j(g) - u_j(g - ij) - \varepsilon$ where ε is chosen so that $\widetilde{t}^i_{ij} + \widetilde{t}^j_{ij} \geq 0$. It follows that $u_i(g(\widetilde{t})) - \sum_{k, ik \in g(\widetilde{t})} \widetilde{t}^i_{ik} = u_i(g - ij) - \sum_{k \neq j, ik \in g(\widetilde{t})} \widehat{t^i_{ik}} + \varepsilon > u_i(g(\widehat{t})) - \sum_{k, ik \in g(\widehat{t})} \widehat{t^i_{ik}}$ and similarly, $u_j(g(\widetilde{t})) - \sum_{k, jk \in g(\widetilde{t})} \widetilde{t}^j_{jk} > u_j(g(\widehat{t})) - \sum_{k, jk \in g(\widehat{t})} \widehat{t^j_{jk}}$, contradicting the definition of pairwise equilibrium.

Finally, let us argue that any network g that is supportable and is also pairwise stable with transfers is supportable as a pairwise equilibrium. Consider an equilibrium $\widehat{t}$ that supports g. We argue that $\widehat{t}$ must also be a pairwise equilibrium. Suppose to the contrary that there exists some $ij \notin g$ such that

$$u_i(g + ij) - \sum_{ik \in g} t^i_{ik} - \widehat{t}^i_{ij} \geq u_i(g) - \sum_{ik \in g} t^i_{ik}$$

and

$$u_j(g+ij) - \sum_{jk \in g} t^j_{jk} - \widehat{t}^j_{ij} \geq u_j(g) - \sum_{jk \in g} t^j_{jk},$$

with one inequality holding strictly, and where $\widehat{t}^i_{ij} + \widehat{t}^j_{ij} \geq 0$ (as otherwise the link ij does not form and the payoffs could not have changed). Thus,

$$u_i(g+ij) - \widehat{t}^i_{ij} + u_j(g+ij) - \widehat{t}^j_{ij} > u_i(g) + u_j(g).$$

Since $\widehat{t}^i_{ij} + \widehat{t}^j_{ij} \geq 0$ it follows that

$$u_i(g+ij) + u_j(g+ij) > u_i(g) + u_j(g),$$

which contradicts the fact that g is pairwise stable with transfers. ∎