Precautionary Bidding in a Dynamic Model of Discrete Non-Cooperative Entry Deterrence*

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Abstract

We consider competitive bidding for a business license, via an *open ascending-price auction*, between two *symmetric* incumbents and a potential entrant, each of whom is *privately informed* about her own valuation of the license. Entry stands to reduce the payoff of each incumbent below that in status quo. The resulting symmetric game of *discrete non-co-operative entry deterrence* has an *asymmetric equilibrium* in which one incumbent always participates as a *precaution*, just in case the other incumbent does not participate. The other incumbent free-rides by participating only if her valuation is sufficiently high. In addition, the game has a symmetric equilibrium (SE). Relative to the SE, the asymmetric (competitive) equilibrium realizes bid data that is more consistent with typical patterns of *collusive bids*.

Keywords: non-cooperative entry deterrence, public good, auction, asymmetric equilibrium, collusion

JEL codes: D44, D62, H41

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1 Introduction

While discussing Europe's recently concluded spectrum license auctions, Klemperer (2001), notes that "What really matters in designing auctions are the same issues that any industry regulator would recognize as his key concerns: discouraging collusive, entry deterring and predatory behavior". The first issue, viz. discouraging bidder collusion, has long been recognized as an important challenge that confronts the seller in an auction. Success in detecting collusive bidder behavior primarily depends on the seller's ability to identify whether observed bid data matches that expected from theoretical considerations of competitive or collusive behavior (see Hendricks and Porter (1989)). An auction format that has been recognized as particularly susceptible to tacit collusion is the *open ascending-price auction* (see, Robinson (1985)).[1] Relative to sealed-bid auction formats, in which bidders must *simultaneously* submit their "final bids", the open ascending-price format facilitates collusion by allowing cartel members to revise their initial bids and thus punish any member whose bid is observed to differ from the cartel's plan.

Suspicions of collusion between a group of bidders can arise if their bids exhibit asymmetries of a kind that is unrelated to the ex-ante asymmetries amongst the bidders. A particularly suspicious *asymmetric* bid pattern is one in which only one bidder in the group submits a serious bid, while all other group members either (i) do not bid or (ii) merely participate by submitting some low bid. As an example of (i), in theoretical work motivated by data from the US Federal Communications Commission's spectrum license auctions, Brusco and Lopomo (2002) have shown that tacitly collusive bidders, seeking to coordinate on who should win which license, may signal, to each other, their license preferences by each bidding only on those licenses which it most prefers. As an example of (ii), in empirical work, Porter and Zona (1993) have found that, in an effort to avoid detection, bidding cartels in the market for highway constructions often resort to "phantom bid" submissions by passive cartel members. Such phantom bids are intended to convince the seller that all cartel members are competing in the auction, while being low enough so as not to inflate the price paid by the active cartel member.

Turning attention to entry deterring behavior, the topic of noncooperative *entry deterrence* by oligopolistic incumbents, and the associated *free-rider problem*, has also been of long-standing

[1] A close approximation of this format, which was introduced into the literature by Milgrom and Weber (1982), is the irrevocable exit ascending-price format or the so-called "button model". In this version, starting from some low initial price, the seller continuously raises the price. Each bidder keeps a button pressed as an indication that she is willing to buy the object at the current price. If a bidder takes her finger off her button, then she is considered to have irrvocably exited from the auction. The auction continues up to the price at which the second last bidder exits, leaving the last standing bidder as the winner of the auction at that price. The identity of each exiting bidder and the associated price is observed instantaneously by all remaining active bidders. Such an auction can be alternatively described as a sequence of sealed-bid submission phases, in which each successive phase includes all but the lowest bidder from the previous phase, with the latter's bid serving as the minimum acceptable bid (reserve price).

regulatory interest.[2] Entry into a market intensifies competition for economic rents, and thus, reduces the payoff earned by each erstwhile incumbent. However, successful entry deterrence is a *public good* amongst the incumbents in that it benefits each incumbent regardless of her individual contribution to it. Therefore, each incumbent prefers that other incumbents be the ones that deter entry.[3] If joint bidding is not permitted in the auction of a business license, then entry deterrence must take the form of a *discrete* public good, one that must be provided by only one of the incumbents. When willingness-to-pay for the license constitutes private information, then in the interim, each incumbent expects to pay the cost of entry deterrence with some probability (to the extent that each incumbent expects to win the auction with some probability). However, ex-post, only one incumbent (the highest bidder) must pay the entire cost.[4,5]

In the context of these two regulatory issues, we examine the auction of a single indivisible license between 2 incumbents (I) and a potential entrant (E). Each of the three bidders *privately* know their (non-negative) intrinsic use-value of the license ("type"). The incumbents are ex-ante symmetric. Allocation of the license to E reduces the payoff to each I by an amount e below her status-quo payoff, whereas allocation to any I leaves E with the same payoff as in status-quo. Therefore, each I has a higher willingness-to-pay with respect to E than with respect to the other incumbent.[6]

Given its collusion-proneness, we study the open ascending-price auction format. Once an incumbent exits the auction, *unlike in a sealed-bid format*, the open ascending-price format then gives the remaining incumbent the option to increase her bid in an effort to pre-empt E. In fact, with one incumbent and E remaining in the auction, it becomes a dominant strategy for the remaining incumbent to not quit the auction until the price exceeds her willingness-to-pay with

[2] For example, see Dixit (1979), Bernheim (1984), Gilbert and Vives (1986), Waldman (1987) and Jehiel and Moldovanu (2002).

[3] An interesting example outside of industrial organization (due to Jehiel et al (1996)) involves the "auction" of nuclear weapons that came to be inherited by Ukraine, a constituent of the erstwhile Soviet Republic. Bidders in that auction included some already nuclear-capable nations like the United States and France, whose principal objective was to contain nuclear proliferation. On the other hand, other bidders consisted of rogue nations, without any confirmed nuclear capability, with the suspected objective of actually using these weapons. Keeping the rogue nations from acquiring these weapons was thus a common objective among the already-nuclear-capable nations.

[4] This discreteness cannot be fully mitigated except in the special case of multi-unit auctions in which the number of *(identical)* licenses being auctioned exactly equals the number of incumbents, or some integral multiple of it. In that case, ex-post symmetric provision of entry deterrence is feasible (see Jehiel and Moldovanu (2002)).

[5] Non-co-operative provision of a discrete public good is examined, among others, by Bliss and Nalebuff (1984) and Palfrey and Rosenthal (1988).

[6] Modelling the entry-induced externality, that is suffered by each incumbent, in this reduced-form fashion results in a perfect positive correlation between an incumbent's willingness-to-pay with respect to E and her "type". Allowing for randomness in the externality suffered by each incumbent does not affect the following arguments as long as that random variable is not too negatively correlated with the incumbent's type.

respect to E. If E's type is sufficiently high relative to that of the remaining incumbent, then, even if she succeeds in pre-empting E, she ends up earning a payoff which is less than her status-quo payoff, as a result of the high price she pays in the process. Therefore, the relatively "low types" of an incumbent would prefer not to inherit the role of being the only remaining incumbent who has to pre-empt E. Consequently, such incumbent types may seek to avoid that fate by choosing not to participate in the auction.

On the other hand, unlike in a sealed-bid auction, the option to raise one's bid in the ascending-price auction offers a low-type incumbent the strategy of participating in the auction with the minimum bid, and then raising her bid only in the event in which her fellow incumbent turns out not to have participated. By using such a *precautionary* strategy a low-type incumbent can seek to be present in the auction so as to pre-empt E just in case the other incumbent is not available to do so. However, if one incumbent is surely going to participate in the auction (i.e. participate regardless of her type), then the second incumbent does not face the risk that E might win the auction unopposed – in that case the relatively low types of the second incumbent will find it profitable to not participate in the auction rather than risk inheriting the entire burden of having to pre-empt E. Thus, in any equilibrium, the low-types of at most one of the two incumbents can adopt the precautionary strategy. Clearly, the *asymmetry* in the precautionary motives of the two incumbents owes itself to the *discrete* nature of the entry pre-emption game. The precautionary motive itself arises because of *private information* and the resulting uncertainty about the other incumbent's participation – uncertainty that is resolved as the auction progresses. The objective of this paper is to investigate whether such *asymmetric* participation can, in fact, be an equilibrium phenomenon, and to what extent the resulting bid realizations can look like the asymmetric patterns of collusive bids that were discussed earlier. To focus attention exclusively on the non-participation incentives created by the public good nature of entry pre-emption, the reserve price in the auction is assumed to be non-binding (zero in this case). We proceed by deriving all perfect Bayesian equilibria of this auction game.

First, we find that there always exists a unique equilibrium within the class of equilibria in which the incumbent bidders use monotonic, symmetric pure strategies (*symmetric equilibrium*; also referred to as SE). In fact, we show that for ex-ante symmetric incumbents, there can be no equilibrium in which they employ asymmetric bid functions in the range of strictly positive bids. The symmetric strategy partitions an incumbent bidder's "type" space into 3 subsets, each of strictly positive measure: (i) "low" values for which the incumbent chooses non-participation; (ii) "intermediate" values for which she participates with the minimum initial bid (zero in this case) with the intention of subsequently raising her bid in order to pre-empt E only in the event that the other incumbent does not participate, and (iii) "high" values, for which she bids with the explicit desire of acquiring the license for its intrinsic use. In that case, her opening bid is determined by a bid function which is strictly increasing in her type. The lowest of these "high" values, at which

I bids according to the strictly increasing bid function is one for which, conditional on outbidding E, she earns a non-negative expected payoff. The intuition for this is described in Section 3.

Interestingly (and as anticipated), even though the incumbents are ex-ante symmetric, as the magnitude of e (the pre-emptive motive) increases, there also arises an equilibrium in which the two incumbents follow asymmetric strategies with regards to their participation decisions. For e *at least* "moderately large" (in a sense to be formalized in Section 3; see also footnote (7)), the participation decisions of the two incumbents become polar opposites (*precautionary equilibrium*; also referred to as PE): one incumbent always participates (i.e. regardless of her valuation), and bids the minimum amount if her valuation happens to be less than the "high" values that make it optimal to submit positive bids. On the other hand, the other incumbent participates only if her valuation is "high" enough for her to submit a positive bid – at all lower valuations, she prefers to not participate in the auction. The "high" types of both incumbents follow the same bidding strategy as in SE.[7]

Two differences between the outcomes in PE and SE are immediate, both of which make the PE outcome a closer match to the "collusive" bid patterns described earlier. First, the event in which none of the incumbents participate in the auction, leaving E to win "without a fight", occurs with zero probability in PE, but with strictly positive probability in the SE. If the seller suspects that the two incumbents are engaged in collusive bidding, then data generated by the PE is less likely than the SE to dispel that suspicion because in the PE, the suspected cartel is represented in every auction. Second, consider the event in which both incumbents participate in the auction, and one of them submits an opening bid that indicates the desire to acquire the license for its intrinsic use (i.e. some bid that is strictly higher than the minimum bid). Conditional on this event, the probability that the other incumbent submits the minimum bid is higher in PE relative to SE.[8] In this sense bidding behavior in the PE resembles "phantom" bidding more often than in the SE. Our findings imply that the phenomenon of precautionary bidding, and the associated asymmetric equilibrium of the symmetric entry deterrence game, may lead to unwarranted inference of collusive bidding. In particular, the higher incidence of asymmetric participation and that of mere participation with no subsequent bid raises can be seen to arise from purely competitive bidding between the incumbents once the discrete nature of entry pre-emption is fully taken into account. Therefore, accurate detection of collusive bidding behavior in such markets may need more sophisticated diagnostics.

Even though the precautionary equilibrium outcomes resemble typical collusive bid patterns, the probability of entry deterrence in the two equilibria cannot be ambiguously ranked (except

[7] Informally, we characterize the value of e as "prohibitively large" if for any realization of bidder types, an incumbent's willingness-to-pay with respect to E is at least as large as E's willingness-to-pay; "moderately large" values are less than "prohibitively large" values.

[8] In PE, this *conditional event* occurs whenever the precautionary bidder's type is less than the "high" type; in SE, this conditional event further requires that the other incumbent have an "intermediate" (but not "low") type.

when e is prohibitively large in which case entry is deterred for sure in PE, but not in SE). In the precautionary equilibrium, coordination failure between the incumbents is avoided in the sense that it is never the case that they both do not participate. However, the resulting higher probability of entry deterrence is offset by the ex-ante (and thus, inefficient) choice of the incumbent who plays the precautionary strategy, and thus seeks to thwart entry more often. This is shown in Section 4.

1.1 Relation to existing literature

Though the study of *dynamic bidding* with *allocative externalities* has started only recently, it has already yielded some interesting insights.[9] Noncooperative entry deterrence represents a commonly observed structure of allocative externalities in which allocation to one bidder (a potential entrant) inflicts a payoff-reducing externality on many bidders (all incumbents). With limited exception (Jehiel and Moldovanu (1996), discussed below), the analysis of dynamic bidding in this environment of *one-to-many* externalities has not received much attention. From an auction-theoretic perspective, this article provides such an analysis by deriving and comparing the equilibria of the open ascending-price auction. The analysis yields new explanation for seemingly collusive bidder behavior.

In a model of *complete information*, Jehiel and Moldovanu (1996) have shown that if bidders experience allocative externalities and if participation can be observed prior to bid submission, then some bidders may engage in *strategic non-participation*. Among other examples, they consider the auction of an innovation amongst oligopolists, only one of whom values the innovation; the remaining firms all suffer negative externalities if the first firm acquires the innovation. The authors show that under plausible conditions, there may be no sub-game perfect equilibrium in which all the oligopolists (potential bidders) participate in the auction.[10]

The contribution of our paper consists of examining the *dynamics* of the *non-participation* incentive when bidders have *private information*. Unlike the complete information setting of Jehiel and Moldovanu (1996), private information makes the equilibrium participation of each bidder uncertain. For a sufficiently high type, a bidder can be expected to bid aggressively in order to acquire the license for its intrinsic use value. On the other hand, with a low type, an incumbent can be expected to bid only if *necessary* (i.e. if the other incumbent does not participate). The open ascending-price format gives the latter the option to participate with the minimum bid, just in case that necessity arises. However, it takes the "low types" of only one of the two incumbents

[9] For example, Jehiel and Moldovanu (1995) examine finite-horizon *bargaining* and show that *delay* can arise in equilibrium in spite of complete information. Das Varma (2002) examines symmetric bidding in the *open ascending-price auction*, and finds that a specific bidder's exit from the auction may bring bidding to an abrupt end by triggering the *en masse exodus* of all remaining bidders.

[10] They also exhibit examples of a different flavor in which, by *credibly commiting* not to participate in bidding, a bidder can "soften" the bid of a rival who would otherwise have won, and whose win would have imposed a severe payoff reducing externality on the former.

to exercise such caution. As a consequence, there exists an asymmetric equilibrium that, for appropriate realization of bidder valuations, generates the full information outcome in Jehiel and Moldovanu (1996). Interestingly, we find that this is not the only equilibrium of the auction. There also exists an equilibrium in which the incumbents follow symmetric strategies.

In a related paper, Jehiel and Moldovanu (2000) have considered the revenue consequences of reserve prices and entry fees when allocative externalities are, in turn, positive and negative. They consider *symmetric equilibria* of second-price sealed-bid auctions in an environment of private information and show, among other results, that with a *binding reserve price* and positive externalities: (i) a symmetric equilibrium in pure strategies may fail to exist, and (ii) when it does exist, it may involve pooling by several bidder types at the reserve price.[11]

Not surprisingly, the symmetric equilibrium in our model shares similarities with that in Jehiel and Moldovanu (2000). Notwithstanding these similarities, important differences in the two environments deserve to be highlighted. First, we consider bidding dynamics in the ascending-price auction (in fact, Jehiel and Moldovanu (2000, footnote (27), p. 773) note that the open ascending-price auction differs from the second-price sealed-bid auction if the number of bidders exceeds two, as is the case here). An important difference which is pertinent to the present comparison is that in the *simultaneous-move* sealed-bid auction game, there is no sense in which, by participating, an incumbent risks inheriting the burden of having to deter the entrant all by herself. As a result, the strategic considerations in the bidders' participation decisions are different. Second, in their set-up, positive externalities result from sale to any one bidder (say, acquisition of a competitor by one of the incumbents). Therefore, the payoff-enhancing externality is enjoyed by the other bidders so long as one bidder bids no lower than the reserve price – in that sense, once the seller announces the reserve price, the cost of the public good becomes common knowledge. In our set-up, the minimum bid by an incumbent that will be required to pre-empt E is not known to the incumbents prior to making their participation decisions – in that sense, the *cost of the public good is uncertain* (see Waldman (1987) as to why uncertainty about the cost of entry deterrence can fundamentally alter incumbents' strategies; see also Nitzan and Romano (1990)). Given the "once and for all" nature of bids in a sealed-bid auction, it may be interesting to explore what effect this cost uncertainty can have on the partial pooling property of its symmetric equilibrium.[12] ¤

The paper continues with Section 2 in which the model is formalized. Section 3 solves the

[11] Non-existence of *symmetric* separating equilibria, and partial pooling at the reserve price are also demonstrated in Haile (2000) where the presence of a resale market creates positive externalities for losing bidders at an auction.

[12] Jehiel and Moldovanu (2002) consider a *complete information* multiple license *sealed-bid* auction in which each incumbent can win at most one license. Each incumbent's willingness-to-pay with respect to any entrant is assumed to exceed the entrant's maximum willingness-to-pay (i.e. the externality is "prohibitively" large). In the *symmetric competitive equilibrium*, each incumbent randomizes between non-participation and the entry pre-empting bid, with the probability of non-participation becoming zero when the number of licenses exactly equals the number of incumbents (recall footnote (4)).

model and presents its equilibria, while Section 4 compares the entry deterrence properties of the precautionary equilibrium to those of the symmetric equilibrium. Section 5 contains some concluding remarks. Mathematical derivation of equilibria is relegated to an Appendix.

2 Model

Consider the auction of a single indivisible license amongst 3 bidders, two of who are "incumbents", denoted by $I \in \{i, j\}$, while the third is a "potential entrant" denoted by E. Each incumbent bidder I's payoff, gross of any payments made by her, depends on the identity of the bidder who gets to acquire the license as a result of winning the auction. Let η denote the identity of the auction winner. Then, bidder I's gross payoff is given by:

$$u_I(\eta) \equiv \begin{cases} v_I & \text{if} \quad \eta = I \\ 0 & \text{if} \quad \eta = \{i, j\} \backslash I \\ -e & \text{if} \quad \eta = E, \end{cases} \tag{1}$$

where $e > 0$. v_I is the use value of the license to I (type) and $-e$ is the externality inflicted on each incumbent when E acquires the license. Each v_I is private information to bidder I, and is commonly known to be the realization of an independent random variable, $\widetilde{v}_I$, which is distributed according to the continuous probability distribution function F. F has support $[0, \overline{v}]$ and density function f, which is strictly positive everywhere on its support. The value of e is common knowledge. Bidder E's gross payoff is given by

$$u_E(\eta) \equiv \begin{cases} v_E & \text{if} \quad \eta = E \\ 0 & \text{if} \quad \eta = I, \end{cases} \tag{2}$$

where v_E is privately known to E, and commonly known to the incumbents to be the realization of an independent random variable, $\widetilde{v}_E$, which is distributed according to F. Therefore, each I has a willingness-to-pay of v_I with respect to the other incumbent, and, $v_I + e$, with respect to the entrant. On the other hand, E has a unique willingness-to-pay, viz. v_E.

Note that the crucial difference with a bidding environment in which there are no externalities is that, here, each incumbent is better off if the other incumbent, rather than the potential entrant, acquires the license. That the former payoff is normalized to be the same as the status-quo payoff (payoff in the event of no sale), or that the externality inflicted by E is negative, are formulations that are convenient to adopt but these do not influence our qualitative findings. Also, as mentioned in the Introduction, allowing the externality suffered by the each incumbent to be random and privately known to her will preserve the results of this model so long as each externality is not too negatively correlated with the corresponding incumbent's type. What is crucial for the results of this model to hold is that an incumbent with a higher type also has a higher (expected) willingness-to-pay with respect to E. A second notable point is that even if we assume $\widetilde{v}_E$ to be distributed according to some distribution other than F, our qualitative results will not change so long as the support of that distribution is not so different from that of F as to rule out any ex-ante uncertainty about whether E can win the auction. Finally, as the reader may already suspect from the previously described intuition for our results, allowing for correlation in bidder types does not affect the set of equilibria or its qualitative character. However, it adds considerably to the amount of notation that is needed for the analysis.

We assume a further technical condition on the distribution F, which is sufficient to insure that the bid function employed by I (whenever I bids a strictly positive amount) is strictly increasing. The condition is that the inverse hazard rate, $\frac{F(v+e)}{f(v+e)}$, is strictly increasing in v, and that it is no less than e for any value of v that satisfies $v \leq \overline{v} - e$. The first part of the assumption is the increasing inverse hazard rate condition which is familiar in the literature on auction theory. The second part can be easily verified as being satisfied by the uniform distribution.

In the open ascending-price auction that we study each bidder can choose whether to participate in bidding. Knowing that in any *sequentially rational* outcome, the other incumbent can be counted upon to bid up to her maximum willingness-to-pay can create a "tie" between the incumbents to exit from the auction, once it is revealed that both incumbents have chosen to participate. To resolve this "tie", we treat the auction as consisting of two phases: an initial phase in which all 3 bidders *simultaneously* submit sealed bids, from which the seller selects the two highest bidders. Ties in the initial phase are resolved through the flip of a fair coin (see below for a discussion about the issue of *re-entry* in the open-auction). The two selected bidders then bid against each other in a final ascending-price phase (or equivalently, in a second-price sealed-bid phase), starting from the price which equals the lowest bid in the initial phase.[13] Each selected bidder knows the identity of her active opponent prior to the start of the second phase. If both selected bidders quit the second phase at the same price, then each one of them is equally likely to be determined as the winner. Bids in the initial phase are restricted to be non-negative.

A natural criticism of the irrevocable exit model stems from the possibility of *re-entry* in real-world open auctions. In our context, the possibility of re-entry amounts to each incumbent's inability to commit not to raise E's high bid when that high bid is strictly below her willingness-to-pay with respect to E. This lack of commitment on the part of both incumbents can result in a "waiting game" in which each incumbent waits for the other to raise E's high bid. A well-defined auction game would need to specify a criterion by which an incumbent can be selected to bid against E in the event in which this "waiting game" ensues. The initial sealed-bid phase, and the associated tie-breaking rule serves that purpose.[14]

[13] Recall from footnote (1) that the sequence of sealed-bid phases is strategically equivalent to the "button model" (see also the next paragraph).

[14] In fact, real world open auctions almost always have *activity rules*, which are designed to move forward the bidding process in a timely manner. Such rules can have the effect of necessitating *simultaneous* bids. A visit to ebay.com reveals that every auction on that site has a specified end time, after which no further bids are entertained. This implies that any bid that is submitted exactly at the deadline must be done *without observing* (any) other such

Let the third highest initial bidder and her initial bid be denoted by χ and p respectively, and the history of bidding at the start of the ascending-price phase by $H \equiv (\chi, p)$. Further, let $\mathcal{H} := \{(\chi, p) \,|\, \chi \in \{i, j, E\}, 0 \leq p\}$ be the set of all possible histories of bidding at the beginning of the ascending-price phase. Let the act of non-participation by a bidder be denoted by ϕ. In the above auction game, a *pure strategy* for any bidder (say bidder i) consists of the following elements: (i) a function that maps her private information into either ϕ, or a non-negative initial bid, denoted by $\sigma_i : [0, \overline{v}] \rightarrow \phi \cup [0, \infty)$, and (ii) a price (no less than the third highest initial bid) at which to quit the ascending-price phase, $\theta_i : [0, \overline{v}] \times \mathcal{H} \rightarrow [p, \infty)$. A Perfect Bayesian Equilibrium (PBE) of the auction game consists of a profile of strategies, one for each bidder, such that each bidder maximizes her expected payoff, given the strategies of her opponents.[15]

The following properties of PBE strategies follow from standard *dominance* arguments. First, in the acending-price phase of the auction, each bidder will quit the auction when the price reaches her willingness-to-pay with respect to her *sole* active opponent. For example, if $H \equiv (j, p)$, then $\theta_i(v_i, H) = \max\{p, v_i + e\}$. On the other hand, if $H \equiv (E, p)$, then $\theta_i(v_i, H) = \max\{p, v_i\}$. Second, E has a private willingness-to-pay, which is invariant with the identities of her opponents. Therefore, E has a dominant strategy given by $\sigma_E(v_E) = \theta_E(v_E, H) = v_E$, for any H. The task of deriving pure strategy PBE of the auction game thus boils down to identifying pairs of functions $\{\sigma_i(.), \sigma_j(.)\}$, such that each function constitutes a best-response to the other. We will limit attention to PBE in strategies that are non-decreasing functions of bidder valuations (i.e. functions $\sigma_i(.)$ and $\sigma_j(.)$ that are non-decreasing in their arguments), and that are strictly increasing whenever they are (strictly) positive. In other words, we will not consider bid functions that involve pooling at strictly positive bids.

3 Equilibria

Consider the mapping $b^*(.)$ which is defined by:

$$b^*(v) \equiv \left\{ b \middle| \int_b^{v+e} (v - z) f(z) \, dz = 0 \right\}. \tag{3}$$

During the course of deriving equilibria, we will prove that under the assumptions made about F, $b^*(.)$ must be one-to-one, strictly increasing and differentiable. The proof consists of (i) verifying that the expression $\int_b^{v+e} (v - z) f(z) \, dz$ is strictly increasing in v, but strictly decreasing in b

bids. Another example is the celebrated procedure of issuing two warnings before declaring the standing high bidder to be the winner (as in "Going, going, gone"). If the high bid is not raised after the first warning, then the second warning offers a final (*simultaneous*) opportunity for the remaining bidders to do so.

[15] Restricting the number of incumbents to 2 helps to keep the equilibrium derivation tractable by enabling closed-form solutions for continuation payoffs following each bidding history. At the same time, 2 incumbents is sufficient to demonstrate the interesting participation consequences of the public good nature of pre-emption.

whenever $b < v$, and (ii) noting that in order for the expression to be equal to 0, it must be that $b < v$.

Let us also define v^* such that $b^*(v^*) = 0$. Note that for any strictly positive value of e, v^* must satisfy the strict inequalities $0 < v^* < \overline{v}$. Let m denote the mean of the distribution F. Then, for later use, note that if $e \geq \overline{v} - m$, then $v^* = m$. To see this, observe that $\int_0^{\overline{v}} (v - z) f(z)\, dz = v - m$, which in turn implies that if $v^* + e \geq \overline{v}$, then $v^* = m$.

The following Proposition characterizes the symmetric equilibrium (which is unique) of the auction game.

Proposition 1 *For each value of e, there exists a unique ω^* with $0 < \omega^* < v^*$ such that the following strategy constitutes the unique symmetric equilibrium of the auction game:*

$$\sigma_I(v_I) = \begin{cases} \phi & \text{if} \quad 0 \leq v_I < \omega^* \\ 0 & \text{if} \quad \omega^* \leq v_I < v^* \\ b^*(v_I) & \text{if} \quad v^* \leq v_I \leq \overline{v}. \end{cases}$$

Proof. All proofs are relegated to a mathematical appendix. ■

The optimal nature of the strategy can be understood intuitively. Let us start with the bid function $b^*(v_I)$, which determines bids in the strictly positive range. First note that in any symmetric equilibrium, the initial bid of i cannot exceed her valuation v_i (which is weakly positive). To see this, note that if i turns out to be the lowest initial bidder, then, by equilibrium symmetry, $v_j > v_i$. As a result, if i could have outbid E, then so can j (i.e. $v_j > v_i \Rightarrow v_j + e > v_i + e$). That is, in a symmetric equilibrium, i, if she places third in the initial round, will never regret not bidding higher if the auction is eventually won by E. On the other hand, i would never pay a price greater than v_i in order to outbid j in the ascending-price phase of the auction. Therefore, an initial bid in a symmetric equilibrium cannot exceed the bidder's use value.

Now, imagine bidder i deciding whether or not to raise her initial bid by a "tiny" amount. If her original bid was among the top two initial bids, then this contemplated increase will not make any difference to either the identities of the two bidders in the subsequent ascending-price phase, or the price at which the ascending-price phase starts. Therefore, if this contemplated increase in her bid amount has any impact on the outcome of the auction, it must be by moving bidder i up from being the lowest to the second highest bidder in the initial round of the auction. Now, if in the process of moving up, i outbids E, then she faces j, who must, by virtue of being the highest initial bidder in a symmetric equilibrium, have a valuation higher than that of i. In this event, bidder j is sure to win the auction, leaving i with a payoff of 0. On the other hand, if in the process of moving up, i just about outbids j, then (i) she faces E in the final ascending price round, and (ii) by equilibrium symmetry, $v_j \approx v_i$. In order for this move-up to be profitable for i, she must make a non-negative expected payoff in the event of outbidding E. This is because by letting j outbid E, i can guarantee herself a payoff of 0. Furthermore, $v_j \approx v_i$ implies that the probability that j can outbid E is no lower than the probability that i can outbid E.

Now, if the starting price of the ascending phase is b_i (which it must be if i just about outbids j in the initial phase), then i's expected payoff, in the event she outbids E in the ascending price phase, is equal to $\int_{b_i}^{v_i+e}(v_i - z) f(z)\, dz$. Based on the reasoning above, i finds it optimal to choose the highest b_i such that this expression is non-negative, which in turn yields $b^*(.)$ as defined in (3).

Since the lowest value of v_E is 0, the lowest type of I, that can earn an expected payoff of 0 while outbidding E, must be strictly higher than 0. This lowest type is v^*. Types of I that lie in the set $[0, v^*)$ must then choose between non-participation (ϕ) on the one hand, and an initial (minimum allowed) bid of 0 on the other. The equilibrium is characterized by a non-degenerate partition of this set into the continuous subset of types $[0, \omega^*)$ that choose ϕ, and the continuous subset of types $[\omega^*, v^*)$ that choose an initial bid 0. The result can be understood by noting that for types below v^*, there are two conflicting incentives, the net effect of which shapes their (non)-participation decision. On the one hand, such a type of i (say) has a relatively low probability of either outbidding j, or earning a positive payoff conditional on outbidding E. Such a type of i would therefore prefer to not participate in the auction. Countering this non-participation incentive is the probability that the other incumbent, j, either may not participate, or may be even less able than i to outbid E (i.e. have a value $\widetilde{v}_j \ll v_i$), in both of which cases E would win the auction, causing i the externality $-e$. The probability of this adverse outcome can be reduced by i by opting to participate (with the minimum bid of 0), because by so doing, she earns a chance to outbid E, in the event that j does not participate in the auction. Notice that the probability that $\widetilde{v}_j \ll v_i$ is increasing in the value of v_i. Therefore, relatively higher values of v_i in the set $[0, v^*)$ have a stronger incentive to participate.

To understand why the marginal type that participates (ω^*) is strictly greater than 0, consider the lowest type $v_i = 0$, and suppose j participates only if $v_j \geq \omega_j$ for some ω_j that belongs to the interval $(0, v^*)$. Whereas participating with an initial bid of 0 does earn i the option to bid against E, that option is of some value to type $v_i = 0$ only when j does not participate, i.e. when $v_j < \omega_j$. If $v_j \in [\omega_j, v^*]$, then participation by type $v_i = 0$ actually hurts i by tying her initial bid with j and thereby eliminating j (with probability 1/2) when in fact j is better able to outbid E (i.e., $v_j + e > v_i + e$). Based on this reasoning, for ω_j low enough, it turns out to be optimal for types of i in the vicinity of $v_i = 0$ to not participate. The ω^* that characterizes the symmetric equilibrium is the fixed point that results from iterating this argument between i and j.

Note that the objective of bidder i (say), when she participates with a minimum initial bid of 0, is purely to have the option to outbid E. This option is exercised by i if j turns out not to participate. Therefore, so long as a set of types of j, of a given positive measure, does not participate, the option value to any given type of i, from participating with an initial bid of 0, increases in the magnitude of the negative externality e. In other words, as the magnitude of e increases, increasingly lower values of v_i prefer to participate with a minimum bid of 0, so long as the set of types of j that participate does not expand with this increase in the magnitude of e.

Reasoning from j's viewpoint, the larger the set of types of i that participate, lower is the value of the option to j from participating; as a result, increasingly higher values of j would prefer to not participate. Then, as the magnitude of e increases, divergent participation decisions by low types of i and j can constitute best-responses to each other. The following Proposition establishes that if $e \geq \overline{v} - m$, then, in fact, such an asymmetric equilibrium exists, and it is unique (up to a permutation of the identities of the two ex-ante symmetric bidders i and j). Interestingly, it is characterized by participation by all types of i, and non-participation by all types of j that lie in the set $[0, m)$.[16]

Proposition 2 *Iff $e \geq \overline{v} - m$, there exists an asymmetric equilibrium (which is unique up to a permutation of the identities of i and j), characterized by the following profile of strategies:*

$$\sigma_i(v_i) = \begin{cases} 0 & \text{if} \quad 0 \leq v_i < m \\ b^*(v_i) & \text{if} \quad m \leq v_i \leq \overline{v}, \end{cases}$$

and

$$\sigma_j(v_j) = \begin{cases} \phi & \text{if} \quad 0 \leq v_j < m \\ b^*(v_j) & \text{if} \quad m \leq v_j \leq \overline{v}. \end{cases}$$

Note that the threshold value of e that supports complete asymmetry in the participation decisions of the two incumbents, viz., $\overline{v} - m$, is not prohibitively large. In particular, as long as $e < \overline{v}$, E enjoys a positive probability of outbidding any I to win the auction. Therefore, the participation decisions of the two incumbents can be completely asymmetric, even though one incumbent, whose types $[0, m)$ choose not to participate, can hardly be certain that the other incumbent will successfully pre-empt E. In this sense, only a "moderately" large value of e is sufficient to support an asymmetric equilibrium, in which one incumbent pools at the minimum bid while the other pools at non-participation, for all of their types except those for which they expect to profitably outbid E.

Whereas Proposition 2 proves that the asymmetric equilibrium exists for $e \geq \overline{v} - m$, it is conceivable that asymmetric equilibria will exist even for values of e smaller than $\overline{v} - m$. Such equilibria will necessarily involve both non-participation, as well as bidding at the minimum of 0 by positive measures of types of both i and j. The equilibrium strategies featured in Proposition 2 have the striking asymmetry that bidder i always participates, whereas bidder j participates only in the event she bids a positive amount. It is worth re-iterating that the symmetric equilibrium featured in Proposition 1 exists for all values of e, in addition to any asymmetric equilibria.

Formally, the auction confronts types $[0, v^*]$ of bidders i and j with an incomplete information version of the classic "battle-of-the-sexes" game (see e.g. Fudenberg and Tirole (1993)). Each of these bidders prefers that E be pre-empted than not; however, each prefers that the other incumbent outbid E. The relevant strategies available to each "sex" are "not participate" and

[16] Recall that if $e \geq \overline{v} - m$, then $v^* = m$.

"participate with the minimum bid". The latter strategy offers a bidder the option to later raise her bid. Now, the complete information battle-of-the-sexes game is known to have two equilibria: one in mixed strategies, and the other in coordinated pure strategies. The symmetric equilibrium characterized in Proposition 1 is the equivalent of that mixed-strategy equilibrium (pure strategy for each type, but different types may play different pure strategies). On the other hand, the asymmetric equilibrium characterized in Proposition 2 corresponds to the coordinated pure-strategy equilibrium (the pooling of all types of i in question, over the minimum bid of 0, and the pooling of all types of j in question, over non-participation).

4 Comparison of equilibria

Given the precautionary nature of bidding in the asymmetric equilibrium (PE), it is of interest to compare its ex-ante entry deterrence probability with that in the symmetric equilibrium (SE). Towards that end, suppose that $e \geq \overline{v} - m$, so that PE, as characterized in Proposition 2, exists. To begin with, let us denote the order statistics, min $\{v_i, v_j\}$, by $v_{(2)}$ and max $\{v_i, v_j\}$, by $v_{(1)}$. Now consider the following 4 cases corresponding to the realizations of $v_{(1)}$ and $v_{(2)}$.

Case 1: $v_{(1)} \in [m, \overline{v}]$: Note that incumbent bidding strategies, in the range of positive bids, is identical in the two equilibria. Therefore, for any given realization of v_i and v_j such that the higher of the two, $v_{(1)}$, lies in the interval $[m, \overline{v}]$, the outcome of SE and PE are identical.

Case 2: $v_{(1)} \in [\omega^*, m)$ and $v_{(2)} \in [\omega^*, m)$: In this case, the SE selects, with equal probability, from among the two incumbents, to bid against E. On the other hand, in PE, the incumbent that ends up bidding against E is the one that plays the precautionary strategy of participating for all its types. With probability 1/2, this is the incumbent whose type is $v_{(1)}$ and with probability 1/2, this is the incumbent whose type is $v_{(2)}$. As a result the distribution of outcomes in SE and PE are identical.

Case 3: $v_{(1)} \in [\omega^*, m)$ and $v_{(2)} \in [0, \omega^*)$: In this case, the outcome in SE is that the only incumbent that participates has type $v_{(1)}$. As a result, she goes on to pre-empt E with probability $F\left(v_{(1)} + e\right)$. On the other hand, in PE, (i) with probability 1/2, the incumbent that always participates is the one who has type $v_{(1)}$ and (ii) with probability 1/2, that incumbent has type $v_{(2)}$. In the event of (i), the probability of entry of deterrence is exactly the same as in SE for case 3. In the event of (ii), the probability of entry deterrence is $F\left(v_{(2)} + e\right)$. Since $v_{(1)} \geq v_{(2)}$, relative to the PE, the SE deters entry with higher probability.

Case 4: $v_{(1)} \in [0, \omega^*)$: In this case, the outcome in SE is that E wins the auction by virtue of being the only bidder, i.e. entry occurs for sure. On the other hand, in PE, (i) with probability 1/2, the incumbent that always participates is the one who has type $v_{(1)}$ and (ii) with probability 1/2, that incumbent has type $v_{(2)}$. In the event of (i), entry is deterred with probability $F\left(v_{(1)} + e\right)$. In the event of (ii), entry is deterred with probability $F\left(v_{(2)} + e\right)$. Thus, relative to the SE, the PE deters entry with a higher probability.

Because of the different equilibrium rankings in cases 3 and 4, and the implicit nature of ω^*, the pursuit of an overall ranking of the two equilibria, in terms of the probability of entry deterrence in each one of them, does not seem to hold much promise. A sharp conclusion can, however, be drawn for the case in which $e \geq \overline{v}$, i.e., the case in which e is "prohibitively" large so that even the lowest type of any I will outbid E. In the SE, the probability that neither i nor j participate, leaving E to win the auction, is $\left[F(\omega^*)^2\right]$, which is strictly positive since $\omega^* > 0$ for any $e > 0$. On the other hand, in the PE, i always participates, thus guaranteeing that E can never win the auction. In other words, when entry lowers the payoffs to I by a sufficiently high margin, then entry is deterred with probability 1 in the PE, but only with probability strictly less than 1 in the SE. In this sense, the PE is more entry-deterring than the SE.[17]

5 Concluding remarks

It may be worth reiterating that in a sealed-bid auction format (e.g., a first-price sealed-bid auction), the equilibrium outcome will typically differ from that in the open auction format considered here. In the first-price sealed-bid format, bidders submit bids simultaneously, and the highest bidder wins the auction. The option to i of submitting the minimum bid so as to secure the *option* to bid against E, in the event j turns out not to have participated in the auction, is not available in the sealed-bid format. Bidders will thus have to take their "best-shot" when submitting bids. The absence of this dynamic aspect to determining one's optimal bid will alter the marginal considerations from those described in Section 3. For that reason, it is unlikely that for "moderately large" values of e, a sealed-bid auction can have a precautionary equilibrium.

We have shown that incumbent preference for entry deterrence may invalidate inference about bidder collusion which may be made by employing standard observations about realized bid patterns. The seemingly-collusive nature of the precautionary equilibrium bids points to the need for more market-specific diagnostics of collusive behavior. For that same reason, the suspected lack of a precautionary equilibrium in the sealed-bid format can only reinforce its reputation as relatively more collusion-proof. An interesting avenue for further research consists of examining joint bidding by incumbents. Joint bidding is a common phenomenon in many auction markets, e.g. in the auction of offshore oil drilling rights (see Hendricks and Porter (1995)). Such a mechanism is a natural way for the incumbents to share the provision of entry pre-emption (in this context, see

[17] To be sure, for "prohibitively" large values of e, even the second-price *sealed-bid* auction will possess the following asymmetric Nash equilibrium: i bids $\overline{v}$, j does not participate (ϕ), whereas E plays her dominant strategy of bidding v_E. In that equilibrium, the winner of the auction is i, regardless of the valuations of i and j. The absence of any uncertainty regarding that equilibrium allocation makes it quite unlike the precautionary equilibrium of the open ascending-price auction. In the latter, j does participate for high realizations of v_j, and wins the auction if $v_j > v_i \geq m$. Therefore, for "prohibitively" large values of e, the precautionary equilibrium of the open ascending-price auction, in addition to deterring entry, realizes a higher surplus.

also Caillaud and Jehiel (1998) for an analysis of why, in the presence of allocative externalities, it may be optimal for the seller to design a mechanism that induces a "collusive" outcome).

6 Appendix: proof of Propositions 1 and 2

The proofs of Propositions 1 and 2 involve the common task of deriving the best-response bidding function. Once this is accomplished, the proofs consist of identifying the fixed points of these best-response functions.

6.1 Additional notation

We will begin by introducing the following notation. Let bidder I's bid amount (conditional on participation) be $b_I(v_I)$. That is, $\sigma_I(v_I) \in \{\phi, b_I(v_I)\}$. Recall, from Section 2, our interest in PBE in strategies that are non-decreasing functions of bidder valuations (i.e. functions $\sigma_i(.)$ and $\sigma_j(.)$ that are non-decreasing in their arguments), and that are strictly increasing, whenever they are (strictly) positive in value. That implies the following form for $\sigma_j(v_j)$ (and similarly for σ_i):

$$\sigma_j(v_j) = \begin{cases} \phi & \text{if} \quad v_j < \omega_j \\ b_j(v_j) & \text{if} \quad v_j \geq \omega_j, \end{cases}$$

where, $b_j(.)$ is a function that is non-decreasing (strictly increasing whenever strictly positive), and ω_j is some number that belongs to the interval $(0, \overline{v}]$.

Now, let $G(z|p)$ denote the probability that $\widetilde{v}_j \leq z$, conditional on $b_j \geq p$ (to limit notation, we will refer to bidder I's initial bid amount by b_I). Let the corresponding probability density be denoted by $g(z|p)$. For any $p > 0$, the following updated beliefs are implied by the strictly increasing nature of $b_j(.)$:

$$G(z|p) = \begin{cases} \frac{F(z)}{1-F\left(b_j^{-1}(p)\right)} & \text{if} \quad z \geq b_j^{-1}(p) \\ 0 & \text{otherwise.} \end{cases}$$

Now, consider bidder i's problem of choosing her optimal strategy. In order to derive σ_i, we will need to first compute i's payoff in the ascending-price phase of the auction game. Towards that end, let us proceed by considering each possible identity of χ in the bidding history.

6.2 Continuation payoffs in the ascending-price phase

First, consider the history $H \equiv (E, p)$ for some $p > 0$. In this case, recall from Section 2, $\theta_I(v_I) = \max\{p, v_I\}$ for each $I = i, j$. Therefore, bidder i's expected payoff in the continuation game is given by:

$$\pi_i((E,p); v_i) = \begin{cases} (v_i - p) G(p|p) + \max\left\{0, \int_{b_j^{-1}(p)}^{v_i} (v_i - z) g(z|p) dz\right\} & \text{if} \quad p \leq v_i \\ \frac{1}{2}(v_i - p) G(p|p) & \text{if} \quad p > v_i. \end{cases} \tag{4}$$

Next, consider the history $H \equiv (j,p)$ for some $p > 0$. Note that since bidder i's maximum willingness-to-pay is $v_i + e$, the strategy $\sigma_i(v_i) = v_i + e$ dominates every strategy in which $\sigma_i(v_i) > v_i + e$. Therefore, from Section 2, we have $\theta_i(v_i) = v_i + e$. Also recall from Section 2 that bidder E follows her dominant strategy of bidding $\sigma_E(v_E) = \theta_E(v_E, H) = v_E$. Therefore, bidder i's updated belief that $\widetilde{v}_E \leq z$, conditional on observing $\sigma_E \geq p$, is given by $\frac{F(z)}{1-F(p)}$. Thus, bidder i's continuation payoff in this case is:

$$\pi_i((j,p);v_i) = \int_p^{v_i+e} (v_i - z) \frac{f(z)\,dz}{1-F(p)} - e\left[\frac{1-F(v_i+e)}{1-F(p)}\right]. \tag{5}$$

Finally, consider the history $H \equiv (i,p)$ for some $p > 0$. Recalling from Section 3, the ascending-price phase strategies of j and E, and noting that i suffers the externality $-e$ only if E emerges the auction winner, we can write bidder i's expected continuation payoff as:

$$\pi_i((i,p);v_i) = -e\int_{b_j^{-1}(p)}^{\overline{v}} \left[\frac{1-F(z+e)}{1-F(p)}\right] g(z|p)\,dz. \tag{6}$$

6.3 Strictly positive bids

The next step is to use the above expressions in order to write down bidder i's expected payoff for some initial candidate bid b_i. Suppose $b_i > 0$, and that $b_j(v_j) \in (b_i - \epsilon, b_i + \epsilon)$ for some $v_j \in [0, \overline{v}]$. Notice that for this candidate bid, the feasible histories are given by $\mathcal{H}(b_i) := \{(\chi, p)\,|\,\chi \in \{i,j,E\}, 0 \leq p \leq b_i\}$. Each feasible history, and the associated probability, is graphically depicted in Figure 1.

Therefore, bidder i's expected payoff is given by:

$$\begin{aligned}\Pi_i(b_i;v_i) \quad = \quad & \int_0^{b_i} \pi_i((E,p);v_i)\left[1-F(b_j^{-1}(p))\right] dF(p) \\ & + \int_0^{b_i} \pi_i((j,p);v_i)\left[1-F(p)\right] dF\left(b_j^{-1}(p)\right) \\ & + \pi_i((i,b_i);v_i)\left[1-F(b_j^{-1}(b_i)\right]\left[1-F(b_i)\right].\end{aligned} \tag{7}$$

Whenever strictly positive, bidder i's optimal bid b_i is chosen to maximize $\Pi_i(b_i;v_i)$. Now, if $b_i > 0$ is in fact the expected payoff maximizing initial bid for i, then the following (interior) first-order necessary condition must be satisfied:

$$\frac{\partial \Pi_i(b_i;v_i)}{\partial b_i} = 0,$$

which works out to

$$\begin{aligned}0 \quad = \quad & \pi_i((E,b_i);v_i)\left[1-F(b_j^{-1}(b_i)\right] f(b_i) \\ & + \pi_i((j,b_i);v_i)\left[1-F(b_i)\right] f\left(b_j^{-1}(b_i)\right) b_j^{-1\prime}(b_i) \\ & + \frac{\partial}{\partial b_i}\left\{\pi_i((i,b_i);v_i)\left[1-F(b_j^{-1}(b_i)\right]\left[1-F(b_i)\right]\right\}.\end{aligned}$$

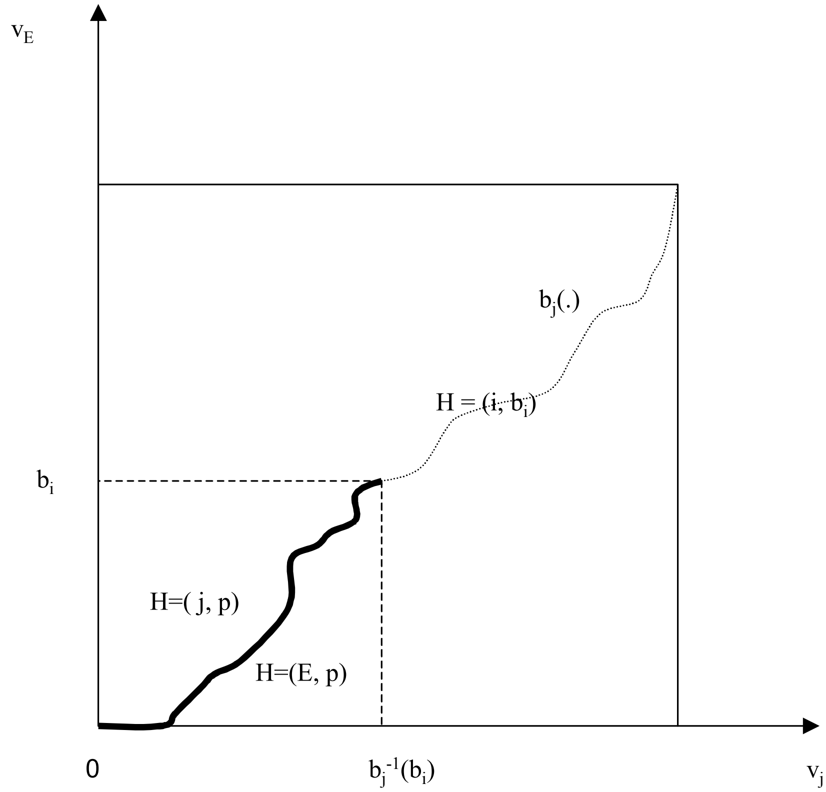

Figure 1: Feasible bidding histories

Using the expression for the updated belief $G(z|p)$, the first-order condition simplifies to

$$\begin{aligned}
0 &= \pi_i\left((E,b_i);v_i\right)\left[1-F(b_j^{-1}(b_i))\right]f(b_i) \\
&+\left\{\int_{b_i}^{v_i+e}(v_i-z)f(z)\,dz-e\left[1-F(v_i+e)\right]\right\}f\left(b_j^{-1}(b_i)\right)b_j^{-1\prime}(b_i) \\
&-e\frac{\partial}{\partial b_i}\left\{\int_{b_j^{-1}(b_i)}^{\overline{v}}\left[1-F(z+e)\right]f(z)\,dz\right\},
\end{aligned}$$

which further simplifies to

$$\begin{aligned}
0 &= \pi_i\left((E,b_i);v_i\right)\left[1-F(b_j^{-1}(b_i))\right]f(b_i) \\
&+\left\{\int_{b_i}^{v_i+e}(v_i-z)f(z)\,dz-e\left[1-F(v_i+e)\right]\right\}f\left(b_j^{-1}(b_i)\right)b_j^{-1\prime}(b_i) \\
&+e\left[1-F\left(b_j^{-1}(b_i)+e\right)\right]f\left(b_j^{-1}(b_i)\right)b_j^{-1\prime}(b_i).
\end{aligned}$$

Collecting together like terms, the first-order condition can be written as:

$$\begin{aligned}
0 &= \pi_i\left((E,b_i);v_i\right)\left[1-F(b_j^{-1}(b_i))\right]f(b_i) \qquad (8)\\
&+\left\{\int_{b_i}^{v_i+e}(v_i-z)f(z)\,dz-e\left[1-F(v_i+e)\right]+e\left[1-F\left(b_j^{-1}(b_i)+e\right)\right]\right\}f\left(b_j^{-1}(b_i)\right)b_j^{-1\prime}(b_i).
\end{aligned}$$

Claim 1 *$b_i(.)$ and $b_j(.)$ must be symmetric.*

Proof. The proof is by contradiction. Suppose there exists a positive bid amount b, such that $b = b_i(v_i) = b_j(v_j)$, where $v_i = b_i^{-1}(b) \neq b_j^{-1}(b) = v_j$. Suppose also, without loss of generality, that $v_i > v_j$. Since $b_j(v_j)$ must satisfy the first-order condition given by (8), we have

$$\begin{aligned} 0 &= \pi_j((E,b);v_j)[1-F(v_i)]f(b) \\ &+ \left\{ \int_b^{v_j+e} (v_j - z) f(z) dz - e[1-F(v_j+e)] + e[1-F(v_i+e)] \right\} f(v_i) b_i^{-1'}(b) \end{aligned} \tag{9}$$

Observe that since $v_i > v_j$, if $v_j < b$, then the expression within the curly brackets in the second line of (9) is strictly negative. Notice also from (4) that $\pi_j((E,b);v_j) < 0$ if $v_j < b$. But then, since $b_j^{-1'}(b) > 0$, the right hand side of (9) must be strictly negative, implying that (9) cannot be satisfied. Therefore, we must have $v_j \geq b$. This implies that $G(p|p) = 0$. Therefore, using the first line on the right-hand side of (4) and the expression for $G(z|p)$, we can rewrite (9) as:

$$\begin{aligned} 0 &= \max\left\{0, \int_{v_i}^{v_j} (v_j - z) f(z) dz\right\} f(b) \\ &+ \left\{ \int_b^{v_j+e} (v_j - z) f(z) dz - e[1-F(v_j+e)] + e[1-F(v_i+e)] \right\} f(v_i) b_i^{-1'}(b). \end{aligned} \tag{10}$$

By symmetry, and the fact that $v_i > v_j$, the corresponding first-order condition for bidder i is:

$$\begin{aligned} 0 &= \max\left\{0, \int_{v_j}^{v_i} (v_i - z) f(z) dz\right\} f(b) \\ &+ \left\{ \int_b^{v_i+e} (v_i - z) f(z) dz - e[1-F(v_i+e)] + e[1-F(v_j+e)] \right\} f(v_j) b_j^{-1'}(b). \end{aligned} \tag{11}$$

First note that if (as assumed) $\frac{F(v+e)}{f(v+e)} \geq e$ for all $v \leq \overline{v} - e$, then the expression $\int_b^{v+e} (v-z) f(z) dz$ must be strictly increasing in v. To verify this, we can simply differentiate the expression with respect to v, which yields $F(v+e) - ef(v+e)$, whereupon the assertion follows. Thus, since $v_i > v_j$, the expression within curly brackets in the second line of (11) must be strictly higher in value than the analogous expression in (10). Therefore, the only way the two first-order conditions can both be satisfied is if $\max\left\{0, \int_{v_i}^{v_j} (v_j - z) f(z) dz\right\} = 0 > \max\left\{0, \int_{v_j}^{v_i} (v_i - z) f(z) dz\right\} > 0$. But this is impossible. Hence, we conclude that our initial hypothesis is wrong and that $b_i(.)$ and $b_j(.)$ are symmetric. QED.

Claim 2 *Whenever strictly positive, the equilibrium bid function (symmetric for i and j) is unique and is given by:*

$$b^*(v) = \left\{ b | \int_b^{v+e} (v-z) f(z) dz = 0 \right\}.$$

Proof. By Claim 1, we can impose symmetry in the necessary condition (8), which yields for bidder i:

$$0 = \pi_i((E, b^*(v_i)); v_i)[1-F(v_i)] f(b^*(v_i)) + \left\{ \int_{b^*(v_i)}^{v_i+e} (v_i - z) f(z) dz \right\} f(v_i) b^{*-1'}(b^*(v_i)). \tag{12}$$

Now, from the second line of (4), we realize that if $b^*(v_i) > v_i$, then $G(b^*(v_i)|b^*(v_i)) = \frac{F(b^*(v_i))-F(v_i)}{1-F(v_i)} > 0$, for which, the first line on the right hand side of (4) would imply that $\pi_i((E, b^*(v_i)); v_i) < 0$. Also, if $b^*(v_i) > v_i$, then the integral $\int_{b^*(v_i)}^{v_i+e}(v_i - z) f(z) dz$ must be strictly negative. In that case, the right hand side of (12) is strictly negative, thus violating the necessary condition. Therefore, we conclude that $b^*(v_i) \leq v_i$.

Now, since $b^*(v_i) \leq v_i$, $G(b^*(v_i)|b^*(v_i)) = 0$. Therefore,

$$\pi_i((E, b^*(v_i)); v_i) = \int_{b^{*-1}(b^*(v_i))}^{v_i} (v_i - z) g(z|p) dz = 0.$$

Inserting this into (12), and recalling that $b^*(v_i)$ is strictly increasing, we get:

$$\int_{b^*(v_i)}^{v_i+e} (v_i - z) f(z) dz = 0. \tag{13}$$

It remains to verify that $b^*(v_i)$ in (13) is in fact a strictly increasing function. But this follows from (i) the fact that the expression $\int_b^{v+e}(v - z) f(z) dz$ is strictly increasing in v for any given b (as proved during the course of Claim 1), and (ii) the fact that the same expression is strictly decreasing in b for any $b < v$. To see (ii), simply differentiate the expression with respect to b, which yields $-(v - b)f(b)$, whereupon, (ii) follows. Note also that in order to satisfy (13), it must be that $b < v$, thus completing the proof. *QED.*

From the definition of the equilibrium bid function b^*, it is clear that the *highest* type of bidder I that will bid an amount 0 is v^* where v^* is given by:

$$\int_0^{v^*+e} (v^* - z) f(z) dz = 0. \tag{14}$$

Clearly, for (14) to hold, it must be that for any value of e, $v^* > 0$, i.e. not all types of I will submit a strictly positive initial bid. In particular, observe that as e increases above $\overline{v} - m$, v^* converges to m, which must be strictly greater than the lower bound of the support of F, which is 0. Our interest hereafter will be on deriving the equilibrium bidding behavior of the set of types $[0, v^*]$ of $I = i, j$. To be precise, we will be interested in knowing which types of $\{i, j\}$ will prefer to not participate in the auction, as opposed to submitting an initial bid of 0.

6.4 Participation v. non-participation

Towards that end, we begin with the conjecture (to be subsequently verified) that the following partition of j's type space delineates her equilibrium strategy of non-participation, from participation with an initial bid of 0. Suppose there exists ω_j such that bidder j's strategy is given by:

$$\sigma_j(v_j) = \begin{cases} \phi & \text{if} \quad 0 \leq v_j \leq \omega_j \\ 0 & \text{if} \quad \omega_j < v_j \leq v^* \\ b^*(v_j) & \text{if} \quad v^* < v_j \leq \overline{v}. \end{cases} \tag{15}$$

Our goal is to find the best response of each type of bidder i in the interval $[0, v^*]$, between the two available strategies ϕ and a bid of 0, to j's strategy ω_j (the only parameter of $\sigma_j(.)$ in (15) that remains to be determined), as hypothesized above. To accomplish this, we need to compare i's expected payoff for each of these two alternative strategies. Let us begin with $\sigma_i(v_i) = \phi$. In this case, if $\omega_j \geq v_j$, then E is the only auction participant (and therefore the winner), whereas if $\omega_j < v_j$, then E wins the auction only if $v_E > v_j + e$. Denoting the payoff by ξ (in order to keep it distinct from payoffs defined for the range of strictly positive bids in which ties do not occur), we have:

$$\xi_i(\phi; v_i, \omega_j) = -eF(\omega_j) - e\int_{\omega_j}^{\overline{v}} [1 - F(z+e)] f(z)\, dz. \tag{16}$$

On the other hand, for $\sigma_i(v_i) = 0$, there are 3 possibilities. Before considering these possibilities, recall from Section 2 that $\sigma_E(v_E) = v_E$. Hence E's bid is strictly positive with probability 1. Now let us turn to the 3 possibilities. First, if $\omega_j \geq v_j$, then i and E compete in the ascending-price phase. Second, if $\omega_j < v_j \leq v^*$, then with equal probability, one of i and j is selected to bid against E. Finally, if $v^* < v_j \leq \overline{v}$, then j and E compete in the ascending-price phase. Therefore,

$$\begin{aligned} \xi_i(0; v_i, \omega_j) &= F(\omega_j)\left\{\int_0^{v_i+e} (v_i - z) f(z)\, dz - e[1 - F(v_i+e)]\right\} \\ &+ \frac{1}{2}[F(v^*) - F(\omega_j)]\left\{\int_0^{v_i+e} (v_i - z) f(z)\, dz - e[1 - F(v_i+e)]\right\} \\ &+ \frac{1}{2}[F(v^*) - F(\omega_j)]\left\{-e\int_{\omega_j}^{v^*} [1 - F(z+e)] \frac{f(z)}{[F(v^*) - F(\omega_j)]} dz\right\} \\ &- e\int_{v^*}^{\overline{v}} [1 - F(z+e)] f(z)\, dz. \end{aligned} \tag{17}$$

Let us denote the best response of type v_i, to j's strategy of ω_j, by $\Omega_i(\omega_j; v_i)$. Also, denote by $P_i(\omega_j; v_i)$ the expression $\xi_i(0; v_i, \omega_j) - \xi_i(\phi; v_i, \omega_j)$, which measures the difference in i's expected payoffs between submitting an initial bid 0, and not participating in the auction. Then i's best response is characterized by

$$\Omega_i(\omega_j; v_i) = \begin{cases} \phi & \text{if} \quad P_i(\omega_j; v_i) < 0 \\ 0 & \text{if} \quad P_i(\omega_j; v_i) \geq 0. \end{cases}$$

Consider the expression for $P_i(\omega_j; v_i)$ that results from using (16) and (17). After algebraic simplification, we get:

$$\begin{aligned} P_i(\omega_j; v_i) &= F(\omega_j)\{e + \pi_i((j,0); v_i)\} \\ &+ \frac{1}{2}[F(v^*) - F(\omega_j)]\left\{\pi_i((j,0); v_i) + e\int_{\omega_j}^{v^*} [1 - F(z+e)] \frac{f(z)}{[F(v^*) - F(\omega_j)]} dz\right\}, \end{aligned} \tag{18}$$

where $\pi_i((j,0); v_i)$ is as defined in (5) with $p = 0$.

We begin by computing the value of $P_i(\omega_j; v_i)$ at $v_i = v^*$ for some arbitrary value of $\omega_j \in [0, v^*]$ (recall the definition of v^* in (14)). First, observe that for any v_i,

$$\begin{aligned} \pi_i\left((j,0);v_i\right) &= \int_0^{v_i+e} (v_i - z) f(z)\, dz - e\left[1 - F(v_i + e)\right] > -e \\ &\Rightarrow \ e + \pi_i\left((j,0);v_i\right) > 0. \end{aligned} \tag{19}$$

Also, since F is a non-decreasing function,

$$\begin{aligned} \int_{\omega_j}^{v^*} \left[1 - F(z+e)\right] \frac{f(z)}{\left[F(v^*) - F(\omega_j)\right]} dz &\geq \left[1 - F(v^* + e)\right] \int_{\omega_j}^{v^*} \frac{f(z)}{\left[F(v^*) - F(\omega_j)\right]} dz \\ &= \left[1 - F(v^* + e)\right]. \end{aligned}$$

Therefore,

$$\begin{aligned} &\pi_i\left((j,0);v^*\right) + e \int_{\omega_j}^{v^*} \left[1 - F(z+e)\right] \frac{f(z)}{\left[F(v^*) - F(\omega_j)\right]} dz \\ &\qquad \geq \pi_i\left((j,0);v^*\right) + e\left[1 - F(v^* + e)\right] \\ &\qquad = \int_0^{v^*+e} (v^* - z) f(z)\, dz \equiv 0, \end{aligned} \tag{20}$$

where the last two lines follows from (5) and (14) respectively.

Hence, each expression, which is enclosed within curly brackets on the right hand side of (18), is non-negative. Therefore, for any value of $\omega_j \in [0, v^*]$, $P_i(\omega_j; v^*) \geq 0$. Moreover, straightforward differentiation yields:

$$\frac{\partial \pi_i\left((j,0);v_i\right)}{\partial v_i} = F(v_i + e) > 0. \tag{21}$$

Therefore, from the expression in the right hand side of (18), we have that for any given ω_j, $P_i(\omega_j; v_i)$ is strictly increasing in v_i. Together with the finding that $P_i(\omega_j; v^*) \geq 0$, we can then state that for a given $\omega_j \in [0, v^*]$, there must exist a unique value of v_i that is given by

$$\varphi_i(\omega_j) \equiv \inf\left\{v_i \in [0, v^*] \,|\, P_i(\omega_j; v_i) \geq 0\right\}. \tag{22}$$

Bidder i's best response then takes the following form:

$$\Omega_i(\omega_j; v_i) = \begin{cases} \phi & \text{if} \quad v_i < \varphi_i(\omega_j) \\ 0 & \text{if} \quad v_i \geq \varphi_i(\omega_j). \end{cases}$$

6.4.1 Participation best-responses

Our immediate objective is to characterize the best response function $\varphi_i(\omega_j)$. Recall from Claims 1 and 2 that in the range of strictly positive bids i and j must employ symmetric bidding strategies. Then, ex-ante symmetry between i and j further implies that the best-response function of bidder i, $\varphi_i(\omega_j)$, must be identical to the best-response function of bidder j that can be analogously defined as $\varphi_j(\omega_i)$. As a consequence, for an equilibrium in symmetric strategies (i.e. an equilibrium in which $\sigma_i(.) = \sigma_j(.)$), there needs to exist a fixed point of the best response function $\varphi_i(\omega_j)$, i.e., there must exist some $\omega^* \in [0, v^*]$ such that:

$$\varphi_i(\omega^*) = \omega^*. \tag{23}$$

Our next objective would be to use the characterization of $\varphi_i(\omega_j)$ so obtained in order to show the existence of such a fixed point.

Towards these objectives, first note from (18) that $P_i(\omega_j; v_i)$ is continuous in each one of its arguments ω_j and v_i. Therefore, by the Implicit Function Theorem, we have that the function $\varphi_i(.)$ must be continuous and differentiable everywhere on the open interval $(0, v^*)$. Furthermore, whenever $P_i(\omega_j; \varphi_i(\omega_j)) > 0$, the following must hold: (i) $\varphi_i(\omega_j) = 0$ and (ii) $\varphi_i'(\omega_j) = 0$, where $\varphi_i'(\omega_j)$ denotes the derivative of $\varphi_i(\omega_j)$ with respect to ω_j. On the other hand whenever $P_i(\omega_j; \varphi_i(\omega_j)) = 0$, we can apply the Implicit Function Theorem to (22) in order to examine the slope of the function $\varphi_i(\omega_j)$ as:

$$\varphi_i'(\omega_j) = -\left(\frac{\partial P_i}{\partial \omega_j}\right) / \left(\frac{\partial P_i}{\partial v_i}\right).$$

Using the expression for $P_i(\omega_j; v_i)$ in (18) to actually calculate this slope, we get:

$$\varphi_i'(\omega_j) = -\frac{f(\omega_j)\left[e + eF(\omega_j + e) + \pi_i((j,0); \varphi_i(\omega_j))\right]}{\left[F(v^*) + F(\omega_j)\right] F(\varphi_i(\omega_j) + e)}, \tag{24}$$

wherein we have used (21) in simplifying expressions.

Now, from (19) we know that $e + \pi_i((j,0); \varphi_i(\omega_j)) > 0$. The right hand side of (24) is then strictly negative, establishing that whenever $\varphi_i(\omega_j)$ lies within the interval $(0, v^*]$, it is a strictly decreasing function.

Together with the slope of $\varphi_i(\omega_j)$, we will also need to develop bounds on its value at the two extreme points of interest, viz. $\omega_j = 0$ and $\omega_j = v^*$, in order to investigate the nature of its fixed points (recall that $v^* > 0$). We begin by evaluating $P_i(\omega_j; v_i)$ at $\omega_j = 0$. In this case, for any v_i, (18) reduces to

$$P_i(0; v_i) = \frac{1}{2} F(v^*) \left\{ \pi_i((j,0); v_i) + e \int_0^{v^*} \left[1 - F(z+e)\right] \frac{f(z)}{F(v^*)} dz \right\}. \tag{25}$$

Now, we already know, from (20), that the term within curly brackets on the right hand side of (25) is non-negative for $v_i = v^*$. In particular, $P_i(0; v^*) > 0$ if $v^* + e < \overline{v}$, whereas, $P_i(0; v^*) = 0$ if $v^* + e \geq \overline{v}$. This in turn implies that $\varphi_i(0) \leq v^*$. Recall also, from Section 3, that $v^* + e \geq \overline{v}$ $\Rightarrow v^* = m$.

On the other hand, consider the expression $P_i(0; 0)$, which (recall (18)), is given by:

$$P_i(0; 0) = \frac{1}{2} F(v^*) \left\{ \pi_i((j,0); 0) + e \int_0^{v^*} \left[1 - F(z+e)\right] \frac{f(z)}{F(v^*)} dz \right\}.$$

However,

$$\int_0^{v^*} [1 - F(z+e)] \frac{f(z)}{F(v^*)} dz \leq [1 - F(e)] \int_0^{v^*} \frac{f(z)}{F(v^*)} dz = [1 - F(e)].$$

Therefore,

$$\begin{aligned} P_i(0;0) &\leq \frac{1}{2} F(v^*) \{\pi_i((j,0);0) + e[1 - F(e)]\} \\ &= \frac{1}{2} F(v^*) \int_0^e (-z) f(z) dz < 0, \end{aligned}$$

where the last line follows from (5). This in turn implies that $\varphi_i(0) > 0$. Collecting together the above findings, we have the following bounds for $\varphi_i(0)$:

$$0 < \varphi_i(0) \leq v^*, \tag{26}$$

where strict equality in the upper bound holds only if $e \geq \overline{v} - m$.

We next turn to evaluating $P_i(\omega_j; v_i)$ at $\omega_j = v^*$ in order compute $\varphi_i(v^*)$. With $\omega_j = v^*$, (18) reduces to

$$P_i(v^*; v_i) = F(v^*) \{e + \pi_i((j,0); v_i)\},$$

which, as already argued in (19), must be strictly positive for all $v_i \in [0, v^*]$. Together with the previously argued property of continuity of the best-response function $\varphi_i(.)$, we conclude that there must exist a non-degenerate interval $(\omega_0, v^*]$ such that,

$$\varphi_i(\omega_j) = 0 \text{ for all } \omega_j \in (\omega_0, v^*]. \tag{27}$$

6.5 Equilibria

Collecting together the properties of the best response developed so far, we have that $\varphi_i(\omega_j) : [0, v^*] \to [0, v^*]$ is a continuous function which is strictly decreasing whenever it is positive, and non-increasing whenever equal to zero. Furthermore $\varphi_i(0) > 0$, whereas $\varphi_i(v^*) = 0$. Such a function must have a unique fixed point ω^*, as defined in (23), where $\omega^* \in (0, v^*)$. Figure 2 illustrates the existence and strictly interior nature of the fixed point. The bold line in Figure 2 denotes i's participation best-response function $\varphi_i(.)$, and the bold dot (point S) in Figure 2 denotes its fixed point. The fixed point represents the equilibrium of the auction game in which bidders i and j employ symmetric strategies. The symmetric equilibrium is unique, and is characterized by a positive measure of types that do not participate, a positive measure of types that pool at the minimum bid 0, and the remaining positive measure of types that employ the bid function $b^*(.)$.

The dotted lines in Figure 2 (need not be straight lines) denote the best response functions $\varphi_i(\omega_j)$ and $\varphi_j(\omega_i)$ when the magnitude of the externality e is at least as large as $\overline{v} - m$. In this case, from (26), we have that $\varphi_i(0) = m$, whereas from the symmetric nature of the best-response functions and (27), we also have $\varphi_j(m) = 0$. Therefore, the best response functions must

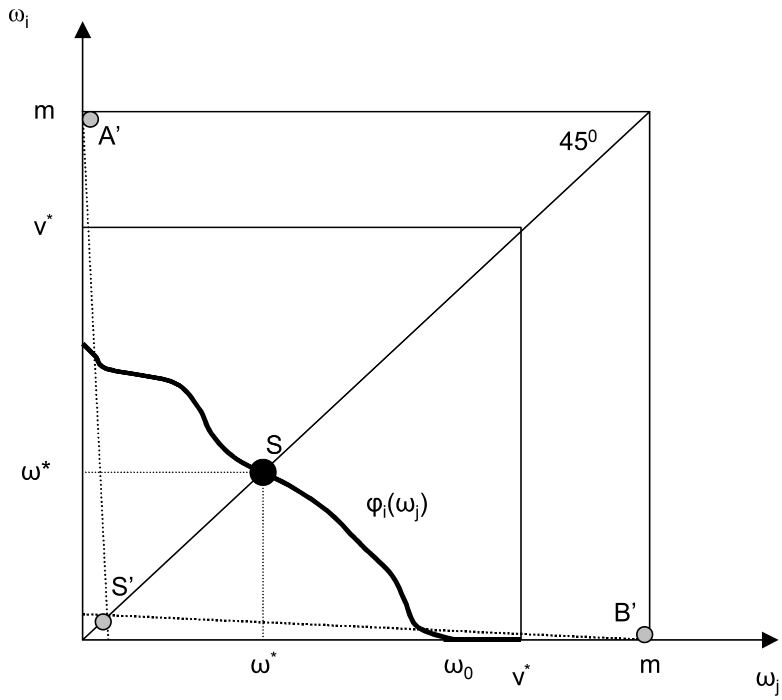

Figure 2: Best-responses and fixed points

also intersect each other at the points depicted by the gray dots (points A', B' in Figure 2), viz. $(\omega_i = m, \omega_j = 0)$ and $(\omega_i = 0, \omega_j = m)$, which represent asymmetric equilibria, where only one of the two bidders $\{i, j\}$ participates in the auction regardless of her realized type, whereas the other participates only if her type is at least m. In other words, the two bidders i, j employ symmetric bid functions $b^*(.)$ for types $[m, \overline{v}]$, but employ different strategies for their remaining types $[0, m)$. One bidder (say i) participates for all her remaining types $[0, m)$ only to pool at bid 0, whereas her opponent j chooses to not participate (ϕ) for all such types. The symmetric equilibrium continues to exist even for $e \geq \overline{v} - m$, and is denoted by the gray dot S' in Figure 2.

6.6 Existence of equilibria

To establish existence of the equilibria characterized above, it remains to be proved that for each of the symmetric and asymmetric candidate equilibria characterized above, each type of bidders i and j in the set $(v^*, \overline{v}]$ in fact prefer to employ the strictly increasing strategy $b^*(.)$ instead of any one of the following possible deviations: (i) some other strictly positive bid, (ii) the minimum initial bid (0) or (iii) non-participation (ϕ).

We begin with (i). That $b^*(.)$ satisfies not only the first-order necessary condition, but is also optimal within the set of strictly positive bids, is checked by verifying that $\Pi_i(b_i; v_i)$, as defined in (7), exhibits increasing differences in $(b_i; v_i)$. Since $\Pi_i(b_i; v_i)$ is differentiable on the set $(v^*, \overline{v})$, this consists of the check that, holding fixed the strategy of bidder j as $b^*(.)$, $\Pi_i(b_i; v_i)$ has a strictly positive cross-partial derivative. To do the check, recall (8) as:

$$\frac{\partial \Pi_i(b_i; v_i)}{\partial b_i} = \pi_i((E, b_i); v_i)\left[1 - F(b_j^{-1}(b_i))\right] f(b_i) \tag{28}$$
$$+\left\{\int_{b_i}^{v_i+e}(v_i - z) f(z)\, dz - e\left[1 - F(v_i + e)\right] + e\left[1 - F\left(b_j^{-1}(b_i) + e\right)\right]\right\} f\left(b_j^{-1}(b_i)\right) b_j^{-1'}(b_i).$$

Now, observe that the derivative, with respect to v_i, of the expression within the curly brackets on the second line of the right hand side of (28) is, after cancellation of terms, $F(v_i + e) - F(b_i)$. The derivative, with respect to v_i, of the term $\pi_i((E, b_i); v_i)$ (as defined in (4)) is given by:

$$\frac{\partial \pi_i((E, b_i); v_i)}{\partial v_i} = \begin{cases} G(b_i|b_j) & \text{if} \quad b_i \leq v_i \\ \frac{1}{2}G(b_i|b_j) & \text{if} \quad b_i > v_i. \end{cases}$$

Pulling these partial derivatives together, we have:

$$\frac{\partial^2 \Pi_i(b_i; v_i)}{\partial v_i \partial b_i} = cG(b_i|b_j)\left[1 - F(b^{*-1}(b_i))\right] f(b_i) + \left[F(v_i + e) - F(b_i)\right] f\left(b^{*-1}(b_i)\right) b^{*-1'}(b_i), \tag{29}$$

where

$$c = \begin{cases} 1 & \text{if} \quad b_i \leq v_i \\ 1/2 & \text{if} \quad b_i > v_i. \end{cases}$$

Now, recall that any $b_i > v_i + e$ is strictly dominated by $b_i = v_i + e$. Also, observe that since $b^*(\overline{v}) = \overline{v}$, the range of $b^*(.)$ is identical to the support of bidder E's bids, viz. $[0, \overline{v}]$. Therefore, while considering (i), we need only contemplate deviations by i to strictly positive bids that do not exceed $\overline{v}$. This implies that for any deviation b_i that needs to be checked against, $f(b_i) > 0$, $f\left(b^{*-1}(b_i)\right) > 0$ and $b^{*-1'}(b_i) > 0$. Therefore, for every v_i except $\overline{v}$, the right hand side of (29) is strictly positive. Hence, for all $v_i \in [v^*, \overline{v}]$, $b^*(v_i)$ dominates every other strictly positive bid. Note also that for $v_i = \overline{v}$, it is a dominant strategy to submit an initial bid equal to $\overline{v}$. To see this, note that for non-participation, type $\overline{v}$'s payoff is bounded from above by 0. On the other hand, an initial bid equal to $\overline{v}$ generates for her a non-negative payoff, which must also be (weakly) higher than her expected payoff from bidding strictly less than $\overline{v}$, regardless of the strategy played by bidder j. The argument is identical to the dominant strategy argument of bidding one's valuation in a second price auction, which insures that a bidder wins in all states in which she earns a non-negative payoff from winning and loses to bidder j in all states in which she would have earned a negative payoff by winning.

While characterizing the participation best-response function $\varphi(.)$, we have already established that for all $v_i \geq \omega^*$, participation yields higher expected payoffs than non-participation (ϕ). Therefore, to complete the proof of existence, we only need to show that all types of bidder i in the set $(v^*, \overline{v})$ prefer to bid according to $b^*(.)$ rather than 0. Note that given the positive probability of ties at bid 0, this needs to be argued in addition to the previous arguments that established the optimality of $b^*(.)$ in the range of strictly positive bids. This will be shown to be true if it can be proved that, holding fixed the strategy of bidder j to be $\sigma_j(.)$, the expected payoff to bidder i

from using $b^*(.)$ exceeds her payoff from bidding 0 when her type is $v^* + \epsilon$ for any $\epsilon > 0$. In other words, we need to show that

$$\lim_{v_i \downarrow v^*} \Pi_i\left(b^*(v_i); v_i\right) \geq \lim_{v_i \uparrow v^*} \xi_i\left(0; v_i, \omega^*\right), \tag{30}$$

where Π_i and ξ_i are as defined in (7) and (17) respectively.

Now, we know by the definition of v^* that $b^*(v^*) = 0$. Therefore, from (7), we have:

$$\lim_{v_i \downarrow v^*} \Pi_i\left(b^*(v_i); v_i\right) = -e \int_{v^*}^{\overline{v}} \left[1 - F(z+e)\right] f(z)\, dz.$$

We also know that $\int_0^{v^*+e} (v_i - z) f(z)\, dz = 0$. Therefore, from (17), we have

$$\begin{aligned}
\lim_{v_i \uparrow v^*} \xi_i\left(0; v_i, \omega^*\right) &= F(\omega^*)\left\{-e\left[1 - F(v_i+e)\right]\right\} \\
&+ \frac{1}{2}\left[F(v^*) - F(\omega^*)\right]\left\{-e\left[1 - F(v_i+e)\right]\right\} \\
&+ \frac{1}{2}\left[F(v^*) - F(\omega^*)\right]\left\{-e \int_{\omega^*}^{v^*} \left[1 - F(z+e)\right] \frac{f(z)}{\left[F(v^*) - F(\omega^*)\right]} dz\right\} \\
&- e \int_{v^*}^{\overline{v}} \left[1 - F(z+e)\right] f(z)\, dz.
\end{aligned} \tag{31}$$

Since the first three terms on the right hand side of (31) are each negative, and the last term equals $\lim_{v_i \downarrow v^*} \Pi_i\left(b^*(v_i); v_i\right)$, the desired inequality is proved.

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