Comparative Statics with Concave and Supermodular Functions

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Abstract: Certain problems in comparative statics, including (but not exclusively) certain problems in consumer theory, cannot be easily addressed by the methods of lattice programming. One reason for this is that there is no order on the choice space which orders choices in a way which conforms with the comparison desired, and which also orders constraint sets in the strong set order it induces. The objective of this paper is to show how lattice programming theory can be extended to deal with situations like these. We show that the interaction of concavity and supermodularity in objective or constraint functions yield a structure that is very useful for comparative statics.

Keywords: lattices, concavity, supermodularity, comparative statics, demand, normality.

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Quite preliminary; comments and suggestions very welcome.

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1. Introduction

In recent years, the methods of lattice programming have been used widely and with considerable success to deal with problems in economic theory.[1] The contribution of these methods are twofold. First, they have turned out be very useful in addressing comparative statics problems which arise in many optimization or game theoretic models. Second, they have contributed to our understanding of these problems because they have helped us to identify the key mathematical features which permit their solution. The success of these methods have highlighted the underlying structural similarity of many of the seemingly different comparative statics problems which arise in economic theory.

In this paper we develop the theory of lattice programming in several directions. Our motivation is to extend the applicability of these techniques to cover an important family of comparative statics problems which have hitherto proved resistant to these methods. To explain this paper's contribution, we first consider a comparative statics problem which standard lattice programming methods can deal with very successfully.

Imagine a small firm producing a single good priced by the market at 1; producing this good requires inputs, represented by some vector x in R^l_+, whose transformation into output is captured by the production function f. Formally, the firm purchases x at prices p (in R^l_{++}) to make a profit of $\Pi(x) = f(x) - p \cdot x$. Imagine that in the short run, the first input is bounded by some number X_1. So the firm's problem is the following:

$$\text{maximize } \Pi(x) = f(x) - p \cdot x \text{ subject to } x \in C = \{x \in R^l_+ : x_1 \leq X_1\}.$$

Suppose that the optimum solution at $X_1 = X'$ is x^* and the optimum solution at $X_1 = X''$

is x^{**}. If $X'' > X'$, when can we say that $x^{**} \geq x^*$?

To this problem, standard lattice programming techniques provide an easy answer: the optimal solution will increase with X_1 if f is supermodular with respect to the *product order* on R^l. For two points x and y in R^l, $x \geq y$ in the product order if $x_i \geq y_i$ for all $i = 1, 2, ...l$. With this order, R^l is a lattice, i.e., it is a partially ordered set where every pair of points has a supremum and an infimum. Assuming f is C^2, *supermodularity* with respect to this order is equivalent to saying that the cross derivative is positive; informally, this means that all inputs are complements in the production process.

With the product order on R^l_+, the constraint sets $C' = \{x \in R^l_+ : x_1 \leq X'\}$ and $C'' = \{x \in R^l_+ : x_1 \leq X''\}$ are related to each other in a very nice way: for any x' in C' and x'' in C'', the supremum of x' and x'' is in C'', while their infimum is in C'. Whenever such a property holds, we say that C'' is greater than C' in the *strong set order* induced by the product order. That the constraint sets in our little problem can be ordered in this way is very convenient because the basic comparative statics result of lattice programming says the following: *whenever the objective function is supermodular, optimal solutions will increase with the constraint sets.* In other words, if we compare the optimal solutions at two constraint sets, with one constraint set greater than another in the strong set order, then the optimal solution at the greater constraint set will be greater than the optimal solution at the lesser constraint set. Applied to our simple example, we see that the optimal choice of *all* inputs will increase as the constraint on input 1 is relaxed.

The reason why the standard result can be applied so successfully over here has to do with the fact that an order on the choice space R^l_+ has been found which satisfies three conditions: (i) the choice variables are ordered in a way which captures the comparative statics relation the modeler is hoping to find; (ii) supermodularity with respect to this order is a reasonable condition to impose on the objective function; and (iii) this order induces a strong set order which in turn successfully orders the two constraint sets being considered. It is a remarkable fact that with so many comparative statics problems in economic theory, an order on the choice space can be found in which these three conditions are simultaneously satisfied.

But not all. A basic problem where the standard results do not apply in any obvious way is the following. Consider a consumer with a utility function $U : R^l_+ \to R$ defined over l goods, with income w and facing prices p (in R^l_{++}). Formally, he maximizes U by choosing x from the budget set $B(p, w) = \{x \in R^l_+ : p \cdot x \leq w\}$. Suppose his utility is maximized at the bundle x^* when income is w' and at the bundle x^{**} when his income is w'', with $w'' > w'$ and prices held fixed at p in both cases. When can we say that the agent's demand is normal, i.e., $x^{**} \geq x^*$?

To apply the standard techniques we must first pick an order on R^l_+. Given that we wish to compare x^{**} and x^* with the product order, the natural order to pick for this problem is again the product order. Furthermore, with this order, the supermodularity of U has a nice interpretation in terms of complementarity, so conditions (i) and (ii) are satisfied. Unfortunately, condition (iii) is not, because the budget sets $B(p, w')$ and $B(p, w'')$ are not ordered in the strong set order: if x' is in $B(p, w')$ and x'' is in $B(p, w'')$, their infimum is indeed in $B(p, w')$ but it is not hard to see that their supremum need not be in $B(p, w'')$. Consequently, at least when using the product order on R^l_+, this basic problem in consumer theory cannot be addressed using the standard comparative statics results.[2]

The contribution of this paper is to extend lattice programming techniques to deal with problems of this sort. The key idea is that the choice spaces in many comparative statics problems, including the utility maximization problem above are not just lattices - they are also vector spaces. On vector spaces, concavity and convexity makes sense. When these properties are added to supermodularity, they interact in way which permits the solution to a large class of comparative statics problems. So, for example, we show that demand is normal if the utility function is both concave and supermodular.

Section 2 is devoted to developing the theory of comparative statics in R^l, considered as a vector space and a lattice with the product order. The principal result of that section is a comparative statics theorem with different features from the one highlighted above. In our result, the conditions on the objective functions are strengthened and in particular, concavity type restrictions have to be imposed, but the requirements on the constraint sets are weakened so they only have to be comparable in what we call the *generalized strong set order*. For example, a sufficient condition for a set C'' to be greater than C' in this order is for there to be an increasing, convex, submodular and continuous function $G : R^l_+ \to R$ such that $C'' = G^{-1}(-\infty, c''])$ and $C' = G^{-1}(-\infty, c'])$ with $c'' > c'$.[3] Clearly the budget sets considered in our example, $B(p, w')$ and $B(p, w'')$, are indeed comparable in this sense since $B(p, w') = E^{-1}((-\infty, w'])$ and $B(p, w'') = E^{-1}(-\infty, w''])$ where $E : R^l_+ \to R$ defined by $E(x) = p \cdot x$ is just the expenditure function.

The sections following Section 2 are devoted to applications: section 3 deals with applications to demand theory, section 4 deals with applications to producer theory, and Section 5 is devoted to other applications.

In this paper we concentrate on developing the theory, and finding applications, in a finite dimensional Euclidean space, but one suspects that many of the theoretical results will go through in Riesz spaces, i.e., in not necessarily finite dimensional vector spaces which also has a lattice structure. This is a potentially fruitful area for future work.

2. The Theory

We endow R^l with the *product order*, which says that $x \geq y$ if $x_i \geq y_i$ for $i = 1, 2, ...l$. With this order, R^l becomes a lattice, i.e., it is a partially ordered set where there is a supremum and an infimum to every pair of points in R^l. We denote the supremum and infimum of x and y by $x \vee y$ and $x \wedge y$ respectively; it is not hard to see that

$$\begin{aligned} x \vee y &= (\max\{x_1, y_1\}, \max\{x_2, y_2\}, ..., \max\{x_l, y_l\}) \text{ and} \\ x \wedge y &= (\min\{x_1, y_1\}, \min\{x_2, y_2\}, ..., \min\{x_l, y_l\}) \end{aligned}$$

A subset X of R^l is a *sublattice* (of R^l) if for every pair of points x and y in X, both $x \vee y$ and $x \wedge y$ are also contained in X. A function $f : X \to R$ is *supermodular* if $f(x \vee y) + f(x \wedge y) \geq f(x) + f(y)$. It is known that supermodularity can be characterized by the property of *increasing differences* (see Topkis (1998)). Consider two pairs of points (x', x'') and (z', z''), both in $X \times X$. We say that (z', z'') *has a greater/larger difference* *than* (x', x'') under f if $f(z') - f(z'') \geq f(x') - f(x'')$. The function f is said to have the *increasing differences property* (or simply has *increasing differences*) if for all $v < 0$ and $v' > 0$ and orthogonal to v, $(x + v', x + v + v')$ has a greater difference than $(x, x + v)$, i.e.,

$$f(x + v') - f(x + v + v') \geq f(x) - f(x + v).$$

(Note that since all the entries in $-v$ and v' are non-negative, a particular entry in v' must be zero if the corresponding entry in v is strictly negative.) When f is C^2 function defined on R^l, the supermodularity of f is equivalent to $\partial^2 f/\partial x_i \partial x_j \geq 0$ for all $i \neq j$ (see Topkis (1978)).

A more broadly conceived notion of increasing differences is also useful in understanding concavity. Assuming that X is convex, the standard definition of concavity says that f is concave if $f(tx + (1-t)y) \geq tf(x) + (1-t)f(y)$ for all t in $[0, 1]$ and x and y in X. For our purpose it is convenient to use a different and equivalent formulation of concavity. We say that f is *concave at x in the direction v* if, for any positive scalar t such that x, $x + v$, $x + tv$ and $x + v + tv$ are all in X, the pair $(x + tv, x + v + tv)$ has a greater difference than $(x, x + v)$, i.e,

$$f(x + tv) - f(x + v + tv) \geq f(x) - f(x + v).$$

We say that the function f is *concave* if it is concave at x in direction v for all x in X and v in R^l. It is not hard to see that this notion of concavity is equivalent to the standard one, but this simple reformulation has the advantage that it emphasizes the formal similarity between concavity and supermodularity: both can be characterized by the behavior of difference terms of the form $f(x) - f(x + v)$. Our first theorem characterizes functions which are both concave and supermodular by identifying all the directions in which the difference term $f(x) - f(x + v)$ is increasing.

Any vector v in R^l can be decomposed into its positive and negative parts, i.e., $v = v_+ + v_-$, where $v_+ = v \vee 0$ and $v_- = -[(-v) \vee 0]$. We define the sets S^v_+ and S^v_- by

$$
\begin{aligned}
S^v_+ &= \{v \in R^l : v \cdot v_- = 0, v \geq 0, \lambda v = v_+, \lambda \in R\} \text{ and} \\
S^v_- &= \{v \in R^l : v \cdot v_+ = 0, v \leq 0, \lambda v = v_-, \lambda \in R\}.
\end{aligned}
$$

In other words, S^v_+ consists of those vectors which are orthogonal to v_-, non-negative, and parallel to v_+. Note in particular that v_+ is in S^v_+. Similarly, S^v_- consists of vectors which are orthogonal to v_+, non-positive and parallel to v_-, which certainly contains v_-.

Theorem 1: *Let X be a convex sublattice of R^l. Then $f : X \to R$ is concave and supermodular if and only if it has the following property: for all x in X, v in R^l, and v' in $S^v_+ \cup S^v_-$, the pair $(x + v', x + v + v')$ has a greater difference than $(x, x + v)$, i.e.,*

$$
f(x + v') - f(x + v + v') \geq f(x) - f(x + v), \tag{1}
$$

provided x, $x + v$, $x + v'$ and $x + v + v'$ are all in X.

Proof: We write

$$
f(x) - f(x + v) = [f(x) - f(x + v_+)] + [f(x + v_+) - f(x + v)].
$$

Consider the first difference term on the right as we add v' in S^v_+. If $v_+ = 0$, it is clear that this term remains unchanged when we add v'. If $v_+ \neq 0$, then v' is some positive multiple of v_+, so the concavity of f guarantees that adding v' will increase the term. Now consider the second difference term. We can re-write it as $f(x+v_+) - f(x+v_+ + v_-)$. Note that $v_- \leq 0$, while $v' \geq 0$ and orthogonal to v_-, so adding v' will increase this term, by the supermodularity of f. This establishes our claim for v' in S_{v+}. The case of v' in S_-^v can be handled in an analogous manner.

This leaves us with the proof of necessity. Let t be a positive scalar. Assume that x, $x+v$, $x+tv$, $x+v+tv$ are all in X. Then with tv_+ and tv_- in S_+^v and S_-^v respectively for any $t > 0$, applying (1) twice gives us

$$\begin{aligned} f(x) - f(x+v) &\leq f(x+tv_+) - f(x+v+tv_+) - \\ &\leq f(x+tv_+ + tv_-) - f(x+v+tv_+ + tv_-) \\ &= f(x+tv) - f(x+v+tv), \end{aligned}$$

which shows that f is concave. (Note that all the elements referred to in the inequalities are in X because it is a convex lattice.) That the supermodularity of f also follows from (1) is obvious, since when $v < 0$, S_+^v consists precisely of all those vectors v' which are non-negative and orthogonal to v. QED

Concavemodular Functions

For the purposes of comparative statics, it is convenient to have a slightly different way of presenting the structure which concavity and supermodularity gives to a function. Let x' and y be two elements in X; then the vectors $v_{x'} = x' \vee y - x'$ and $w_{x'} = y - x' \vee y$ are both nonzero, and positive and negative respectively. In Figure 1, we have the picture of a rectangle, with x' and y being opposite corners, whose other corners are $x' \vee y$ and $x' \wedge y$, and with the vectors $v_{x'}$ and $w_{x'}$ forming the sides. Supermodularity is equivalent to saying that $f(x'+v_{x'})-f(y) \geq f(x')-f(x'+w_{x'})$ since $x' \vee y = x'+v_{x'}$ and $x' \wedge y = x+w_{x'}$, while the property represented by (1) implies that $f(x'+v_{x'}-\lambda v_{x'})-f(y) \geq f(x')-f(x+w_{x'}+\lambda v_{x'})$ where λ is in $[0,1]$. This is easily obtained from (1) by substituting $v = w_{x'} + \lambda v_{x'}$ and $v' = (1-\lambda)v_{x'}$. Since $x' + v_{x'} - \lambda v_{x'} = x' \vee y - \lambda v_{x'}$ and $x + w_{x'} + \lambda v_{x'} = x' \wedge y + \lambda v_{x'}$, we can rewrite this inequality as

$$f(x' \vee y - \lambda v_{x'}) - f(y) \quad \geq \quad f(x') - f(x' \wedge y + \lambda v_{x'}); \tag{2}$$

in other words, for any λ in $[0,1]$, the pair $(x' \vee y - \lambda v_{x'}, y)$ has a greater difference than $(x', x' \wedge y + \lambda v_{x'})$. Now it is clear that the two pairs of points form a parallelogram rather than a rectangle: it is this little twist to the geometry which is at the heart of the comparative statics results in this paper.

We call the function f *i-concavemodular* if for any x' and y in X with $x'_i > y_i$, the inequality (2) holds for all λ in $[0,1]$. Note that (2) holds trivially if $x' > y$ since in this case $v_{x'} = x' \vee y - x' = 0$. So checking for i-concavemodularity really involves checking for (2) for x' and y which are unordered. The next result states formally the connection between i-concavemodularity and the generalized notion of increasing differences as represented by inequality (1). We omit the straightforward proof.

PROPOSITION 1: *Let X be a convex lattice. The function $f : X \to R$ is i-concavemodular if and only if for all x in X, v in R^l, with $v_i < 0$ and $v \not< 0$, and v' in $S^v_+ \cup S^v_-$, the pair $(x+v', x+v+v')$ has a greater difference than $(x, x+v)$, i.e., inequality (1) holds, provided x, $x+v$, $x+v'$ and $x+v+v'$ are all in X.*

It follows immediately from this proposition and Theorem 1 that the function f will be i-concavemodular if it is concave and supermodular. But notice that these conditions are probably a bit stronger than necessary since in Proposition 1 the generalized increasing differences property is only required for those vectors v satisfying $v_i < 0$ and $v \nless 0$ (rather than for all v as in Theorem 1). Indeed a careful examination of the proof of Theorem 1 will show that to guarantee (1) for v satisfying $v_i < 0$ and $v \nless 0$ it is sufficient that at every x in X, the function f be concave in all directions $\bar{v}$ such that $\bar{v} > 0$ and $\bar{v}_i = 0$. The next result restates the sufficiency part of Theorem 1 under this weakened concavity assumption.

PROPOSITION 2: *Let $X \subset R^l$ be a convex sublattice. Then $f : X \to R$ is i-concavemodular if it is supermodular and satisfies the following concavity assumption: for every x in X, f is concave in every direction $\bar{v}$ satisfying $\bar{v} > 0$ and $\bar{v}_i = 0$.*

We call the function f *concavemodular* if it is i-concavemodular for all $i = 1, 2, ..l$. We say that f is *partially concave* if it is concave in x_{-i} for $i = 1, 2, ...l$. (Note that 'concave in x_{-i}' has the standard meaning, namely, that f is concave when viewed as a function of the other $l-1$ variables, with the ith variable held fixed.) Note that partial concavity is certainly different from concavity. The function $f : R^l_{++} \to R$ given by $f(x_1, x_2) = x_1 x_2$ is partially concave but not concave. For differentiable functions defined on open and convex subsets of R^2, checking partial concavity is especially convenient since it only requires checking that the second derivatives of the function with respect to each argument is negative.

The next result follows immediately from Proposition 2.

COROLLARY 1: *Let $X \subset R^l$ be a convex sublattice. Then the function $f : X \to R$ is* *concavemodular if it is supermodular and partially concave.*

Given Proposition 2, one may get the impression that partial concavity is a stronger than necessary to guarantee concavemodularity, so that Corollary 1 is a little crude. But that is wrong. In the next two results, we establish the full implications of concavemodularity for concavity. We first show that the concavity assumption in Corollary 1 is, in essence, also necessary for i-concavemodularity.

LEMMA 1: *Let $X \subset R^l$ be a convex and open sublattice and suppose that $f : X \to R$ is an i-concavemodular function which is also continuous in x_i. Then for every x in X, f is concave in every direction $\bar{v}$ satisfying $\bar{v}_i = 0$ and either $\bar{v} > 0$ or $\bar{v} < 0$.*

Proof: Suppose, by way of contradiction, that there is $\bar{v} > 0$ with $\bar{v}_i = 0$ such that $f(x) - f(x + \bar{v}) > f(x + t\bar{v}) - f(x + \bar{v} + t\bar{v})$. Since f is continuous in x_i, there is $\delta > 0$ and sufficiently close to zero such that $f(x) - f(x + \bar{v} - \delta e_i) > f(x + t\bar{v}) - f(x + \bar{v} + t\bar{v} - \delta e_i)$, where e_i is the unit vector pointing in direction i. This is a violation of i-concavemodularity: we see that (2) is violated once we set $x' = x$ and $y = x + \bar{v} + t\bar{v} - \delta e_i$, and $\lambda = 1/(1 + t)$. (Note that in this case, $x' \vee y = x + \bar{v} + t\bar{v}$, $x' \wedge y = x - \delta e_i$, and $v_{x'} = (1 + t)\bar{v}$.) The case of $v < 0$ with $v_i = 0$ can be proven in a similar way. QED

The next result is the converse of Corollary 1.

PROPOSITION 3: *Let $X \subset R^l$ be a convex and open sublattice and suppose that $f : X \to R$ is a concavemodular function which is also continuous in each of its arguments (but not necessarily jointly continuous). Then f is supermodular and has the following concavity property: for all x in X, f is concave at x in all directions v satisfying $v \not\gg 0$ and $v \not\ll 0$.*

In particular, f must be partially concave.

Proof: That concavemodularity implies supermodularity is obvious, so we need only establish that the concavity condition also follows. For $v > 0$ (or < 0) such that $v_i = 0$ for some i, we can appeal to Lemma 1. (Note that this is the only place where the continuity property on f is used.) For v that satisfies $v \not> 0$ and $v \not< 0$, we can use the characterization of concavemodularity in Proposition 1 and then repeat that part of the proof in Theorem 1 which establishes concavity. QED

As a simple illustration, consider again the function $f : R^2_{++} \to R$ given by $f(x_1, x_2) = x_1 x_2$. As we had pointed out, this function is partially concave; clearly, it is also supermodular. By Corollary 2, it is concavemodular, which means by Proposition 3 that it is concave in all directions except possibly those which are strictly positive or strictly negative. To check this, consider the behavior of the function along the ray emanating from the point $(\bar{x}_1, \bar{x}_2)$ and in the direction (a, b): $f(\bar{x}_1 + at, \bar{x}_2 + bt) = \bar{x}_1\bar{x}_2 + (b\bar{x}_1 + a\bar{x}_2)t + abt^2$, which is a concave function of t whenever a and b are of different signs, but convex whenever a and b are both strictly positive or strictly negative.

Quasiconcavemodular Functions

It has been emphasized by Milgrom and Shannon (1994) in their wide ranging and influential study of comparative statics that the core comparative statics theorems rely not on supermodularity as such, but rather on an ordinal version of that property which they refer to as *quasisupermodularity*: the function $f : X \to R$ is quasisupermodular if $f(x') \geq (>) f(x' \wedge y)$ implies $f(x' \vee y) \geq (>) f(y)$. Following Milgrom and Shannon (1994), we say that the function f is *i-quasiconcavemodular* if for any x' and y in X with $x_i^{'} > y_i$, and for any λ in $[0,1]$,

$$f(x') \geq (>) f(x' \wedge y + \lambda v_{x'}) \Longrightarrow f(x' \vee y - \lambda v_{x'}) \geq (>) f(y) \tag{3}$$

(Recall that $v_{x'} = x' \vee y - x'$.) We call a function *quasiconcavemodular* if it is i-quasiconcavemodular for $i = 1, 2, ...l$. Clearly, quasiconcavemodularity is stronger than quasisupermodularity. It is also clear that any i-quasiconcavemodular function is i-concavemodular; the former is an ordinal property in the sense that if f is i-quasiconcavemodular then so is $\phi \circ f$, for any strictly increasing function $\phi : R \rightarrow R$.

Since i-concavemodularity is preserved by addition, we know that, for any w in R^l, the map $g_w : X \rightarrow R$ given by $g_w(x) = f(x) - w \cdot x$ is also an i-concavemodular function provided f is i-concavemodular. The next result shows that i-concavemodularity of the functions g_w imply the i-concavemodularity of f. This result is analogous to Theorem 10 in Milgrom and Shannon (1994).

PROPOSITION 4: *Let X be a convex sublattice of R^l and let f be a map from X to R. (i) Then f is i-concavemodular if for all w_i in R, the map g_{w_i}, bringing x in X to $f(x) - w_i x_i$ is i-quasiconcavemodular. (ii) Provided f is increasing, f is i-concavemodular if for all w in R^l_+, the map g_w, bring x in X to $f(x) - w \cdot x$ is i-quasiconcavemodular.*

Proof: Suppose that there is x' and y, with $x_i' > y_i$ and λ such that (2) is violated, so

$$f(x' \vee y - \lambda v_{x'}) - f(y) \quad < \quad f(x') - f(x' \wedge y + \lambda v_{x'}). \tag{4}$$

Choose $\bar{w}_i$ such that $\bar{w}_i[x_i' - (x' \wedge y + \lambda v_{x'})_i] = f(x') - f(x' \wedge y + \lambda v_{x'})$. Furthermore, since $x' - (x' \wedge y + \lambda v_{x'}) = (x' \vee y - \lambda v_{x'}) - y$, we have $\bar{w}_i[x_i' - (x' \wedge y + \lambda v_{x'})_i] = \bar{w}_i[(x' \vee y - \lambda v_{x'})_i - y_i]$.

Deducting this term from both sides of (4), we obtain

$$g_{\bar{w}_i}(x' \vee y - \lambda v_{x'}) - g_{\bar{w}_i}(y) < g_{\bar{w}_i}(x') - g_{\bar{w}_i}(x' \wedge y + \lambda v_{x'}) = 0.$$

So $g_{\bar{w}_i}$ violates i-quasiconcavemodularity and we have a contradiction.

The proof of (ii) is similar. Note firstly that if (4) is true for $\lambda = 0$, then the right hand side of (4) is nonnegative (since f is increasing), while $x'_i - (x' \wedge y + \lambda v_{x'})_i > 0$, so that, in the proof above, one could choose $\bar{w}_i \geq 0$ and we are done. (The other entries of the vector $\bar{w}$ can be chosen to be 0). So we consider the case when (4) is true for $\lambda > 0$. This implies that x' and y must be unordered, and with $\lambda > 0$, x' and $(x' \wedge y + \lambda v_{x'})$ must also be unordered. Therefore, $x' - (x' \wedge y + \lambda v_{x'})$ has both positive and negative entries, and there is $\bar{w}$ in R^l_+ such that $\bar{w} \cdot [x' - (x' \wedge y + \lambda v_{x'})] = f(x') - f(x' \wedge y + \lambda v_{x'})$. Now repeating the steps in our proof of (i), we see that $g_{\bar{w}}$ must violate i-quasiconcavemodularity. QED

The significance of this proposition is that in those situations where we require quasiconcavemodularity for all functions in the class $\{g_w\}_{w \in R^l}$ or $\{g_w\}_{w \in R^l_+}$, we must necessarily impose concavemodularity on f. Of course these classes of functions do indeed arise naturally in comparative statics problems, since it can be interpreted as a profit function, with $f(x)$ as the revenue of the firm when it produces the output vector x and with w_i as the unit cost of producing good i (so $w \cdot x$ is the total cost of producing x).

The Generalized Strong Set Order

Given that our ultimate goal is to obtain results which say how optimal solutions vary with parameters and constraints, we must first develop some way of comparing constraint sets. In standard monotone comparative statics, the order typically used is the *strong set order* introduced by A. Veinott (see Topkis (1998)). In this order, a set V'' is greater than V' if for any y in V'' and x' in V', $x' \vee y$ is in V'' and $x' \wedge y$ is in V'. As we had indicated in the introduction, the strong set order is, in a sense, too strong because it does not always successfully order pairs of constraint sets whose optimal solutions we wish to compare. What we need is a weaker notion of order, which we now define.

Let C' and C'' be subsets of the convex sublattice X. We say that C'' *is* i*-greater than* C' *in the generalized strong set order* (and write $C'' >_i C'$) if for any x' be in C' and y in C'', with $x'_i > y_i$, there is λ in $[0, 1]$ such that $x' \wedge y + \lambda v_{x'}$ is in C' and $x' \vee y - \lambda v_{x'}$ is in C''. Pictorially, this condition just means that one can find two other points, in addition to x' and y, one in C' and one in C'' such that the four points form a parallelogram. For the special case of $x' > y$, $v_{x'} = 0$, so this condition requires that y be in C' and x' be in C''. We say that C'' *is greater than* C' *in the generalized strong order* (and write $C'' > C'$) if $C'' >_i C'$ for all $i = 1, 2, ...l$.

Notice that the point $x' \vee y - \lambda v_{x'}$ which lies in C'' is greater than x', and that the point $x' \wedge y + \lambda v_{x'}$ which lies in C' is smaller than y. Our next claim is then obvious.

PROPOSITION 5: *Let C' and C'' be nonempty subsets of a convex sublattice X in R^l.*

(i) If $C'' >_i C'$, then for any x' in C', there is x'' in C'' such that $x''_i \geq x'_i$ and for any x'' in C'' there is x' in C' such that $x''_i \geq x'_i$.

(ii) If $C'' > C'$, then for any x' in C', there is x'' in C'' such that $x'' \geq x'$ and for any x'' in C'' there is x' in C' such that $x'' \geq x'$.

As a simple illustration, let $C'' = \{(1+t,2),(2+t,1)\}$ and $C' = \{(1,2),(2,1)\}$. For any $t > 0$ it is easy to see that $C'' >_2 C'$, though for t in $(0,1)$, $C'' \not>_1 C'$. We do have $C'' >_1 C'$ if $t \geq 1$ so in this case $C'' > C'$. Thus for $t \geq 1$, $C'' > C'$. Note that C'' is certainly not a superset of C', so a set can be greater than another in the generalized strong set order without it being a superset of the other set. Having said that, the constraint sets one encounters in applications are often ordered in the set-theoretic sense, or at least can be understood in that manner. The next result gives sufficient conditions for $C'' >_i C'$ when C'' contains C'.

PROPOSITION 6: *Let C' and C'' be subsets of a convex sublattice X of R^l. Then $C'' >_i C'$ if the following conditions hold:*

(i) $C' \subset C''$,

(ii) C' is closed;

(iii) C' satisfies free disposal, i.e., if $y < x$ and x is in C' then y is in C',

(iv) let x and u be positive vectors with $u_i = 0$, x in C', $x + u$ in C'' and $x + tu \notin C'$ for all t in $(0,1]$; then for any $\mu > 0$, and $u' > 0$ with $u'_i > 0$ and orthogonal to u,

$$x - \mu u + u' \in C' \Longrightarrow (x+u) - \mu u + u' \in C''.$$

Proof: Let x' be in C' and y be in C'' with $x'_i > y_i$. If $x' > y$, the condition for $C'' >_i C'$ requires x' to be in C'' and y to be in C', which follows from (i) and (iii) respectively. So we assume that x' and y are unordered. If y is in C', the condition for $C'' >_i C'$ holds with $\lambda = 1$. This leaves us with the case of x and y are unordered, with y not in C'. Since $x' \wedge y$

is in X and less than x', we know that it is in C'. By (ii) and (iii) there λ^* in $[0,1)$ such that $x' \wedge y + \lambda^* v_{x'}$ is in C' and $x' \wedge y + \lambda v_{x'}$ is not in C' for λ in $(\lambda^*, 1]$. Define $u = (1-\lambda^*)v_{x'}$. Choose $\mu = \lambda^*/(1-\lambda^*)$ and $u' = x' - x' \wedge y$. Note that $u_i = 0$ and $u'_i > 0$. We then have $x' \wedge y + \lambda^* v_{x'}$ in C', $(x' \wedge y + \lambda^* v_{x'}) - \mu u + u' = x$ in C', and $x + u = y$ in C''; so by (iv), $(x+u) - \mu u + u' = x' \vee y - \lambda^* v_{x'}$ must also be C''. So we conclude that $C'' >_i C'$. QED

Conditions (i), (ii) and (iii) in the proposition are quite standard. Condition (iv) is the more substantive condition, but it also seems entirely natural, given that we are working towards monotone comparative statics. It says, in a specific formal sense, that the set of substitution possibilities which favor good i in the constraint set C'' is larger than the set of substitution possibilities which favor good i in the constraint set C': if at the point x, it possible to substitute λu with u' and still stay within the constraint set C' then it is possible to make the *same substitution* at the point $x+u$ and stay within the constraint set C''.

The next theorem gives us a simple way of generating a class of ordered sets via quasiconvexmodular functions. A real-valued function f defined on a convex sublattice X is *i-quasiconvexmodular* if for any x' and y in X, with $x'_i > y_i$, and for any λ in $[0,1]$,

$$f(x') \leq (<) f(x' \wedge y + \lambda v_{x'}) \Longrightarrow f(x' \vee y - \lambda v_{x'}) \leq (<) f(y) \tag{5}$$

We say that f is *quasiconvexmodular* if it is i-quasiconvexmodular for $i = 1, 2, ...l$.

THEOREM 2: *Let $X \subset R^l$ be a convex sublattice and let S be a interval in R. Suppose that the function $C : X \times S \to R$ is decreasing in s, and as a function of x, i-quasiconvexmodular, increasing and continuous. Suppose also that it has the following property: whenever $x'_i > y_i$*

for x' and y in X, the expression $C(x', s) - C(y, s)$ decreases with s. Then the sets $C' = \{x \in X : C(x, s') \leq k'\}$ and $C'' = \{x \in X : C(x, s'') \leq k''\}$ will satisfy conditions (i)-(iv) in Proposition 6 provided $(k'', s'') > (k', s')$. Consequently, $C'' >_i C'$.

Proof: Condition (i) of Proposition 1 is true because C is a decreasing function of s. Condition (ii) follows from the continuity of C, while (iii) holds because C is an increasing function. This leaves us with condition (iv). Suppose now that the vectors x and u satisfy x in C', $u > 0$, $u_i = 0$, $x+u$ in C'', and $x+tu \notin C'$ for all t in $(0, 1]$. In this case, $C(x, s') = k'$, by the continuity of C. If for $\mu > 0$, and u', with $u' > 0$, $u'_i > 0$, and orthogonal to u, we have $x - \mu u + u'$ in C', then $C(x - \mu u + u', s') \leq k'$. Since C is i-quasiconvexmodular in x, and $C(x-\mu u+u', s')-C(x, s') \leq 0$, we also have $C((x+u)-\mu u+u', s')-C(x+u, s') \leq 0$. Since the difference term decreases with s, we have $C((x+u)-\mu u+u', s'')-C(x+u, s'') \leq 0$. So $C((x + u) - \mu u + u', s'') \leq k''$, as required. QED

We wish to identify a class of functions C which obey the conditions of Theorem 2. To this end, we first prove the next lemma, which we will be useful in other places as well. The lemma refers to the set X_i: given any set X in R^l, X_i is the set $\{r \in R : x_i = r \text{ for some } x \in X\}$. Provided X is convex, this set will be an interval.

LEMMA 2: *Let $X \subset R^l$ be a convex set, S an interval of R, and suppose that $C : X \times T \to R$ is given by $C(x, s) = \bar{C}(x) + c(x_i, s)$ where $\bar{C}$ is any real valued function defined on X and $c : X_i \times S \to R$ is supermodular (submodular) in (x_i, s). Provided $x'_i \geq y_i$, $C(x', s) - C(y, s)$ increases (decreases) with s.*

Proof: We write

$$C(x', s) - C(y, s) = [C(x', s) - C(x' \vee y, s)] + [C(x' \vee y, s) - C(y, s)].$$

Since $x'_i = (x' \vee y)_i$, the term in the first square brackets simply equals $\bar{C}(x') - \bar{C}(x' \vee y)$ and does not vary with s, while the second term equals $\bar{C}(x \vee y) - \bar{C}(y) + c((x \vee y)_i, s) - c(y_i, s)$, which increases with s when c is supermodular and decreases with s when f is submodular.

. QED

Proposition 7: *Let $X \subset R^l$ be a convex sublattice and let S be an interval in R. The function $C : X \times S \to R$ defined by $C(x, s) = \bar{C}(x) + c(x_i, s)$ will satisfy all the conditions of Theorem 2, provided the following holds:*

(a) the function $\bar{C} : X \to R$ is submodular, increasing and continuous in x, and convex in x_{-i} and

(b) the function $c : X_i \times S \to R$ is submodular in (x_i, s), and increasing and continuous in x_i, and decreasing in s.

Proof: The fact that C is increasing and continuous in x is obvious from the assumptions on $\bar{C}$ and c. It is also decreasing in s by assumption. By Proposition 2 (or rather its obvious analog) $\bar{C}$ is i-convexmodular, which also means that C is i-convexmodular. Finally, from Lemma 2 we know that $C(x', s) - C(y, s)$ decreases with s when $x'_i \geq y_i$. QED

The next very useful corollary follows immediately from Theorem 2 and Proposition 7.

Corollary 2: *Let $C : X \to R$ be a continuous, increasing, and quasiconvexmodular function. (A sufficient condition for the latter property is that C is partially convex, and submodular.) Then $C^{-1}((-\infty, k'']) > C^{-1}((-\infty, k'])$ if $k'' > k'$.*

Another way of generating comparable sets is given in the next result.

Corollary 3: *Let $\widetilde{X}$ be a convex sublattice of R^{l-1} and I an interval of R, and let $G : \widetilde{X} \to R$ be a continuous, supermodular, concave and decreasing function. Then if $s'' > s' > 0$,*

$$\{(\tilde{x}, x_l) \in R^{l-1} \times I : \tilde{x} \in \widetilde{X}, x_l \leq s''G(\tilde{x})\} >_l \{(\tilde{x}, x_l \in R^{l-1} \times I : \tilde{x} \in \widetilde{X}, x_l \leq k'G(\tilde{x})\}.$$

Proof: Define the function C acting on $\tilde{X} \times I \times R_+$ by $C(\tilde{x}, x_l, s) = x_l/s - G(\tilde{x})$. Notice that $-G$ is a continuous, submodular, convex and increasing function. Furthermore, the map from x_l to x_l/s is submodular in (x_l, s), increasing and continuous in x_l and decreasing in s. By Proposition 7 and Theorem 2, the set $\{(\tilde{x}, x_l \in R^{l-1} \times I : C(\tilde{x}, x_l, k'') \leq 0\}$ is l-greater than the set $\{(\tilde{x}, x_l \in R^{l-1} \times I : C(\tilde{x}, x_l, k') \leq 0\}$, exactly as the corollary claims . QED

Our final result shows that the quasiconvexmodularity condition in Corollary 2 is, in essence, a necessary condition.

Proposition 8: *Let X be a convex sublattice of R^l and let $C : X \to R$ be a continuous and strictly increasing function. If $C^{-1}((-\infty, k'']) >_i C^{-1}((-\infty, k'])$ whenever $k'' > k'$, then C is i-quasiconvexmodular.*

Proof: Consider x' and y, unordered, with $x'_i > y_i$ and suppose that $C(x') = k'$ and $C(y) = k''$. If $k'' < k'$, then by the fact that C is strictly increasing, $C(x' \wedge y + \lambda v_{x'}) \leq C(y) = k'' < k' = C(x')$ for all λ in $[0, 1]$, which means that (5) is vacuously true for all λ in $[0, 1]$. If $k'' = k'$, (5) is vacuously true for λ in $[0, 1)$, while it is trivially true at $\lambda = 1$.

So we assume that $k'' > k'$; since $C^{-1}((-\infty, k'']) >_i C^{-1}((-\infty, k'])$ we know that there is $\bar{\lambda}$ such that $x' \wedge y + \bar{\lambda} v_{x'}$ is in $C^{-1}((-\infty, k'])$ and $x' \vee y - \bar{\lambda} v_{x'}$ is in $C^{-1}((-\infty, k''])$. Since C is continuous and increasing, there is $\lambda^* \geq \bar{\lambda}$ such that $C(x' \wedge y + \lambda^* v_{x'}) = k'$ and $C(x' \vee y - \lambda^* v_{x'}) \leq k''$. Furthermore, since C is strictly increasing, for $\lambda < \lambda^*$, we have $C(x' \wedge y + \lambda v_{x'}) < k'$ and for $\lambda > \lambda^*$, we have $C(x' \wedge y + \lambda^* v_{x'}) > k'$ and $C(x' \vee y - \lambda^* v_{x'}) < k''$. Together, this means that (5) holds. QED

Comparative Statics Theorems

Let X be a convex sublattice of R^l and let F be a real valued function defined on X. We say that F has the *monotonic property in variable i* (respectively, *monotonic property*) if whenever $C'' >_i (>) C'$ we also have $\arg\max_{x \in C''} F(x) >_i (>) \arg\max_{x \in C'} F(x)$.[4] If F has the monotonic property, then Proposition 5 tells us the following:

(a) whenever $C'' >_i C'$ ($C'' > C'$) and x' is in $\arg\max_{x \in C'} F(x)$, and $\arg\max_{x \in C''} F(x)$ is nonempty, then there is x'' in $\arg\max_{x \in C''} F(x)$ such that $x''_i \geq x'_i$ ($x'' \geq x'$);

(b) whenever $C'' >_i C'$ ($C'' > C'$) and x'' is in $\arg\max_{x \in C''} F(x)$, and $\arg\max_{x \in C'} F(x)$ is nonempty, then there is x' in $\arg\max_{x \in C'} F(x)$ such that $x''_i \geq x'_i$ ($x'' \geq x'$).

The main comparative statics result of this paper says that the monotonic property in variable i is equivalent to the i-quasiconcavemodularity of the objective function. This result, once we have laid out the relevant groundwork, is very easy to prove - a feature it shares with the standard comparative statics theorems. It is well known that greater sets (in the sense of the strong set order) lead to greater solution sets (with respect to the same set order) when the objective function is supermodular. (The first version of this result is due to A. Veinott; see Topkis (1978) for an early statement of this result). More precisely, the quasisupermodularity of the objective function is both sufficient and necessary for this property (see Milgrom and Shannon (1994)). The proof of our comparative statics theorem has a similar structure to those earlier proofs. Indeed, we have developed the theory in the way we did precisely so that we can now adopt the arguments they employed in their proofs, subject to certain natural modifications. We begin with a fundamental lemma needed for the main theorem.

THEOREM 3: *Let X be a convex sublattice of R^l and let F be a real valued function defined on X. Then F is i-quasiconcavemodular if and only if it has the monotonic property in variable i.*

Proof: We first proof sufficiency. Assume that $C'' >_i C'$ and let x' be in $\arg\max_{x \in C'} F(x)$ and let y be in $\arg\max_{x \in C''} F(x)$. Suppose that $x'_i > y_i$; there is some $\tilde{\lambda}$ in $[0,1]$ such that $x' \wedge y + \tilde{\lambda} v_{x'}$ is in C' and $x' \vee y - \tilde{\lambda} v_{x'}$ is in C''. By revealed preference, $F(x') \geq F(x' \wedge y + \tilde{\lambda} v_{x'})$ and by i-quasiconcavemodularity, $F(x' \vee y - \tilde{\lambda} v_{x'}) \geq F(y)$, so $x' \vee y - \tilde{\lambda} v_{x'}$ is in $\arg\max_{x \in C''} F(x)$. If $F(x') > F(x' \wedge y + \tilde{\lambda} v_{x'})$, then i-quasiconcavemodularity implies that $F(x' \vee y - \tilde{\lambda} v_{x'}) > F(y)$ which contradicts the assumption that y maximizes F in C''. So we must also have $x' \wedge y + \tilde{\lambda} v_{x'}$ in $\arg\max_{x \in C'} F(x)$.

We prove the necessity part of the theorem by contradiction. Let x' and y be elements in X with $x'_i > y_i$. There are two possible violations of i-quasiconcavemodularity. One possibility is that there is λ^* in $[0,1]$ such that $F(x') \geq F(x' \wedge y + \lambda^* v_{x'})$ but $F(x' \vee y - \lambda^* v_{x'}) < F(y)$. In this case, let C' be the set with elements x' and $x' \wedge y + \lambda^* v_{x'}$ and let C'' be the set with elements $x' \vee y - \lambda^* v_{x'}$ and y. Then, clearly, $C'' >_i C'$, x' maximizes F in C' and y uniquely maximizes F in C''. This violates the monotonic property since $x'_i > y_i$.

The other possible violation of quasiconcavemodularity is that there is λ^* in $[0, 1]$ such that $F(x') > F(x' \wedge y + \lambda^* v_{x'})$ but $F(x' \vee y - \lambda^* v_{x'}) = F(y)$. In this case, with C' and C'' defined as above, y maximizes F in C'' while x' is the unique maximizer of F in C'. Again this violates the monotonic property. QED

The next result follows immediately from Theorem 3 and Corollary 2. Note also that by Corollary 1 we can easily modify the assumptions in the next result: instead of quasiconcavemodularity, we can assume that F is partially concave and supermodular, while we can replace the quasiconvexmodularity of C by its partial convexity and submodularity.

Corollary 4: *Let $F : X \to R$ be a quasiconcavemodular function and let $C : X \to$ be a continuous, increasing and quasiconvexmodular function. Then the optimal solutions to* $\max_{x \in C^{-1}(-\infty,k])} F(x)$ vary monotonically with respect to k *in the following sense: whenever $k'' > k'$, we have* $\arg\max_{x \in C''} F(x) > \arg\max_{x \in C'} F(x)$.

In certain comparative statics problems both the constraint sets and the objective functions are allowed to change. The next result addresses those situations. Loosely speaking, it captures the idea that if the change in the objective function and the constraint set both favor variable i, then the optimal value of i will rise.

Theorem 4: *Let X be a convex sublattice in R^l and T and S be intervals on R. The functions F and C (representing families of objective and constraint functions respectively) are defined in the following way:*

(i) F maps $X \times T$ to R, with $F(x,t) = \bar{F}(x) + f(x_i, t)$ where $\bar{F} : X \to R$ is supermodular and concave in x_{-i} and $f : X_i \times T \to R$ is supermodular in (x_i, t);

(ii) C maps $X \times S$ to R, with $C(x,s) = \bar{C}(x) + c(x_i, s)$ where $\bar{C} : X \to R$ is submodular, increasing and continuous in x, and convex in x_{-i}, and $c : X_i \times S \to R$ is submodular in (x^1, s), increasing and continuous in x^1, and decreasing in s.

Then the i-value of the optimal solution varies monotonically with (k,t,s) *in the sense that whenever $(k'', s'', t'') > (k', t', s')$, we have*

$$argmax_{\{x \in X : C(x,s'') \leq k''\}} F(x,t'') >_i argmax_{\{x \in X : C(x,s') \leq k'\}} F(x,t').$$

Proof: By Theorem 2 and Proposition 7, $C'' >_i C'$, where $C'' = \{x \in X : C(x,s'') \leq k''\}$ and $C' = \{x \in X : C(x,s') \leq k'\}$. Let y be in $\arg\max_{C''} F(x,t'')$ and let x' be in $\arg\max_{C'} F(x,t')$ and assume that $x'_i > y_i$. There is some $\tilde{\lambda}$ in $[0,1]$ such that $x' \wedge y + \tilde{\lambda} v_{x'}$ is in C' and $x' \vee y - \tilde{\lambda} v_{x'}$ is in C''. By revealed preference, $F(x',t') \geq F(x' \wedge y + \tilde{\lambda} v_{x'}, t')$. Since $F(\cdot, t')$ is i-concavemodular, $F(x' \vee y - \tilde{\lambda} v_{x'}, t') \geq F(y,t')$. Note that $(x' \vee y - \tilde{\lambda} v_{x'}, t')_i = x'_i > y_i$. By Lemma 2, we have $F(x' \vee y - \tilde{\lambda} v_{x'}, t'') \geq F(y,t'')$, so $x' \vee y - \tilde{\lambda} v_{x'}$ also maximizes $F(\cdot, t'')$ in C''.

We claim that $x' \wedge y + \tilde{\lambda} v_{x'}$ maximizes $F(\cdot, t')$ in C'. If not, $F(x',t') > F(x' \wedge y + \tilde{\lambda} v_{x'}, t')$, which implies, by the i-concavemodularity of $F(\cdot, t')$, that $F(x' \vee y - \tilde{\lambda} v_{x'}, t') > F(y,t')$. By Lemma 2, we obtain $F(x' \vee y - \tilde{\lambda} v_{x'}, t'') > F(y,t'')$, contradicting the assumption that y maximizes $F(\cdot, t'')$ in C''. QED

Our final comparative statics result applies specifically to R^2. Loosely speaking, it captures the idea that if the objective function changes in a way which is unfavorable to variable 1, while the constraint set expands but in a way which raises the marginal cost of variable 1, then the optimal value of variable 2 will rise.

THEOREM 5: *Let X be a convex lattice in R^2 and T and S be intervals on R. The functions F and C (representing families of objective and constraint functions respectively) are defined in the following way:*

(i) F maps $X \times T$ to R, with $F(x,t) = \bar{F}(x) + f(x_1, t)$ where $\bar{F} : X \to R$ is supermodular in (x_1, x_2) and concave in x_1, $f : X_1 \times T \to R$ is submodular in (x_1, t), and F is increasing in x_1;

(ii) C maps $X \times S$ to R, with $C(x,s) = \bar{C}(x) + c(x_1, s)$ where $\bar{C} : X \to R$ is submodular, increasing and continuous in x and convex in x_1, and $c : X_1 \times S \to R$ is supermodular in (x^1, s), increasing and continuous in x_1, and decreasing in s.

Then the 2-value of the optimal solution varies monotonically with (k, s, t) (in the sense defined in Theorem 4).

The proof of the theorem relies on the next lemma.

LEMMA 3: *Let X be a convex lattice in R^2 and S an interval on R. The function C maps $X \times S$ to R and satisfies the assumptions (under (ii)) in Theorem 5. Then $C'' >_2 C'$, where $C' = \{x \in X : C(x, s') \leq k'\}$, $C'' = \{x \in X : C(x, s'') \leq k''\}$ and $(k'', s'') \geq (k', s')$.*

Proof: The claim is trivially true if $(k'', s'') = (k', s')$, so we assume that $(k'', s'') > (k', s')$. Assume that x' is in C' and y is in C'', with $x'_2 > y_2$. Since C is decreasing in s, we can easily show that $C' \subset C''$, while the fact that C and c are both increasing in x

guarantees that C' satisfies free disposal. This means that if $y < x'$ then y is also in C'. If $y < x'$, then the generalized strong set order requires precisely that y be in C' and x' be in C'', both of which are certainly satisfied. So we assume that x' and y are not ordered. If y is in C', the condition for $C'' >_2 C'$ holds for $\lambda = 1$.

So we assume that y is not in C', which means in particular that $y_1 > x'_1$. Since C is increasing in x, $C(x' \wedge y, s') \leq C(x', s') \leq k'$. Again because C is increasing in x and also because it is continuous in x, there is $\tilde{\lambda}$ in $[0, 1]$ such that $C(x \wedge y + \tilde{\lambda} v_{x'}, s') = k'$. Note that $C(x', s') \leq k'$, so the 2-quasiconvexmodularity of $C(\cdot, s')$ guarantees that $C(x' \vee y - \tilde{\lambda} v_{x'}, s') \leq C(y, s')$. Since $(x' \vee y - \tilde{\lambda} v_{x'})_1 \leq y_1$, applying Lemma 2, the supermodularity of c implies that $C(x' \vee y - \tilde{\lambda} v_{x'}, s'') \leq C(y, s'')$. With y in C'', we have $C(y, s'') \leq k''$, so $x' \vee y - \tilde{\lambda} v_{x'}$ is also in C''. QED

Proof of Theorem 5: By Lemma 3, $C'' >_2 C'$. Let x' maximize $F(x, t')$ for x in C' and let z maximize $F(\cdot, t'')$ in C'', and assume that $x'_2 > z_2$. Note that we can always find y which maximizes $F(\cdot, t'')$ in C'' such that $y_1 \geq x'_1$ and $y_2 = z_2 < x'_2$. If $z_1 \geq x'_1$ simply let $z = y$. If $z_1 < x'_1$, let $y = (x'_1, z_2)$. Since F is increasing in x_1, $F(y) \geq F(z)$. Furthermore, by the free disposal property on C', y is in C' and therefore in C''.

We now assume that $x'_2 > y_2$ and $x'_1 \geq y_1$. Since $C'' >_2 C'$, there is $\tilde{\lambda}$ in $[0, 1]$ such that $x' \wedge y + \tilde{\lambda} v_{x'}$ is in C' and $x' \vee y - \tilde{\lambda} v_{x'}$ is in C''. By revealed preference, $F(x', t') \geq F(x' \wedge y + \tilde{\lambda} v_{x'}, t')$. Since $F(\cdot, t')$ is 2-concavemodular, $F(x' \vee y - \tilde{\lambda} v_{x'}, t') \geq F(y, t')$. Note that $(x' \vee y - \tilde{\lambda} v_{x'})_1 \leq y_1$ and f is submodular, so Lemma 2 implies that $F(x' \vee y - \tilde{\lambda} v_{x'}, t'') \geq F(y, t'')$. So $x' \vee y - \tilde{\lambda} v_{x'}$ also maximizes $F(\cdot, t'')$ in C''.

We claim that $x' \wedge y + \tilde{\lambda} v_{x'}$ maximizes $F(\cdot, t')$ in C'. If not, $F(x', t') > F(x' \wedge y + \tilde{\lambda} v_{x'}, t')$, which implies, by the 2-concavemodularity of $F(\cdot, t')$, that $F(x' \vee y - \tilde{\lambda} v_{x'}, t') > F(y, t')$. By Lemma 2, we obtain $F(x' \vee y - \tilde{\lambda} v_{x'}, t'') > F(y, t'')$, contradicting the assumption that y maximizes $F(\cdot, t'')$ in C''. QED

To motivate the formal results we have developed so far, we will now consider their applications, beginning with their applications to demand theory.

3. Applications to Classical Demand Theory

We have in mind a consumer who maximizes a utility function $U : R^l_+ \to R$, while facing a budget constraint. At the price p in R^l_+, and income $w > 0$, we denote his budget set by $B(p, w)$, where $B(p, w) = \{x \in R^l_+ : p \cdot x \leq w\}$. A solution to maximizing U in $B(p, w)$ is referred to as *a demand at* (p, w).

Example 1. We say that the agent has *normal demand* if the demand for all goods increase with his income. It is natural to ask when demand will be normal, but this is not a question to which standard monotone comparative statics theorems can be straightforwardly applied to yield an answer.[5] This is because to apply those theorems, budget sets have to be ordered in the strong set order, but with the usual order on R^l_+, budget sets are clearly not ordered in this sense. Specifically, consider two budget sets $B(p, w')$ and $B(p, w'')$ with $w' < w''$; if x' is in $B(p, w')$ and y is in $B(p, w'')$, we know that $x \wedge y$ is in $B(p, w')$, but $x \vee y$ need not be in $B(p, w'')$.

On the other hand, the theorems developed in the last section can easily address this question. First we note that the map $C : R^l_+ \to R$ given by $C(x) = p \cdot x$ is continuous, increasing, convex and submodular, and $B(p, w) = C^{-1}(-\infty, w]$. By Corollary 4, we know that provided U is supermodular and partially concave, then demand will be normal in the following sense: assuming that demand at (p, w'') exists, then if x' is a demand bundle at (p, w'), where $w'' > w'$, there is a demand x'' at (p, w'') such that $x'' \geq x'$; analogously, assuming that demand at (p, w') exists, then for any demand x'' at (p, w''), where $w'' > w'$, there is a demand x' at (p, w') such that $x'' \geq x'$.

It is worth saying a bit about what we have *not* assumed to arrive at this conclusion. Firstly we have not made any of the assumptions needed to guarantee the existence of demand, since our result is a statement on the monotone response of demand to income change, *if* demand exists. In particular, U need not be continuous and the budget set need not be compact since we allow for some prices to be zero. (Of course, demand can still exist in a noncompact budget set provided U is not strictly increasing in all arguments.) Because we have not assumed that U is increasing in all arguments, or more generally, that U obeys local non-satiation, demand need not obey the budget identity, i.e., demand may be valued by p at strictly less than income.

Strengthening our assumptions with other assumptions usually made in demand theory will lead to slightly stronger results. We know that if U is strictly quasi-concave, demand must be unique if it exists. So if we add this assumption to the concavity and supermodularity of U, we can say that if x' is the demand at (p, w') and x'' is the demand at (p, w''), with $w'' > w'$, then $x'' \geq x'$. If we also know that demand obeys the budget identity (for example, because U obeys local non-satiation) then we can say that $x'' > x'$.

As a special case of our result, we know that demand is normal if U is additive and concave, i.e., $U(x) = \sum_{i=1}^{l} u_i(x_i)$, where $u_i : R_+ \to R$ is a concave function, for $i = 1, 2, ..l$. Normality in this special case is well known, though a standard proof will assume that the u_is are differentiable and increasing; as we have shown, while these assumptions may serve other useful purposes, they are not crucial to the comparative statics as such.

The conditions we have imposed on U are not the weakest possible - for example, we could just require U to be quasiconcavemodular - but they are quite natural in some sense. In demand theory, it is typical to assume that preferences are quasiconvex to guarantee that the demand correspondence is convex valued, or even strongly quasiconvex to guarantee that demand at any particular price-income situation is unique. With these assumptions (and conditional on certain technical assumptions like smoothness), preferences are always representable by concave (rather than just quasiconcave) utility functions (see Mas-Colell (1985)). Clearly it follows that concave utility functions alone do not guarantee normality; but the property is guaranteed by utility functions which are *both* concave and supermodular.

Example 2. Another basic question in demand theory is whether the law of demand holds. We would like to say that as the price of good 1 falls, with other prices and income held fixed, that the demand for 1 rises.[6] More generally, one ought to be able to identify conditions under which, holding all other prices fixed, the demand for i increases if all or any of the following occur: the price of 1 falls, income goes up, and tastes change in a way which is favorable to good 1. Those conditions are identified by Theorem 4.

To capture the change in tastes in favor of - say - good 1, we construct a family of utility functions by defining U, which maps $R^l_+ \times T$ to R, with $U(x,t) = \bar{U}(x) + u(x_1, t)$. We assume that T is an interval in R, $\bar{U} : R^l_+ \to R$ is supermodular and concave in x^{-1}, and $u : R_+ \times T \to R$ is supermodular in (x_1, t), so the conditions on the objective functions in Theorem 4 are satisfied. The family of constraint functions is $C : R^l_+ \times R_+ \to R$, where $C(x,s) = (x_1/s) + \sum_{i=2}^l p_i x_i$. Note that an increase in s corresponds to a fall in the price of good 1 and it is also not hard to see that C in this case does satisfy the conditions on the constraint functions in Theorem 4. So we conclude that demand for good 1 increases with (t, s, w) in the sense of that theorem.[7]

Example 3. A demand function is said to exhibit the gross substitutability property if a fall in the price of good i causes the demand for all other goods to decrease. This property is important because, amongst other things, it helps to guarantee the uniqueness and stability of the equilibrium price in general equilibrium models (see, for example, Mas-Colell et al (1995)). The most well known conditions guaranteeing gross substitutability are the following. Let $U : R^l_{++} \to R$ be of the form $U(x) = \sum_{i=1}^l u_i(x_i)$ where each $u_i : R_+ \to R$ is C^2, with $u_i'(x_i) > 0$ and $u_i'' \leq 0$ for $i = 1, 2, ...l$. Then if $f : R^l_{++} \times R_+ \to R^l_{++}$ is the demand function generated by U, f will obey gross substitutability if $-x_i u_i''(x_i)/u_i'(x_i) < 1$ for all i and $x_i > 0$.

One can easily obtain this result using the techniques developed here. Assume that income is held fixed at w and consider a price change from p' to p'', where $p_i'' = p_i'$ for $i \geq 2$ and $p_1'' < p_1'$. Suppose that demand exists at both prices, with x' being a demand at p'. We wish to show that there is a demand at p'' in which the demand for good i rises and that of all other goods fall.

First, observe that x^* solves the following problem: (i) maximizing $\sum_i^l u_i(x_i)$ subject to x satisfying $p \cdot x = w$ if and only if $(s_1^*, x_2^*, ..., x_l^*)$, where $s_1^* = p_1 x_1^*$, solves the following problem: (ii) maximizing $u_1(s_1/p_1) + \sum_{i=2}^l u_i(x_i)$ subject to $s_1 + \sum_{i=2}^l p_i x_i = w$. So we can focus on problem (ii).

Since x' solves (i) at $p = p'$ we know that $(s_1', x_2', x_3', ..., x_l')$, with $s_1' = p_1' x_1'$ is a solution to (ii) at $p = p'$. Provided the map from $(x_1, 1/p_1)$ to $u_1(x_1/p_1)$ is supermodular, and since demand exists at p'' by assumption, we know from Lemma 2 that there is a solution $(s_1'', x_2,'', ..., x_l'')$ to (ii) at $p = p''$ such that $s_1'' \geq s_1'$. In other words, there must be a demand at $p = p''$ in which the expenditure on good 1 is higher than that at $p = p'$. In particular, $x_1'' > x_1'$.

Since U is additive, we know that $(x_2', x_3', ...x_l')$ maximizes $\bar{U}(x_2, x_3, ...x_l) = \sum_{i=2}^l u_i(x_i)$ subject to $\sum_{i=2}^l p_i x_i \leq w - s_1'$. If u_is are concave, so is $\bar{U}$; furthermore, $\bar{U}$ is additive and therefore supermodular. From our discussion in Example 1, we know that $\bar{U}$ generates normal demand. When more is spent on good 1, the expenditure available for other goods is reduced from $w - s_1'$ to $w - s_1''$, and so there must be $(x_2''', x_3''', ..., x_l''')$ which maximizes $\bar{U}(x_2, x_3, ...x_l)$ subject to $\sum_{i=2}^l p_i x_i \leq w - s_1''$ such that $x_i''' \leq x_i'$ for $i \geq 2$. Furthermore, $(s_1'', x_2''', x_3''', ..., x_l''')$ solves (ii) at $p = p''$, which establishes gross substitutability.

It remains for us to point out what it means for the map from (x_1, a) in R^2_{++} to $u_1(ax_1)$ to be supermodular. It is not hard to check that this is equivalent to the convexity of the map $\tilde{u}_1 : R \to R$ given by $\tilde{u}_1(z_1) = u_1(e^{z_1})$. In short, we have shown that the additive utility function U will generate demand satisfying gross substitutability if for all $i \geq 1$, u_i is concave and $\tilde{u}_i$ is convex. It is also not hard to check that when u_i is C^2 with $u_i' > 0$, then $\tilde{u}_i$ is convex if and only if $-x_i u_i''(x_i)/u_i'(x_i) \leq 1$ for all $x_i > 0$. In other words, we have obtained the non-differentiable version of the well known result.

4. Applications to Producer Theory

We begin with the most basic and obvious application of Theorem 3.

Example 4. A producer chooses the production vector $\bar{q}$, drawn from the production possibility set Q' in R^l. The vector $\bar{q}$ is chosen to maximize the firm's profit. To each good i is associated a price, which we assume is a function of $\bar{q}$, so we write it as $p^i(\bar{q})$. We denote the vector of l prices by $p(\bar{q})$. The firm's problem is to maximize profit, $\Pi(\bar{q}) = p(\bar{q}) \cdot \bar{q}$, subject to $\bar{q}$ in Q'. If we write the revenue derived from good i as R_i, so $R^i(\bar{q}) = p_i(\bar{q})\bar{q}^i$, the profit function may also be written as $\Pi(\bar{q}) = \sum_{i=1}^{l} R_i(\bar{q})$.

Theorem 3 tells us that if Π is quasiconcavemodular then it has the monotonic property. For Π to be quasiconcavemodular, it is sufficient that it is supermodular and partially concave. This will certainly occur if perfect competition is assumed, so $p(\bar{q})$ is identically constant and, in particular, independent of $\bar{q}$. More general it will be true if the price of each good is a function only of its output level, i.e., p_i is a function only of $\bar{q}_i$, and the revenue function R_i is a concave function of $\bar{q}_i$. In this situation, Π is additive, hence supermodular, and concave. More generally, a sufficient condition for Π to be supermodular and concave is for each R^i to satisfy these properties.

Consider now an expansion of the firm's production possibility set, to Q'' , satisfying the assumptions of Proposition 6 for every i. So $Q'' > Q'$ and Theorem 3 will tells us that the optimal production vector will increase with this technological change. The word 'increase' here has to be interpreted correctly: typically, $\bar{q}$ will have positive and negative entries, corresponding to outputs and inputs. The 'increase' in q means that outputs will increase and inputs will fall.

For a precise example of a technological change with an effect of this kind, assume that there are n inputs, collectively denoted by the vector l in R_+^n (the letter 'l' being suggestive of 'labor', to produce m outputs, to be denoted by the vector q in R_+^m. We assume that there is a continuous, increasing, convex, and submodular function $\phi : R_+^m \to R$, and a continuous, increasing, concave and supermodular function $\psi : R_+^n \to R$, so that the firm's production possibility set, Q_K is given by the elements $\bar{q} = (q, -l)$ such that $\phi(q) \leq \psi(l) + K$.

Varying K will vary the firm's production possibilities. In fact we have chosen an example where K has a very simple interpretation. One can think of the vector l of inputs being converted into a single composite input, whose level is given by $\psi(l) + K$. With this level of composite input, the possible output vectors is given by those q for which $\phi(q) \leq \psi(l) + K$. An increase in K of - say - δ, corresponds to a technological change which raises the level of the composite input by a constant amount δ for every input vector l.

Notice that $Q_K = \Gamma^{-1}(-\infty, K]$ where the function $\Gamma : R_+^m \times R_-^n \to R$ is given by $\Gamma(q, -l) = \phi(q) - \psi(l)$. With our assumptions on ϕ and ψ, it is not hard to see that Γ is a continuous, increasing, convex, and submodular function. So by Corollary 4, $\bar{q}$ increases with K.

Example 5. Consider a firm producing just one good, whose revenue when q units of output are produced is given by $R(q, a)$, where a is parameter drawn from an interval A contained in R. Producing this good requires n inputs; we denote the typical input vector by l, the vector of input prices by w, and let $F : R^n_+ \to R_+$ be the firm's production function. The firm's objective is to maximize profit; formally, it chooses (q, l) to maximize $\Pi(q, l, a) = R(q, a) - w \cdot l$ subject to (q, l) in the set $\{(q, l) \in R_+ \times R^n_+ : q \leq F(l)\}$. We wish to identify conditions under which we can sign the effect of a on q and l.

Let (q', l') be a solution when $a = a'$ and (q'', l'') be a solution when $a = a''$, with $a'' > a'$. Assume that R is a supermodular function of (q, a). Our first claim is that if $q'' \leq q'$, then (q', l') also maximizes Π at $a = a''$. Revealed preference says that $\Pi(q', l', a') \geq \Pi(q'', l'', a')$. Note that R enters additively in the objective function, so that if $q'' \leq q'$, Lemma 2 tells us that $\Pi(q', l', a'') \geq \Pi(q'', l'', a'')$, which means that (q', l') also maximizes Π at $a = a''$.

So we assume that $q'' > q'$. Let $\tilde{F} : R^n_+ \to R$ be some representation of the firm's isoquants in R^n_+; note that $\tilde{F}$ may or may not be the real production function F. Since profit maximization implies cost minimization, l' and l'' must minimize $w \cdot l$ subject to l satisfying $q' \leq \tilde{F}(l)$ and $q'' \cdot \tilde{F}(l)$ respectively, or equivalently, maximize $-w \cdot l$ subject to $-\tilde{F}(l) \leq -q'$ and $-\tilde{F}(l) \leq -q''$. Applying Corollary 4, we know that $-l$ increases monotonically with $-q$ (equivalently, l increases with q) provided the map from $-l$ in R^n_- to $-\tilde{F}(l)$ is submodular, partially convex, increasing and continuous. This is equivalent to having $\tilde{F}$ supermodular, partially concave, increasing, and continuous.

In short, assuming that the revenue function R is supermodular in (q, a), and that the firm's isoquants have a supermodular, concave, increasing and continuous representation will guarantee that the profit maximizing choice of q and l both rise with a.

Example 6. Consider a firm with a single input, whose level we denote by $L > 0$, and m outputs, whose output level is denoted by the vector q in R^m_+. The firm's goal is to choose (q, L) to maximize profit, given by $\Pi(q, L, w) = \sum_{i=1}^m R^i(q) - wL$, where R^i is the income derived from good i if the output vector is q and w is the unit cost of the input. We assume there are increasing functions $\phi : R^m_+ \to R$ and $\psi : R_+ \to R$ constraining the firm's choice of $(q, -L)$ to those satisfying $\phi(q) \leq \psi(L)$.

We can employ arguments similar to those in the previous example to sign the change in q and L following a change in w. Assume that w rises from w' to w''. Firstly, by applying Lemma 2 again, we may restrict our attention to the case when the optimal level of output falls strictly from L' to L''. To determine its impact on output, we must compare the optimal choices at two constrained maximization problems: in the first case, total revenue $R(q) = \sum_{i=1}^m R^i(q)$ is maximized while constraining q to the set $Q' = \{q \in R^n_+ : \phi(q) \leq \psi(L')\}$ and in the second case, $R(q)$ is maximized with q constrained to $Q'' = \{q \in R^n_+ : \phi(q) \leq \psi(L'')\}$. Since ψ is increasing, $\psi(L'') \leq \psi(l')$ and $Q'' \subset Q'$. Corollary 4 then identifies sufficient conditions for q to vary monotonically with L: R should be partially concave and supermodular, and, in addition to being increasing, ϕ should be partially convex, submodular and continuous. Of course, a sufficient condition for R to be partially concave and supermodular is for R_i to be a concave function of q_i.

Example 7. Consider a firm who employs n inputs to produce one or several output goods. When the vector of inputs is l in R^n_+ , the firm produces goods which generate revenue of $R(l)$. The unit cost of inputs is given by the vector w, so the firm's profit function is $\Pi(l) = R(l) - w \cdot l$, which it maximizes by choosing the vector l. Suppose, in addition, that the firm faces a constraint on the level of some input (let us say it is input 1) that it can employ. Using standard monotone comparative statics techniques, one can show that relaxing this constraint will cause *all* inputs to go up provided R is a supermodular function of l. But suppose the constraint is of the form $\sum_{i=1}^{k} l^i \leq L$. A constraint of this type will make sense if, for example, goods 1 to k in fact represent different ways of deploying a particular type of labor within the firm, whose *total number* in the short run cannot be increased beyond L. Provided R is partially concave and supermodular, Corollary 4 tells us that a relaxation of this constraint, i.e., an increase in L, will cause the demand for *all* inputs to increase.

5. Other Applications

In Section 2, we developed our general monotone comparative statics results, and then applied them, in ways which are in a sense quite obvious, to problems in demand and producer theory in Sections 3 and 4. In this section, to demonstrate the usefulness of our basic results, we will consider three somewhat less obvious applications, all taken from the profit maximization problem of a monopolist.

Example 8. Consider a monopolist who produces a single good priced at p and maximizes profit, $\Pi(p, q) = pq - c(q)$, subject to the demand condition, $\phi(p, q) \leq k$. The function

$\Pi : R_+ \times R_+ \to R$ is supermodular in (p, q) and obviously concave in p; it will also be concave in the output q if the cost function c is a convex function of q. Provided ϕ is increasing, continuous, submodular, and partially convex, Corollary 4 tells us that an increase in demand corresponding to an increase in k will cause both price and output of the monopolist to rise.

For a specific example of the function ϕ, suppose that the demand curve has the form $p = G(k - H(q))$ where H is increasing, continuous, and convex, and G is strictly increasing, continuous, and concave. Then $\phi(p, q) = G^{-1}(p) + H(q)$ is additive (hence submodular), increasing, continuous and convex.

Example 9. We consider a single product monopolist again, but this time we are interested in how the profit margin varies with unit cost, which is assumed to be constant (over output). We write his profit function as $\Pi(m, q) = mq$, where m is the margin over the unit cost c and q is the output level. He faces a demand function, $p = \phi(q)$, so we can think of the monopolist as maximizing $\Pi(m, q)$ subject to $m + c \leq \phi(q)$. We can write this constraint in a more familiar way as $m - \phi(q) \leq -c$. It is clear that Π is supermodular, and concave in m and q separately. Provided ϕ is continuous, decreasing and concave, the function $C(m, q) = m - \phi(q)$ will be continuous, increasing and convex. Since C is additive, it is also submodular. Therefore, all the conditions of Corollary 4 are satisfied, and we may conclude that m and q will both fall as c increases.

Example 10. We consider a profit-maximizing monopolist who produces two goods, 1 and 2 and we allow the price of one good to affect the demand for another. Formally, if $x_1 \geq 0$ is the price of good 1 and $x_2 \geq 0$ the price of good 2, the demand for good i ($i = 1, 2$) is given by $\bar{D}_i(x_1, x_2) \geq 0$. For reasons which will make themselves clear later, it is convenient to re-write demand as a function of the price of good 1, x_1 and the negative of the price of good 2, y_2; formally, we define new functions D_i ($i = 1, 2$) such that $D_i(x_1, y_2) = \bar{D}_i(x_1, -y_2)$. We assume that the marginal cost of producing goods 1 and 2 are constant over output levels, and equal c_1 and c_2 respectively. We wish to consider the impact of a change in c_1 on the optimal choice of y_2; in other words, we want to know how a change in the marginal cost of producing 1 affects the profit maximizing price of 2. We assume that the goods are substitutes in the sense that demand for good 1 *falls* with the price of good 2; for that reason, the price of 2 will never be chosen to be below the marginal cost of c_2, since raising it to c_2 will unambiguously increase profits. Hence, without loss of generality, we may restrict the domain of D_1 and D_2 to the set $R_+ \times (-\infty, -c_2]$.

In this case, it is instructive to think of the monopolist as choosing $x_1 \geq 0$, y_2 in $(-\infty, -c_2]$, and the output level of good 1, denoted by $d_1 \geq 0$, to maximize

$$\Pi(x_1, y_2, d_1) = x_1 d_1 - y_2 D_2(x_1, y_2) - c_1 d_1 - c_2 D_2(x_1, y_2)$$

subject to the demand constraint on good 1, $d_1 \leq D_1(x_1, y_2)$. We wish to identify conditions under which y_2 increases with c_2 (in other words, that the price of good 2 falls as the marginal cost of good 1 increases).

Let (x_1', y_2', d_1') and (x_1'', y_2'', d_1'') be solutions at $c_1 = c_1'$ and $c_1 = c_1''$ respectively, with $c_1'' > c_1'$. If $d_1'' \geq d_1'$, one can argue, by appealing to Lemma 2, that (x_1', y_2', d_1') also maximizes profit at $c_1 = c_1''$ and we are done. (The argument is similar to that in Example 4; note also that this conclusion requires no assumptions at all on the demand functions.)

So we assume that $d_1'' < d_1'$. The additive structure of the profit function means that (x_1', y_2') must also maximize the function $G' : R_+ \times (-\infty, -c_2] \to R$ given by

$$G'(x_1, y_2) = x_1 d_1' + (-y_2 - c_2) D_2(x_1, y_2)$$

while subject to the constraint $-D_1(x_1, y_2) \leq -d_1'$. Analogously, (x_1'', y_2'') maximizes the function $G'' : R_+ \times (-\infty, -c_2] \to R$ given by

$$G''(x_1, y_2) = x_1 d_1'' + (-y_2 - c_2) D_2(x_1, y_2)$$

while subject to the constraint $-D_1(x_1, y_2) \leq -d_1''$. We are now effectively in the setting of Theorem 5. To guarantee that y_2 falls with d_1, i.e., y_2 increases with $-d_1$, Theorem 5 requires the constraint function $-D_1$ to be submodular, continuous, and increasing in (x_1, y_2) and convex in x_1; more familiarly this means that the demand function $\bar{D}_1$ (which you recall is a function of prices (x_1, x_2)) is submodular and continuous in both variables, concave in x_1, decreasing in x_1, and increasing in x_2. For the objective function, Theorem 5 requires that the map from (x_1, d_1) to $x_1 d_1$ be increasing in x_1, decreasing in $-d_1$ and submodular in $(x_1, -d_1)$, all of which certainly hold. It also requires that the function mapping (x_1, y_2) in $R_+ \times (-\infty, -c_2]$ to $(-y_2 - c_2) D_2(x_1, y_2)$ be supermodular in (x_1, y_2) and concave and increasing in x^1. It is not hard to check that this is true if the function $\Pi : R_+ \times (c_2, \infty) \to R$ given by $\Pi_2(x_1, x_2) = (x_2 - c_2) \bar{D}_2(x_1, x_2)$ is submodular in (x_1, x_2) and concave and increasing in x_1.

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Footnotes:

1. For a textbook introduction to these methods see Topkis (1998) or Vives (1999).

2. The product order is of course not the only possible order. For certain types of problems, it can be helpful to turn to some other ordering of the Euclidean space. For a recent discussion of this issue, with special reference to problems in consumer theory see the interesting paper of Mirman and Ruble (2003). In particular, by ordering the Euclidean space differently, they identify conditions under which a particular good is normal. In contrast, our emphasis in this paper is on finding conditions under which all goods are simultaneously normal. The work of Mirman and Ruble (2003) builds on Antoniadou (1995), which was the first serious attempt at applying lattice programming techniques to problems in consumer theory.

3. The function G is *submodular* if $-G$ is supermodular.

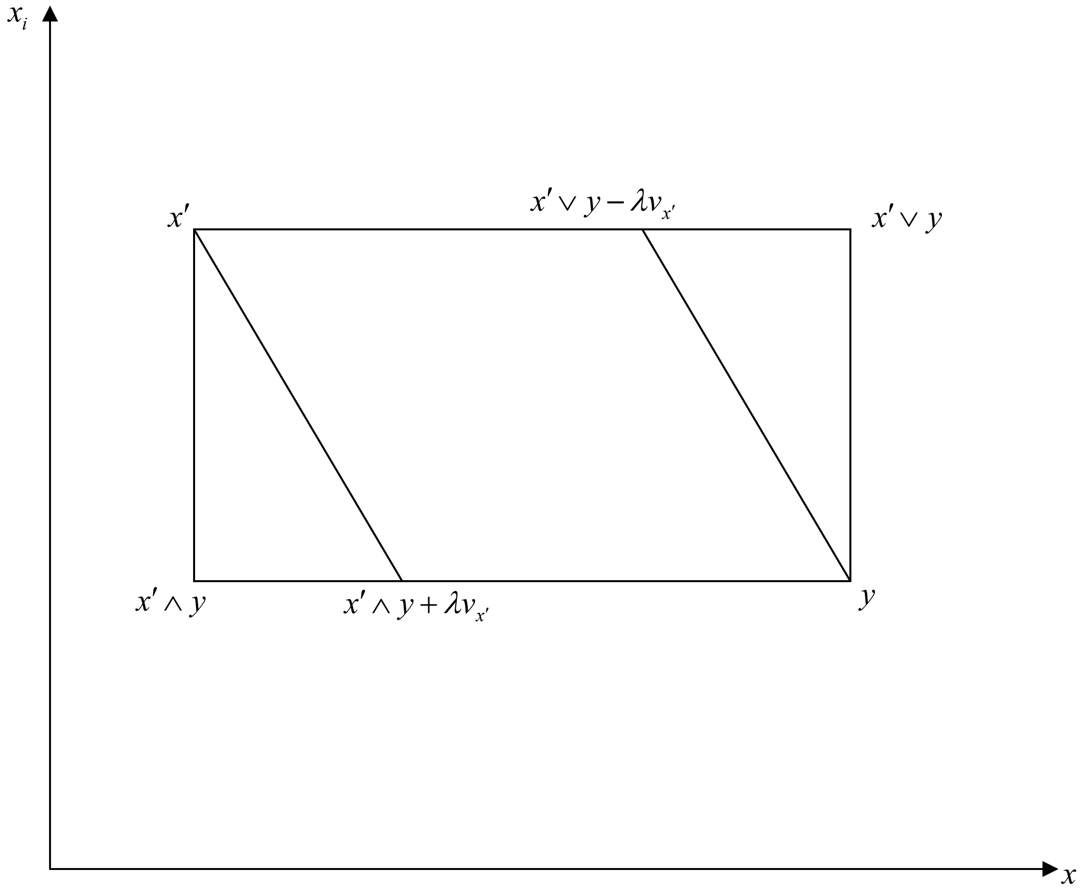
4. According to our definitions, for F to have the monotonic property is *not* equivalent to F having the monotonic property for all $i = 1, 2, ..l$; the latter property is stronger.

5. For a discussion of the role of normality in general equilibrium comparative statics, see Quah (2003).

6. Note that this one good version of the law of demand is not the same as the multi-good version, which requires the inner product of the price change and demand change vectors to be negative. For more on this stronger version of the law of demand see Mas-Colell et al (1995). It is an interesting question (to which we have no answer) how one may derive the well known conditions on the utility function for this stronger property (due to Milleron,

Mitjuschin and Polterovich) from lattice programming techniques.

7. There is a familiar and simple argument which shows that good i obeys the law of demand if it is normal. The idea is to decompose the change in demand from the old to the new price into changes arising from the substitution and income effects. With a fall in the price of i, both the substitution and income effects act to increase the demand for good i - the first follows from revealed preference and the second by the assumption of normality - so the demand for i rises. Notice that this argument requires that demand be defined at every price-income situation; in particular, it has to defined when the agent is given just enough income to purchase his original demand bundle at the new price. Since we do not make this assumption, we cannot simply repeat this argument in our example.


x_i
x'
$x' \vee y - \lambda v_{x'}$
$x' \vee y$
$x' \wedge y$
$x' \wedge y + \lambda v_{x'}$
y
x_j