Nonconvex Production Technology and Price Discrimination

Bing Jing and Roy Radner
Stern School of Business, New York University

January 8, 2004

Abstract

A modern firm often employs multiple production technologies based on distinct engineering principles, causing non-convexities in the firm's unit cost as a function of product quality. Extending the model of Mussa and Rosen (1978), this paper investigates how a monopolist's product line design may crucially depend on the non-convexities in the unit cost function. We show that the firm does not offer those qualities where the unit cost exceeds its convex envelope. Consequently, there are "gaps" in its optimal quality choice. When the firm is only permitted to offer a limited number of quality levels (due to possible fixed costs associated with offering each quality), the optimal location of quality levels still lies within those regions of the quality domain where the unit cost function coincides with its convex envelope. We further show that the firm's profit is a supermodular function of its quality levels, and characterize a necessary condition for the optimal quality location.

1 Introduction

In a familiar context of vertical differentiation, this paper examines how a monopolist firm's product line design may be driven by its production technologies. In this market consumers have heterogeneous willingness to pay for quality, and the unit cost as a function of product quality is technology specific. For a given production technology, the unit production cost tends to rise more rapidly as quality increases, and an increasing, convex cost function effectively captures such decreasing returns. Nevertheless, firms in many product markets frequently use multiple types of technology to produce the same generic kinds of good. Based on a distinct engineering principle, each type of technology has its own cost advantage with respect to a certain range of product performance parameters. Even though the cost function for each individual technology may be convex, the superposition of multiple convex technologies may lead to a situation where the minimum of the unit cost functions is not convex over the entire quality domain. A central task of the current paper is to examine the effects of the non-convexities in the firm's unit cost function on its price discrimination and product choice.

For example, currently ink-jet, laser, and dye sublimation are the three dominant print technologies. The print head of an inkjet printer contains up to hundreds of nozzles (each thinner than a human hair) through which tiny drops of ink are propelled onto the paper, and resolution is determined by the number and size of nozzles. A laser printer focuses a laser beam onto a photoelectric belt, creating an electrical charge in areas where toner is to adhere. For dye sublimation printers, the solid dye is vaporized into gas, which deposits itself onto the directed areas of the media and solidifies. Ink-jet has the lowest cost and is most suitable for low-to-intermediate qualities of black-and-white printing. Laser printers require higher unit costs than ink-jet and are good for high-end black-and-white printing. Dye sublimation is the most expensive but provides the best quality, especially for color or photographic printing. Many printer manufacturers install all three types of technologies to meet the demand of customers with various quality requirements.

There exist many other examples of firms' adopting multiple production technologies. While digital machines have an edge in making products with relatively high precision requirements, conventional analog machines are still more efficient in settings with low precision requirements. In markets where custom product features are valuable (e.g., application specific integrated circuits (ASICs), apparel, and furniture), a firm may use flexible machines (e.g., FMS) to make products with higher degrees of customization and use more dedicated machines (e.g., assembly lines) for those with low degrees of customization. Likewise, LCD (Liquid Crystal Display) and CRT (Cathode Ray Tube) technologies are more efficient in generating digital displays with high and low degrees of steadiness and clarity, respectively. Sutton (1998) provides additional vivid empirical evidence on firms' pursuing multiple "technological trajectories". In addition, firms also add new technologies over time (due to capacity expansion or technical progress), and therefore possess multiple "vintages" of technologies (Arrow 1962). When the most recent know-how in production is incorporated into the latest technology, each in the firm's "fleet" of technologies may have different cost limitations with regard to certain product performance attributes.

To highlight the role of a non-convex unit cost function in a monopolist's product line design, we employ a modeling framework in which bunching does not arise and both quality and consumer type are unidimensional. Each consumer has a unitary demand and a quasi-linear Mussa-Rosen utility function, i.e., the net utility is separable in quality and price, and each consumer has a constant marginal utility of quality. To simplify the analysis, we further assume that consumer types are uniformly distributed on the unit real interval. (However, see the comments on this assumption in Section 6.) The firm's unit cost as a function of quality may exhibit non-convexities, and consequently, there are subintervals in the quality domain where the unit cost exceeds its convex envelope; such subintervals are called *anomalous*. Solving for the optimal pricing procedure directly is complex, but we present a simple shortcut solution to the firm's problem. The crux of the analysis throughout this paper is to regard the convex envelope of the true unit cost as a "virtual" unit cost function. Clearly, the maximum profit attainable with this "virtual" unit cost function is no less than that attainable with the true unit cost function. The virtual unit cost function is convex by construction, and is linear in the anomalous intervals. We show that if the firm were able to produce according to this virtual unit cost function, its optimal price schedule would be given by a simple formula, and moreover the firm would not sell any quality in the intervals where the virtual unit cost function is linear, and hence not in any anomalous interval corresponding to the true unit cost function. That is, the firm would only sell those qualities where the true unit cost equals its convex envelope. This implies that the firm's optimal profit with the true cost function equals that with the virtual cost function. Therefore, the price schedule optimal for the virtual cost function is also optimal for the true cost function.

We separately consider two cases according to whether the firm faces a constraint on the number of quality levels it may offer. In the first case, the firm is free to offer as many quality levels as it wishes from an exogenously given quality space, which may be either discrete or continuous. In this case we assume that there is no fixed cost associated with offering each quality level, as is common in the literature. We show that the firm's optimal price and product policies have the same spirit for both discrete and continuous quality spaces. As noted above, the optimal price policy is what it would be if the unit cost schedule were the convex envelope of the firm's true unit cost schedule. Thus there may be "holes" in the firm's optimal quality choice, since it offers only those quality levels where the unit cost schedule coincides with its convex envelope. For a firm with multiple production technologies, the above result also specifies the optimal utilization of each individual technology. Over time, the unit production cost on some technologies may decrease due to technical progress or process innovation. We show that cost reduction on one technology will reduce the range of qualities produced using each of its neighboring technologies. In our model, a technology may become obsolescent even when it still has a cost advantage over some quality region.

In the second case, the firm is only permitted to offer a limited number of quality levels (e.g., because of fixed costs) from a continuous quality space, and the firm has to decide where to locate these quality levels. Here again, we show that none of the optimal quality levels is located in an anomalous subinterval. The firm's profit is a supermodular function of these quality levels. We concisely characterize a necessary condition of the optimal quality levels, which has the usual interpretation that the marginal cost of each quality equals its marginal revenue.

When the consumer distribution is continuous and has positive density everywhere on its support, existing research has reached a standard conclusion that the monopolist's optimal quality choice is continuous, i.e., the optimal quality assignment schedule does not jump as consumer type increases (see p. 310-311 of Mussa and Rosen 1978, and Proposition 6 of Rochet and Chone 1998). The economic rationale offered by Mussa and Rosen is that jumps in the quality assignment imply that the monopolist would not be able to fully exercise its power to discriminate among different types of consumers. To the contrary, when the unit cost function is nonconvex, the present paper indicates that jumps in the quality assignment are necessary for maximizing the extent of price discrimination.

Some intuition for such a contrast may be found by examining the efficient quality allocation. Suppose each quality were made available at its unit cost. If the unit cost function is strictly convex, then a consumer of a slightly higher type would choose a slightly higher quality, and thus the efficient quality allocation is continuous. If the unit cost function has nonconvexities, however, no consumer would choose a quality in an anomalous subinterval, and the efficient quality allocation necessarily has jumps. Our result may be extrapolated to shed light on the nonlinear pricing literature as well, where the validity of the conclusion that the optimal spectrum of *quantities* offered by the monopolist is continuous may also be sensitive to the cost specifications in their models.

Section 2 presents a brief literature review. Sections 3 and 4 derive the optimal price and product policies when the firm's exogenous quality space is discrete and continuous, respectively. In Section 5, we analyze the optimal location of a limited number of quality levels in a continuous quality domain. Section 6 comments on the modeling assumptions and concludes.

2 Related Literature

Product line design by a monopolist has been studied by Mussa and Rosen (1978), Gabszewicz et al (1986), Matthews and Moore (1987), and more recently by Rochet and Chone (1998). The key contribution of Mussa and Rosen (1978) is to realize that inducing different consumers to purchase the same product (also known as "incomplete sorting" or "bunching") may be optimal for the monopolist, depending on the distribution of consumer types, and to design an "ironing" procedure to address bunching. Gabszewicz et al (1986) examine how a "natural" monopolist's product line choice may critically depend on the scope of the distribution of consumer income. The natural monopolist prices its product line in a manner that keeps out potential entry, and its output is thus fixed. Matthews and Moore (1987) extend the Mussa-Rosen analysis into a two dimensional product space composed of both warranties and qualities (while consumer types remain unidimensional), and surprisingly, they show that the optimal quality assignment is not always monotonic in consumer type. Rochet and Chone (1998) devise a "sweeping" procedure to deal with bunching in multi-dimensional spaces of consumer types and product attributes. As we can see, the focus of these articles is mainly on demand-side features such as the distribution of consumer tastes (or income) in a uni- or multi- dimensional space. However, they only adopt specific forms of unit cost functions and thus do not fully consider the role of production in designing an optimal screening procedure. The unit cost is an increasing, strictly convex function of quality in Mussa and Rosen (1978) and Rochet and Chone (1998), and is zero for all qualities in Gabszewicz et al (1986). Shaked and Sutton (1983) consider a general production technology. However, they consider an oligopolistic setting and their interest is to identify cost conditions under which only a limited number of firms may survive at a Nash price equilibrium (a phenomenon they call "natural oligopolies"). The unit cost of production is suppressed to zero in many other competitive models of vertical differentiation such as Gabszewicz and Thisse (1979, 1980) and Shaked and Sutton (1982).

Closely related to product line design is another prominent stream of research in the screening literature—nonlinear pricing (e.g. Mirman and Sibley 1980, Spence 1980, Maskin and Riley 1984, McAfee and McMillan 1988, Wilson 1993, and Armstrong 1996, Sibley and Srinagesh 1997), but there the authors' main concern is still about inducing the right kind of consumer choice under various demand conditions, rather than cost conditions, with most of the analytical effort devoted to tackling "bunching" and higher dimensional product and consumer spaces.

3 Exogenous Finite Set of Qualities

Consider a monopolist firm that produces a service. The quality of the service is scalar-valued, and the service can be provided at various levels of quality. Let Q denote the exogenously given finite set of quality levels, $s_k, \quad k = 0, ..., K$. Assume that $s_0 = 0$, and (without loss of generality) that the qualities are distinct and increasing in k. The marginal cost of providing a service of quality s_k is a constant c_k, and there is no fixed cost. We assume that $c_0 = 0$ and $c_k > 0$ for $k > 0$. Let $c = (c_0, c_1, ..., c_K)$. The firm charges a price, p_k, per unit of service of quality s_k. A convention is that the zero-quality service, s_0, is offered at $p_0 = 0$.

We now turn to the model of demand. There is a continuum of consumers, indexed by the real variable θ, uniformly distributed on the interval $[0, 1]$. An individual consumer either does not purchase, or purchases exactly one unit of the service. If consumer θ purchases the service of quality s_k at price p_k, her net utility is

$$U(s_k, p_k; \theta) = \theta s_k - p_k. \tag{1}$$

Note that this is the familiar Mussa-Rosen utility function, where θ measures the consumer's marginal valuation of quality. Given the price vector p, each consumer chooses a quality level that maximizes her utility. To be definite, if a consumer is indifferent between any two quality levels, we assume that she will choose the lower quality level. In particular, the purchase of s_0 generates zero utility, and thus represents not purchasing the service at all.

We next derive the demand for each quality level. Let R denote the set of pairs (s_k, p_k) with s_k in Q, let H' be the convex hull of R, and let L' be the lower boundary of H'. Note that L' is the graph of a convex function. For any given consumer θ, her net utility function is linear in quality. Hence her utility-maximizing quality-price pair must be in L', and so any quality-price pair in the interior of H' will not be chosen by any consumer. In fact, because we have assumed that, among the pairs with highest utility she chooses the one with lowest quality, her optimal choice is an extreme point ("vertex") of H'. (A point of H' is extreme if it is not a nondegenerate convex combination of two distinct points of H'. Note that (s_0, p_0) is such an extreme point.) Besides, since the highest value of θ is 1, any extreme point of L' whose left-hand slope of L' exceeds 1 is not chosen by any consumer.

Define

$$d_k = \frac{p_{k+1} - p_k}{s_{k+1} - s_k}, \quad 0 \leq k \leq K - 1. \tag{2}$$

The variables d_k are the "slopes" of the price vector. Observe that all of the points in R are in L' if and only if d_k is a nondecreasing function of k. Furthermore, for $p_1 \geq 0$ we need $d_0 \geq 0$. Hence the price vector p is called *admissible* if it satisfies

$$p_0 = 0, \tag{3}$$

$$0 \leq d_0 \leq ... \leq d_k \leq ...d_{K-1} \leq 1. \tag{4}$$

For an admissible price vector p, it is straightforward that, for $1 \leq k \leq K-1$, consumer θ chooses (s_k, p_k) if and only if

$$d_{k-1} < \theta \leq d_k.$$

Consumer θ chooses (s_0, p_0) if and only if

$$0 \leq \theta \leq d_0.$$

Finally, consumer θ chooses (s_K, p_K) if and only if

$$d_{K-1} < \theta < 1.$$

Therefore, given the admissible price vector p, the demand for each quality level is

$$\begin{aligned} D_0(p) &= d_0, \\ D_k(p) &= d_k - d_{k-1}, \text{ for } 1 \leq k \leq K-1, \\ D_K(p) &= 1 - d_{K-1}. \end{aligned} \tag{5}$$

Note that, for an admissible price vector and $1 \leq k \leq K-1$, the pair (s_k, p_k) is an extreme point of L'- and hence demand for that quality level is strictly positive - if and only if $d_k > d_{k-1}$. In addition, the demand for the K'th quality level is strictly positive if and only if $d_{K-1} < 1$. The firm chooses the price vector p to maximize his profit

$$P(p) = \sum_k (p_k - c_k) D_k(p). \tag{6}$$

In what follows, it will be useful to express the profit formula in terms of the variables d_k, as done in Lemma 1 below. Define

$$g_k = \frac{c_{k+1} - c_k}{s_{k+1} - s_k}, \quad 0 \leq k \leq K-1. \tag{7}$$

These are the “slopes" of the cost vector. The cost vector c is called "quasi-convex" if g_k is nondecreasing in k.

Lemma 1

$$P(p) = \sum_{0}^{K-1} (s_{k+1} - s_k) \left[-d_k^2 + (1 + g_k) d_k \right] - c_K. \tag{8}$$

The proof of Lemma 1 is relegated to the Appendix.

We are now in a position to characterize the firm's optimal price vector. Let H be the convex hull of the pairs (s_k, c_k), $s_k \in Q$, let L be the lower boundary of H, and let h be the function whose graph is L (see Figure 1). Call a quality level s_k *anomalous* if $c_k > h(s_k)$. For instance, in Figure 1 u is an anomalous quality but v is an extreme point of H.

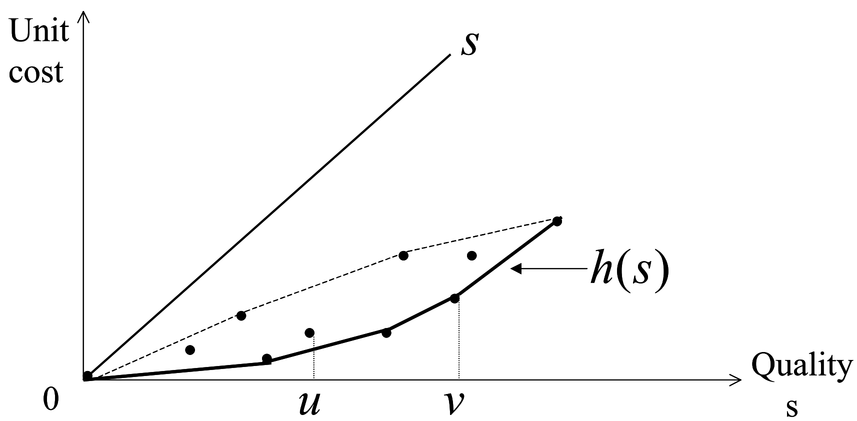

Figure 1.

Define the price vector π by

$$\begin{aligned} \pi_0 &= 0, \\ \pi_k &= \frac{s_k + h(s_k)}{2}, \quad 1 \le k \le K. \end{aligned} \tag{9}$$

Theorem 2 *If, in addition to the above assumptions, $c_K < s_K$, then π is an optimal price vector, and the monopolist only sells*

$$Q^* = \{s_k \in Q | c_k = h(s_k),\ g_{k-1} < g_k,\ \text{and}\ g_{k-1} < 1\}.$$

In particular, the demand for an anomalous quality level is zero.

Proof: Let P_c denote the maximum profit obtainable with the cost vector c. With an abuse of notation, let h denote the vector with coordinates $h(s_k)$. The plan of the proof is to show that if the true cost vector were h (instead of c), then the firm's maximum profit, P_h, would be at least as large as P_c, and π would be an optimal price vector. We then show that in that case the demand for every anomalous quality level would be zero. Hence demand would be positive only for quality levels such that $h(s_k) = c_k$, so that the profit P_h is attainable with the cost vector c and price vector π.

Lemma 3 $P_c \leq P_h$.

The proof of this lemma is given in the Appendix.

From (6), the firm's profit from an admissible price vector p is

$$P(p) = \sum_{1}^{K-1} (p_k - c_k)(d_k - d_{k-1}) + (p_K - c_K)(1 - d_{K-1}). \tag{10}$$

Recall that for an admissible price vector the slopes, d_k, must be nondecreasing. By Lemma 1, if we were to maximize the profit without the constraint that the slopes be nondecreasing, the optimal values of the slopes would be

$$\frac{1 + g_k}{2}, \tag{11}$$

and the corresponding prices would be π_k. If the cost vector c is quasi-convex, then the price slopes d_k are nondecreasing, and the price vector π is admissible. Hence, the conclusion of the theorem is verified in the case in which the cost vector is quasi-convex.

In the general case, the vector h is quasi-convex, and so the price vector π would be optimal if the cost vector were h. A maximal set of contiguous anomalous quality levels will be called an *anomalous region*. If $\{s_m, s_{m+1}, ..., s_n\}$ is an anomalous region, then h is linear on the interval $[s_{m-1}, s_{n+1}]$, and hence for the price vector π the slopes $d_{m-1}, ..., d_n$, are all equal. As noted above, the demand for quality level s_k is strictly positive if and only if $d_k > d_{k-1}$. Hence, *for the price vector that is optimal for the cost vector h, the demand for an anomalous quality level is zero.* It follows that, *for the price vector that is optimal for the cost vector h, demand for quality level s_k is strictly positive only if $h(s_k) = c_k$, $g_{k-1} < g_k$* and $g_{k-1} < 1$. Hence, by (10), the profit P_h is attainable with the cost vector c, and so, by Lemma 3, $P_c = P_h$, and π is optimal for the cost vector c, which completes the proof of the theorem. *Q.E.D.*

According to Theorem 2, demand is positive only for those quality levels s_k where (s_k, c_k) is an extreme point of L and where the cost slope $g_{k-1} < 1$.

4 Exogenous Interval of Qualities

In this section we characterize the monopolist firm's optimal price and product policy when the set of available quality levels is an interval, say $Q = [0, b]$. We shall see that the results for this model are quite similar to those for the case of a finite set of quality levels. The method of analysis is also similar, except that we shall be dealing with an infinite-dimensional space of price functions, instead of a finite-dimensional space of price vectors, so some extra machinery is needed.

We also discuss the implications of the optimal product policy on technological utilization when the monopolist employs multiple production technologies.

We first specify the production side of the model. The unit cost of producing at quality level s is $c(s)$, which is independent of the output level at quality s. Again, there is no fixed cost. We assume that the unit cost function has the following properties:

$$c \text{ is nonnegative and continuously twice-differentiable, except at finitely many points (or none);} \quad (12)$$

$$c(0) = 0, \quad c(b) < b. \quad (13)$$

The behavior and distribution of consumers are as in the preceding section. Consumers are uniformly distributed on $[0, 1]$, and if a consumer of type θ purchases a quality level s at price $p(s)$, then her net utility is

$$U(s, p(s); \theta) = \theta s - p(s). \quad (14)$$

Given the price function p, a consumer of type θ chooses a quality level that maximizes her net utility. If the utility-maximizing quality level is not unique, then she chooses the minimum such level. The firm's price policy is a real-valued function p on Q, which we assume satisfies the following conditions:

$$p \text{ is Lebesgue measurable, nonnegative, and nondecreasing.} \quad (15)$$

$$p(0) = 0, \text{ and } p \text{ is continuous from the right.} \quad (16)$$

(The reason for requiring the measurability of the price function will become apparent below.)

Adapting the notation of Section 3, for a given price function, p, let R denote the graph of p in the quality-price plane, let H' denote the convex hull of R, and let $L^{'}$ denote the lower boundary of $H^{'}$ As in Section 3, because of the linearity of a consumer's net utility in quality level and price, a quality level s such that $(s, p(s))$ is in the interior of $H^{'}$ will not be purchased by any consumer. In fact, again as in Section 3, quality levels such that $(s, p(s))$ is not an extreme point of $H^{'}$ in $L^{'}$ will not be purchased by any consumer. Hence, without loss of generality, we can confine our attention to price functions that are convex.

The following standard proposition about convex functions will be useful (see, e.g., Royden, 1988, Prop. 17, pp. 113-114).

Proposition 4 *If p is convex on $[0, b]$, then it is absolutely continuous. Its right-and left-hand derivatives exist at each point, and are equal to each other except possibly on a countable set. The left- and right-hand derivatives are monotone nondecreasing functions, and at each point the left-hand derivative is less than or equal to the right-hand derivative.*

In view of the Proposition, we shall make the convention that the derivative of the price function, which we shall denote by F, is its right-hand derivative, and so it is continuous from the right. With this convention,

$$p(s) = \int_0^s F(t)dt = \int_0^s F(t-)dt, \tag{17}$$

(where $F(t-)$ denotes the left-hand derivative),

We next characterize the measure of the consumers who purchase a quality not exceeding s. For the moment, fix a convex price function, say p, and let F denote its derivative. Since p is convex, the net utility of consumer θ is concave as a function of the quality level, s, and the derivative of her net utility function at s is

$$\theta - F(s).$$

There are three cases to consider. First, suppose that there is an interior s^* such that

$$F(s^*-) \leq \theta \leq F(s^*),$$

and s^* is the minimum of such values. Then the consumer will purchase s^*, provided

$$\theta s^* - p(s^*) > 0.$$

Second, if $\theta \leq F(0)$; then the consumer will purchase quality level 0. Finally, if $F(b-) \leq \theta \leq 1$, then the consumer will purchase quality level b, provided

$$\theta b - p(b) > 0.$$

In this case, the mass of consumers purchasing quality level b is $1 - F(b-)$.

A price function p is called *admissible* if it satisfies

$$p(0) = 0, F(0) \geq 0, F(b) = 1, \text{ and} \tag{18}$$

$$F(s) \text{ is nondecreasing in } s. \tag{19}$$

The assumption that $F(b) = 1$ is only a convention, since it does not affect the price function, but with this convention the mass of consumers purchasing quality level b is equal to $F(b) - F(b-)$.

It now follows that, since θ is distributed uniformly on $[0, 1]$, for an admissible price function the measure of the set of consumers who purchase a quality level not exceeding s is equal to $F(s)$. Therefore, the firm's profit is

$$P(p) = \int_0^b [p(s) - c(s)]dF(s). \tag{20}$$

[Cf. equation (6) of Section 3.]

Again, it will be useful to express the profit in terms of the function F.

Lemma 5

$$P(p) = V(F) - c(b), \tag{21}$$

where

$$V(F) \equiv \int_0^b \left\{-[F(s)]^2 + [1 + c'(s)]F(s)]\right\} ds. \tag{22}$$

The proof of Lemma 5 is given in the Appendix.

We can now characterize the firm's optimal price and product policies. Let H denote the convex hull of the graph of the cost function c, let L be the lower boundary of H, and let h be the function whose graph is L. The function h will be called the *lower convex envelope* of the cost function (see Figure 2). Call a quality level s *anomalous* if $h(s) < c(s)$. The properties of the cost function assumed above imply that the anomalous quality levels constitute a finite set of open intervals, which we shall call *anomalous intervals.* In Figure 2, (u, v) is an anomalous interval.

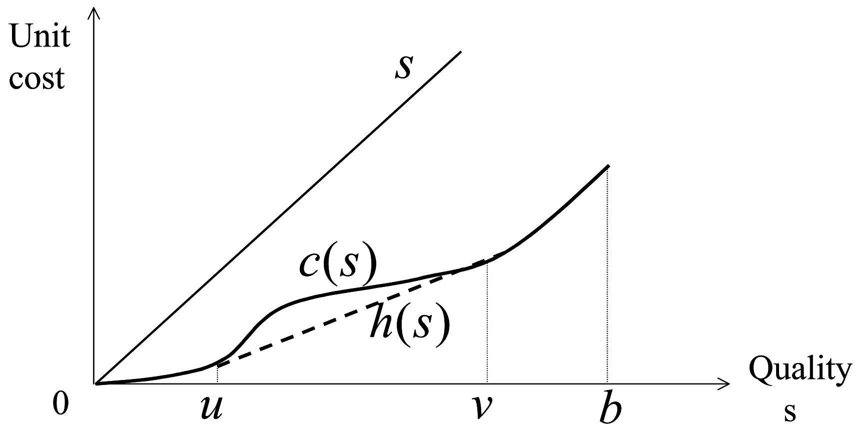

Figure 2.

Define the price function π by

$$\pi(s) \equiv \frac{s + h(s)}{2}. \tag{23}$$

Theorem 6 *Under assumptions (12)-(13), π is an optimal price function for the monopolist, and the monopolist sells only qualities in the set*

$$Q^* = \{s \in Q | c(s) = h(s),\ c''(s) > 0,\ \text{and } c'(s) \leq 1\}. \tag{24}$$

In particular, the demand for an anomalous quality is zero.

Proof: The proof mirrors that for the case of a finite set of qualities. First, for any cost function f satisfying (12)-(13), define P_f to be the supremum of the profit over the set of all admissible price functions. Clearly, P_f is bounded. As in Section 3, since $c(s) \leq h(s)$ on $[0, b]$, $P_c \leq P_h$.

If the firm's cost function were h (instead of c), its profit function would be as given in Lemma 5 with the function c replaced with h, and pointwise maximization of (22) would directly give the optimal $F(s)$:

$$\frac{1 + h'(s)}{2},$$

and π would be the optimal price function. Notice, h is linear in the anomalous intervals, and so is π. Consumer utilities are also linear in quality by assumption. Therefore, given the price function π, the demand for each anomalous quality level (where $c(s) > h(s)$) is zero, and demand is positive only where $h(s) = c(s)$, $c''(s) > 0$, and $c'(s) < 1$. By (20), the profit P_h is thus attainable with the cost function c and price function π, i.e., $P_c = P_h$. Therefore π must be optimal for the cost function c. *Q.E.D.*

A striking implication of Theorem 6 is that, when the unit cost is not convex, the monopolist's optimal quality choice may not be a continuous interval in the quality domain. In particular, if there are multiple anomalous intervals given the unit cost function, then the monopolist will skip offering these anomalous intervals. This contrasts with the existing result that the quality choice of a monopolist with a convex cost function is a continuous spectrum(Mussa and Rosen 1978, and Rochet and Chone 1998). Perhaps more importantly, Theorem 6 lays the foundation for understanding the behavior of a firm employing multiple convex production technologies, which we turn to next.

Corollary 7 *Suppose the monopolist employs I production technologies, and the unit cost function under technology i is c_i, where $c_i(s) > 0$, $c_i''(s) > 0$ on $[0, b]$. Then its optimal quality choice is*

$$Q^* = \bigcup [u_k, v_k], \tag{25}$$

such that $c_i(s) = h(s)$ and $c_i'(s) \leq 1$ on $[u_k, v_k]$ for some $1 \leq i \leq I$, where h is the lower convex envelope of

$$c(s) = \min\{c_i(s), i = 1, ..., I\}.$$

The proof is immediate in light of Theorem 6 and hence is omitted. This corollary specifies how each production technology of the monopolist should be utilized: The firm should use technology i to produce those quality intervals $[u_k, v_k]$ where $c_i(s) = h(s)$. Therefore, technology i is actively utilized only if there exists at least one nondegenerate such interval $[u_k, v_k]$, and is called *obsolescent* otherwise. Note that, in our model *a technology may become obsolescent even if it still possesses a cost advantage over certain quality levels; this happens for technology i if $c(s) > h(s)$ holds where $c(s) = c_i(s)$.*

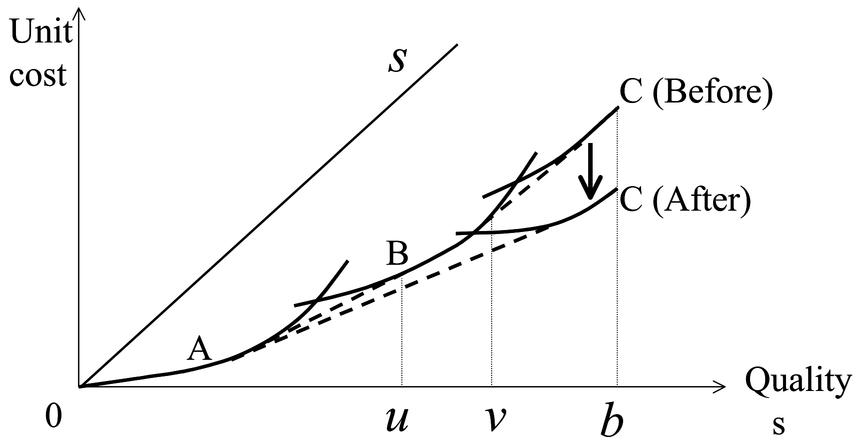

Figure 3.

Corollary 7 allows useful dynamic interpretations, even though it is obtained in a static model. Over time, the unit costs under some production technologies are likely to fall, due to either external technical improvements or in-house cost-reduction initiatives such as process innovation. While some articles have argued that the firm should offer new brands as consumer tastes change (e.g., Schmalensee 1978, Jovanovic and Rob 1989), Corollary 7 suggests that, as its production technologies change, the firm should also adjust its product line accordingly. In particular, cost reduction on one technology usually leads to reduced utilization of the adjacent active technologies in the sense that the quality spectrums produced using these adjacent technologies will become narrower. Cost reduction on one technology may even render a formerly active technology obsolescent. As shown in Figure 3, technology B was used to produce qualities $[u, v]$ before the cost reduction on technology C, but becomes obsolescent afterwards.

5 Endogenous Choice of Product Qualities

So far we have considered which quality levels the firm should offer from an exogenously given (discrete or continuous) set of qualities. Consistent with much of the literature, the preceding analysis ignores those factors that may limit the number of quality levels the firm offers (other than the quality set itself), such as any likely fixed costs required for offering each quality. In this Section, we shall take such factors into account, and examine the firm's quality choice when it can offer only a limited number of qualities.

The basic setting remains the same as in Section 4, except that we require the unit cost $c(s)$ to be twice continuously differentiable on the quality domain $Q = [0, b]$. However, the firm now faces a "variety constraint" and we assume that it can offer only finitely many quality levels, say N. The firm locates these N quality levels in $[0, b]$ to maximize profit. To simplify the analysis, assume that the firm would sell more than N quality levels without the variety constraint (in the spirit of Section 4). Consequently, for the optimal location of N quality levels, none is degenerate.

We have seen that, absent the variety constraint, the firm would never offer a quality level in an anomalous interval (Theorem 6). As the next theorem shows, this conclusion continues to hold even when the variety constraint is present, although this does not follow directly from the previous results.

Theorem 8 *Suppose* $\mathbf{s}^* = (s_1^* ..., s_N^*)$ $(0 \leq s_1^* < s_2^* < ... < s_N^* \leq b)$ *is an optimal vector of quality levels. Then none of the qualities* s_n^* *is in the interior of an anomalous interval.*

Proof. The proof is based on two Lemmas. The first is an analogue of Lemma 3 of Section 3, which we state without proof. For any unit-cost function f on $[0, b]$, define

P_f = the maximum profit from locating N quality levels in $[0, b]$.

Recall that h is the lower convex envelope of the cost function c.

Lemma 9

$$P_c \leq P_h. \tag{26}$$

Since we assume that c is twice continuously differentiable, h is convex on $[0, b]$ and twice continuously differentiable except possibly at finitely many points. As before, there exist a finite number of anomalous quality intervals, where h is linear.

Lemma 10 *Suppose that the true cost function equals* h*, and that* h *is linear on a interval* $[u, v]$ *of* $[0, b]$*. Then for the optimal (for* h*) location of* N *quality levels, no quality level will be in the open interval* (u, v)*.*

Proof of Lemma. Consider a quality vector $\mathbf{s} = (s_1, ..., s_N)$. By Theorem 2 in Section 3, when the cost function h is convex, the optimal price vector is $\mathbf{p} = (p_1, ..., p_N)$, where

$$p_n = \frac{s_n + h(s_n)}{2}.$$

Therefore, the profit from the quality vector $\mathbf{s}$ can be expressed as

$$P(\mathbf{s}) = \frac{1}{4}\sum_{n=1}^{N}(s_n - h(s_n))\,(e_{n+1} - e_n)\,, \tag{27}$$

where $s_0 = h(s_0) = 0$, $e_{N+1} = 1$, and

$$e_n = \frac{h(s_n) - h(s_{n-1})}{s_n - s_{n-1}}, 1 \leq n \leq N.$$

We can verify that

$$\frac{\partial P}{\partial s_n} = \frac{1}{4}e_{n+1}^2 - \frac{1}{4}e_n^2 - \frac{1}{2}h'(s_n)\left(e_{n+1} - e_n\right), 1 \le n \le N,$$

$$\begin{aligned}\frac{\partial^2 P}{\partial s_n^2} &= \frac{1}{2(s_{n+1} - s_n)}\left(e_{n+1} - h'(s_n)\right)^2 + \frac{1}{2(s_n - s_{n-1})}\left(h'(s_n) - e_n\right)^2 \\ &\quad -\frac{1}{2}h''(s_n)\left(e_{n+1} - e_n\right),\end{aligned}$$

for $1 \le n \le N-1$, and

$$\frac{\partial^2 P}{\partial s_N^2} = \frac{1}{2(s_N - s_{N-1})}\left(h'(s_N) - e_N\right)^2 - \frac{1}{2}h''(s_N)\left(1 - e_N\right).$$

For s_n located in (u, v), we have $h''(s_n) = 0$, and thus

$$\frac{\partial^2 P}{\partial s_n^2} > 0.$$

Therefore locating s_n in (u, v) cannot be optimal. This completes the proof of the lemma.

The remainder of the proof mirrors that of Theorem 2, and is omitted. *Q.E.D.*

According to Theorem 8, the optimal quality levels must be located in the regions where the unit cost function $c(s)$ coincides with its convex envelope $h(s)$. Therefore, Theorem 8 essentially converts the problem of locating N qualities for the true cost function c into one of locating them for its convex envelope h, as stated in (27).

The next Theorem shows that the firm's profit is a supermodular function of the quality levels. When the firm employs multiple production technologies, the supermodularity of P has an interesting interpretation. Technological progress such as process innovation is likely to lower the unit cost in a certain region of the quality domain, thus causing some quality level(s) located in that region to rise. If so, then the location of the remaining quality levels will also increase, even though the unit cost function stays unchanged in the regions where these remaining quality levels are located.

Theorem 11 *The profit function P in (27) is supermodular.*

Proof: Since the unit cost c is twice continuously differentiable (by assumption), P is also twice continuously differentiable. We only need to show that

$$\frac{\partial^2 P}{\partial s_n \partial s_m} \ge 0,$$

for $n \neq m$ (Topkis 1978, Milgrom and Roberts 1990). We can readily verify that

$$\begin{aligned} \frac{\partial^2 P}{\partial s_n \partial s_{n+1}} &= \frac{1}{2(s_{n+1} - s_n)} \left(e_{n+1} - h'(s_n)\right) \left(h'(s_{n+1}) - e_{n+1}\right) \\ &\geq 0, \end{aligned}$$

(where $e_{n+1} - h'(s_n) \geq 0$ and $h'(s_{n+1}) - e_{n+1} \geq 0$ since h is convex,) and that

$$\frac{\partial^2 P}{\partial s_n \partial s_m} = 0,$$

for $|n - m| \geq 2$. *Q.E.D.*

Because the true cost function c may demonstrate non-convexities (which is the central concern of this paper), h is linear where $c(s) > h(s)$. From the proof of Lemma 10 above, the second derivative of the profit function with respect to each quality is positive wherever h is linear. Therefore, the firm's profit function is not concave in general.

The nonconcavity of the profit function is most clearly seen in the following example. Consider the simplest case of locating one quality level, i.e., $N = 1$. Let $b = 2$, i.e., the quality domain is $[0, 2]$. Consider a unit cost function with the following lower convex envelope

$$h(s) = \begin{cases} 0.35s^2 & \text{in } [0,1], \\ 0.7s - 0.35 & \text{in } [1, 1.5], \\ 0.1s^2 + 0.4s - 0.125 & \text{in } [1.5, 2]. \end{cases} .$$

The firm's profit function is

$$P(s) = \frac{1}{4}(s - h(s))(1 - \frac{h(s)}{s}).$$

It is easily verified that $P''(s)$ is negative in $[0, 1]$, positive in $[1, 1.5]$, and negative in $[1.5, 2]$. In addition, $P'(1) < 0$, $P'(1.5) > 0$, and $P'(2) < 0$. Therefore $P(s)$ has two peaks in $[0, 2]$.

Even though the firm's profit function is not concave for a non-convex unit cost function, a profit-maximizing quality vector always exists because the profit function is continuous on the finite interval $[0, b]$.

We note that the maximum profit obtainable with N qualities is bounded above by the profit when the firm faces no constraint on the number of quality levels it may offer, which is derived in Section 4.

The following Theorem identifies a necessary condition for the optimal quality levels, and the respective conditions under which corner and interior solutions obtain for the highest quality s_N.

Theorem 12 *A). If*

$$h'(s) \leq \frac{1}{2}\left(1 + \frac{h(s)}{s}\right), \quad 0 \leq s \leq b,$$

then $s_N^* = b$ *and the remaining optimal qualities satisfy*

$$h'(s_n^*) = \frac{1}{2}\left(\frac{h(s_{n+1}^*) - h(s_n^*)}{s_{n+1} - s_n} + \frac{h(s_n^*) - h(s_{n-1}^*)}{s_n - s_{n-1}}\right)$$

for $1 \leq n \leq N-1$, *and* $s_0^* = h(0) = 0$.

B). If

$$h^{'}(b) > \frac{1}{2}\left(1 + \frac{h(b)}{b}\right),$$

then $s_N^* < b$ *and the optimal qualities satisfy*

$$h'(s_N^*) = \frac{1}{2}\left(1 + \frac{h(s_N^*) - h(s_{N-1}^*)}{s_N - s_{N-1}}\right),$$

$$h'(s_n^*) = \frac{1}{2}\left(\frac{h(s_{n+1}^*) - h(s_n^*)}{s_{n+1} - s_n} + \frac{h(s_n^*) - h(s_{n-1}^*)}{s_n - s_{n-1}}\right)$$

for $1 \leq n \leq N-1$.

The proof is given in the Appendix. In Theorem 12, the LHS of the equation for quality level s_n is the marginal cost, and the RHS is the *average* marginal utility gain of its buyers. Each equation thus has the usual interpretation that the marginal cost of each quality equals its marginal revenue.

6 Concluding Remarks

Clearly, the optimal screening procedure and product line choice of a monopolist are jointly determined by both demand and supply side factors. While prior research has emphasized demand-side factors such as the consumer preference distribution and multi-dimensional contexts, this paper highlights the effect of unit production costs on price discrimination. A core result is that the monopolist should avoid offering those quality levels where the unit cost function exceeds its lower convex envelope. In what follows, we discuss the robustness of this result in light of the various assumptions used in our model.

From Sections 3 and 4, we see that this result does not rely on whether the firm's quality space is discrete or continuous. In Section 4, we assume that the cost function is smooth except at (at most) finitely many points. The smoothness assumption facilitates exposition but does not appear essential. In fact, our result holds for some cost functions with kinks or jumps, as long as the number of such kinks or jumps is finite. For instance, the analysis of the case of a step cost function can be accomodated by a straightforward extension of the argument in Section 3.

We have assumed that consumer types are uniformly distributed. Our explorations indicate that the core result holds for a more general class of distributions, as long as a familiar hazard rate condition is satisfied so that bunching does not arise.

Finally, we admit that our core result relies on the linearity of consumer utilities. It would be interesting to consider how a nonlinear utility function would affect our current results. Nevertheless, such a direction would tremendously complicate the analysis. Largely due to the same reason, a majority of the previous articles (e.g. Mussa and Rosen 1978, Gabszewicz et al. 1986, Rochet and Chone 1998) have also employed such linear forms.

7 Appendix

Lemma 1.

$$P(p) = \sum_{0}^{K-1} (s_{k+1} - s_k) \left[-d_k^2 + (1 + g_k) d_k \right] - c_K.$$

Proof : Let

$$x_k = p_k - c_k.$$

With this notation, (6) becomes

$$P(p) = \sum_{1}^{K-1} x_k (d_k - d_{k-1}) + x_K (1 - d_{K-1}).$$

By a rearrangement of terms, one has

$$\sum_{1}^{K-1} x_k (d_k - d_{k-1}) = \sum_{0}^{K-2} (x_k - x_{k+1}) d_k + x_{K-1} d_{K-1}.$$

(This is an analogue for finite sums of "integration by parts.") Hence

$$P(p) = -\sum_{0}^{K-1} (x_{k+1} - x_k) d_k + p_K - c_K.$$

Observe that

$$\begin{aligned} (x_k - x_{k+1}) &= -(s_{k+1} - s_k) d_k - (c_k - c_{k+1}), \\ p_K &= \sum_{0}^{K-1} (s_{k+1} - s_k) d_k. \end{aligned}$$

These, together with the preceding equation for the profit, lead immediately to the conclusion of the lemma. *Q.E.D.*

Lemma 3. $P_c \leq P_h$.

Proof: For any cost vector f, let $P(p, f)$ denote the profit from a price vector p, as given by equation (6). Since $c \geq h$, it follows from (6) that $P(p, c) \leq P(p, h)$, and so

$$P(p, c) \leq \max_{p'} P(p', h) = P_h.$$

Since this last inequality holds for all p, it follows that

$$P_c = \max_p P(p, c) \leq P_h.$$

Q.E.D.

Lemma 5.

$$P(p) = V(F) - c(b),$$

where

$$V(F) \equiv \int_0^b \left\{-[F(s)]^2 + [1 + c'(s)]F(s)]\right\} ds.$$

Proof: Applying integration by parts to (19), we get

$$P(p) = -\int_0^b [p'(s) - c'(s)]F(s)ds + [p(b) - c(b)]F(b) - [p(0) - c(0)]F(0).$$

However,

$$\begin{aligned} p'(s) &= F(s), \\ F(b) &= 1, \\ p(0) &= c(0) = 0, \end{aligned}$$

and

$$p(b) = \int_0^b F(s)ds.$$

Hence the firm's profit from the admissible price function p is

$$\begin{aligned} P(p) &= -\int_0^b [F(s) - c'(s)]dF(s) + \int_0^b F(s)ds - c(b) \\ &= \int_0^b \left\{-[F(s)]^2 + [1 + c'(s)]F(s)]\right\} ds - c(b) \qquad (28) \\ &= V(F) - c(b), \end{aligned}$$

where

$$V(F) \equiv \int_0^b \left\{-[F(s)]^2 + [1 + c'(s)]F(s)]\right\} ds.$$

Q.E.D.

Theorem 12. *A). If*

$$h'(s) \leq \frac{1}{2}\left(1 + \frac{h(s)}{s}\right), \quad 0 \leq s \leq b,$$

then $s_N^* = b$ and the remaining optimal qualities satisfy

$$h'(s_n^*) = \frac{1}{2}\left(\frac{h(s_{n+1}^*) - h(s_n^*)}{s_{n+1} - s_n} + \frac{h(s_n^*) - h(s_{n-1}^*)}{s_n - s_{n-1}}\right),$$

for $1 \leq n \leq N-1$, *and* $s_0^* = h(0) = 0$.

B). If

$$h^{'}(b) > \frac{1}{2}\left(1 + \frac{h(b)}{b}\right),$$

then $s_N^* < b$ *and the optimal qualities satisfy*

$$h'(s_N^*) = \frac{1}{2}\left(1 + \frac{h(s_N^*) - h(s_{N-1}^*)}{s_N - s_{N-1}}\right),$$

$$h'(s_n^*) = \frac{1}{2}\left(\frac{h(s_{n+1}^*) - h(s_n^*)}{s_{n+1} - s_n} + \frac{h(s_n^*) - h(s_{n-1}^*)}{s_n - s_{n-1}}\right)$$

for $1 \leq n \leq N-1$.

Proof: A). We can verify that

$$\frac{\partial P}{\partial s_N} = \frac{1}{4}\left(1 - \frac{h(s_N) - h(s_{N-1})}{s_N - s_{N-1}}\right)\left(1 + \frac{h(s_N) - h(s_{N-1})}{s_N - s_{N-1}} - 2h'(s_N)\right),$$

By Theorem 2, the cost slope at each quality must be less than 1, and so

$$1 - \frac{h(s_N) - h(s_{N-1})}{s_N - s_{N-1}} > 0.$$

When

$$h'(s) \leq \frac{1}{2}\left(1 + \frac{h(s)}{s}\right)$$

holds on $[0, b]$, we have

$$1 + \frac{h(s_N) - h(s_{N-1})}{s_N - s_{N-1}} - 2h'(s_N) > 0,$$

and so

$$\frac{\partial P}{\partial s_N} > 0,$$

which implies that corner solution is optimal for the highest quality ($s_N^* = b$). The property of the remaining quality levels follows from rearranging the respective first-order necessary conditions.

B). When

$$h'(b) > \frac{1}{2}\left(1 + \frac{h(b)}{b}\right)$$

holds, we have

$$\frac{\partial P}{\partial s_N}|_{s_N = b} < 0.$$

Therefore a corner solution can never be optimal for the highest quality ($s_N^* < b$). As in part A), the property of the optimal quality levels follows directly from the first order conditions. *Q.E.D.*

8 References

Armstrong, M., "Multiproduct Nonlinear Pricing, " *Econometrica*, 64, (1996), 51-75.

Arrow, K.J., "The Economic Implications of Learning by Doing," *Review of Economic Studies*, 29, (1962), 155-173.

Gabszewicz, J.J., A. Shaked, J. Sutton, and J.-F. Thisse, "Segmenting the Market: The Monopolist's Optimal Product Mix," *Journal of Economic Theory*, 39, (1986), 273-289.

Gabszewicz, J.J., and J.-F. Thisse, "Price Competition, Quality and Income Disparities," *Journal of Economic Theory*, 20, (1979), 340-359.

Gabszewicz, J.J., and J.-F. Thisse, "Entry (and Exit) in a Differentiated Industry," *Journal of Economic Theory*, 22, (1980), 327-338.

Jovanovic, B., and R. Rob, "Demand-Driven Innovation and Spatial Competition over Time," *Review of Economic Studies*, 54, (1987), 63-72.

Maskin, E., and J. Riley, "Monopoly with Incomplete Information," *The Rand Journal of Economics*, 15, (1984), 171-196.

Matthews, S., and J. Moore, "Monopoly Provision of Quality and Warranties: An Exploration of the Theory of Multidimensional Screening," *Econometrica*, 55, (1987), 441-467.

McAfee, R.P., and J. McMillan, "Multidimensional Incentive Compatibility and Mechanism Design," *Journal of Economic Theory*, 46, (1988), 335-354.

Milgrom, P., and J. Roberts, "The Economics of Modern Manufacturing: Technology, Strategy, and Organization," *American Economic Review*, 80, (1990), 511-528.

Mirman, L.J., and D.S. Sibley, "Optimal Nonlinear Prices for Multiproduct Monopolies," *Bell Journal of Economics*, 15, (1980), 659-670.

Mussa, M., and S. Rosen, "Monopoly and Product Quality," *Journal of Economic Theory*, 18, (1978), 301-317.

Rochet, J.-C., and P. Chone, "Ironing, Sweeping, and Multidimensional Screening," *Econometrica*, 66, (1998), 783-826.

Shaked, A., and J. Sutton, "Relaxing Price Competition through Product Differentiation," *Review of Economic Studies*, 49, (1982), 3-14.

Shaked, A., and J. Sutton, "Natural Oligopolies," *Econometrica*, 51, (1983), 1469-1493.

Schmalensee, R., "Entry Deterrence in the Ready-to-Eat Breakfast Cereal Industry," *The Bell Journal of Economics*, 9, (1978), 305-327.

Sibley, D.S., and P. Srinagesh, "Multiproduct Nonlinear Pricing with Multiple Taste Characteristics," *The Rand Journal of Economics*, 28, (1997), 684-707.

Spence, A. M., "Multi-product Quantity-dependent Prices and Profitability Constraints," *Review of Economic Studies*, 47, (1980), 821-842.

Sutton, J., *Technology and Market Structure*, Cambridge: The MIT Press, (1998).

Topkis, D., "Minimizing a Submodular Function on a Lattice," *Operations Research*, 26, (1978), 305-321.

Wilson, R., *Nonlinear Pricing*, Oxford: Oxford University Press, (1993).