Monotone Equilibrium in Multi-Unit Auctions

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Abstract

Existence of monotone pure strategy equilibrium is established in the discriminatory and uniform $S + \alpha$-th price ($\alpha \in [0, 1]$) auctions of S identical objects when bidders are risk-neutral with independent signals. The model requires discrete price / quantity grids and allows for multi-dimensional signals, interdependent values, increasing marginal values, allocative externalities, and two-sided trading. Given no externalities, further, *all* mixed-strategy equilibria in these auctions must be ex post allocation- and interim expected payment-equivalent to some monotone pure strategy equilibrium. Thus, for standard expected surplus / revenue analysis, there is no loss in restricting attention to monotone strategies.

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1 Introduction

The U.S. Treasury's bond issue auctions and the NYSE's opening batch auctions are just two among many examples of real-world auctions of multiple identical objects (so-called "multi-unit auctions"). A bid in such an auction is a demand and/or supply *schedule*, specifying a price (or "unit-bid") for each unit, and a bidder receives or keeps a given unit if his bid on that unit is among the highest unit-bids. There are a wide variety of payment rules for such auctions, but two sorts are the most common in practice: discriminatory and uniform-price auctions. Until the late 1990s, the U.S. Treasury used a discriminatory auction to issue bonds: bidders' paid their bids for each unit that they won. Recently the Treasury changed its auction to a uniform $S+1$-st price format: all bidders pay a price equal to the highest losing bid (S is the number of units, so this is the $S+1$-st highest unit-bid). Every morning at the beginning of trading, NYSE market-makers run a two-sided auction (bids are submitted overnight) and all trades at that time execute at the same price. The Paris and Amsterdam exchanges also open trading with batch auctions which may be thought of, at least approximately, as uniform $S+1/2$-th price auctions: bidders pay (if buying) or receive (if selling) the average market-clearing price.[1]

Despite their practical importance, the theory of multi-unit auctions is quite incomplete. This would be troubling but not of great concern if insights from the well-developed theory of single-object auctions applied in multi-unit settings. Unfortunately, single-object auction theory provides an unreliable guide in settings with multi-unit demand / multi-unit supply. For example, critics of the Treasury's plan to switch to the uniform price auction noted that, even though this auction appears similar to a second-price auction (which is an $S+1$-price auction when $S=1$), bidders will not submit truthful demand schedules but rather shade their bids down in equilibrium. Indeed, Back and Zender (1993) expanded an example in Wilson (1979) showing that in some models the uniform $S+1$-st price auction has a multitude of equilibria with so much bid-shading that revenues can be arbitrarily low!

Even more basic insights drawn from single-object auction theory regarding the structure of equilibria can run afoul in multi-unit settings. To see

[1] To the best of my knowledge, NYSE does not have a formally defined price-setting rule. Yet "specialists must maintain a *fair*, competitive, orderly and efficient market" (NYSE website, italics added), so the $S+1/2$-price auction is a natural modelling candidate since it treats buyers and sellers symmetrically.

this, consider the simplest possible scenario in which bidders are risk-neutral with independent private values (IPV). In the first-price and second-price auctions, it is well-known that all equilibria are monotone, i.e. equilibrium bids are non-decreasing in bidders' values.[2] Yet all equilibria may be non-monotone in the uniform $S+1$-st price auction in the same case, i.e. bidders may lower their bids on some units as their values increase. (See Example 5.) Since intuitions drawn from single-object auction theory can be very deceptive, we need to develop a free-standing multi-unit auction theory.

In this paper, I begin to address some of the basic issues surrounding existence and monotonicity of equilibrium in multi-unit auctions. In particular, is it safe to assume that an equilibrium exists at all?, that bidders adopt pure strategies in equilibrium?, and that bidders adopt monotone strategies in equilibrium? The paper's two main results provide a "qualified Yes" to these questions for the discriminatory and uniform $S+\alpha$-th price ($\alpha \in [0,1]$) auctions. "Yes", a monotone pure strategy equilibrium (MPSE) exists and, "Yes", there is a sense in which *all* equilibria are equivalent to MPSE. "Qualified" in two ways. First, extra assumptions need to be made: Both the existence and characterization results require that bidders receive independent signals (or "types"), that they are risk-neutral, and that a specific tie-breaking rule ("priority rationing") is used. Unfortunately, each of these conditions is essential. As Examples 3, 4, and 5 show, respectively, all equilibria may be non-monotone given risk-averse bidders and IPV, given risk-neutral bidders and affiliated private values, or given risk-neutrality, IPV, and "proportional rationing" to break ties instead of priority rationing. Furthermore, the characterization result requires that there be no externalities. Example 6 shows that this restriction also can not be relaxed.

Second, equilibria need only be "equivalent" to some MPSE. To be specific, every mixed strategy equilibrium is both ex post allocation- and interim expected payment-equivalent to some MPSE. (These terms are discussed below and defined formally on page 27.) Some such qualification is necessary since one must account for the fact that bidders are indifferent between all demand / supply schedules ("bids") that always lead to the same allocation and the same payment for them. For example, in the uniform $S+1$-st price

[2] In the second-price auction, bidding one's true value is a weakly dominant strategy, so that (trivially) all equilibria in weakly undominated strategies are monotone. Other equilibria can involve randomized or non-monotone bidding, but such behavior is not crucial to any equilibrium: in terms to be defined later, any such equilibrium is ex post allocation- and interim expected payment-equivalent to a monotone pure strategy equilibrium.

auction, consider a situation in bidder 1 never wins a first unit with any unit-bid less than $\underline{p}$ and a range of types $[0, \underline{t}]$ all submit first unit-bids less than $\underline{p}$ in equilibrium. Since these bidder 1 types always lose, they are indifferent between all bids in the range $[0, \underline{p}]$ and hence may randomize or submit lower first unit-bids as type increases, causing a non-monotonicity. The interim expected payment-equivalence result demonstrates, however, that it is always possible to "monotonize" bidder 1's strategy in such a way that preserves equilibrium and such that every other bidder's interim expected payment (i.e. conditional on his type) is preserved. A priori, there also is the possibility that randomizations or non-monotonicities may support an equilibrium whose allocation differs from that of any MPSE. The ex post allocation-equivalence result rules that out.

While our results are cast in a setting with $S \geq 1$ units, the characterization result appears to be new even in the well-studied case of $S = 1$. Specifically, in the second-price auction, this paper appears to be the first to recognize that every mixed strategy equilibrium is ex post allocation- or interim expected payment-equivalent to a monotone pure strategy equilibrium. (Even given risk-neutral bidders and independent types, it is not obvious that all equilibria must be equivalent in these ways to MPSE when bidders have general interdependent values.)

Establishing monotonicity of equilibria in auctions is a common thread in much of the auction literature (see below). Why is monotonicity important? Probably most significant, in the special case of symmetric models, is that strict monotonicity – when combined with symmetry – guarantees efficiency of the auction outcome. This is why, for instance, Reny and Perry (2003) focus so intensively on establishing monotonicity in large uniform-price auctions. There are also some potential practical benefits[3] as well as technical reasons why it's easier to study auctions in which bidders are known to adopt monotone strategies. Another important reason is that monotonicity serves as an intuition check as we learn what distinguishes multi-unit from single-object auctions. If we are expecting equilibria to be monotone but they fail to be so, then we need to understand why. From this point of view, the several examples presented here are interesting as they show

[3] A bidder (or an econometrician) computing best responses faces a simpler problem if attention can be restricted to the smaller space of monotone strategies. Monotonicity also creates an additional layer of struture in data generated by an auction. To be specific, higher bids correlate with higher values. This may help with identification of empirical models (though it is by no means necessary nor sufficient for identification).

how monotonicity can *fail*. Sometimes this failure is due to a detail in the rules. For instance, Example 5 shows that equilibria may fail to be monotone given the most commonly-studied tie-breaking rule, whereas the paper's main results show that such possibilities disappear if we choose a different rule. This suggests that modellers and auction designers may want to consider more carefully their choice of tie-breaking rule. Other times the failure of monotonicity points out a faulty intuition carried over from our more well-developed understanding of single-object auctions. Examples 3, 4 both fit in this category, showing why, respectively, affiliation and risk-aversion play very different sorts of roles in multi-unit auctions than in single-object auctions.

Related Literature: This paper complements a growing literature studying existence and monotonicity of equilibrium in multi-unit auctions. There are several important dimensions to models of such auctions so, perhaps not surprisingly, the models in this literature are typically incomparable in their generality. Each paper makes a contribution by relaxing certain assumptions but at the cost of requiring a relatively strong assumption elsewhere. In the brief (and non-exhaustive) review to follow, I italicize the key aspects in each paper that are less general than in the others, summarizing at the end how this paper relates. Reny (1999) shows existence of MPSE in the one-sided discriminatory auction given multi-dimensional *independent private values*. Using an entirely different approach, Jackson and Swinkels (2001) (hereafter "JS") establish existence of equilibrium in *distributional strategies* in a wide variety of one- and two-sided auctions when bidders have multi-dimensional correlated *private values*. In two-sided auctions, JS also show how to guarantee existence of a non-trivial equilibrium, i.e. one having positive probability of trade. In yet another vein, Fudenberg, Mobius, and Sziedl (2003) ("FMS") and Reny and Perry (2003) ("RP") establish existence of MPSE in *large enough* two-sided *uniform price auctions* when bidders have *single-unit demand*. FMS further require that bidders have correlated *private values*, while RP allow for interdependent values given affiliated signals drawn from a *symmetric distribution*. This paper takes yet another type of approach, establishing existence of MPSE in one- and two-sided versions of the uniform-price and discriminatory auctions when bidders receive multi-dimensional *independent signals* and there may be just a few bidders having multi-unit demand and interdependent values. I also allow for certain sorts of allocative externalities à la Jehiel, Moldavanu, and Stacchetti (1996), i.e.

bidders may care who else wins how many units. In short, this paper relies on independence while JS and FMS rely on private values and FMS and RP on there being sufficiently many bidders.

Another related literature is that proving existence and characterizing monotonicity properties of equilibria in the first-price auction (which is a discriminatory auction with $S = 1$). Bajari (1997) showed that, given independent private values, there is a unique equilibrium which is in monotone pure strategies. Several other papers establish existence of MPSE or uniqueness among the class of all MPSE but, unlike this paper, do nothing to rule out the possibility of non-MPSE.[4] More to the point of this paper, Rodriguez (2000) (for two bidders) and McAdams (2003a) (for n bidders) establish that *all* equilibria must be equivalent to MPSE (in the stronger sense of ex post allocation- and ex post payment-equivalence) in asymmetric models with affiliated signals and interdependent values. Lastly, McAdams and Persico (2003) apply this result to show that the efficient MPSE demonstrated by Milgrom and Weber (1982) in their n-bidder symmetric model is its unique equilibrium.

This first-price auction literature, however, grapples primarily with settings in which bidders have *affiliated but not independent* signals. Indeed, proving our characterization results for the first-price auction is fairly straightforward when bidder signals are independent. This paper grapples with something totally different, the multidimensionality of bids. Bidders typically are indifferent between many different bids and even if the *set* of bidders' best responses is increasing in the strong set order (see page 32) higher types may have a best response that is not comparable with or strictly less than a lower type's best response. (Two bids are incomparable if each specifies a higher unit-bid for some quantity.) The challenge of our characterization work is to show that submitting incomparable bids is never vital to supporting an equilibrium allocation nor expected payments. To manage this inherent multidimensionality problem, I exploit a novel representation of mixed strategies as weighted planar graphs in which bids can be interpreted as paths.

Lastly, the existence part of the paper applies and extends techniques developed in McAdams (2003b) so it is important to distinguish the two

[4] In an asymmetric two-bidder model with affiliated signals and interdependent values, Lizzeri and Persico (2000) show that there is a unique MPSE. Maskin and Riley (2000), Athey (2001), and Reny and Zamir (2002) establish MPSE existence in models of varying levels of generality but all allowing for n asymmetric bidders, interdependent values, and affiliated signals.

papers. McAdams (2003b) has no characterization result but there is some overlap of results: he proves MPSE existence in the uniform S-th and $S+1$-st price auctions given independent signals (but not requiring risk-neutrality). As shall be discussed more completely later (pages 15-18), the uniform S-th and $S+1$-st price auctions have a special structure – not shared by the discriminatory or other uniform price auctions – that makes the key technical observation that payoffs are modular in own bid especially clear. Furthermore, this difference has significant implications. Modularity holds in the S-th and $S+1$-st price auctions regardless of bidders' risk preference, fails in the discriminatory and $S+\alpha$-th price auctions ($\alpha \in (0,1)$) when bidders are risk-averse, and fails in the $S+2$-nd price auction even when bidders are risk-neutral. In short, this paper establishes modularity beyond the uniform S-th and $S+1$-st price auctions making the existence result here more general than that in McAdams (2003b).

The remainder of the paper is organized as follows: Section 2 lays out the model of multi-unit auctions. Section 3 discusses and sketches the proof that bidders' expected payoffs are modular in own bid in the discriminatory and uniform $S+\alpha$-th price auctions ($\alpha \in [0,1]$), a key technical contribution. Sections 4, 5 leverage modularity to conclude that a MPSE exists and that, as long as there are no externalities, all mixed strategy equilibria must be ex post allocation- and interim expected payment-equivalent to MPSE. A few remarks and an Appendix containing some proofs concludes the paper.

2 Model: Multi-Unit Auctions

This paper studies one- and two-sided discriminatory and uniform-price auctions. Unless otherwise specified, all results in the paper maintain the following framework and assumptions.

Values and Information: Bidder i receives *value* $V_i(\mathbf{q},\mathbf{t})$ from the allocation $\mathbf{q}=(q_1,...,q_n)$ in the state $\mathbf{t}=(t_1,...,t_n)$, where *types* $t_i=(t_i^1,...,t_i^h)$ are multi-dimensional with support $T_i=[0,1]^h$ and $\{t_i^j\}_{j=1,...,h}^{i=1,...,n}$ are independent. V_i is bounded and piecewise continuous in $\mathbf{t}$ and $V_i(\mathbf{q}',\mathbf{t})-V_i(\mathbf{q},\mathbf{t})$ is increasing in t_i and non-decreasing in $\mathbf{t}_{-i}$ whenever $q_i'>q_i$ and $q_j'\le q_j$ for all $j\neq i$. Bidders are risk-neutral, i.e. seek to maximize expected surplus, the difference between their value and payment. To avoid notational confusion, vectors are bolded (including vectors of quantities, types, bids, etc...). The main exception to this rule are individual bidder types t_i and individual bids $P_i(\cdot)$ which each are multi-dimensional objects but remain unbolded.

Notes: The proof of existence (Theorem 2) requires the relatively weak assumption that $\{t_i\}^{i=1,...,n}$ are independent while the proof of the characterization result (Theorem 3) uses the stronger property that $\{t_i^j\}_{j=1,...,h}^{i=1,...,n}$ are independent. The model includes "interdependent values" (in which $V_i(\mathbf{q}, \mathbf{t}) = V_i(q_i, \mathbf{t})$, i.e. value depends on others' information but not on what others' receive) and allows for allocative externalities, with the caveat that "own incremental values" are non-decreasing in own type. Such a monotonicity assumption is present, for instance, in Jehiel, Moldavanu, and Stacchetti (1996)'s study of single-object auctions with externalities. As an example, consider a common value stock IPO model. If the total value of the equity being auctioned is $Z(\mathbf{t})$, the value of allocation $\mathbf{q}$ to bidder i depends not only on the quantity i receives but on how much others receive:

$$V_i(\mathbf{q}, \mathbf{t}) = Z(\mathbf{t}) \frac{q_i}{\sum_{j=1}^{n} q_j}$$

(This would not be an issue if bidders knew the total quantity of shares when formulating their bids. Companies going public in the United States, however, have the "Greenshoe Option" to issue up to 10% more shares than initially planned.) When $q_i' \geq q_i$ and $q_j' \leq q_j$ for all $j \neq i$, $V_i(\mathbf{q}', \mathbf{t}) - V_i(\mathbf{q}, \mathbf{t})$ is increasing in t_i since $Z(\mathbf{t})$ is increasing in t_i and

$$\frac{q_i'}{q_i' + \sum_{j \neq i} q_j'} \geq \frac{q_i}{q_i + \sum_{j \neq i} q_j'} \geq \frac{q_i}{q_i + \sum_{j \neq i} q_j}$$

Also, marginal values may be non-monotone in q_i, so the model applies to procurement settings in which suppliers have increasing returns to scale. (Still, we require that bids be non-increasing schedules.)

Auction Framework: A potentially important structural aspect of the model is that there is a *discrete grid* of prices $\mathfrak{p}$ and of quantities $\mathfrak{q}$:

$$\mathfrak{p} = \left\{\{\emptyset\}, p^{min}, p^{min} + 1, ..., p^{max} - 1, p^{max}, \infty\right\}$$
$$\mathfrak{q} = \{-E, -E + 1, ..., E - 1, E\} \text{ where } E \text{ finite integer}$$

The "null prices" $\{\emptyset\}$ and ∞ are meant to allow bidders to make requirements that they not receive more or less than some quantity. ($\{\emptyset\} = \max P_i(q)$ (or

$\infty = \min P_i(q)$) "means" that bidder i is unwilling to buy more (or less) quantity than q at any permissible price.) Discrete grids are "potentially important" since it remains an open question[5] (even given independent signals!) whether a convergent sequence of equilibria in these discrete-grid auctions can be chosen so that, as the grid of prices becomes arbitrarily fine, it's limit is an equilibrium in the corresponding continuum-grid auction.

A *bid* $P_i(\cdot)$ is a demand correspondence that maps each quantity $q \in \mathfrak{q}$ to a set of prices (or "unit-bids") $P_i(q) \subset \mathfrak{p}$. Not every possible bid is *permissible.* To be permissible, $P_i(\cdot)$ must satisfy four requirements. (These guarantee that a market-clearing allocation exists.) (i) *Non-empty valued:* $P_i(q) \neq \emptyset$ for all $q \in \mathfrak{q}$. (ii) *Inverse non-empty valued:* Let $q \in D_i(p)$ iff $p \in P_i(q)$. Then $D_i(p) \neq \emptyset$ for all $p \in \mathfrak{p}$.[6] (iii) *Order interval-valued:* $\{p', p''\} \subset P_i(q)$ implies that the order interval $[p', p''] \subset P_i(q)$ for all $q \in \mathfrak{q}$ and $\{q', q''\} \subset D_i(p)$ implies $[q', q''] \subset D_i(p)$ for all $p \in \mathfrak{p}$. (iv) *Non-decreasing:* $\max P_i(q') \leq \min P_i(q)$ for all $q' > q$.

For each subset of bidders $I \subset \{1, ..., n\}$, define the "aggregate demand" of I as $D_I(\cdot) = \sum_{j \in I} D_j(\cdot)$. That is to say, $Q \in D_I(p)$ iff there exists $\{q_j\}_{j \in I}$ such that $\sum_{j \in I} q_j = Q$ and $q_j \in D_j(p)$ for all $j \in I$. Let $P_I(\cdot)$ be the corresponding inverse demand correspondence. (Note that unbolded $P_I(\cdot)$ refers to an aggregate inverse demand correspondence whereas $\mathbf{P}_I(\cdot)$ refers to a vector of individual bids.)

The set of permissible bids, $\mathcal{P}$, forms a lattice with respect to the *product order.* $P^2(\cdot) \geq_P P^1(\cdot)$ in the product order iff, for all $q \in \mathfrak{q}$, $\max P^2(q) \geq \max P^1(q)$ and $\min P^2(q) \geq \min P^1(q)$. The type spaces $[0, 1]^h$ are also endowed with the product order $\geq_P$ (For simplicity, I will usually refer to $\geq_P$ as $\geq$.) The meet and join of any two bids $P^2(\cdot), P^1(\cdot)$ are their lower- and

[5] In work-in-progress, Eiichiro Kazumori appears to have made some progress on this hard problem, but to my knowledge his proof remains incomplete. (Following this paper, Kazumori studies a simplified version of our model, takes as given our proof that an equilibrium exists given any discrete grid, then aims to fashion the limiting argument to extend this paper's results to auctions with a continuum of prices.) The most fundamental problem with continuum grids, of course, is that bidders may fail to have a best response.

[6] For most of the analysis, I employ the inverse demand correspondence notation $P_i(\cdot)$ to represent a given bid. The inverse of that, $D_i(\cdot)$, provides the demand correspondence representation of the same bid. Unfortunately, neither sort of notation can comfortably handle all of the arguments in the paper. I will use them interchangeably, depending on which is more convenient, with the default being $P_i(\cdot)$.

upper envelopes:

$$\max P^2 \vee P^1(q) = \max\{\max P^2(q), \max P^1(q)\}$$
$$\max P^2 \wedge P^1(q) = \min\{\max P^2(q), \max P^1(q)\}$$

for all $q \in \mathfrak{q}$.

Each bidder i has an *endowment* $e_i \in \mathcal{Z}$ where endowments sum to $S \geq 0$. Each bidder may potentially receive more than his endowment ("buy units") or less than his endowment ("sell units") in a final allocation and may receive a negative number of units. $P_i(\cdot)$ is announced inverse demand, if you will, over gross quantities *not* over net quantities.

A *strategic bidder* is one who, in equilibrium, submits a bid that is a best response to others' bidding behavior given his own private information. A *non-strategic bidder* is one whose bid is a function of his private information but is not necessarily a best response. In typical two-sided auction models, there may be some non-strategic "noise traders" but otherwise all bidders are strategic. In most typical one-sided auction models, there are strategic bidders $i = 1, ..., n-1$ and the auctioneer who can be thought of as non-strategic bidder n. If the auctioneer specifies supply correspondence $S(\cdot)$, then we can think of $S \equiv \min S(\infty)$ as the maximal possible supply and of the auctioneer's "bid" as being defined by $D_n(p) = S - S(p)$.

Notes: In a two-sided auction (also called "double auction"), bidders may submit bids to buy or sell units. Following Jackson and Swinkels (2001) our model of two-sided auctions differs from most of the literature (such as Rustichini, Satterthwaite, and Williams (1994), Fudenberg, Mobius, and Sziedl (2003), and Reny and Perry (2003)) in that bidders have multi-unit demand and may wind up being buyers *or* sellers.

Allocation Rule: The *lowest market-clearing price* $\underline{p}$ is the highest price such that supply does not exceed demand or equivalently $P_{1,...,n}(S+1)$, the $S+1$-st highest unit-bid submitted by any bidder. Similarly, the *highest market-clearing price* $\overline{p}$ is the lowest price such that demand does not exceed supply or equivalently $P_{1,...,n}(S)$. These prices are not necessarily linked to bidder payment, but are useful in describing market-clearing allocations.

If $\overline{p} = \{\emptyset\}$ or $\underline{p} = \infty$, then the auction is cancelled. Else each bidder receives a *market-clearing quantity*, i.e. a quantity in $D_i(\overline{p})$. ($\overline{p} \neq \underline{p}$ implies that there is a unique market-clearing allocation. Without any ambiguity, then, we can use just price $\overline{p}$ to determine market-clearing quantities.) When there is more than one set of market-clearing allocations (i.e. a tie), we will use a specific *"priority rationing rule"*. Each bidder i is assigned a rank $\rho(i)$ where $(\rho(1), ..., \rho(n))$ is a permutation of $(1, ..., n)$.[7] The auctioneer first determines $\overline{p}$ and the minimal and maximal quantities $\underline{q}_i(\mathbf{P}(\cdot))$ (shorthand $\underline{q}_i$) and $\overline{q}_i(\mathbf{P}(\cdot))$ (shorthand $\overline{q}_i$) that each bidder could receive in a market-clearing allocation:

$$\underline{q}_i \equiv \min\left(D_i\left(\overline{p}\right) \cap RS_i\left(\overline{p}\right)\right), \quad \overline{q}_i \equiv \max\left(D_i\left(\overline{p}\right) \cap RS_i\left(\overline{p}\right)\right)$$

where $RS_i(p) \equiv S - D_{-i}(p)$ is the *residual supply* facing bidder i given others' bids. (Recall from footnote 6 that $P_i(\cdot), D_i(\cdot)$ are equivalent representations of the same bid.) Each bidder receives at least $\underline{q}_i$. Next, bidders are rationed additional quantity one-at-a-time in order (first bidder $\rho^{-1}(1)$ then $\rho^{-1}(2)$, etc...), where each bidder receives all of the remaining supply or the maximal quantity that he demanded at $\overline{p}$, whichever is less. This process leads to a unique *allocation* $\mathbf{q}(\mathbf{P}(\cdot); \rho) = (q_1(\mathbf{P}(\cdot); \rho), ..., q_n(\mathbf{P}(\cdot); \rho))$. (To simplify notation, we will usually drop reference to ρ. See the discussion in Section 4 on the role of the tie-breaking rule.)

Payment in uniform $S + \alpha$-th price auction: Each bidder pays (or receives) a market-clearing price on all units that he buys (or sells). Given the profile of bids $\mathbf{P}(\cdot)$,

$$Z_i^{UP}(\mathbf{P}(\cdot)) = (q_i(\mathbf{P}(\cdot)) - e_i)\left(\alpha \min\{\overline{p}, p^{max}\} + (1 - \alpha) \max\{\underline{p}, p^{min}\}\right)$$

for some $\alpha \in [0, 1]$. (We need to use $\min\{\overline{p}, p^{max}\}$ instead of $\overline{p}$ since it is possible that $\overline{p} = \infty$ and bidders can't pay ∞ per unit, and likewise the lowest market-clearing price can't be the low null bid $\{\emptyset\}$.)

Payment in discriminatory auction: Each bidder pays (or receives) his marginal bid on each unit that he buys (or sells), interpreted here as meaning

[7] It doesn't matter how these ranks are assigned. In particular, they may be assigned before the auction begins or after the bids are submitted. In the first case, a natural interpretation is that some bidders are favored over others. In the second case, this tie-breaking rule is a generalization of the standard coin-flip rule for breaking ties (if each bidder is equally likely to get first priority, second priority, etc...).

the maximal price that he was willing to pay for that unit:[8]

$$
\begin{aligned}
Z_i^D(\mathbf{P}(\cdot)) &= \sum_{q=e_i+1}^{q_i(\mathbf{P}(\cdot))} \max P_i(q) \text{ if } q_i(\mathbf{P}(\cdot)) > e_i \\
&= \sum_{q=q_i(\mathbf{P}(\cdot))+1}^{e_i} -\max P_i(q) \text{ if } q_i(\mathbf{P}(\cdot)) < e_i \\
&= 0 \text{ if } q_i(\mathbf{P}(\cdot)) = e_i
\end{aligned}
$$

Payoff and Equilibrium: A *pure strategy* $P_i : \mathfrak{q} \times T_i \to \mathfrak{p}$ for bidder i specifies a set of unit-bids (in $\mathfrak{p}$) for each unit in $\mathfrak{q}$ and each type $t_i \in T_i$ or, equivalently, a permissible bid $P_i(\cdot; t_i)$ for each t_i. Let $\mathcal{S}_i$ be the set of bidder i's pure strategies and $\mathcal{S} \equiv \Pi_{j=1}^n \mathcal{S}_j$ and $\mathcal{S}_{-i} \equiv \Pi_{j \neq i} \mathcal{S}_j$ the sets of pure strategy profiles of all bidders and of bidders $-i$, respectively.

Given bids $\mathbf{P}(\cdot; \mathbf{t}) = (P_1(\cdot; t_1), ..., P_n(\cdot; t_n))$ in state $\mathbf{t}$, bidder i's *ex post payoff* depends on his valuation for the allocation and his payment:

$$\Pi_i^{post}(\mathbf{P}(\cdot; \mathbf{t}), \mathbf{t}) = V_i\left(\mathbf{q}(\mathbf{P}(\cdot; \mathbf{t})), \mathbf{t}\right), Z_i\left(\mathbf{P}(\cdot; \mathbf{t})\right)$$

Given others' pure strategies $\mathbf{P}_{-i}(\cdot; \cdot)$, bidder i's *interim expected payoff* depends on his own bid and type (as well as, of course, on the conditional distribution of other types):

$$\Pi_i(P_i(\cdot; t_i), t_i; \mathbf{P}_{-i}(\cdot; \cdot)) = E_{\mathbf{t}_{-i}}\left[\Pi_i^{post}\left(\mathbf{P}(\cdot; \mathbf{t}), \mathbf{t}\right) \middle| t_i\right]$$

Let $BR_i(t_i; \mathbf{P}_{-i}(\cdot; \cdot)) \equiv \arg\max_{P_i(\cdot) \in \mathfrak{D}} \Pi_i(P_i(\cdot), t_i; \mathbf{P}_{-i}(\cdot; \cdot))$ be bidder i's set of best response bids to the pure strategy profile $\mathbf{P}_{-i}(\cdot; \cdot)$ given type t_i. Let $BR_i(\mathbf{P}_{-i}(\cdot; \cdot))$ be bidder i's *best response correspondence*, mapping others' pure strategy profiles into sets of own pure strategies. A profile $\mathbf{P}^*(\cdot; \cdot) \in \mathcal{S}$ is a *pure strategy equilibrium* iff

$$P_i^*(\cdot; \cdot) \in BR_i\left(\mathbf{P}_{-i}^*(\cdot; \cdot)\right) \text{ for all } i.$$

A *mixed strategy* $\lambda_i : \mathcal{P} \times T_i \to [0, 1]$ specifies a mixture over demand correspondences for each type $t_i \in T_i$, i.e. $\lambda_i\left(P_i(\cdot), t_i\right)$ is the probability that

[8] To be completely correct, we should replace $\max P_i(q)$ with $\min\{\max P_i(q), p^{max}\}$ to rule out the possibility of bidders paying ∞ on some unit, but in equilibrium this issue will not arise.

type t_i bids $P_i(\cdot)$. A *mixed strategy equilibrium* is a profile of mixed strategies such that, for all t_i and each element $P_i(\cdot)$ in the support of $\lambda_i(\cdot; t_i)$, $P_i(\cdot)$ is a best response to strategies $\lambda_{-i}(\cdot;\cdot)$ for type t_i.

Monotone Strategies: A pure strategy $P_i(\cdot;\cdot)$ is *monotone* iff $P_i(\cdot; t_i') \geq P_i(\cdot; t_i)$ whenever $t_i' > t_i$. A mixed strategy $\lambda_i(\cdot; t_i)$ is monotone iff $P_i'(\cdot) \geq P_i(\cdot)$ for *all* $P_i'(\cdot)$, $P_i(\cdot)$ in the support of $\lambda_i(\cdot; t_i')$, $\lambda_i(\cdot; t_i)$ (respectively) for all $t_i' > t_i$. Any monotone mixed strategy can involve randomization by at most a zero measure set of types and, hence, every selection from a monotone mixed strategy equilibrium must be a monotone pure strategy equilibrium.

3 Modularity in Multi-Unit Auctions

The key observation that allows our analysis to proceed is that, holding others' bids fixed, each bidder's ex post valuation and payment are *modular* (or additively separable) in own bid.

Definition (Modularity in x). Let $(X, \geq, \vee, \wedge)$ be a lattice. $f : X \to \Re$ is *modular* in x iff

$$f(x' \vee x) + f(x' \wedge x) = f(x') + f(x)$$

for all $x', x \in X$.

When X is a subset of Euclidean space, modularity of f is equivalent to its being additive separable in its first component, second component, and so on. In the multi-unit auction context, any bid $P_i(\cdot)$ can be associated with the vector $(\max P_i(q))^{q\in\mathfrak{q}}$, so modularity implies that the incremental return to varying one's bid on unit q does not depend on the level of bids on other units.[9]

Why does the incremental return to increasing one's bid on unit q not depend on the level of one's bids on other units? For some multi-unit auctions, such as the discriminatory auction, this result seems rather intuitive. Slightly decreasing one's bid on unit q has just three effects, each of which does not depend on the level of one's bid for other units: (a) if unit q is inframarginal, then decreasing my bid decreases my total payment; (b) if

[9]Of course, $P_i(\cdot)$ can also be associated with the vector $(\max D_i(p))^{p\in\mathfrak{p}}$, so modularity also implies that the incremental return to increasing the quantity one is willing to buy at price p does not depend on the levels of one's demand at other prices.

unit q is extramarginal, then decreasing my bid has no effect; and (c) if unit q is marginal, then I no longer win unit q nor pay anything on that unit. For other auctions such as uniform price auctions, however, additively separability can seem more counter-intuitive. After all, the amount that I pay for my q-th unit depends on my bid on the q'-th unit if my bid on the q'-th unit affects the price I pay on the q-th unit. On the other hand, the extent to which changing my bid on the q'-th unit affects the amount that I pay on the q-th unit does not depend on the level of my bid for the q-th unit. These intuitions are only suggestive, however, and mask some important subtleties. (See the discussion below of the $S+2$-nd price auction.)

Theorem 1. *In the discriminatory and uniform $S+\alpha$-th price auctions ($\alpha \in [0,1]$), each bidder's interim expected payoff function is modular in own bid regardless of the strategies that others adopt.*

Proof sketch and discussion. (Proof details in the Appendix.) First, since weighted sums of modular functions are modular, it suffices to show that, for all $\mathbf{t}$ and $\mathbf{P}_{-i}(\cdot)$ (and ρ), *ex post* surplus $\Pi_i^{post}(\mathbf{P}(\cdot),\mathbf{t}) = V_i(\mathbf{q}(\mathbf{P}(\cdot)),\mathbf{t}) - Z_i(\mathbf{q}(\mathbf{P}(\cdot)))$ is modular in $P_i(\cdot)$. We break the argument into two parts: (a) valuation and (b) payment are modular in $P_i(\cdot)$.

Ex Post Valuation: This part of the argument is the same as part of the argument in McAdams (2003b). Modularity of valuation arises from the structure imposed on allocations by market-clearing (see Figure 1). The join of $P_i^1(\cdot)$ and $P_i^2(\cdot)$ in the product ordering, $P_i^1(\cdot) \vee P_i^2(\cdot) \equiv P_i^{1\vee 2}$, is traced by unfilled circles. Note that the allocation is the same when bidder i submits bid $P_i^2(\cdot)$ or $P_i^{1\vee 2}(\cdot)$ and the same when he submits bid $P_i^1(\cdot)$ or $P_i^{1\wedge 2}(\cdot)$, given that others have submitted the profile of bids $\mathbf{P}_{-i}(\cdot)$, since the allocation only depends on where i's demand schedule crosses the residual supply schedule. (In Figure 1, I have represented bids as demand rather than inverse demand, since it is more convenient to describe residual supply in the demand formulation.) Since ex post valuation only depends on the allocation and the state, this implies modularity directly:

$$\left\{V_i\left(\mathbf{q}^1,\mathbf{t}\right), V_i\left(\mathbf{q}^2,\mathbf{t}\right)\right\} = \left\{V_i\left(\mathbf{q}^{1\vee 2},\mathbf{t}\right), V_i\left(\mathbf{q}^{1\wedge 2},\mathbf{t}\right)\right\}$$

implies that

$$V_i\left(\mathbf{q}^1,\mathbf{t}\right) + V_i\left(\mathbf{q}^2,\mathbf{t}\right) = V_i\left(\mathbf{q}^{1\vee 2},\mathbf{t}\right) + V_i\left(\mathbf{q}^{1\wedge 2},\mathbf{t}\right)$$

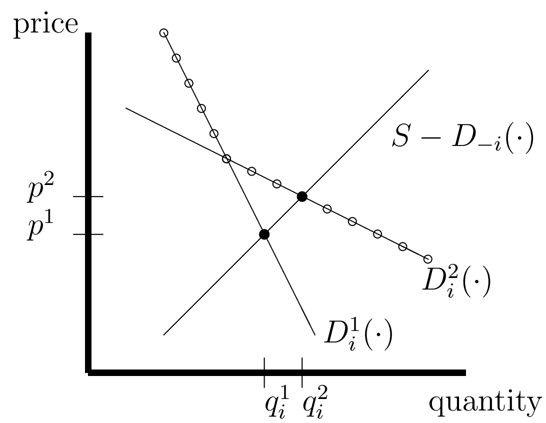

Figure 1: Modularity of ex post valuation given signals $\mathbf{t}_{-i}$

where we use shorthand $\mathbf{q}^1 \equiv \mathbf{q}\left(P_i^1(\cdot), \mathbf{P}_{-i}(\cdot)\right)$ and so on. (Note that only bidder i's bid varies.) Of course, Figure 1 is only suggestive since there is a continuum price-quantity grid and no rationing occurs. Indeed, the choice of rationing rule is vital for the result, as discussed in detail in Section 4.

Ex Post Payment: Modularity of bidders' payment functions $Z_i(\mathbf{P}(\cdot))$ in $P_i(\cdot)$ is more complex and does not follow from McAdams (2003b)'s proof technique for the uniform S-th and $S+1$-st price auctions. Those particular auctions have a special property that the two price/allocation combinations that occur on the main-diagonal[10] are identical to the price/allocation combinations that occur on the off-diagonal. (To parse this, see again Figure 1. The statement boils down, loosely, to the observation that the two demand schedules $P_i^1(\cdot)$, $P_i^2(\cdot)$ cross the residual supply at the same price/quantity as their upper- and lower-envelopes.) In other uniform-price auctions, from the $S+\alpha$-th varieties studied here to the $S+2$-nd price variety discussed below, there is a similar-sounding but crucially different property. The two prices that occur on the main-diagonal are identical to the prices that occur

[10]When I speak of the "main- and off-diagonal" when considering two bids and their meet and join, I invoke a geometrical interpretation of the sublattice $\{P_i^1(\cdot), P_i^2(\cdot), P_i^{1\wedge 2}(\cdot), P_i^{1\vee 2}(\cdot)\}$ as a "rectangle". $P_i^{1\wedge 2}(\cdot), P_i^{1\vee 2}(\cdot)$ comprise the main diagonal while the original bids comprise the off-diagonal. Of course, when $P_i^1(\cdot)$ and $P_i^2(\cdot)$ are comparable, this rectangle is degenerate and there is no difference between the main- and off-diagonal.

on the off-diagonal and the two allocations that occur on the main diagonal are the same as on the off-diagonal. But the price/allocation *combinations* need no longer match! Indeed, despite the apparent similarity, this difference can destroy modularity of bidder payments, making the current effort a much more subtle exercise. Example 1 shows why ex post payment fails to be modular in the $S+2$-nd price auction. (In fact, ex post payment is supermodular in this auction making ex post surplus submodular.)

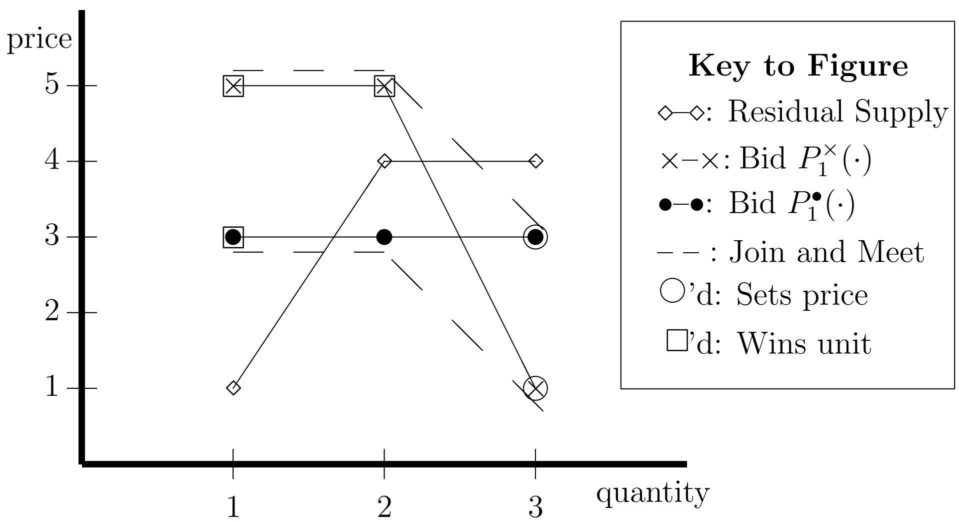

Figure 2: Payment *supermodular* in uniform $S+2$-nd price auction

Example 1 (Uniform $S+2$-price auction). There are two bidders, three units, and permissible prices $\{0,1,2,3,4,5\}$ in a *fifth*-price auction. We will focus on bidder 1, holding as fixed a bid for bidder 2 of $P_2(\cdot)=(4,4,0)$ or, equivalently, holding as fixed the residual supply $(0,4,4)$, traced in Figure 2 with solid lines connecting diamonds. (Here for notational ease we use the representation of bids as vectors of maximal unit-bids, $(\max P_i(q))^{q\in\mathfrak{q}}$.) Consider now two incomparable bids for bidder 1, labeled as $P_1^{\times}=(5,5,1)$ and $P_1^{\bullet}=(3,3,3)$ (solid lines connecting $\times$'s and $\bullet$'s, respectively). Their join and meet are $P_1^{\times\vee\bullet}=(5,5,3)$ and $P_1^{\times\wedge\bullet}=(3,3,1)$ (dashed lines). Now, in the Figure, unit-bids that lead bidder 1 to win are boxed whereas unit-bids that set the price are circled. Thus, for example, bidding $P_1^{\times\vee\bullet}$ leads bidder 1 to win two units at price 3 for payment 6. Similarly, $P_1^{\times}$ leads to two units at price 1 and payment 2, $P_1^{\bullet}$ to one unit for payment 3, and $P_1^{\times\wedge\bullet}$ to one unit for payment 1. Overall, the sum of payments on the main-diagonal of the bid-space (7) exceeds the sum on the off-diagonal (5) i.e. payment is supermodular. As hinted at in the previous discussion, the source of this failure of modularity is a "mismatch" of price/allocation combinations generated on the main- and off-diagonals. On each diagonal, quantity is either one or two units and price is either 1 or 3. But on the main-diagonal the price is 1 for one unit and 3 for two units, whereas on the off-diagonal the price is 3 for one unit and 1 for two units.

In uniform $S+\alpha$-th price auctions, there can be a mismatch of another sort. The pair of payments made on the main diagonal need not be the same as the pair of payments made on the off-diagonal. The extra bit of structure that allows us to salvage modularity, however, is that this can only happen when the allocation is constant over the whole rectangle. Or, for a deeper way to appreciate what's going on that doesn't involve ad hoc arguments, note that *each unit-bid's pair of marginal contributions* to overall payment are the same on the main- and off-diagonals. An example again is the clearest way to illustrate this idea:

Example 2 (Uniform $S+1/3$-price auction). There are two bidders and two units in a uniform 7/3-rd price auction, with permissible bids $\{0,1,2,3,4\}$. Holding fixed bidder 2's bid of $(4,0)$, consider two bids for bidder 1, $(3,0)$ and $(2,1)$, as well as their upper- and lower-envelopes, $(3,1)$ and $(2,0)$. Given any of these bids, bidder 1 receives onle unit. Bid $(3,1)$ yields price 7/3, $(3,0)$ price 2, $(2,1)$ price 5/3, and $(2,0)$ price 4/3. Overall, total payment is 11/3 both on the main- and off-diagonal so modularity is not violated. The "marginal contribution" of his first-unit bid to the price is 2/3(3)=2 when he bids 3 and 2/3(2)=4/3 when he bids 2, and similarly the marginal contribution of his second-unit bid is either 1/3 or 0. There is a mismatch of these marginal contributions (on the main-diagonal, $2+1/3$ and $4/3+0$; on the off-diagonal, $2+0$ and $4/3+1/3$) but this mismatch is not fatal to modularity.

Discriminatory auction: Proving that payment is modular in the discriminatory auction might seem at first blush even more difficult than in the uniform $S+\alpha$-th price auctions, since there can be a mismatch of total payments on the main- and off-diagonals even in situations in which different allocations are realized in the rectangle generated by $P_1^1(\cdot)$, $P_1^2(\cdot)$. See Figure 3. Bidding $P_1^{1\vee 2}(\cdot)$ yields high quantity and payment $A+B+C$, $P_1^1(\cdot)$ low quantity

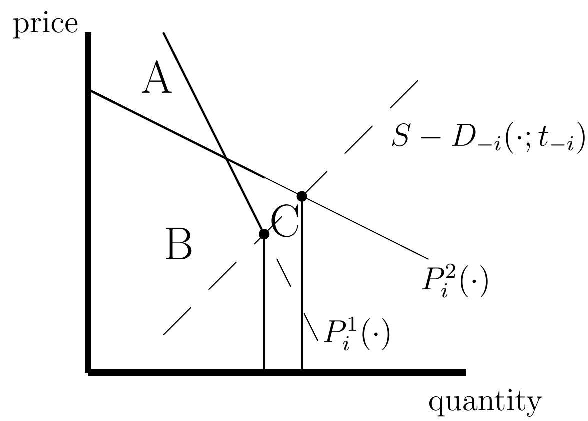

Figure 3: Modularity of ex post payment in discriminatory auction

and payment $A + B$, $P_1^2(\cdot)$ high quantity and payment $B + C$, and $P_1^{1\wedge 2}(\cdot)$ low quantity and payment A. In the Figure, of course, modularity is not violated since the sum of payments on both the main- and off-diagonals is $A + 2B + C$. At a deeper level, the reason for this is the same as for the uniform $S + \alpha$-th price auctions: the marginal contributions of each unit-bid are the same on both diagonals (the marginal payment on unit q is $P(q)$, and $\{P^1(q), P^2(q)\} = \{P^{1\vee 2}(q), P^{1\wedge 2}(q)\}$ by definition of the product order) and total payment is a sum of these marginal contributions.

4 Monotone equilibrium exists

Given that we have established that expected payoffs are modular in own bid, it is routine to apply McAdams (2003b)'s existence theorem to conclude that monotone pure strategy equilibria exist. The key remaining condition to check is Athey's so-called "single-crossing condition" (SCC).

Definition (Single-crossing in (x, t)). Let X, T be partially ordered sets. The function $f : X \times T \to R$ satisfies the single-crossing property in (x, t) if, for all $x' > x$ and $t' > t$, $f(x', t) > (\geq) f(x, t)$ implies $f(x', t') > (\geq) f(x, t')$, i.e. strict inequality implies strict, weak inequality implies weak.

In a single-object or multi-unit auction, each bidder i's expected payoff is a function of own bid (from a lattice), own type (from a partially ordered set), and others' strategies. SCC is satisfied when each bidder i's *expected* payoff satisfies the single-crossing property in own bid and types *whenever others' adopt monotone strategies*. Given independence and risk-neutrality, we show that in fact expected payoffs satisfy non-decreasing differences in own bid and type (NDD) for any given strategies by others.

Definition (Non-decreasing differences in $(x;t)$). Let X, T be partially ordered sets. The function $f : X \times T \to R$ has non-decreasing differences in (x,t) iff $f(x',t') - f(x,t') \geq f(x',t) - f(x,t)$ for all $x' > x$, $t' > t$.

Theorem. *[McAdams (2003b)] Let G be a game in which each player's action a_i is chosen from a finite lattice, each player's type $t_i \in [0,1]^h$, and each player's interim expected payoff function satisfies quasisupermodularity in a_i and has single-crossing in (a_i, t_i) whenever others adopt monotone strategies.*[11] *Then G has a monotone pure strategy equilibrium.*

Theorem. *[McAdams (2003b)] A monotone pure strategy equilibrium exists in the uniform S-th price and the uniform $S+1$-st price auctions.*

Theorem 2. *A monotone pure strategy equilibrium exists in the discriminatory and uniform $S+\alpha$-th price ($\alpha \in (0,1)$) auctions.*

Proof. Theorem 1 implies that expected payoffs are modular in own bid no matter what strategies others' adopt, so certainly they are quasisupermodular in own bid when others adopt monotone strategies. (Risk-neutrality is crucial here. Theorem 1 fails when bidders are risk-averse, as discussed in detail below.) No matter what others bid, submitting a higher bid causes one to win (weakly) more quantity and others to win less. (Consult equations (3, 4) in the Appendix.) Since incremental values are assumed to be non-decreasing, each bidder's ex post surplus thus satisfies non-decreasing differences in own bid and type (NDD). By risk-neutrality, then, ex post *payoff* satisfies NDD and, by independence, *expected* payoff satisfies NDD. (This is the *only* place where independence is used in the proof but it is vital, as discussed in detail below.) We may therefore apply McAdams (2003b)'s general existence result. □

[11] See McAdams (2003b) for the formal definition of quasisupermodularity in a_i, for which modularity in a_i is sufficient.

Discussion of Assumptions

Three assumptions stand out in this paper's analysis and merit further discussion: independence, risk-neutrality, and the rationing rule. In each case, it is illuminating to compare their role in this paper's multi-unit auction analysis with their more limited role in single-object auction theory.

Independence. In the first-price auction, each bidder's ex post payoff satisfies a weak form of complementarity across bids: if bidding higher increases my payoff when others bid lower, then bidding higher can not decrease my payoff when others bid higher. When bidder signals are positively correlated, this complementarity across bids (loosely speaking) *increases* bidders tendency to adopt monotone strategies in response to monotone strategies by their opponents: when I receive a higher signal, you are more likely to have received a higher signal and hence be bidding higher, which in turn makes me want to bid no lower by strategic complementarity. Indeed, this virtuous cycle of monotonicity breeding monotonicity is precisely what drives the literature on monotone equilibrium in single-object auctions (such as Athey (2001), McAdams (2003a), and Reny and Zamir (2002)).

In multi-unit auctions, however, complementarity across bids and hence SCC tend to fail in wholesale fashion. For instance, in the uniform $S+1$-st price auction example below, suppose that bidder 1 has value 100 for both units and compare two bids $(50,50)$ vs. $(50,20)$. If bidder 2 bids $(25,0)$, then bidder 1 prefers $(50,50)$ over $(50,20)$ since the former gives him payoff 100 whereas the latter only 80. But if bidder 2 increases his bid to $(100,0)$, bidder 1 now prefer $(50,20)$ over $(50,50)$ since the former gives him 80 whereas the latter only 50. Thus, when bidders receive positively correlated types, the virtuous cycle is broken: my having a higher type implies that I am more likely to face high bids if my opponents are following monotone strategies, but this may lead me to prefer to bid lower and hence not adopt a monotone strategy myself. While the example employs the uniform $S+1$-st price rule, similar examples exist for discriminatory and other uniform price rules. See McAdams (2002a).)

Example 3 (Symmetric, uniform $S+1$-st price). Two bidders and two units in a third-price auction. Bids are allowed in non-negative dollar increments. Bidders receive types $t_1, t_2 \in [0,1]$ and have private values $v_i(1,t_i) = v_1(2,t_i) = 200t_i$ for all $t_i \in [0,1/2]$. $v_i(1,t_i) = 200t_i$ and $v_i(2,t_i) = 100$ for all $t_i \in (1/2,1]$. The joint density of $\mathbf{t}$ is given by

$f(\mathbf{t}) = 4(1-t_1)(1-t_2)$ when $t_1, t_2 > 1/2$ or $t_1, t_2 < 1/2$ and $f(\mathbf{t}) = 0$ otherwise. In other words, $t_1 > (<)1/2$ implies that $t_2 > (<)1/2$ and $\mathbf{t}$ are independent conditional on both being greater than or less than 1/2. In particular, bidders have affiliated private values.

Proposition 1. *There is a unique equilibrium* $\mathbf{P}^*(\cdot;\cdot)$ *in weakly undominated strategies in this example, and bidder 2 adopts a non-monotone strategy in this equilibrium.*

Proof. Conditional on $t_1, t_2 < 1/2$ or $> 1/2$, t_1, t_2 are independent. Applying Theorem 2 to each of these "sub-auctions" in which the types are less than or greater than 1/2, we conclude that a pure strategy equilibrium exists in which bids are non-decreasing over the type range $[0, 1/2)$ and over the type-range $(1/2, 1]$. (Non-monotonicities will arise at $t_i = 1/2$.) In any equilibrium in weakly undominated strategies, $P_1^*(1, t_1), P_2^*(1, t_2) \geq 100$ and $P_1^*(2, t_1) = P_2^*(2, t_2) = 0$ for all $t_1, t_2 > 1/2$: The bidders are certain to each win one unit, so someone's second-unit bid will set the price. Thus each bidder's weakly dominant strategy is to bid zero on the second unit. On the other hand, $P_i^*(2, t_i) > 0$ in any equilibrium for some $t_i < 1/2$: Since the other bidder adopts a weakly undominated strategy, bidder 1 (conditional on $t_1 < 1/2$) faces probability at least $p(\triangle) = \Pr(t_2 < \triangle/100 | t_2 < 1/2)$ that bidder 2 will bid less than $\triangle$ on both units. Consequently, for $t_1 \approx 1/2$, bidding $\triangle$ rather than 0 allows him to capture approximately $(100-\triangle)p(\triangle)$ extra expected surplus from winning the second unit while paying at most $\triangle$ extra on the first unit. In this example, $p(\triangle) = \int_0^{\triangle/100}(1-x)dx / \int_0^{1/2}(1-x)dx = 8/3(\triangle/100) - 4/3(\triangle/100)^2$. As long as $\triangle << 100$, then, the benefit of raising one's bid from zero to $\triangle$ on the second unit outweighs the cost. $\square$

In this symmetric example, all bidders reduce their bids on some unit as type increases. Other examples with affiliated private values exist in which some bidders reduce their bids on all units. (See McAdams (2002a).)

When bidders receive negatively correlated signals, further, there is an obvious reason why some bidder's best response may be non-monotone even in single-object auctions when others adopt monotone strategies: Others are more likely to receive lower signals and hence bid lower when I receive a higher signal. See Jackson and Swinkels (2001) for a first-price auction example along these lines in which every equilibrium is non-monotone. Thus, we can not possibly hope for a general MPSE existence result *except* in the case of independent types (or many bidders each having little impact on price).

Risk-neutrality. In the first-price auction the presence of risk-aversion also (loosely speaking) *increases* bidders tendency to adopt monotone strategies in response to monotone strategies by others. For instance, consider the benchmark case of independent private values (IPV). Let $p(b)$ be bidder i's probability of winning given bid b and others' strategies. The incremental upside $(p(b') - p(b))u_i(v_i - b')$ of winning with b' when b would have lost is increasing in the private value v_i, whereas the incremental downside $p(b)(u_i(v_i - b) - u_i(v_i - b'))$ of paying more when b would have won is decreasing in v_i. (Given risk-neutrality, this downside to bidding higher is constant in v_i.) In the discriminatory auction, however, risk-aversion can cause SCC to fail and hence cause bidders to have non-monotone best responses to monotone strategies. Assume again IPV and just two bidders competing for two units. Here the discriminatory auction is akin to two separate first-price auctions, where my first-unit bid competes against your second-unit bid and vice versa, with the caveat that second-unit bids can not exceed first-unit bids. Holding our bids fixed, suppose that my value for the first unit increases. If I am winning the first unit, this increases my "wealth" going into the second auction. Consequently, *if* I have decreasing absolute risk aversion (DARA), then I will tend to be "less risk averse" and hence bid less in the second auction. Of course, this logic depends crucially on my having DARA utility and does not generalize to all sorts of risk-averse bidders. Furthermore, it applies only to the discriminatory auction. It remains an open question whether SCC is satisfied in uniform-price auctions for some sorts of risk-averse bidders.[12]

In short, with risk-aversion there is an interaction between different unit-bids: bidding higher on one unit, in and of itself, may lead a bidder to prefer bidding lower on other units. Example 4 shows how this can lead *all* equilibria to be non-monotone in the discriminatory auction when bidders are risk-averse, even given independent private values. Both effects of risk-aversion – on SCC and modularity – are at play in this example.[13]

[12]In particular, I remains unclear to me whether SCC fails in the uniform $S+1$-st price auction given DARA bidders. This is an interesting area for future research. If SCC can be established in the $S+1$-st price auction in environments with risk-aversion, then the analysis in this paper will imply that MPSE exists in such environments and probably that all equilibria are monotone. This would expand the current literature in another direction. With the exception of Jackson and Swinkels (2001) and papers following it for the case of private values, the literature on multi-unit auctions has been limited by the need to assume risk-neutrality.

[13]Bidder 1 has increasing absolute risk aversion in the example, but his bid on the first unit increases by more than his value increases, so that his effective "wealth" when bidding

Example 4 (Risk-averse, Discriminatory). Two units are auctioned to two bidders in a discriminatory auction. In order to avoid confusing issues due to risk-aversion with issues related to the possibility of tying (see Example 5), each bidder is given a disjoint set of non-null prices that he is permitted to bid on each unit. In particular, bidder 1's permissible prices are $\mathfrak{p}_1 = \{\{\emptyset\}, 20, 60, 100, \infty\}$ whereas $\mathfrak{p}_2 = \{\{\emptyset\}, 40, 80, \infty\}$.

Bidder 1 receives signal $t_1 \in \{L, H\}$ and bidder 2 receives signal $t_2 \in \{L, M, H\}$, where t_i are independent. $\Pr(t_1 = L) = \Pr(t_2 = H) = 1/2$ while $\Pr(t_2 = L) = 1/4$, $\Pr(t_2 = M) = 1/8$, and $\Pr(t_2 = H) = 5/8$. (It is straightforward to modify this example so that each bidder receives a signal from $[0, 1]$ with values increasing in own signal: all relevant preferences amongst various bids are strict.) Bidder 1 is risk-averse with utility over surplus $X_1 = V_1 - Z_1$ of the form $u_1(X_1) = X_1$ for all $X_1 < 100$ and $u_1(X_1) = 100$ for all $X_1 \geq 100$; bidder 2 is risk-neutral.

Bidder 1 is the interesting bidder here, who will adopt a non-monotone strategy in equilibrium. His marginal value for a second unit does not depend on his signal, $v_1(2; L) = v_1(2; H) = 99$, but his marginal value on the first unit increases with his signal, $v_1(1; L) = 110$, $v_1(1; H) = 130$. Bidder 2 has equal marginal value for the first and second unit: $v_2(\cdot; L) = 0$, $v_2(\cdot; M) = 60$, and $v_2(\cdot; H) = 200$.

Proposition 2. *There is a unique equilibrium $\mathbf{P}^*(\cdot; \cdot)$ in weakly undominated strategies in this example, and bidder 1 adopts a non-monotone strategy in this equilibrium.*

Proof. When $t_2 = L$, bid $(\{\emptyset\}, \{\emptyset\})$ is weakly dominant for bidder 2 since his marginal values are less than 40. Similarly, bid $(40, 40)$ is weakly dominant when $t_2 = M$. Bidder 2's marginal values are so high when $t_2 = H$, finally, that bid $(80, 80)$ is his best response unless $P_1(1; L), P_1(1; H) \leq 20$. In that case, bidder 2's best response would be $(40, 40)$. But this leads to a contradiction, since bidder 1 would then prefer to bid 60 over either 20, $\{\emptyset\}$ on the first unit given either type L, H. In summary, $P_2^*(\cdot; L) = (\{\emptyset\}, \{\emptyset\})$, $P_2^*(\cdot; M) = (40, 40)$, and $P_2^*(\cdot; H) = (80, 80)$.

Since $v_1(1; \cdot) > 20$ and $v_1(2; \cdot) < 100$, bidder 1 always bids at least 20 on both units and less than 100 on the second unit. This leaves five bids that might be best responses for bidder 1. Consider first bidder 1's best response

on the second unit has decreased. Submodularity also clearly plays a role since raising his bid on the first unit, in and of itself, leads him to prefer a lower bid on the second unit.

when $t_1 = H$. Note that bidder 1's incremental surplus from winning the first unit at price 100 is 30, at price 60 is 70, and at price 20 is 110; and from winning the second unit at price 60 is 39 and at price 20 is 79. Since bidder 1 is assumed satiated at surplus 100, we can express bidder 1's expected utility from each possible bid as follows:

$$\begin{aligned}
(100, 60) &\mapsto 5/8(30) + 1/8(69) + 1/4(69) = 44.625 \\
\mathbf{(100, 20)} &\mapsto \mathbf{5/8(30) + 1/8(30) + 1/4(100) = 47.5} \\
(60, 60) &\mapsto 5/8(0) + 1/8(100) + 1/4(100) = 37.5 \\
(60, 20) &\mapsto 5/8(0) + 1/8(70) + 1/4(100) = 33.75 \\
(20, 20) &\mapsto 5/8(0) + 1/8(0) + 1/4(100) = 25
\end{aligned}$$

So, $P_1^*(\cdot; H) = (100, 20)$. Now consider bidder 1's best response when $t_1 = L$, when his value for the first unit is 20 less.

$$\begin{aligned}
(100, 60) &\mapsto 5/8(10) + 1/8(49) + 1/4(49) = 24.625 \\
(100, 20) &\mapsto 5/8(10) + 1/8(10) + 1/4(89) = 29.75 \\
\mathbf{(60, 60)} &\mapsto \mathbf{5/8(0) + 1/8(100) + 1/4(89) = 34.75} \\
(60, 20) &\mapsto 5/8(0) + 1/8(50) + 1/4(100) = 31.25 \\
(20, 20) &\mapsto 5/8(0) + 1/8(0) + 1/4(100) = 25
\end{aligned}$$

So, $P_1^*(\cdot; L) = (60, 60)$ and bidder 1 reduces his bid on the second unit as his type increases. □

Priority rationing rule. Tie-breaking rules have received a lot of attention in the single-object auction literature but the issues surrounding tie-breaking become much more complex in multi-unit environments. For one thing, a bidder may not just "win" or "lose" a tie but also "win *some of*" a tie when bidders express excess demand for multiple units. Furthermore, details of the rationing rule can induce bidders to adopt non-monotone strategies in ways that just can't happen in a single-object setting.

An important feature of the rationing rule used in this paper, obviously, is that the quantity a bidder wins is non-decreasing in his bid. One could conceive of (non-standard) rationing rules in which the amount that a bidder is rationed at market-clearing price p^* decreases as $\min D_i(p^*)$ and/or $\max D_i(p^*)$ increase.[14] Not surprisingly, given such a rationing rule a bidder

[14] The market-clearing price depends on the bids. The intention here is to restrict

may choose to decrease his bids on some units as his valuation increases, if decreasing his bid allows him to win more!

A more subtle and interesting aspect of priority rationing is that changing one's bid on unit q does not change one's likelihood of winning some other unit. (For a formal verification, consult equations (3, 4) in the Appendix.) This property of priority rationing is not obvious and fails in other rationing rules. Consider for instance so-called "proportional rationing", the most commonly studied type of rationing rule.

Definition (Proportional Rationing). Let $\underline{q}_i$ be as defined on page 11. The rationed quantity $R = S - \sum_i \underline{q}_i$ is split amongst the bidders in proportion to their excess demands.[15] For instance, bidder 1 receives quantity $q_1^* = \underline{q}_1 + R\frac{\max D_1(p^*)-\underline{q}_i}{\sum_{i=1}^n \max D_i(p^*)-\underline{q}_1}$.

As a simple example, suppose that there are twelve units and two bidders. Bidder 1 bids \$10 on all twelve units while bidder 2 bids \$10 on the first six and \$5 on the last six. $\underline{q}_1 = 6$ and $\underline{q}_2 = 0$ in this example, so given proportional rationing bidder 1 gets nine units while bidder 2 gets three units. If bidder 2 were to raise his bids on all of his units to \$10, however, he would receive six. Thus, his likelihood of getting units 4-6 depends on his bids on units 7-12. Under priority rationing, on the other hand, given the original bids bidder 1 gets six (or twelve) units and bidder 2 gets six (or zero) if bidder 2 (or bidder 1) has priority. If bidder 2 raises his bids on all units 7-12, he now either wins zero or twelve ... but his chances of winning units 4-6 have not changed.

Why does this matter, that unit-bids on units 7-12 may affect one's winning chances on units 4-6? The reason is that now there is an *interaction* between bids on units 4-6 and bids on 7-12. Raising your bid from \$9 to \$10 on units 4-6 leads to a *greater* increase in your probability of winning those units if you also raise your bids on units 7-12 to \$10. Consequently, ex post valuation will typically fail to be modular. Example 5 shows that this

attention to those bids by i after which the market-clearing price is p^* and explore how the quantity that bidder i receives depends on the range of quantities that he demands at price p^*.

[15] There are other variations of the proportional rationing rule, as when bidders are awarded $\min D_i(p^*)$ plus an amount proportional to $\max D_i(p^*) - \min D_i(p^*)$. All such variations share the property that additive separability fails and that bidders may adopt non-monotone strategies in equilibrium.

failure of modularity can lead all equilibria to be non-monotone even given independent private values.

Example 5 (Proportional Rationing, $S + 1$-st price). Two strategic bidders with permissible bids from the set $\emptyset \cup \{10, 20\}$. Auctioneer has 10 units for sale but requires that all 10 sell for at least 10 or else the auction is cancelled. (Supply $S(\emptyset) = \{0, ..., 10\}$, $S(10) = S(20) = 10$.) Proportional rationing. Uniform $S + 1$st-price payment rule. (Variations on this example apply also to uniform S-st price, discriminatory, and Vickrey payment rules.) Each bidder has either a high (H) or low (L) type, and these types are i.i.d. with $\Pr(t_1 = L) = \Pr(t_2 = L) = 1/2$. When $t_1 = L$, bidder 1 has value 11 for the first four units and value 0 for all subsequent units; when $t_1 = H$, she has value 110 for the first four units and value 0 for subsequent units. When $t_2 = L$, bidder 2 has value 11 for the first two units and value 0 for all subsequent units; when $t_2 = H$, he has value 11 for the first eight units and value 0 for subsequent units.

Proposition 3. *There is a unique equilibrium $\mathbf{P}^*(\cdot;\cdot)$ in weakly undominated strategies in this example, and bidder 1 adopts a non-monotone strategy in this equilibrium.*

Proof. Let $\mathbf{P}^*(\cdot)$ be a pure strategy equilibrium in weakly undominated strategies. (Without loss we may disregard mixed strategies.) First, the requirement that bidders adopt weakly undominated strategies completely nails down bidder 2's strategy: $P_2^*(q; t_2) = 10$ for $(q; t_2) = (1, L)$, $(2, L)$, ... , $(1, H)$, $(8, H)$ and $P_2^*(q; L) = \emptyset$ otherwise.

When $t_1 = L$, bidder 1's unique best response is to bid

$$P_1^*(\cdot; L) = (20, 20, \emptyset, \emptyset, ..., \emptyset).$$

This guarantees that she wins two objects at price 10 in the event that $t_2 = H$ and that the auction will be cancelled when $t_2 = L$.

When $t_1 = H$, bidder 1's unique best response is to bid

$$P_1^*(\cdot; H) = (10, ..., 10, \emptyset, \emptyset).$$

When $t_2 = L$, this leads her to win 8 objects at price 10 for surplus 2*110 - 8*10 = 140; when $t_2 = H$, this leads her to be rationed $4 = 8/2$ units for surplus 2*110 - 4*10 = 180 (and expected surplus 160). $(10, ..., 10, \emptyset, \emptyset)$ is the best bid among those that guarantee that the auction is not cancelled when

$t_2 = L$: she must bid at least 10 for the 8th unit to avoid the cancellation, and by bidding 10 for all units $q = 0, ..., 8$ she *minimizes* the amount that she wins in the case that $t_2 = H$. Compare that to the best bid that doesn't avoid cancellation, again $(20, 20, \emptyset, ..., \emptyset)$ for expected surplus $1/2(2*110 - 2*10) = 100 < 160$. Bidder 1's strategy is non-monotone since $P_1^*(1; L) = 20$ and $P_1^*(1; H) = 10$. □

5 All equilibria are monotone

Not only do monotone pure strategy equilibria (MPSE) exist but *all* mixed-strategy equilibria[16] in the discriminatory and uniform $S + \alpha$-th price ($\alpha \in [0, 1]$) auctions are "equivalent" to MPSE in the sense of ex post allocation- and interim expected payment-equivalence defined below. Thus, at least insofar as one is concerned with expected surplus / revenue analysis, there is no loss in restricting attention to monotone strategies. A caveat, to be discussed after the proof, is that these results hold only when there are no externalities. That is to say, bidder values take the simpler form $V_i(\mathbf{q}, \mathbf{t}) = V_i(q_i, \mathbf{t})$. (All other assumptions remain the same.)

The following definitions are made for pure strategy profiles to simplify the exposition, but the extension to mixed strategy profiles should be clear:

Definition (Ex post allocation-equivalence). Two pure strategy profiles $\mathbf{P}'(\cdot; \cdot)$, $\mathbf{P}(\cdot; \cdot)$ are *ex post allocation-equivalent* if the induced allocation (and hence total ex post surplus) is the same with probability one:

$$\Pr{}_{\mathbf{t}} \left(\mathbf{q}(\mathbf{P}'(\cdot; \mathbf{t})) = \mathbf{q}(\mathbf{P}(\cdot; \mathbf{t})) \right) = 1$$

Similarly, two *bids* $P_i'(\cdot)$, $P_i(\cdot)$ are ex post allocation-equivalent when

$$\Pr{}_{\mathbf{t}_{-i}} \left(\mathbf{q}(P_i'(\cdot), \mathbf{P}_{-i}(\cdot; \mathbf{t}_{-i})) = \mathbf{q}(P_i(\cdot), \mathbf{P}_{-i}(\cdot; \mathbf{t}_{-i})) \right) = 1$$

Definition (Interim expected payment-equivalence). Two pure strategy profiles $\mathbf{P}'(\cdot; \cdot)$, $\mathbf{P}(\cdot; \cdot)$ are *interim expected payment-equivalent* if each

[16] The present analysis does not apply to correlated equilibrium. Indeed, whenever there are multiple (non-equivalent) MPSE there also exists a correlated equilibrium in which bidders observe a public coin-flip and play the first (or second) equilibrium upon seeing heads (or tails). Such a correlated equilibrium will typically involve higher types playing some bids that are not greater than some bids played by lower types.

bidder i makes the same interim expected payment (conditional on his own type) with probability one:

$$\Pr{}_{\mathbf{t}}\left[E_{\mathbf{t}_{-i}}\left(Z_i(P'_i(\cdot;t_i),\mathbf{P}'_{-i}(\cdot;\mathbf{t}_{-i}))\right)=E_{\mathbf{t}_{-i}}\left(Z_i(P_i(\cdot;t_i),\mathbf{P}_{-i}(\cdot;\mathbf{t}_{-i}))\right)\text{ for all }i\right]=1$$

Interpretation: An observer of *the state of the world* $\mathbf{t}$ and all realized allocations would not be able to distinguish the data generated by two ex post allocation-equivalent strategy profiles. Similarly, an observer of the state of the world and each bidder's interim expected payment would not be able to distinguish the data generated by two interim expected payment-equivalent strategy profiles. Consequently, auctioneer revenue and expected bidder surplus (for each bidder type) induced by two such strategy profiles are identical.

Theorem 3. *Under the extra assumption of no externalities, all mixed strategy equilibria are ex post allocation- and interim expected payment-equivalent to a monotone pure strategy equilibrium in the discriminatory and uniform $S+\alpha$-th price ($\alpha\in[0,1]$) auctions.*

Proof. To simplify the exposition, I deal here with the less notationally complex case of one-dimensional types. A proof for the multi-dimensional case is available from the author in an expanded version of the paper.[17] I also refer more simply to "allocation-equivalence" or "payment equivalence" rather than ex post allocation-equivalence and interim expected payment equivalence. Similarly, reference to "expected payment" means each bidder's interim expected payment conditional on own type.

Strategies as graphs: The proof leverages a novel (many-to-one) mapping G from the space of all mixed strategies to a space of weighted directed planar graphs (or simply "graphs"), $\lambda_1(\cdot;\cdot)\mapsto G(\lambda_1(\cdot;\cdot))$.[18] A weighted directed graph consists of a set of nodes, directed edges connecting pairs of nodes, and scalar weights for each edge. A planar graph is one whose nodes and edges are embedded in the plane (nodes as points, edges as straight line-segments). All graphs that are in the range of this mapping have common

[17] This multi-dimensional extension uses the assumption that $\{t_i^j\}_{j=1,\dots,h}^{i=1,\dots,n}$ are independent, whereas the existence result (Theorem 2) required the weaker condition that $\{t_i\}^{i=1,\dots,n}$ are independent.

[18] Shorthand notation G is used to refer both to the mapping from strategies to graphs and to a representative graph in the range of that map. Specifically, when I write "any graph G" I mean any graph that is mapped to by some strategy. All such graphs have the property that the total weight of all in-edges to a node equals the total weight of all out-edges from that node.

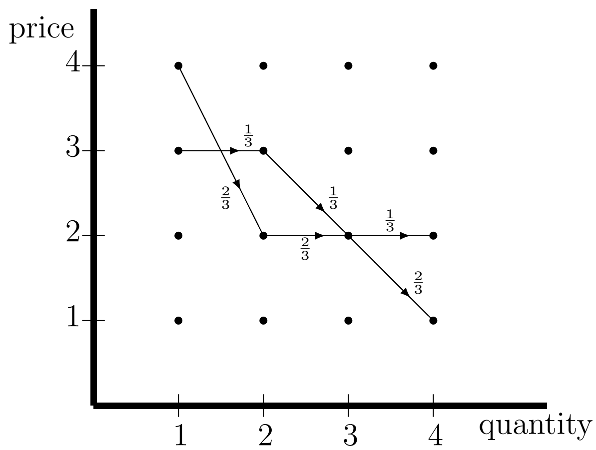

Figure 4: Planar graph not corresponding to a monotone strategy

node-set $\{(p, q) : q \in \mathfrak{q}, p \in \mathfrak{p}\}$ corresponding to points in the quantity/price grid, and common edge-set $\{((b', q) \to (b, q',)) \colon \ q' = q + 1, b' \geq b\}$. The weight put on edge $((b', q) \to (b, q + 1))$ in the graph $G(\lambda_1(\cdot; \cdot))$ equals the probability $a_{(q,q+1)}((b', b); \lambda_1(\cdot; \cdot))$ that a bid $P_1(\cdot)$ is played in strategy $\lambda_1(\cdot; \cdot)$ with the property that both unit-bids $P_1(q) = b'$ and $P_1(q+1) = b$. Formally, for any $Q \subset \mathfrak{q}$ and $b_Q = (b_q)^{q \in Q}$,

$$a_Q(b_Q; \lambda_1(\cdot; \cdot)) = \int_{t_1} \sum_{P_1(\cdot) : P_1(q) = b_q \forall q \in Q} \lambda_1(P_1(\cdot); t_1) dt_1$$

I also assign weight $w((p, q))$ to each node equal to the total weight of all its in-edges to that node: $w((p, q)) = \sum_{p' \geq p} w\left((p', q - 1) \to (p, q)\right)$. Next, define a *path* in graph G as a sequence of nodes connected by edges having non-zero weights, including one node for each quantity in $\mathfrak{q}$. The set of all possible paths is isomorphic with the set of all permissible bids, and it inherits the product order. It is worth noting, however, that the set of graph $G(\lambda_1(\cdot; \cdot))$'s paths is not necessarily identical with the set of bids played with positive probability in $\lambda_1(\cdot; \cdot)$ (though all played bids are among G's paths). Similarly, a graph's paths need not have a maximal element, just as the set of bids played in a strategy need not have a maximal element.

For example, consider the following non-monotone pure strategy $P_1(\cdot; \cdot)$

in a setting in which $t_1 \sim U[0,1]$: $P_1(\cdot;t_1) = (4,2,2,2)$ when $t_1 \in [0,1/3)$, $P_1(\cdot;t_1) = (4,2,2,1)$ when $t_1 \in [1/3,2/3)$, $P_1(\cdot;t_1) = (3,3,2,1)$ when $t_1 \in [2/3,1]$. The graph mapped to by this strategy is illustrated in Figure 4. Only those edges having positive weight are displayed. The edge $((2,2) \to (2,3))$ has weight $2/3$ since all types $t_i \in [0,2/3)$ submit a bid such that $P_i(2;t_1) = 2$ and $P_i(3;t_i) = 2$). There are four paths in this graph but only three played bids; path $((3,1)),(3,2),(2,3),(2,4))$ does not correspond to a played bid.

There are three nice things about this representation of strategies as graphs. (1) The total weight of all in-edges to (or out-edges from) (p,q) in G is the probability that bidder 1 makes unit-bid p on unit q in the strategy mapping to G. (2) Any graph G whose nonzero-weight edges never intersect always has a maximal path. (3) Any graph G whose nonzero-weight edges never intersect is mapped to by some monotone strategy whereas no monotone strategy maps to any graph that has such intersecting edges. Point (1) is evident from the construction.

Proof of (2): For given graph G, let $\overline{p}(q) \equiv \max\{p \in \mathfrak{p} : w((p,q)) > 0\}$. Clearly, the path $\overline{P} \equiv ((\overline{p}(q),q) \to (\overline{p}(q+1),q+1))^{q \in \mathfrak{q}}$ is greater than all paths in G. It suffices to show that, as long as there are no intersecting edges having nonzero weight, $\overline{P}$ is a path in G. Suppose otherwise. Then for some $q \in \mathfrak{q}$, $(\overline{p}(q),q) \to (\overline{p}(q+1),q+1)$ has zero weight. By definition of $\overline{p}(\cdot)$, however, both nodes $(\overline{p}(q),q)$ and $(\overline{p}(q+1),q+1)$ have positive weight. This means that $(\overline{p}(q),q)$ has an out-edge which must be going to some node $(p,q+1)$ with $p < \overline{p}(q+1)$, whereas $(\overline{p}(q+1),q+1)$ has an in-edge which must be coming from some node (p,q) with $p < \overline{p}(q)$. Any such two edges must intersect, a contradiction.

Proof of (3): Suppose that a graph G has edges $((p^1,q) \to (p_1,q+1))$ and $((p^2,q) \to (p_2,q+1))$ of non-zero weight that intersect in the plane. That these intersect means that $p^1 > p^2 \geq p_2 > p_1$ (or symmetrically that $p^2 > p^1 \geq p_1 > p_2$). But then bidder 1 sometimes plays two incomparable bids: in one his unit-bids are (p^1,p_1) on units $(q,q+1)$ and in the other these unit-bids are (p^2,p_2). No incomparable bids can ever be played in a monotone strategy, however, so such a graph can not represent any monotone strategy. Now suppose that the graph has no such intersecting edges (and that the total weight of all in-edges equals that of out-edges for each node). We construct a monotone strategy that maps to this graph recursively, by assigning bids to types beginning with the highest types and proceeeding to lower types. Since no edges cross, there is always a maximal path $\overline{P}$ by (2). Let $q(\overline{P})$

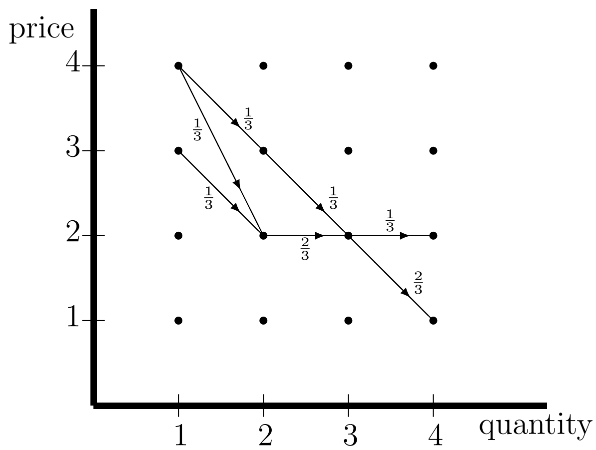

Figure 5: Planar graph corresponding to a monotone strategy

be the *minimal* weight of any of the edges connecting nodes in $\overline{P}$. Now we assign the bid corresponding to path $\overline{P}$ to $q(\overline{P})$-probability mass of the highest remaining types, reduce the weights of each edge in $\overline{P}$ by $q(\overline{P})$, and repeat the process with the new graph. ($\overline{P}$ will no longer be a path in this new graph since one of its edges now has zero weight. Also, the maximal path in this new graph was a path in the old graph and so is less than $\overline{P}$.) Once this process is complete and all types have been assigned a bid to play, higher types always play a bid that is greater than or equal to that played by lower types. Thus, we have constructed a monotone pure strategy that maps to this graph. For example, given the graph illustrated in Figure 5, we would construct the monotone pure strategy in which types $t_1 \in [2/3, 1]$ bid $(4, 3, 2, 2)$, types $t_1 \in [1/3, 2/3)$ bid $(4, 2, 2, 1)$, and types $t_1 \in [0, 1/3)$ bid $(3, 2, 2, 1)$.

Allocation-equivalence: Let $\lambda^*(\cdot;\cdot)$ be a mixed-strategy equilibrium. Arguments here apply to bidder 1 and, hence, to all bidders. Suppose that $\lambda_1^*(\cdot;\cdot)$ is non-monotone. Let $P_1'(\cdot), P_1(\cdot)$ be any pair of bids such that $P_1'(\cdot) \not\geq P_1(\cdot)$ and there exist types $t_1' > t_1$ such that $\lambda_1^*(P_1'(\cdot), t_1') > 0$ and $\lambda_1^*(P_1(\cdot), t_1) > 0$. Define as shorthand $P_1^{\vee}(\cdot) \equiv P_1'(\cdot) \vee P_1(\cdot)$ and $P_1^{\wedge}(\cdot) \equiv P_1'(\cdot) \wedge P_1(\cdot)$.

By the definition of the product order, disjoint sets $Q_*'(P_1'(\cdot), P_1(\cdot))$ and

$Q_*(P_1'(\cdot), P_1(\cdot)) \subset \mathfrak{q}$ (shorthand Q_*', Q_*) exist so that

$$P_1^{\vee}(\cdot) = \left(b'_{Q_*' \cup Q_*}, b_{-Q_*' \cup Q_*}\right), P_1'(\cdot) = \left(b'_{Q_*'}, b_{-Q_*'}\right),$$
$$P_1(\cdot) = \left(b'_{Q_*}, b_{-Q_*}\right), P_1^{\wedge}(\cdot) = b_{\mathfrak{q}}$$

where $b'_q \gneq b_q$ for all $q \in Q_*' \cup Q_*$.

Since bidders' expected payoffs are modular in own bid and have non-decreasing differences in own bid and own type, Milgrom and Shannon (1994)'s Monotonicity Theorem implies that each bidder i's set of best response bids is a lattice for all types t_i and that the correspondence mapping bidder types into best response bids is increasing in the strong set order.

Definition (Increasing in strong set order (ISSO)). Let X be a lattice, T a partially ordered set, and $\phi : T \to X$ a correspondence. ϕ is *increasing in the strong set order* if, for all $t' > t$, $x \in \phi(t)$ and $x' \in \phi(t')$ implies that $x \wedge x' \in \phi(t)$ and $x \vee x' \in \phi(t')$.

In our case, ISSO implies that the lower type t_1 must find both $\left(b'_{Q_*}, b_{-Q_*}\right)$ and $b_{\mathfrak{q}}$ to be best responses, while $\left(b'_{Q_*' \cup Q_*}, b_{-Q_*' \cup Q_*}\right)$ and $\left(b'_{Q_*'}, b_{-Q_*'}\right)$ must be best responses for the higher type t_1'. In particular, type t_1 is indifferent to raising its bid on units in Q_* from b_{Q_*} to b'_{Q_*} when its other bids are at level $\left(b_{Q_*'}, b_{-Q_* \cup Q_*'}\right)$, and type t_1' is indifferent to raising its bid on units in Q_* from b_{Q_*} to b'_{Q_*} when its other bids are at the higher level $\left(b'_{Q_*'}, b_{-Q_* \cup Q_*'}\right)$.

By modularity in own bid, we can express bidder 1's interim expected payoffs as a sum of functions that each depend only on an individual unit-bid, the bidder's type, and others' strategies:

$$\Pi_1\left((b_q)^{q \in \mathfrak{q}}, t_1; \lambda_{-1}(\cdot;\cdot)\right) = \sum_{q \in \mathfrak{q}} \Pi_1^q\left(b_q, t_1; \lambda_{-1}(\cdot;\cdot)\right)$$

For every $q \in Q_*$ consider the two unit-bid levels $b'_q > b_q$. Since the priority rationing rule is being used, changing one's unit-bid on q can not change one's likelihood of being awarded any other unit $q' \neq q$ (see discussion on page 25). Thus, if bidder 1 wins the q-th unit more often with unit-bid b'_q, then the incremental return to bidding b'_q vs. b_q is *strictly increasing* in own type. Since two *different* types t_1, t_1' are indifferent to raising these unit-bids (albeit with their other bids at different levels), it must be that bidder 1's likelihood of winning each unit $q \in Q_*$ does not change as he raises his bids on that unit from b_q to b'_q. Furthermore, variations in one's own bid can only affect others' allocations if they change one's own allocation. We may conclude that, for each $q \in Q_*$, if bidder 1 were to modify his strategy by sometimes submitting unit-bid b'_q when he originally had bid b_q on that unit or vice versa, then the allocation would remain the same with probability one given others' strategies *no matter what he bids on other units.*

Preserving expected payments & winning probabilities: Now we replace bidder 1's equilibrium strategy $\lambda_1^*(\cdot;\cdot)$ with a monotone strategy $\tilde{P}_1(\cdot;\cdot)$ so that (a) each bidder's probability of winning at least quantity q and expected payment from any bid (whether an equilibrium bid or deviation) is the same as given bidder 1's equilibrium strategy.

Constuction of the monotone pure strategy $\tilde{P}_1(\cdot;\cdot)$: First, we replace $\lambda_1^*(\cdot;\cdot)$ with a strategy $\tilde{\lambda}_1(\cdot;\cdot)$ that need not be monotone but which maps to a graph that has no intersecting nonzero-weight edges. Second, we use the procedure described in the proof of (3) above to construct a monotone pure strategy that maps to this graph. For this first part, then, suppose that the graph corresponding to strategy $\lambda_1^*(\cdot;\cdot)$ has a pair of intersecting edges. In particular, there must be consecutive quantities $q, q+1$ such that $E_1 = (p^1, q) \to (p_1, q+1)$ and $E_2 = (p^2, q) \to (p_2, q+1)$ have non-zero weight and $p^1 > p^2 \geq p_2 > p_1$ (or symmetrically $p^2 > p^1 \geq p_1 > p_2$). The weight $w(E_1)$ on edge E_1 is the probability that bidder 1 submits a bid $P_1(\cdot)$ with the property that $P_1(q) = p^1$ and $P_1(q+1) = p_1$, and similarly for $w(E_2)$.

Now we adjust the graph, creating new weights $w'(\cdot)$ as follows, with two cases. In the first case, $w(E_1) \geq w(E_2)$. Set $w'(E_1) = w(E_1) - w(E_2)$, $w(E_2) = 0$, $w'((p^1, q) \to (p_2, q+1)) = w'((p^1, q) \to (p_2, q+1)) + w(E_2)$, and $w'((p^2, q) \to (p_1, q+1)) = w'((p^2, q) \to (p_1, q+1)) + w(E_2)$. This eliminates the intersection by putting zero weight on E_2, maintains the same total in-edge weight into every node, and puts weight onto two other edges, one above and one below both E_1 and E_2. The second case, $w(E_2) \geq w(E_1)$, is symmetrical, with the difference that we put zero weight on E_1. Repeating this process for all such intersections yields a graph that puts the same weight on each node as the original graph and which is mapped to by some monotone strategy $\tilde{P}_1(\cdot; t_1)$. Figure 5 provides an example of the graph that results from "monotonizing" Figure 4, where $q = 1$, $E_1 = ((4,1) \to (2,2))$, $E_2 = ((3,1) \to (3,2))$, $p^1 = 4$, $p^2 = p_2 = 3$, and $p_1 = 2$.

The weight on node (p, q) is the probability that bidder 1 will submit a bid having unit-bid p for quantity q. Since these weights are preserved, the *distribution* of bidder 1's unit-bids are also preserved. Consequently, for every bidder $i \neq 1$ (say bidder 2), his probability of winning at least q units and his expected payment conditional on winning q units *after any bid* (whether an equilibrium bid or a deviation) are preserved, for all q. To see why, note that bidder 2's probability of winning at least q units with any given bid depends only on the probability that his q-th unit bid is greater than, equal to, or less than the $(S - q + 1)$-st unit-bid submitted by any other bidder. (This uses again the property of the priority rationing rule discussed on page 25.) Holding strategies fixed for all bidders $3, ..., n$, these probabilities do not change if the distribution of bidder 1's unit-bids stay the same. Similarly, bidder 2's expected payment with any given bid depends on the quantity that he wins, his own bid, and (potentially) the distribution of the S-th and $S + 1$-st highest unit-bids overall. Holding strategies fixed for all bidders $3, ..., n$, this distribution does not change if the distribution of bidder 1's unit-bids stays the same.

Preserving best responses: While the *probability* that bidder 2 wins at least q units with any given bid remains the same whether bidder 1 plays strategy $\lambda_1^*(\cdot;\cdot)$ or $\tilde{P}_1(\cdot;\cdot)$, we must account for the fact that bidder 2's expected value from winning may be different, since that value depends on which types of opponents he wins against. Formally, for every profile $\mathbf{t}_{-2}$ of others' types, define $\beta^*(\mathbf{t}_{-2}; (p, q))$ to be the probability that bidder 2 wins at least q units with q-unit-bid p when others receive type profile $\mathbf{t}_{-2}$ and follow strategies $\lambda_{-2}^*(\cdot;\cdot)$. Bidder 2's expected value for the q-unit conditional on having type t_2 and winning at least q units, then, may be expressed as

$$\frac{\int_{\mathbf{t}_{-2}} v_2(q; \mathbf{t})\beta^*(\mathbf{t}_{-2}; (p, q))d\mathbf{t}_{-2}}{\int_{\mathbf{t}_{-2}} \beta^*(\mathbf{t}_{-2}; (p, q))d\mathbf{t}_{-2}}$$

when bidders $1, 3, ..., n$ adopt strategies $\lambda_{-2}^*(\cdot;\cdot)$. Suppose now that bidder 1 adopts monotone pure strategy $\tilde{P}_1(\cdot)$ instead (holding fixed the strategies of $3, ..., n$). Let $\tilde{\beta}(\mathbf{t}_{-2}; (p, q))$ be the probability that bidder 2 wins at least q units with q-unit-bid p when others receive type profile $\mathbf{t}_{-2}$ and follow strategies $\left(\tilde{P}_1(\cdot), \lambda_{-1,2}^*(\cdot;\cdot)\right)$. Since the distribution of bidder 1's unit-bids remains the same,

$$\int_{\mathbf{t}_{-2}} \beta^*(\mathbf{t}_{-2}; (p, q))d\mathbf{t}_{-2} = \int_{\mathbf{t}_{-2}} \tilde{\beta}(\mathbf{t}_{-2}; (p, q))d\mathbf{t}_{-2} \text{ for all } \mathbf{t}_{-2}$$

By monotonicity of bidder 1's strategy, however, $\tilde{\beta}(\mathbf{t}_{-2}; (p, q))$ is *non-increasing* in t_1. Thus, for all t_2 and all (p, q)

$$\frac{\int_{\mathbf{t}_{-2}} v_2(q; \mathbf{t})\beta^*(\mathbf{t}_{-2}; (p,q))d\mathbf{t}_{-2}}{\int_{\mathbf{t}_{-2}} \beta^*(\mathbf{t}_{-2}; (p,q))d\mathbf{t}_{-2}} \geq \frac{\int_{\mathbf{t}_{-2}} v_2(q; \mathbf{t})\tilde{\beta}(\mathbf{t}_{-2}; (p,q))d\mathbf{t}_{-2}}{\int_{\mathbf{t}_{-2}} \tilde{\beta}(\mathbf{t}_{-2}; (p,q))d\mathbf{t}_{-2}} \tag{1}$$

Denominators are equal. The first numerator is weakly greater than the second by a basic fact from real analysis: Take any $f, g, h : [0,1] \to R$ such that f is non-decreasing, h non-increasing, and $\int g(x)dx = \int h(x)dx$. Then $\int f(x)g(x)dx \geq \int f(x)h(x)dx$. Thus, *for all fixed* $\mathbf{t}_{-2}$, and fixed (p, q), $v_2(q; \mathbf{t})$ is a non-decreasing function of t_1 and $\tilde{\beta}(\mathbf{t}_{-2}; (p, q))$ is a non-increasing function of t_1. Consequently,

$$\int_{t_1} v_2(q; \mathbf{t})\beta^*(\mathbf{t}_{-2}; (p,q))dt_1 \geq \int_{t_1} v_2(q; \mathbf{t})\tilde{\beta}(\mathbf{t}_{-2}; (p,q))dt_1$$

for all (p, q) and all $\mathbf{t}_{-2}$, implying the desired inequality of numerators in equation (1).

In short, bidder 2 is *no better off* submitting any bid after the modification of bidder 1's strategy than before. Lastly, by ex post allocation-equivalence, bidder 2 is *equally well off* now as before after submitting any equilibrium bid. (Allocation equivalence guarantees that the event in which he wins is the same.) All together, this proves that $(\tilde{\lambda}_1(\cdot;\cdot), \lambda^*_{-1}(\cdot;\cdot))$ is itself an equilibrium. Repeating this process for each bidder, finally, yields MPSE $\tilde{\lambda}(\cdot;\cdot)$. □

In the proof of Theorem 3, it is vital that each bidder's marginal value for each unit, $v_i(q; \mathbf{t})$, is well-defined. Indeed, Theorem 3 does not apply to general settings with payoff externalities. The following example shows how fundamentally non-monotone equilibria (i.e. not ex post allocation-equivalent to MPSE) can exist given externalities.

Example 6 (Externalities, uniform $(S + 1)$-st price). There are three bidders and four units in a uniform fifth-price auction. The grid of permissible non-null prices is $\{0, 1, 2, ..., 99, 100\}$. Bidder 1 receives a signal $t_1 \in \{L, H\}$ each with probability 1/2; the other bidders get no signal. (It is straightforward to modify this example so that each bidder receives a signal from $[0, 1]$ with values increasing in own signal: all relevant preferences are strict.) Bidders 1,2 have private values exceeding 100 for the first two units and no value for additional units regardless of t_1. Bidder 3 has preferences over allocation vectors (q_1, q_2, q_3) that depend on t_1.

- When $t_1 = L$, allocations $\{(2, 1, 1), (0, 1, 3)\}$ are most preferred (say value 1000), allocation $(2, 2, 0)$ next most preferred (say value 100) and all others are least preferred (say value 0).
- When $t_1 = H$, allocations $\{(1, 2, 1), (1, 0, 3)\}$ are most preferred (say value 1000), allocation $(2, 2, 0)$ next most preferred (say value 100) and all others are least preferred (say value 0).

Proposition 4. *The following strategies constitute a non-monotone equilibrium that is not outcome-equivalent to any MPSE:* $P_1(\cdot; L) = (90, 10, 0, 0)$ *and* $P_1(\cdot; H) = (50, 48, 0, 0)$*;* $P_2(\cdot) = (70, 30, 0, 0)$*; and* $P_3(\cdot) = (0, 0, 0, 0)$*.*

Proof. Bidders 1,2 each receive two units at price zero, so clearly they are adopting a best response. Next, $(0, 0, 0, 0)$ is a best response for bidder 3 for all t_1, so it must maximize bidder 3's expected payoff as well. First, consider the case in which $t_1 = L$. Note that bidder 1 has the highest unit-bid, followed by bidder 2's two unit-bids, followed by bidder 1's second unit-bid. Thus, by varying his bid, bidder 3 can induce allocation $(2, 2, 0)$, $(1, 2, 1)$, $(1, 1, 2)$,$(1, 0, 3)$, $(0, 0, 4)$ but not $(2, 1, 1)$, $(0, 1, 3)$ or any other. By the assumptions on values when $t_1 = L$, $(2, 2, 0)$ is the allocation that gives bidder 3 the greatest value among these (and he pays nothing) so bidding $(0, 0, 0, 0)$ is a best response. Next, consider the case in which $t_1 = H$. Now, bidder 3 can induce allocation $(2, 2, 0)$, $(2, 1, 1)$, $(1, 1, 2)$,$(0, 1, 3)$, $(0, 0, 4)$ but not $(1, 2, 1)$, $(1, 0, 3)$ or any other. Again, then, $(0, 0, 0, 0)$ is a best response and these strategies constitute an equilibrium.

The more interesting part of this example is that there is no outcome-equivalent monotone pure strategy equilibrium (in which bidders 1,2 always receive two units at price zero). The key observation is that, to induce bidder 3 to stay out of the bidding competition, the *ranking* of bidder 1,2's unit-bids was crucial. In particular, it is essential that $P_1(1; L) > P_2(1), P_2(2) > P_1(2; L)$ but also that $P_2(1) > P_1(1; H), P_1(2; H) > P_2(2)$. Of course, these inequalities can only be satisfied if $P_1(1; L) > P_1(1; H)$.

To see why these inequalities must be satisfied in any equilibrium in which bidders 1,2 each always win two units at price zero, consider bidder 3's incentives to deviate otherwise. If $P_2(1) > P_1(1; L) + 1$, then bidder 3 can get surplus of $1000 - 3(P_1(1; L) + 1) \geq 700$ by submitting bid $(P_1(1; L) + 1, P_1(1; L) + 1, P_1(1; L) + 1, 0)$.[19] The key here is that bidder 3 can now induce the allocation $(0, 1, 3)$ in the state $t_1 = L$ in which that allocation is highly

[19]The cases in which there is no bid between these two and ties must be considered –

valued. Since the surplus from this bid is at worst -300 when $t_1 = H$, its expected surplus is at least $(700 - 300)/2 > 0$. Thus, $(0, 0, 0, 0)$ is definitely not a best response for bidder 3. Similarly, if $P_1(1; L) > P_2(2) + 1$, then bidder 3 can induce the allocation $(2, 1, 1)$ (also highly valued when $t_1 = L$) by submitting bid $(P_2(2) + 1, 0, 0, 0)$, and this bid is preferred to $(0, 0, 0, 0)$. The argument for the $t_1 = H$ inequalities are entirely symmetrical, so I omit those details. □

By Theorem 2, of course, we know that there also is a MPSE.

Equilibria with positive probability of trade: In two-sided auctions, there are always "no-trade equilibria" in which sellers demand extremely high prices and buyers demand extremely low prices. Naturally, we are more interested in equilibria in which trade occurs. In the context of private-value auctions, Jackson and Swinkels (2001)'s ("JS") showed that an equilibrium with positive probability always exists by exploiting the fact that no-trade equilibria require bidders to play weakly dominated strategies: If some bidder(s) are assumed to sometimes exogenously submit bids in the relevant range, then other bidders now strictly prefer to bid seriously. (By a serious bid, I mean one that has positive probability of leading to trade given the distribution of others' bids.) JS then prove that, as the probability of such exogenous bidding vanishes, an equilibrium with trade persists. A key observation is that all equilibria in settings with exogenous bidding remain equilibria if we (artifically) disallow bidders from submitting weakly dominated bids. Thus, as the probability of exogenous bidding goes to zero, these equilibria converge to an equilibrium in which bidders do not adopt weakly dominated strategies.

In our context, this same approach applies, albeit some extra assumptions appear necessary. For instance, the following appears to be a sufficient (though by no means necessary) condition for JS's analysis to apply:

$$\Pr_{\mathbf{t}}\left(\max_i \underline{v}_i^b(t_i) > \min_i \overline{v}_i^s(t_i)\right) > 0 \tag{2}$$

when $P_2(1) = P_1(1; L)$ or $= P_1(1; L) + 1$ – make verification more tedious but do not pose a substantive problem, as long as the benefit that bidder 3 gets from outcomes $(2, 1, 1)$ and $(0, 1, 3)$ are sufficiently high. Therefore I leave out these cases.

where for each bidder i we define

$$\underline{v}_i^b(t_i) \equiv \inf_{\mathbf{t}_{-i};\mathbf{q}',\mathbf{q}:q_i'=e_i+1,q_i=e_i,q_j'\leq q_j\forall j\neq i} (V_i(\mathbf{q}',\mathbf{t}) - V_i(\mathbf{q},\mathbf{t}))$$
$$\overline{v}_i^s(t_i) \equiv \sup_{\mathbf{t}_{-i};\mathbf{q}',\mathbf{q}:q_i'=e_i,q_i=e_i-1,q_j'\leq q_j\forall j\neq i} (V_i(\mathbf{q}',\mathbf{t}) - V_i(\mathbf{q},\mathbf{t}))$$

$\underline{v}_i^b(t_i)$ is a lower bound on bidder i's marginal value for the first unit that she might buy, given her own type t_i. (The infimum is taken with respect to all other bidders' types and all relevant pairs of allocations: she may care who receives one less unit when she wins that first unit.) Similarly, $\overline{v}_i^b(t_i)$ is an upper bound on bidder i's marginal value for the first unit that she might sell. Condition (2) then requires that there *would* be positive probability of trade under the presumption that all bidders submitted bids equal to their lowest possible (if buyers) or highest possible (if sellers) marginal values for each unit.

6 Concluding Remarks

This paper introduces a new style of analysis for the discriminatory and uniform $S+\alpha$-th price multi-unit auctions ($\alpha \in [0,1]$).[20] This lattice-theoretical approach leverages the fact that, in these auctions, each bidder's expected surplus is modular (i.e. additively separable) in own bid. When bidders ar risk-neutral, this powerful property allows us to "reduce" the issue of monotone pure strategy equilibrium (MPSE) existence in these multi-unit auctions to that in single-object auctions: the sufficient condition for MPSE existence in single-object auctions, Athey (2001)'s "single-crossing condition" (SCC), is also sufficient in multi-unit auctions.[21]

Even more importantly, there is a sense in which *all* equilibria have monotone pure strategies in these auctions. Any given mixed-strategy equilbrium

[20]There are several sorts of auctions of multiple identical objects to which this paper's analysis does *not* apply. Most prominently among them are (i) combinatorial auctions in which bidders may demand (say) "q units or nothing", (ii) sequential auctions in which the sale of units is spread out over several rounds, and (iii) auctions in which some of the bidders choose their bids after observing others' bids. For instance, Back and Zender (2001) and McAdams (2002b) study auctions in which the auctioneer (a non-strategic bidder) decides how much to supply after receiving the bids.

[21]More precisely, Reny and Zamir (2002)'s weaker version of SCC, so-called "best-reply single-crossing condition" (BR-SCC) is sufficient, both for single-object and multi-unit auctions.

is ex post allocation- and interim expected payment-equivalent to some monotone pure strategy equilibrium. Thus, at least insofar as one is concerned with expected surplus / revenue analysis, there is no loss in restricting attention to monotone strategies.

Furthermore, our focus on the ordinal conditions necessary to generate monotone bidding behavior allows us to relax several common assumptions in the existing literature and study models with (among other things) multi-dimensional types, interdependent values, increasing marginal values, allocative externalities, and two-sided trading. Verifying SCC is more difficult in multi-unit auctions, however, than in single-object auctions such as the first-price auction. Indeed, though there may be some hope of extending our results to some settings with risk-aversion (see footnote 12), they fail to generalize if we allow for affilated rather than independent types, if we substitute risk-averse for risk-neutral bidders, or if we employ a different rationing rule.

Overall, the several counter-examples to monotonicity (Examples 1, 3, 4, 5, 6) may leave some readers with the negative impression that little structure can be established in multi-unit auctions outside of the very special case of independent types. While such examples tell a cautionary tale that equilibria in multi-unit auctions can possess unusual and subtle structures not seen in single-object auctions, I do not share this perspective. Especially in auctions with many "small" bidders, I expect that future research will establish that ordinal conditions (akin to SCC) are satisfied sufficient to apply the powerful style of ordinal analysis introduced here. Indeed, some initial progress has already been made on this front: Reny and Perry (2003) have established that such a condition is satisfied for uniform-price auctions when bidders have interdependent values, affiliated one-dimensional types, and single-unit demand.

Appendix: Proof of Theorem 1

It suffices to show that each bidder's ex post valuation and ex post payment are modular in own bid. In the following, consider bidder 1 only and fix the profile of others' bids $\mathbf{P}_{-1}(\cdot)$, the priority ranking ρ, and the state $\mathbf{t}$. The analysis focuses on properties of the realized allocation and payment when bidder 1 submits one of two bids $P_1^1(\cdot)$ or $P_1^2(\cdot)$ or their join $P_1^{1\vee 2}(\cdot) \equiv P_1^1(\cdot) \vee P_1^2(\cdot)$ or meet $P_1^{1\wedge 2}(\cdot) \equiv P_1^1(\cdot) \wedge P_1^2(\cdot)$.

Shorthand notation: $q_j^1 \equiv q_j\left(P_1^1(\cdot), \mathbf{P}_{-1}(\cdot); \rho\right)$ and so on for the other bids $P_1^2(\cdot)$, $P_1^{1\vee 2}(\cdot)$, and $P_1^{1\wedge 2}(\cdot)$. (Note that while bidder 1's bid varies, others' bids are held fixed.) Similarly, define shorthand $b_S^1 \equiv b_S\left(P_1^1(\cdot), \mathbf{P}_{-1}(\cdot)\right)$, $b_{S+1}^1 \equiv \left(P_1^1(\cdot), \mathbf{P}_{-1}(\cdot)\right)$ and so on, where $b_S(\mathbf{P}(\cdot))$ and $b_{S+1}(\mathbf{P}(\cdot))$ are the Sth and $S+1$st highest unit-bids given the profile of bids $\mathbf{P}(\cdot)$.

Characterizing the allocation: Define bidder 1's *rationing function* to be

$$R_1^\rho(p) \equiv S - \sum_{\rho(j)=1}^{\rho(j)=\rho(1)-1} \max D_j(p) - \sum_{\rho(j)=\rho(1)+1}^{\rho(j)=n} \min D_j(p).$$

$R_1^\rho(p)$ is the amount that would be left for bidder 1 if all ahead of him in the ranking ρ were given their maximum demand at price p and all behind him were given their minimal demand at that price. By design of the assumed rationing rule,[22]

$$\begin{aligned} q_1 &= \min D_1(b_S) \text{ if } R_1^\rho(b_S) \le \min D_1(b_S) \\ &= R_1^\rho(b_S) \text{ if } R_1^\rho(b_S) \in [\min D_1(b_S), \max D_1(b_S)] \\ &= \max D_1(b_S) \text{ if } R_1^\rho(b_S) \ge \max D_1(b_S). \end{aligned}$$

or, *equivalently*,

$$\begin{aligned} q_1 &= \min D_1(b_{S+1}) \text{ if } R_1^\rho(b_{S+1}) \le \min D_1(b_{S+1}) \\ &= R_1^\rho(b_{S+1}) \text{ if } R_1^\rho(b_{S+1}) \in [\min D_1(b_{S+1}), \max D_1(b_{S+1})] \\ &= \max D_1(b_{S+1}) \text{ if } R_1^\rho(b_{S+1}) \ge \max D_1(b_{S+1}). \end{aligned}$$

Both approaches, based on b_S and on b_{S+1}, lead to the same rationing outcome because rationing only occurs when $b_S = b_{S+1}$: $b_{S+1}(\mathbf{P}(\cdot)) \neq b_S(\mathbf{P}(\cdot))$ implies that there is a unique market clearing allocation, and both approaches

[22] $R_1^\rho(b_S) \le \min D_1(b_S)$ ($R_1^\rho(b_S) \ge \max D_1(b_S)$) if and only if the rationing process described on page 11 ends before 1 is reached (after 1 is fully served) in the rationing queue. Similarly, $R_1^\rho(b_S) \in (\min D_1(b_S), \max D_1(b_S))$ if and only if 1 can only be partially served after those ahead of him have been fully served.

lead to that allocation. More explicitly,

$$q_1(\mathbf{P}(\cdot);\rho) = \max\left\{\min D_1(b_S(\mathbf{P}(\cdot))), \min\left\{R_1^\rho(b_S(\mathbf{P}(\cdot))), \max D_1(b_S(\mathbf{P}(\cdot)))\right\}\right\} \tag{3}$$

$$= \max\left\{\min D_1(b_{S+1}(\mathbf{P}(\cdot))), \min\left\{R_1^\rho(b_{S+1}\mathbf{P}(\cdot)), \max D_1(b_{S+1}(\mathbf{P}(\cdot)))\right\}\right\}. \tag{4}$$

Modularity of ex post valuation: A bidder's valuation for what he wins does not depend on the auction's payment rule. Consequently, I may cite McAdams (2003b)'s proof of existence in the uniform S-th and $S+1$-st price auctions for this part of our proof. The key fact (which directly implies modularity of ex post valuation in own bid since values take the form $V_i(\mathbf{q}, \mathbf{t})$) is that

$$\left\{\mathbf{q}^1, \mathbf{q}^2\right\} = \left\{\mathbf{q}^{1\vee 2}, \mathbf{q}^{1\wedge 2}\right\}$$

(The priority rationing rule is crucial to this result; neither risk-neutrality nor independence is needed.)

Modularity of ex post payment in discriminatory auction: By definition of the $\vee, \wedge$ operations,

$$\left\{\max P_1^1(q), \max P_1^2(q)\right\} = \left\{\max P_1^{1\vee 2}(q), \max P_1^{1\wedge 2}(q)\right\} \text{ for all } q \tag{5}$$

In the discriminatory auction, bidder 1's payment may be broken down as the sum of marginal payments on each unit:

$$\begin{aligned}
Z_i^D(\mathbf{P}(\cdot)) &= \sum_{q\in\mathsf{q}} Z_i^{q,D}(\mathbf{P}(\cdot)) \text{ where} \\
Z_i^{q,D}(\mathbf{P}(\cdot)) &= 0 \text{ if } q \le \min\{e_1, q_1(\mathbf{P}(\cdot))\} \\
&= \max P_1(q) \text{ if } e_1 < q \le q_1(\mathbf{P}(\cdot)) \\
&= -\max P_1(q) \text{ if } q_1(\mathbf{P}(\cdot)) < q \le e_1
\end{aligned}$$

It suffices to show that each of these marginal payment functions is modular in $P_1(\cdot)$. Equation (5) directly implies that $Z_i^{q,D}(\mathbf{P}(\cdot))$ is modular for all q except (in the case when $q_1^2 > q_1^1$) for those q such that $q_1^1 < q \le q_1^2$: either all terms are zero or

$$\begin{aligned}&Z_i^{q,D}(P_1^1(\cdot),\mathbf{P}_{-1}(\cdot))+Z_i^{q,D}(P_1^1(\cdot),\mathbf{P}_{-1}(\cdot))=\max P_1^1(q)+\max P_1^2(q)\\&=\max P_1^{1\wedge 2}(q)+\max P_1^{1\vee 2}(q)=Z_i^{q,D}(P_1^{1\wedge 2}(\cdot),\mathbf{P}_{-1}(\cdot))+Z_i^{q,D}(P_1^{1\vee 2}(\cdot),\mathbf{P}_{-1}(\cdot))\end{aligned}$$

Since bidding $P_1^2(\cdot)$ always bidder 1 to win strictly more quantity, furthermore, it must be that $\max P_1^2(q) > \max P_1^1(q)$ for all $q_1^1 < q \leq q_1^2$. Thus, for such q,

$$\begin{aligned}&Z_i^{q,D}(P_1^1(\cdot),\mathbf{P}_{-1}(\cdot))+Z_i^{q,D}(P_1^1(\cdot),\mathbf{P}_{-1}(\cdot))=0+\max P_1^2(q)\\&\quad=0+\max P_1^{1\vee 2}(q)=Z_i^{q,D}(P_1^{1\wedge 2}(\cdot),\mathbf{P}_{-1}(\cdot))+Z_i^{q,D}(P_1^{1\vee 2}(\cdot),\mathbf{P}_{-1}(\cdot))\end{aligned}$$

Modularity of ex post payment in uniform $S+\alpha$-th price auction: Again by definition of the $\vee,\wedge$ operations,

$$\left\{b_S^1, b_S^2\right\} = \left\{b_S^{1\vee 2}, b_S^{1\wedge 2}\right\} \text{ and } \left\{b_{S+1}^1, b_{S+1}^2\right\} = \left\{b_{S+1}^{1\vee 2}, b_{S+1}^{1\wedge 2}\right\}$$

Since we know already that $\{q_1^1, q_1^2\} = \{q_1^{1\vee 2}, q_1^{1\wedge 2}\}$ and payment in the $S+\alpha$-th price auction equals $q_1(\mathbf{P}(\cdot))\left(\alpha b_S(\mathbf{P}(\cdot)) + (1-\alpha)b_{S+1}(\mathbf{P}(\cdot))\right)$, to establish modularity it will suffice to show that[23]

$$\left\{\left(b_S^1, b_{S+1}^1, q_1^1\right), \left(b_S^2, b_{S+1}^2, q_1^2\right)\right\} = \left\{\left(b_S^{1\wedge 2}, b_{S+1}^{1\wedge 2}, q_1^{1\wedge 2}\right), \left(b_S^{1\vee 2}, b_{S+1}^{1\vee 2}, q_1^{1\vee 2}\right)\right\}.$$

Without loss, suppose that $b_S^1 \leq b_S^2$ and that $b_S^1 = b_S^2$ implies $b_{S+1}^1 \leq b_{S+1}^2$. If $b_S^1 = b_S^2$ and $b_{S+1}^1 = b_{S+1}^2$, then we are obviously done. If $b_S^1 \leq b_S^2$ and $b_{S+1}^1 \leq b_{S+1}^2$, then we are also done: $q_1^1 \leq q_1^2 = q_1^{1\vee 2}$ and $(b_S^2, b_{S+1}^2) = (b_S^{1\vee 2}, b_{S+1}^{1\vee 2})$.

As the final (and most difficult) case, suppose that $b_S^2 > b_S^1 \geq b_{S+1}^1 > b_{S+1}^2$. It will suffice to show that $q_1^1 = q_1^2$ in this case. Note that, when bidder 1 submits bid $P_1^1(\cdot)$, some bidder submits unit-bids equal to b_S^1, b_{S+1}^1 but that no bidder submits such unit-bids when he submits bid $P_1^2(\cdot)$. (In the latter case, b_S^2, b_{S+1}^2 are *consecutive* unit-bids across all those submitted by any bidder for any quantity, and others' bids are fixed.) Thus, we may conclude that these are bidder 1's unit-bids when he bids $P_1^1(\cdot)$ and, in particular, that $P_1^1(q_1^1) = b_S^1$ and $P_1^1(q_1^1+1) = b_{S+1}^1$.

[23] The reason why modularity fails in Example 1 is that the analogue to this condition fails. The bids that set the price need not "match up" with the quantities that bidder 1 wins upon submitting bids $P_1^1(\cdot), P_1^2(\cdot)$ or from bids $P_1^{1\wedge 2}(\cdot), P_1^{1\vee 2}(\cdot)$. The proof here relies crucially on the fact that the bids setting price (highest-selling bid and lowest-buying bid) are also the bids that, in a sense, determine the allocation.

Now, consider unit-bid b_S^2. Since this is the S-th highest unit-bid when bidder 1 submits bid $P_1^2(\cdot)$ it must be that $P_1^2(q_1^1) \geq b_S^2$. Otherwise any unit-bid at level b_S^2 would be, at lowest, the $S-1$-st highest. Loosely speaking, we need bidder 1 to bid high enough on all of the units $1, ..., q_1^1$ that previously won in order to make it possible that b_S^2 might be the S-th highest price. Conversely, consider unit-bid B_{S+1}^2. Since this is the $S+1$-st highest unit-bid when bidder 1 submits bid $P_1^2(\cdot)$ it must be that $P_1^2(q_1^1+1) \leq b_{S+1}^2$. Otherwise any unit-bid at level b_{S+1}^2 would be, at highest, the $S+2$-st highest. Consequently, $q_1^1 = q_1^2$ and we are done. $\square$

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Appendix: For Referees Only

Extending proof of Theorem 3 for multi-dimensional types (page 28)

Single-dimensionality of types is used vitally in two spots in the proof.

(A) Construction of monotone pure strategy that maps to a given planar graph having no intersecting edges (page 33). "We construct a monotone strategy that maps to this graph recursively, by assigning bids to types beginning with the highest types and proceeeding to lower types." This recursive process is not well-defined when types are multi-dimensional since the order on types is not complete. For this purpose, endow each type-space $T_i = [0,1]^h$ with the lexicographic order: $(t_i^{'1}, ..., t_i^{'n}) \geq (t_i^1, ..., t_i^n)$ iff $t_i^{'1} > t_i^1$ or $t_i^{'1} = t_i^1, t_i^{'2} > t_i^2$, or etc.. The process of constructing a monotone strategy from a graph therefore "assigns" highest bids to types on $(h-1)$-dimensional hyperplanes with highest first coordinate, and so on for lower types.

(B) Bidder 2 gets weakly lower payoff from any bid after bidder 1 adopts the new, monotone strategy (page 35), i.e.

$$\frac{\int_{\mathbf{t}_{-2}} v_2(q;\mathbf{t})\beta^*(\mathbf{t}_{-2};(p,q))d\mathbf{t}_{-2}}{\int_{\mathbf{t}_{-2}} \beta^*(\mathbf{t}_{-2};(p,q))d\mathbf{t}_{-2}} \geq \frac{\int_{\mathbf{t}_{-2}} v_2(q;\mathbf{t})\tilde{\beta}(\mathbf{t}_{-2};(p,q))d\mathbf{t}_{-2}}{\int_{\mathbf{t}_{-2}} \tilde{\beta}(\mathbf{t}_{-2};(p,q))d\mathbf{t}_{-2}}$$

since

$$\int_{t_1} v_2(q;\mathbf{t})\beta^*(\mathbf{t}_{-2};(p,q))dt_1 \geq \int_{t_1} v_2(q;\mathbf{t})\tilde{\beta}(\mathbf{t}_{-2};(p,q))dt_1 \text{ for all } t_2, \mathbf{t}_{-1,2} \quad (6)$$

$$\int_{\mathbf{t}_{-2}} \beta^*(\mathbf{t}_{-2};(p,q))d\mathbf{t}_{-2} = \int_{\mathbf{t}_{-2}} \tilde{\beta}(\mathbf{t}_{-2};(p,q))d\mathbf{t}_{-2} \quad (7)$$

While equation (7) continues to hold for all strategies $\tilde{P}_1(\cdot;\cdot)$ that map to such a graph, equation (6) relies on one-dimensionality in an apparently crucial way: Take any $f, g, h : [0,1] \to R$ such that f is non-decreasing, h non-increasing, and $\int g(x)dx = \int h(x)dx$. Then $\int f(x)g(x)dx \geq \int f(x)h(x)dx$. Here, for fixed $t_{-1} = (t_2, \mathbf{t}_{-1,2})$ and fixed (p,q), $v_2(q_2; t_1, t_{-1})$ corresponds to the non-decreasing one-dimensional function $f(x)$, $\tilde{\beta}(t_1, \mathbf{t}_{-1,2}; (p,q))$ corresponds to the non-increasing one-dimensional function $h(x)$, and $\beta^*(t_1, \mathbf{t}_{-1,2}; (p,q))$ to the function $g(x)$.

Once t_1 is multi-dimensional, of course, these functions are too. One can get around this problem by recasting bidder 1's type-space as a one-dimensional space consisting of (h_1-1)-dimensional hyperplanes, $T_1 = [0,1] \times [0,1]^{h_1-1}$. More formally, re-define the primitive "bidder 1 type" as an equivalence class (hyperplane) of original types: $T_1(x) = \{(t_1^1, t_1^{-1} \in T_1 : t_1^1 = x\}$. Under this re-definition, bidder 1's set of types is $T_1 = \{T_1(x) : x \in [0,1]\}$. A strategy, similarly, maps each such hyperplane of original types to the (probability density-weighted) mixture of bids played by types in that hyperplane. While monotone pure strategies with respect to the original definition of types may correspond to mixed strategies under this new formulation, the same planar graph corresponds to both such strategies. (The planar graph only encodes *overall* probabilities that various sorts of bids are played, and the overall probability that any given bid is played does not depend on whether we use the original or new formulation for types.) Now we can construct a pure strategy $\tilde{P}_1(\cdot;\cdot)$ corresponding to this planar graph that is monotone with respect to the lexicographic total order on original types described above in (A). This yields a monotone strategy with respect to the *new* formulation of types as hyperplanes. (Higher bids are assigned to higher equivalence classes of types, i.e. to types on hyperplanes having higher first coordinate.)

To complete the proof, we need to show that (for fixed $t_{-1} = (t_2, \mathbf{t}_{-1,2})$ and fixed (p,q))

$$v_2(q_2; T_1(x), t_{-1}) \equiv E_{t_1^{-1}}[v_2(q_2; (x, t_1^{-1}, t_{-1}))]dt_1^{-1}$$

is non-decreasing in x and that

$$\tilde{\beta}(T_1(x), \mathbf{t}_{-1,2}; (p,q)) \equiv E_{t_1^{-1}}[\tilde{\beta}(x, t_1^{-1}, \mathbf{t}_{-1,2})]dt_1^{-1}$$

is non-increasing in x. The first observation follows from the independence assumption. (Here is where we use the stronger independence condition that $\{t_i^j\}_{j=1,\ldots,h}^{i=1,\ldots,n}$ are independent.) The second condition, finally, follows from the fact that types on higher hyperplanes play higher actions in our reconstructed strategy $\tilde{P}_1(\cdot;\cdot)$.