# Heterogeneous Information about the Term Structure of Interest Rates, Least-Squares Learning and Optimal Interest Rate Rules for Inflation Forecast Targeting


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First Draft 12 September 2003


## Abstract


In this paper we incorporate the term structure of interest rates in a standard inflation forecast targeting framework. Learning about the transmission process of monetary policy is introduced by having heterogeneous agents - i.e. the central bank and private agents - who have different information sets about the future sequence of short-term interest rates. We analyse inflation forecast targeting in two environments. One in which the central bank has perfect knowledge, in the sense that it understands and observes the process by which private sector interest rate expectations are generated, and one in which the central bank has imperfect knowledge and has to learn the private sector forecasting rule for short-term interest rates. In the case of imperfect knowledge, the central bank has to learn about private sector interest rate expectations, as the latter affect the impact of monetary policy through the expectations theory of the term structure of interest rates. Here following Evans and Honkapohja (2001), the learning scheme we investigate is that of least-squares learning (recursive OLS) using the Kalman filter. We find that optimal monetary policy under learning is a policy that separates estimation and control. Therefore, this model suggests that the practical relevance of the breakdown of the separation principle and the need for experimentation in policy may be limited.



Keywords**:** Learning, Rational Expectations, Separation Principle, Kalman Filter, Term Structure of Interest Rates

JEL Codes**:** C53, E43, E52, F33

## INTRODUCTION [1]

As pointed out by Bullard (1991), in the three decades since the publication of the seminal work on rational expectations (RE) in the early 1960s, a steely paradigm was forged in the economics profession regarding acceptable modelling procedures. Simply stated, the paradigm was that economic actors do not persist in making foolish mistakes in forecasting over time.

Since the late 1980s researchers have challenged this paradigm by examining the idea that *how* systematic forecast errors are eliminated may have important implications for macroeconomic policy. Researchers who have focused on this question have been studying what is called 'learning', because any method of expectations formation is known as a learning mechanism. Thus, since the late 1980s a learning literature, or learning paradigm, developed. An excellent introduction to – and survey of – this paradigm is presented in Evans and Honkapohja (2001).

A different strand of literature in the economics profession has been dealing with optimal control or dynamic optimisation.

In general there are few papers in the literature that combine the themes of *learning* and (optimal) *control.* An exception is recent and important work by Wieland (2000a,b). Wieland (2000a) analyses the situation where a central bank has limited information concerning the transmission channel of monetary policy. Then, the CB is faced with the difficult task of simultaneously *controlling* the policy target and estimating (*learning*) the impact of policy actions. Thus, the so-called separation principle does not hold, and a trade-off between estimation and control arises because policy actions influence estimation (learning) and provide information that may improve future performance. Wieland analyses this trade-off in a simple model with parameter uncertainty and conducts dynamic simulations of the central bank's decision problem.

In this paper we incorporate the term structure of interest rates in a standard inflation forecast targeting framework. Learning about the transmission process of monetary policy is introduced by having heterogeneous agents - i.e. the central bank and private agents - who have different information sets about the future sequence of short-term interest rates. We analyse inflation forecast targeting in two environments. One in which the central bank has perfect knowledge, in the sense that it understands and observes the process by which private sector interest rate expectations are generated, and one in which the central bank has imperfect knowledge and has to learn the private sector forecasting rule for short-term interest rates. In the case of imperfect knowledge, the central bank has to learn about private sector interest rate expectations, as the latter affect the impact of monetary policy through the expectations theory of the term structure of interest rates. Here following Evans and Honkapohja (2001), the learning scheme we investigate is that of least-squares learning (recursive OLS) using the Kalman filter.

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We find that optimal monetary policy under learning is a policy that separates estimation and control. Therefore, this model suggests that the practical relevance of the breakdown of the separation principle and the need for experimentation in policy may be limited.

The remainder of this paper is organized as follows. Section 2 discusses the basic inflation targeting framework and the term structure of interest rates. In Section 3 we solve for the optimal monetary policy rule under perfect knowledge. Imperfect knowledge and the Kalman filter are introduced in Section 4. We conclude in Section 5. The appendices contain the derivation of results for convergence and the optimal policy rules under perfect knowledge and learning.

## 2 THE ENVIRONMENT

Monetary policy is conducted by a central bank that controls a short-term nominal interest rate $i_t$, and that has an exogenously given inflation target, $\pi^*$. The authorities aim to minimize deviations of inflation from its assigned target. Consequently, the central bank will choose a sequence of current and future short-term nominal interest rates to meet the objective

$$\underset{\{i_\tau\}_{\tau=t}^{\infty}}{Min} E_t \sum_{\tau=t}^{\infty} \delta^{\tau-t}[\frac{1}{2}(\pi_\tau - \pi^*)^2] \qquad \textbf{(2.1)}$$

Here $\pi$ is the inflation (rate) in year $t$, $\pi^*$ is the central bank's inflation target, while the parameter δ (which fulfils 0 < δ < 1) denotes the discount factor (i.e. a measure of the policy horizon). The expectations operator $E$ refers to the policymaker's expectations. This expectation is conditional on the central bank's information set in period t.

As in Rudebusch and Svensson (2002), inflation and output are linked by the following short-term Phillips-curve relationship:[2]

$$\pi_{t+1} = \pi_t + \alpha_1 z_t - \eta_{t+1} \qquad \textbf{(2.2)}$$

The variable $z$ represents the (log of the) output gap in period t where potential output has been normalized to zero, finally $\eta$ is a i.i.d productivity (supply) shock

The output gap is determined by the following dynamic relationship:

$$z_{t+1} = \beta_1 z_t - \beta_2 R_t + d_{t+1} \qquad \textbf{(2.3)}$$

where $R$ is the long-term real interest rate and $d$ is an i.i.d. demand shock. Again, this relationship is similar to Rudebusch and Svensson (2002). The differences are that here the output gap depends on the long-term real interest rate rather than the short-term real interest rate, and that they consider an additional lagged $z$ term.

[2] Rudebusch and Svensson (2002) consider additional lags of inflation.

We assume that the short real rate ($r_t$) and the long real rate ($R_t$) are related by the following version of the Pure Expectations Hypothesis (PEH)

$$r_t = R_t - D(\hat{E}_t R_{t+1} - R_t) \qquad \textbf{(2.4')}$$

The expectations operator $\hat{E}$ refers to the private sector's expectations. Here $r_t$ represents the real yield to maturity on a one-period bond which is traded on the interbank money market. The expectations operator $\hat{E}$ refers to private agents' (possibly subjective) expectations. The LHS denotes the (one-period) real holding period return on a long-term bond. The latter's real yield to maturity ($R_t$) is the long-term real interest rate. The parameter D is defined such that D + 1 is equal to Maccaulay's duration.[3]

For our purposes it turns out to be convenient to rewrite this equation to express the current long real rate as a convex combination of the current short real rate and the expected long real rate in the next period:

$$R_t = (1-k)r_t + k\hat{E}_t R_{t+1} \; ; \; k \equiv \frac{D}{1+D} \qquad \textbf{(2.4)}$$

Note that the long and short real interest rates will be equal if the parameter k is equal to zero. In that case the duration of the long-term bond will be equal to one and there is no distinction between short and long term interest rates. Note that this equation can be rewritten as

$R_t = (1-k)\sum_{\tau=t}^{\infty} k^{\tau-t} \hat{E}_t r_\tau$ . Or alternatively, as $R_t = (1-k)\, r_t + (1-k)\sum_{\tau=t+1}^{\infty} k^{\tau-t} \hat{E}_t r_\tau$. It follows that

$$\hat{E}_t R_{t+1} = \left(\frac{1-k}{k}\right)\sum_{\tau=t+1}^{\infty} k^{\tau-t} \hat{E}_t r_\tau \qquad \textbf{(2.5)}$$

Thus, in equation **(2.4)** the long term real interest rate is a weighted average of the current ex ante real short rate and the expected future sequence of future short real rates over the $t+1$ - infinity horizon.

The current short-term real interest rate will be equal to:

$$r_t = i_t - E_t \pi_{t+1} \qquad \textbf{(2.6)}$$

Here $i_t$ is the instrument of the central bank (i.e. the nominal interest rate on the inter-bank money market) and $E_t \pi_{t+1}$ represents the expected rate of inflation in period t+1 conditional on the information set in period t.

[3] For more details see Eijffinger, Schaling and Verhagen (2000), hereafter ESV.

## 3 IMPLEMENTING INFLATION TARGETING UNDER PERFECT KNOWLEDGE

To get some straightforward results, we assume that the central bank understands and observes the process by which private sector inflation expectations are generated. This is the benchmark case of *perfect knowledge*. We model least-squares learning by the central bank in section 4.

*3.1 Timing of Events*

The timing is that <u>first</u> the private sector (PS) sets its expectation about the sequence of future short real rates – that is, it chooses $\hat{E}_t R_{t+1}$ **-** and the central bank (CB) then chooses $r_t$ (through the choice of $i_t$, given $\hat{E}_t \pi_{t+1} = E_t \pi_{t+1}$). The policymaker's best response, $r_t\left(\pi_t, z_t, E_t\left[\hat{E}_t R_{t+1}\right]\right)$ maximizes the monetary authority's payoff given $E_t\left[\hat{E}_t R_{t+1}\right]$. The model is completed by imposing rational expectations on the policymaker, namely, $E_t\left[\hat{E}_t R_{t+1}\right] = \hat{E}_t R_{t+1}$. More specific, in the case of perfect knowledge the CB understands and observes the process by which PS interest rate expectations are generated.[4] How those expectations are generated remains to be specified. The equilibrium ex ante real interest rate is given by the solution of this equation, and is denoted by $r_t^*\left(\pi_t, z_t, E_t\left[\hat{E}_t R_{t+1}\right]\right)$. To summarize, the above case of discretion can be represented as follows. Then, also at time $t$ the CB sets the interest rate based on strict inflation targeting (SIT) and on its <u>correct observation</u> of PS expectations. Figure 3.1 illustrates.

**Figure 3.1 Discretion: Timing of Events**

| Time $t$ | |
|---|---|
| **Stage 1:**<br><br>• PS forecasts sequence of future interest rates, i.e. sets $\hat{E}_t R_{t+1}$. | **Stage 2:**<br><br>• CB decides on monetary policy according to strict IFT , i.e. sets $r_t^*\left(\pi_t, z_t, \hat{E}_t R_{t+1}\right)$. |

*3.2 Optimality*

In Appendix B.1 we show that the first-order condition of this optimization problem is[5]

$$E_t \pi_{t+2} = \pi^* \qquad \textbf{(3.1)}$$

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[4] So, the central bank knows how much policy 'is in the pipeline' according to financial markets**.**

[5] See Bullard and Schaling (2001) and Schaling (2004) for examples of the method of solving for the optimal policy.

Substituting from the constraints it can easily be established that can be easily established that the closed form solution for the ex ante real interest rate is

$$r_t = \frac{1}{\alpha_1\beta_2(1-k)}\left(\pi_t - \pi^*\right) + \frac{(1+\beta_1)}{\beta_2(1-k)} z_t - \frac{k}{(1-k)} E_t \hat{R}_{t+1} \qquad \textbf{(3.2)}$$

The difference with standard Taylor-type monetary policy rules is that now the CB responds to *three* state variables: inflation, output and the private sector forecast of the long real rate. If $k \to 0$ there is no term structure, and the policy rule collapses to Svensson's (1997) version of the Taylor rule (hereafter the Svensson-Taylor rule).

An interesting characteristic of this solution is that the central bank's optimal level of short-rates is inversely related to PS expectations about its future short rates (because of the minus sign on the term $E_t R_{t+1}$). For example, if the PS expects rates to go up in the future, as a consequence (ceteris paribus; given its inflation target) rates can be lower today (and vice versa). The latter (reverse case), i.e. the PS expects rates to go down, and as a consequence the CB raises (or talks about raising them) reminds us of the old joke about the Bundesbank: 'The BuBA is just like cream, the more you stir it, the thicker it gets'.[6] The reason for this is that the central bank's inflation forecast - given other state variables such as the present inflation rate and the present output gap – depends on the present level of the real <u>long term</u> interest rate, $R_t$. So, an optimal forecast implies an optimal level of this variable. Since the optimal long-term rate (that is consistent with strict IT) is a weighted average of the present ex ante optimal real rate and $\hat{E}_t R_{t+1}$, i.e. $R_t^* = (1-k) r_t^* + k \hat{E}_t R_{t+1}$ , the higher $\hat{E}_t R_{t+1}$ the lower the optimal ex ante real rate can be. Similarly lower expected interest rates necessitate a tighter policy stance today to compensate.

An important limiting case of equation **(3.2)** is when $k \to 0$ and there is no difference between long-term and short-term interest rates. Then the policy rule collapses to

$$r_t = \frac{1}{\alpha_1\beta_2}\left(\pi_t - \pi^*\right) + \frac{(1+\beta_1)}{\beta_2} z_t \qquad \textbf{(3.3)}$$

which – as in Svensson (1997, p 1119) – is essentially a version of the simple policy rule popularized by Taylor (1993).[7] This result will be referred to as the 'Svensson-Taylor' rule.

*3.3 The Rational Expectations Solution*

It remains to present the rational expectations solution. This is the case where private sector interest rate expectations are formed rationally; i.e. private agents think that the CB will implement inflation forecast targeting in each and every period. That is, they think that the CB will set policy according to $E_t \pi_{t+2} = \pi^* \; \forall t$ .

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[6] In addition, the Deutsche Bundesbank always considered the long-term interest rate as a reflection of the credibility of its monetary policy.

[7] Taylor rules are often written in terms of nominal interest rates, but given the definition of $r_t$ the rules in equations **(3.2)** and **(3.3)** can easily be interpreted in these terms.

As a first step, solve for the long-term real interest rate. Substituting **(3.2)** into equation **(2.4)** we obtain

$$R_t = \frac{1}{\alpha_1 \beta_2}\left(\pi_t - \pi^*\right) + \frac{\left(1+\beta_1\right)}{\beta_2} z_t \qquad \textbf{(3.4)}$$

The solution for the output gap and inflation under strict IT is

$$z_{t+1} = -z_t - \frac{1}{\alpha_1}\left(\pi_t - \pi^*\right) + d_{t+1} \qquad \textbf{(3.5)}$$

Leading equation **(3.4)** by one period and substituting from equations **(2.2)** and **(3.5)** we get

$$R_{t+1} = -\frac{\beta_1}{\alpha_1 \beta_2}\left(\pi_t - \pi^*\right) - \frac{\beta_1}{\beta_2} z_t + u_{t+1} \qquad \textbf{(3.7)}$$

where $u_{t+1} = \frac{\left(1+\beta_1\right)}{\beta_2} d_{t+1} - \frac{1}{\alpha_1 \beta_2}\eta_{t+1}$ is a composite white noise shock, i.e. a linear combination of the demand and supply shocks (both white noise). Rearranging and taking PS expectations at time $t$ gives

$$\hat{E}_t R_{t+1} = -\frac{\beta_1}{\alpha_1 \beta_2}\left(\pi_t - \pi^*\right) - \frac{\beta_1}{\beta_2} z_t \qquad \textbf{(3.8)}$$

This is now the solution for the rational expectation of next period's long real rate from the perspective of the private sector. Note that the benchmark case of the model, that of perfect knowledge, relates to the situation where both the <u>policymaker</u>, and <u>the private sector</u> have rational expectations. More specific, in the case of perfect knowledge the CB understands and observes the process by which PS interest rate expectations are generated. In turn those expectations are consistent with the solution for the long-term real interest rate implied by strict IT.

Plugging expression **(3.8)** into the CB's optimal policy rule **(3.2)** yields

$$r_t = \frac{\left(1+k\beta_1\right)}{\alpha_1 \beta_2 (1-k)}\left(\pi_t - \pi^*\right) + \frac{\left(1+k\beta_1\right)+\beta_1}{\beta_2(1-k)} z_t \qquad \textbf{(3.9)}$$

This equation is the equivalent of equation (3.6) of Eijffinger, Schaling and Verhagen (2000) (ESV).[8] However, the equations are not strictly comparable because the optimal real interest rate in ESV is in ex post terms, whereas in **(3.9)** it is in ex ante terms.

*3.5 Comparing the Rules*

[8] However, the equations are not completely identical because the optimal real interest rate in ESV is in ex post terms, whereas in **(3.9)** it is in ex ante terms.

It is interesting to compare the optimal rule with the Svensson-Taylor rule. Table 3.1 summarizes the parameter values used in our calibrated economy. We use standard, illustrative values for $\alpha_1$, $\beta_1$ and $\beta_2$. We chose the shocks $\eta$ and $d$ from a normal distribution with mean zero and variance $\sigma_\eta^2 = \sigma_d^2 = 0.078$.

**Table 3.1 Parameter Configuration**[1]

| Parameter | Controls | Value |
|---|---|---|
| $\alpha_1$ | Response of inflation to the output gap | 1 |
| $\beta_1$ | Output persistence | 0.7 |
| $\beta_2$ | Elasticity of the output gap with respect to the long-term real interest rate | 1 |
| $k$ | Duration of the long bond | 0.5 |
| $\sigma_d^2$ | Variance of the shock to the output gap | 0.078 |
| $\sigma_\eta^2$ | Variance of the supply shock | 0.078 |
| $\pi^*$ | Policymaker's inflation target | 0 |

[1] We illustrate our analytical findings using these calibrations.

In Figures 3.1 and 3.2 we display the last 100 of 10.000 observations on the short-term ex ante real interest rate for both the optimal rule ($k = 0.5$) and the Svensson-Taylor rule ($k = 0$). Both systems are calculated based on the same realized sequence of shocks. We use 100 observations to keep the Figure relatively clear. The primary feature of the optimal rule is that the interest rate appears to be more volatile than according to the Svensson-Taylor rule. Figure 3.2 clearly shows that the mean-squared deviation of the interest rate is higher for the optimal rule as compared with the Svensson-Taylor rule. We will now provide some intuition for this result.

From equation **(3.9)** we see that the optimal response of the short-term interest rate to its determinants becomes stronger if the duration of the long bond ($D$) increases - that is the parameter $k$ becomes larger. This result is driven by a decrease in policy leverage over the long real rate since the latter will now to a greater extent be determined by *expected future* short real rates at the expense of the *present* short real rate. However, provided central bank preferences are constant over time, a change in duration will not alter the central bank's optimal intermediate target as expressed in equation **(3.1)**. Therefore, the central bank will have to manipulate its instrument more aggressively in order to attain the same desired effect on the long-term real interest rate.

**Figure 3.1 Short-Term Interest Rate - Svensson-Taylor Rule**

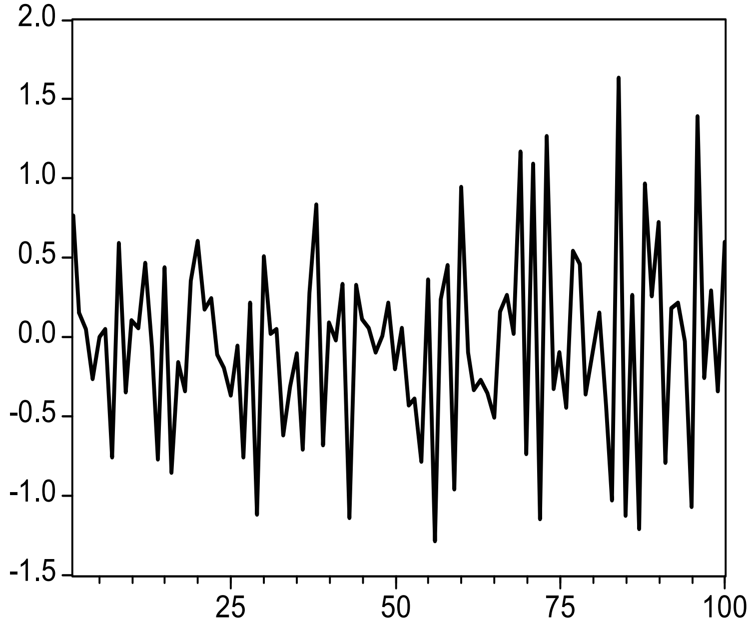
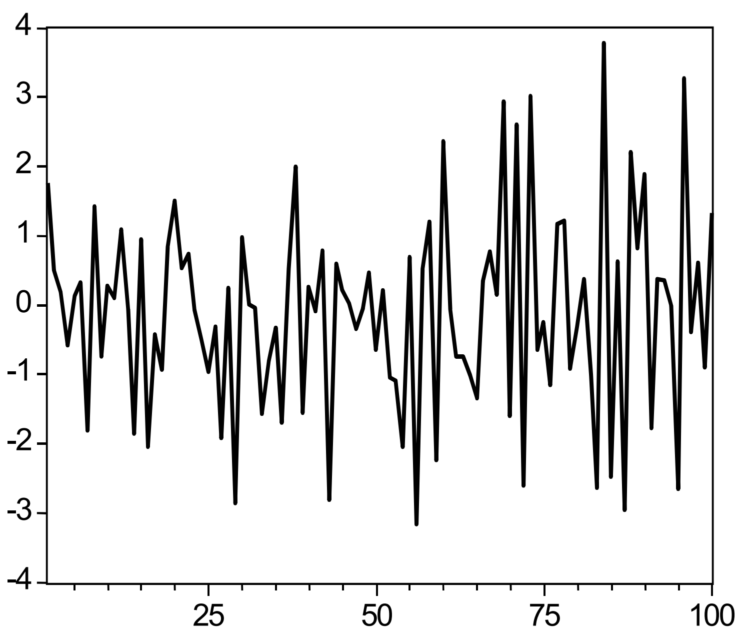



**Figure 3.2 Short-Term Interest Rate - Optimal Rule**




## 4 IMPLEMENTING INFLATION TARGETING UNDER LEARNING

The case of perfect knowledge can be represented as follows. First, at time $t$ the central bank sets its expectation (forecast) for private sector interest rate expectations. Next, also at time $t$ the private sector sets its forecast, $\hat{E}_t R_{t+1} = x_{t+1}$, of the long term real interest rate for period $t+1$. Then, the CB sets the interest rate at time $t$ based on its own forecast of the long-term real interest rate, where - importantly - the forecast turns out to be correct.

The main problem in practice for a central bank is that private sector agents - financial analysts, investment bankers, institutional investors, etc. - have become more and more sophisticated in analyzing and predicting future monetary policy actions of the central bank. This increased degree of sophistication of private sector agents makes it harder for the central bank to understand private sector expectations. Hence, the idea that the CB can forecast or - what is actually equivalent – observe $x_{t+1} = \hat{E}_t R_{t+1}$ without error is hardly realistic. This assumption will now be relaxed.

*4.1 The Kalman Filter*

Suppose the CB can no longer forecast private agents' interest rate expectations $x_{t+1}$ without error. Assume that the CB has a forecast $E_t y_{t+1}$ at time $t$ of $x_{t+1}$ which it subsequently uses to set the short-term interest rate $r_t$ at time $t$.

More specifically, let $y_t$ be the CB's noisy signal on $x_t$

$$y_t = x_t + \varepsilon_t \tag{4.1}$$

where $y_t$ is the central bank's signal of $x_t$, and $\varepsilon_t$ is its measurement error. [9] The only information available to the CB when it sets policy at time $t$ is its forecast of $y$ which is conditional on past values of $y$; i.e. $E[y_{t+1}|t] = E[y_{t+1}|y_{t+1-n}, n = 1,2,...]$. Even ex post, the CB cannot observe separately the two components of $y$, $x$ and $\varepsilon$. [10] We assume the measurement error is normally distributed with mean zero and variance $\sigma_\varepsilon^2$. So, the central bank's signal is unbiased, but not without error. An important limiting case of **(4.1)** is when $\sigma_\varepsilon^2 \to 0$ and we are back to the previous case of perfect knowledge, i.e. $y_t = x_t$.

To make the problem more tractable we set $\alpha_1 = 1$ and $\pi^* = 0$. These assumptions have the advantage of reducing the dimension of the state space in the central bank's optimal filtering problem. In this way we avoid what Ljungqvist and Sargent (2000) call the 'curse of dimensionality'. [11]

Under the above simplifying assumptions equation **(3.8)** reduces to

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[9] Subscript '$t$' denotes variables that are observed or determined at time $t$, except for the variables $x$ and $y$ where the subscript $t$ refers to the time period for which the expectation, $x$, or its noisy observation, $y$, is held.

[10] A real world counterpart of our signal processing can be that CBs may get data on PS interest rate expectations (say interest rate futures), which is then taken as a signal of the true PS expectation. Here we focus on one learner: the CB (whose rationality is thus bounded), and let's assume the PS has rational expectations (rationality not bounded). In future we may want to look at two-sided learning where both the PS and the CB are learning.

[11] For the technical details see Appendix D of Schaling (2003).

$$x_t = \frac{\beta_1}{\beta_2} w_{t-1} = \gamma w_{t-1} \qquad \text{where } \gamma \equiv \beta_1 / \beta_2 \text{ and } w_{t-1} \equiv (-\pi_{t-1} - z_{t-1}) \qquad \textbf{(3.8')}$$

Note that the situation above can be represented as the case where the CB believes that private sector interest rate expectations follow the stochastic process

$$y_t = \gamma w_{t-1} + \varepsilon_t \qquad \textbf{(4.2)}$$

corresponding to the true (actual) law of motion of PS interest rate expectations, but that $\gamma$ is unknown to them (this can be seen by substituting the expression for private sector interest rate expectations **(3.8')** into equation **(4.1)**). Thus, here we assume that the central bank employs a reduced form of the expectations formation process that is correctly specified.[12]

So, we assume that equation **(4.2)** is the *perceived law of motion* (PLM) of the central bank and that the policymaker attempts to estimate $\gamma$. Following Evans and Honkapohja (2001), this is our key bounded rationality assumption: we back away from the rational expectations assumption, replacing it with the assumption that, in forecasting private sector inflation expectations, the central bank acts like an econometrician.

The central bank's estimates will be updated over time as more information is collected. Letting $c_{t-1}$ denote its estimate through time $t-1$, the central bank's one-step-ahead forecast at $t-1$, is given by

$$E_{t-1}[y_t] = c_{t-1} w_{t-1} \qquad \textbf{(4.3)}$$

Under this assumption we have the following model of the evolution of the economy. Let $\Omega_t$ be the central bank's information set for time $t$. Suppose that at time $t-1$ the central bank has data on the economy from periods $\tau = t-1,...,t-n$. Thus the time $t-1$ information set is $\Omega_{t-1} = \{y_\tau, w_\tau\}_{\tau=t-n}^{t-1}$. Imagine that we have already calculated the ordinary least squares estimate $c_{t-1}$ of $\gamma$ in the model $(y_{t-1}; w_{t-2}\gamma, \sigma_\varepsilon^2)$. Given the new information, which is provided by the observations $y_t, w_{t-1}$, we wish to form a revised or updated estimate of $\gamma$. Here $c_t$ is the CB's OLS estimate of $\gamma$ in the model $(y_t; w_{t-1}, \gamma, \sigma_\varepsilon^2)$.

The timing of events is summarized in Figure 4.1 below.

[12] Instead - as pointed out by Orphanides and Williams (2002) - the learner may be uncertain of the correct from and estimate a more general specification, for example, in our case a linear regression with additional lags of expected inflation which nests **(4.2)**.

**Figure 4.1 Imperfect Knowledge: Timing of Events**

| Time $t$ | | Time $t+1$ |
|---|---|---|
| **Stage 1:** | **Stage 2:** | **Stage 3:** |
| • CB forecasts PS interest rate expectations using $c_t$ and $w_t$; i.e. sets $E_t[x_{t+1}] = E_t[y_{t+1}]$. | • 2a) PS forecasts long-term real interest rate, i.e. sets $x_{t+1} = \gamma w_t$ <br>• 2b) CB decides on monetary policy, i.e. sets $r_t^*(\pi_t, z_t, E_t[y_{t+1}])$. | • 3a) Nature chooses $\varepsilon_{t+1}$, and $y_{t+1} = x_{t+1} + \varepsilon_{t+1}$ realizes. <br>• 3b) CB observes the signal $y_{t+1}$ and forms a revised estimate $c_{t+1}$. <br>• Back to stage 1, for time $t = t+1$ etc. |

Using data through period $t$, the least squares regression parameter for equation **(4.2)** can be written in recursive form (see Appendix D of Schaling (2003) for details)

$$c_t = c_{t-1} + \kappa_t \left( y_t - w_{t-1} c_{t-1} \right) \tag{4.4}$$

$$p_t = p_{t-1} - \kappa_t w_{t-1} p_{t-1} \tag{4.5}$$

$$\kappa_t = p_t w_{t-1} (\sigma_\varepsilon^2)^{-1} \tag{D.10}$$

The method by which the revised estimate of $\gamma$ is obtained may be described as a filtering process, which maps the sequence of prediction errors into a sequence of revisions; and $\kappa_t = p_t w_{t-1} (\sigma_\varepsilon^2)^{-1}$ may be described as the gain of the filter, i.e. the Kalman gain.

Equations **(4.4)** and **(4.5)** are known as the updating, or smoothing equations. These updating equations represent the *learning channel*, through which the current realizations of inflation and the output gap[13] affects next period's estimate or *beliefs* $b_{t+1}$, where $b$ is a 1 x 2 row vector of state variables containing the mean and variance of the estimate, i.e. $b_t = [c_t \quad p_t]'$.

*4.2 The Case of Passive Learning*

In order to get some analytical results, we now consider the case of *passive learning*. This is the case where the central bank disregards the effect of current policy actions on future estimation and prediction. In this case the policy maker treats control and estimation separately.

The central bank will first choose $r_t$ to minimise the expected loss based on its current parameter estimate (its belief about $\gamma$). Then, a white noise shock $\varepsilon_{t+1}$ occurs and a new realization $y_{t+1}$ can be observed. Before choosing next period's control

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[13] Note that $w_t \equiv (-\pi_t - z_t)$.

$r_{t+1}$ the central bank will proceed by updating its estimate (belief) using the new information $(w_t, y_{t+1})$.

In case of imperfect knowledge –and a passive learning policy in which the central bank separates estimation and control, see Wieland (2000b, pp. 506-507) - that is a central bank who does not internalize the effect of current policy actions on future beliefs - in stage 2b) of Figure 4.1 we have

$$r_t = \frac{1}{\beta_2(1-k)}\pi_t + \frac{(1+\beta_1)}{\beta_2(1-k)}z_t - \frac{k}{(1-k)}E_t[E_t \hat{R}_{t+1}] \tag{4.6}$$

where the central bank's forecast of market expectations of future rates is governed by

$$E_t\left[\hat{E}_t\, R_{t+1}\right] = c_t w_t \tag{4.7}$$

Plugging **(4.7)** in equation **(4.6)**, we get the solution for the central bank's policy rule under *passive learning*

$$r_t = \frac{1}{\beta_2(1-k)}\pi_t + \frac{(1+\beta_1)}{\beta_2(1-k)}z_t + \frac{k}{(1-k)}c_t(\pi_t + z_t) \tag{4.8}$$

(For a proof see Appendix C.1). Note that now we have *four* state variables in the policy rule: inflation, output, the existing parameter estimate, and nominal GDP.[14] In addition to raising interest rates in response to inflation and output being above target and trend, respectively, the central bank now also responds to the level of nominal GDP. Note that the GDP term in the interest rate rule does not occur because the level of GDP enters the central bank's loss function.[15]

If nominal GDP $\pi_t + z_t$ is above (below) its target level (of zero), the central bank raises (lowers) short-term interest rates. The reason it does this, is that the central bank's optimal level of short-rates is inversely related to its expectation of the PS expectations about its future rates $\gamma w_t$, which in turn is inversely related to the level of GDP ( $w_t = (-\pi_t - z_t)$).[16] Thus, the central bank's optimal level of short-rates is inversely related to its *expectation* of the PS expectations about its future short rates. For example, if the CB *expects* the PS to expect that rates go up in the future, as a consequence (ceteris paribus) short-term interest rates can be lower today (and vice versa).

We also find that – in so far as nominal GDP is concerned - the policy rule now becomes *state-contingent*, as the parameter $c$ is in general unequal to $\gamma$, and moves

[14] Or three state variables, if we split-up nominal GDP in inflation and output.
[15] For a recent paper where the central bank targets nominal income *growth*, see Mitra (2003).
[16] The occurrence of the third GDP term in the policy rule is not, however, a general result. It depends on the specific simplifying assumptions made about the slope of the Phillips curve and the level of the inflation target.

in real time. This means that the central bank's optimal response to the deviation of nominal GDP and its target level also becomes state-contingent. Over time the estimate converges (for a proof, see Appendix A) to the true parameter and the policy under passive learning converges to optimal monetary policy under perfect knowledge **(3.9)**.[17]

*4.3 Optimal Monetary Policy under Learning*

We now examine how the nature of optimal monetary policy is affected by learning considerations. Under imperfect knowledge the central bank chooses $\{r_\tau\}_{\tau=t}^{\infty}$ so as to maximize

$$E_t\left[\sum_{\tau=t}^{\infty} -\frac{\delta^{\tau-t}}{2}\pi_\tau^2\right] \qquad \textbf{(B.1')}$$

subject to **(2.2)** and

$$z_{t+1} = (\beta_1 + \beta_2 k c_t) z_t - \beta_2 (1-k) r_t + \beta_2 k c_t \pi_t + d'_{t+1} \qquad \textbf{(C.1.3)}$$

$$y_{t+1} = \theta_t \qquad \textbf{(C.2.2)}$$

$$c_{t+1} = (1 + \varpi_t(\pi_t + z_t))c_t + \varpi_t \theta_t \qquad \textbf{(C.3.1)}$$

$$\kappa_{t+1} = \varpi_t \qquad \textbf{(C.3.3)}$$

$$\kappa_{t+1} = -\psi_t(\pi_t + z_t)\left(\sigma_\varepsilon^2\right)^{-1} \qquad \textbf{(C.4.1)}$$

$$p_{t+1} = (1 + \omega_t(\pi_t + z_t))p_t \qquad \textbf{(C.4.2)}$$

$$p_{t+1} = \psi_t \qquad \textbf{(C.4.3)}$$

Note that nonlinearities enter the problem in five places. First, the output equation becomes nonlinear. This can be seen from the presence of the 'product terms' $c_t z_t$ and $c_t \pi_t$ on the right-hand side of **(C.1.3)**. There is a third nonlinearity, as the product terms $c_t z_t$ and $c_t \pi_t$ on the right-hand side of the updating equation **(C.3.1)** are multiplied by the Kalman gain $\kappa_{t+1} = \varpi_t$. Since the prediction variance also moves over time, we have two more nonlinearites. First, the Kalman gain in equation **(C.4.1)** depends on the *products* of the prediction variance $p_{t+1} = \psi_t$ and the inflation and output realizations $\pi_t$ and $z_t$. Second, the prediction variance itself is governed by the nonlinear first-order difference equation **(C.4.2)**.

In Appendix C.2 we show that the first-order condition can be expressed as

[17] With $\alpha_1 = 1$, $\pi^* = 0$, and $E_t \hat{R}_{t+1} = \gamma w_t = -\gamma(\pi_t + z_t)$, where $\gamma = \beta_1 \beta_2^{-1}$ .

$$E_t \pi_{t+2} = 0 \qquad \textbf{(B.9)}$$

which is identical to the FOC for the cases of *passive learning* (see Appendix C.1) and perfect knowledge (see Appendix B.1). This means that the optimal policy is identical to the passive policy. Put differently, estimation and control can be separated and the so-called separation principle holds. Therefore, we can now study the optimal rule under imperfect knowledge in terms of equation **(4.8)**.

*4.4 Comparing Optimal Policy under Perfect Knowledge and Learning*

Now we compare the optimal rule under learning with the optimal rule under prefect knowledge. Table 4.1 summarizes the parameter values (in addition to the values already mentioned in Table 3.1).

**Table 4.1 Additional Parameter Values**

| **Parameter** | **Controls** | **Value** |
|---|---|---|
| $\sigma_\varepsilon^2$ | Variance of the measurement error | 1 |
| $p_0$ | Initial value of the prediction variance | 3 |
| $c_0$ | Initial value of the parameter estimate | 0.75 |
| $\gamma$ | True value of the parameter | 0.7 |

In Figures 4.2 and 4.3 we display the first 100 of 10.000 observations on the short-term ex ante real interest rate for both the optimal rule under perfect knowledge and the optimal rule under learning.[18] As can be seen from Table 4.2, the primary feature of the optimal rule under learning is that the interest rate exhibits less persistence than the interest rate under perfect knowledge. This feature reflects the phenomenon that the latter rule is linear in state with constant parameters, whereas the optimal rule under learning is state-contingent, i.e. has time-dependent coefficients that move with the updating of the parameter estimate.

**Table 4.2 Summary Statistics of first 100 Observations**

| **Case** | **Rule under Perfect Knowledge** | **Rule under Learning** |
|---|---|---|
| Mean | -0.03579 | 0.12795 |
| Variance | 1.94518 | 1.89131 |
| Coefficient for AR(1) term | -0.3988 | -0.2666 |

Due to convergence to the true parameter in the learning case, the two rules are almost identical in the last 100 observations. Therefore those observations are not shown. We illustrate convergence with Figures 4.4 and 4.5.

[18] Due to convergence to the true parameter in the learning case, the two rules are almost identical in the last 100 observations.

**Figure 4.2 Short-Term Interest Rate – Optimal Rule under Perfect Knowledge**

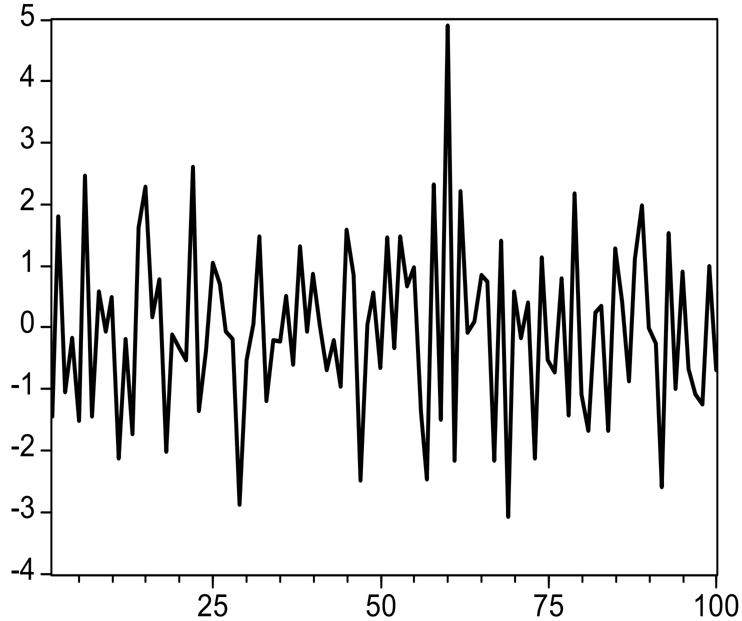



**Figure 4.3 Short-Term Interest Rate – Optimal Rule under Learning**

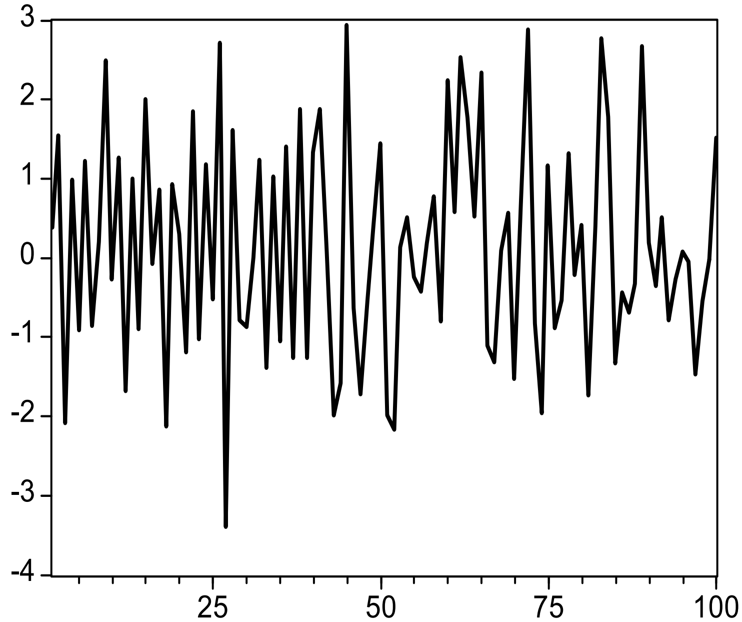


**Figure 4.4 Optimal Learning- Convergence of the Parameter Estimate to the True Value 0.7**

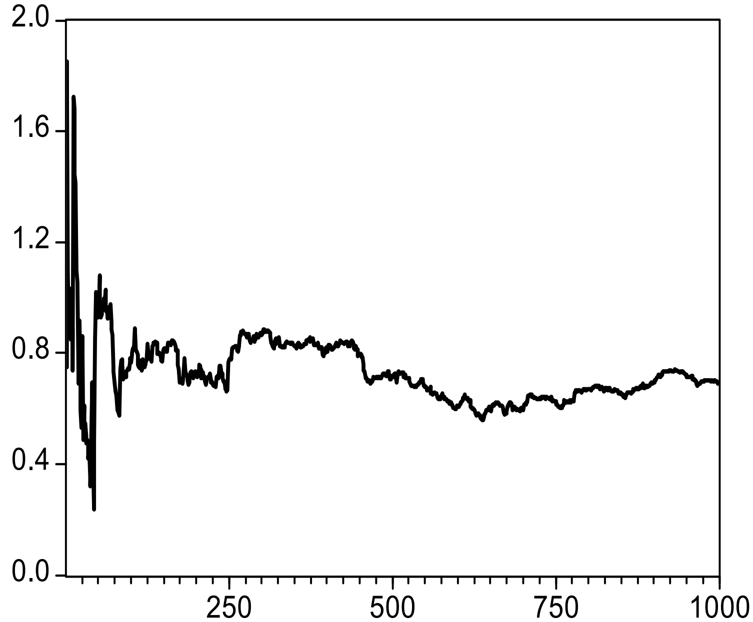
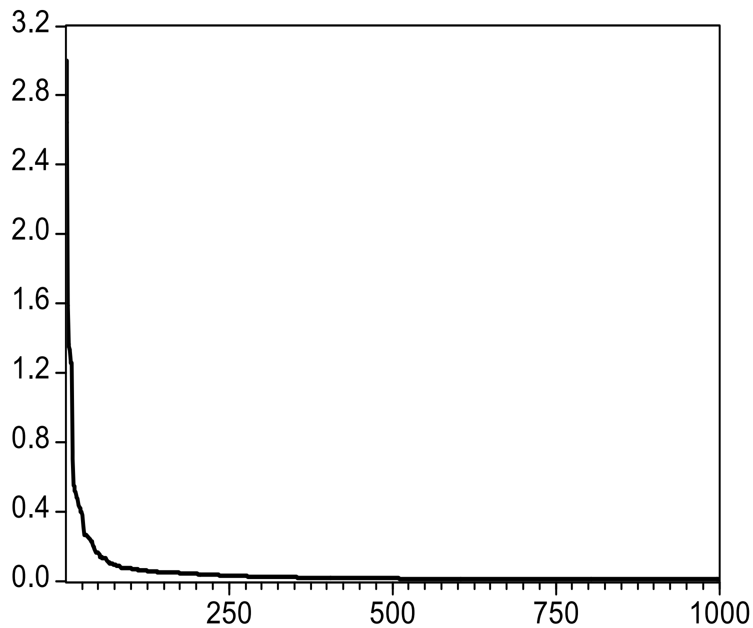



**Figure 4.5 Optimal Learning - Prediction Variance of the Parameter Estimate**



## 5 EVALUATION AND CONCLUDING REMARKS

In this Section we put our main result – that the optimal policy under learning coincides with the passive policy, i.e. that the optimal policy can separate estimation and control – in the context of the dual control literature. We informally discuss how a trade-off between estimation and control might resurface in our model, but find the argument unconvincing.

It is interesting to observe that our one-period objective function differs from Wieland (2000b, p. 506). Using the notation of this paper, Wieland considers

$$L(y_t, r_t) = (y_t - y^*)^2 + \lambda(r_t - r_t^*)^2 \qquad \textbf{(5.1)}$$

(where he sets $r_t^* = 0$ ). Thus, in his set-up both the control $r_t$ and the signal $y_t$ affect the agent's pay-off. A similar objective function is used by Beck and Wieland (2002) and Kiefer and Nyarko (1989). Apparently, the loss function **(5.1)** is standard in the dual control literature.

In the literature on learning and control, the stochastic process to be controlled is usually static in nature. Using the notation of this paper

$$y_t = \alpha + \beta r_t + \varepsilon_t \qquad \textbf{(5.2)}$$

where $y_t$ is the target variable and $r_t$ is the control variable.[19] Under perfect knowledge of $\alpha$ and $\beta$ , $r_t$ will be a function of these parameters and as such will be constant. With this type of constraint the optimal value of the control variable under passive learning is a function of the estimates of parameters (certainty-equivalence rule) or a function of the parameter estimates and their variances and covariances (myopic rule). Kiefer and Nyarko (1989) show that beliefs and actions $r_t$ converge in the limit. However, the risk is that if $r_t$ converges too quickly, then beliefs may also converge to incorrect values. The need for policy experimentation is therefore relevant here.

The story is different if the process to be controlled is dynamic and the RHS includes the lagged dependent variable, say

$$y_t = \alpha + \beta u_t + \delta y_{t-1} + \varepsilon_t \qquad \textbf{(5.3)}$$

In this case, under learning the optimal value of $r_t$ will be a function of the parameter estimates and $y_{t-1}$. In essence at any point in time, $r_t$ will be reacting to past shocks hitting $y_{t-1}$ and thus actions $r_t$ will never converge (or settle down). The upshot is that as long as the policy maker chooses to react to the state variable $y_{t-1}$, the control variable $r_t$ will also be stochastic. Kiefer and Nyarko (1989) have shown that beliefs

[19] An example is Wieland (2000a,b) who studies the problem of a single decision maker, who attempts to control a linear stochastic process with two unknown parameters, just like **(5.2)**.

would converge with probability 1 to the truth if actions $r_t$ do not converge. This finding implies that with the RHS variables ($r_t$ and $y_{t-1}$) showing variations, beliefs will converge with probability 1 to the truth.

In our paper, instead of **(5.2)** and **(5.3)** we have

$$y_t = \gamma w_{t-1} + \varepsilon_t \tag{4.2}$$

Thus, it appears that we also need to control a dynamic process. However, our one-period objective function differs markedly from the standard objective function of the dual control literature **(5.1)**.

Note that intuitively strict inflation targeting in our model can be re-interpreted in terms of the dual control literature. For, strict inflation targeting can be thought of as the case where the CB is only interested in minimizing

$$L(y_t, r_t) = (r_t - r_t^*)^2 \tag{5.4}$$

that is, in stabilizing the interest rate at its - time-dependent - target level, where the target level is given by

$$r_t^* = \frac{1}{\beta_2(1-k)}\pi_t + \frac{(1+\beta_1)}{\beta_2(1-k)}z_t + \frac{k}{(1-k)}c_t(\pi_t + z_t) \tag{4.8}$$

Of course, this is simply the case where the central bank –in implementing inflation targeting under imperfect knowledge of the term structure of interest rates - follows its optimal monetary policy rule under learning (with coincides with the passive policy). Thus**,** the main difference with the standard literature on learning and control is that in our case the regression to be estimated is different from the process to be controlled, as $y_t$ does not appear in our utility function.[20]

Conversely, if we set $\lambda = 0$ in **(5.1)** the central bank's objective function coincides with the input-target model that is often used in studies of learning by doing such as Jovanovich and Nyarko (1996) and Foster and Rosenzweig (1995). In this case it is also possible to back out an optimal level of the control $r_t$, i.e. a certain monetary policy setting that now does not minimizes deviations of inflation from the target, but tries to minimize the deviation between the signal $y_t$ and a certain target level.

Since there is an intuitive correspondence between this target level and the state variable $x_t = E_t \hat{R}_{t+1}$, we can think of this case as the case where the central bank is

[20] That is, we have the central bank which is estimating the process of private sector expectation formation with $w_t$ on the RHS of the regression. Since $w_t$ never settles down due to supply and demand shocks, we have central bank beliefs converging with probability 1 to the truth, i.e. Therefore in our model, the central bank's passive learning policy will lead in the limit to the full information rational expectations solution.

interested in learning the true value of the level of market expectations per se. Thus, here the CB is not targeting inflation, but is targeting 'knowledge about the markets'.

Then the question is what could be the value of 'experimentation' in our model? The answer is: probably a higher sample variance of $w_t$ leads to a more precise estimate. That is the central bank can engineer a higher volatility in $w_t$ by 'not stabilizing the shocks too well'. This means that the central bank should deviate from the optimal reaction function **(4.8)** for the case of strict inflation targeting, rule such that the process for $w_t$ becomes more volatile. Some intuition for this is provided by Figure 5.1.

**Figure 5.1 Constant Interest Rate Rule - Convergence of the Parameter Estimate to the True Value 0.7**

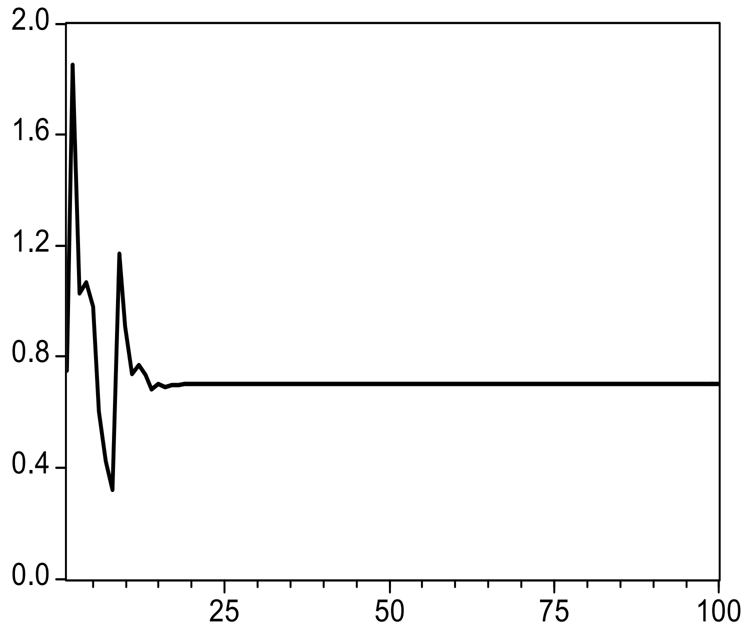



Figure 5.1 displays the first 100 of 10.000 observations on the convergence of the parameter estimate under a constant real interest rate rule, more specifically for the case where the real interest rate stays at is equilibrium or neutral level – which here is normalized at zero. If we compare Figure 5.1 with Figure 4.4 – which illustrates convergence under the optimal interest rate rule – we have an important result. Thew speed of convergence under a constant interest rate rule is much higher than under the optimal rule!

This suggests that indeed there is a trade-off between learning and control *if the signal enters the utility function*. So, in the case that $\left(y_t - y^*\right)$ is an argument in the central bank's loss function – that is if $\lambda \neq 0$ in equation **(5.1)**, we are back in the neighbourhood of the standard dual control literature and the separation principle will break down, indicating that estimation and control cannot be separated.

However, it is hard to see how the objective function **(5.1)** can be justified. Inflation targeting has become the dominant monetary policy strategy for the major central banks in the world since the late 1990s.[21] So, it is clear that the deviation of inflation from its assigned target should be in their objective function. Why, however, would the central bank be interested in limiting the deviation between its noisy observations of market expectations, $y_t$, and the true values of those expectations.? The only clear rationale would be if it needs to know those market expectations as an essential part of the monetary transmission mechanism. This is the avenue we have followed in this paper. However, apart from the need to learn these expectations *for the sake of controlling inflation* there is no reason whatsoever why understanding those expectations in itself should be one of the goals of monetary policy. Therefore, there is no good case for having $\lambda$ different from 0 and therefore also no good case supporting a trade-off between estimation and control. Therefore, this model suggests that the practical relevance of the breakdown of the separation principle and the need for experimentation in policy may be limited.

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[21] Some central banks - like the Bank of Canada, the Bank of England and the Reserve Bank of Australia - have *explicitly* chosen inflation (forecast) targeting framework as their disciplinary framework. Other central banks - such as the European Central Bank and the US Federal Reserve System - have *implicitly* incorporated many elements of inflation targeting in their monetary analysis and policy strategy.

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## APPENDIX A CONVERGENCE OF STOCHASTIC RECURSIVE ALGORITHMS

### A.1 SAMPLE HAS BEEN GENERATED BY NATURE

Let $R_t = t^{-1}\sum_{i=1}^{t}(w_{i-1})^2$ **(A.1.1)**

Then in estimating the model $y_t = \gamma w_{t-1} + \varepsilon_t$, the least squares formula for the parameter estimate $c_t$ can be written in recursive form

$$c_t = c_{t-1} + t^{-1}R_t^{-1}w_{t-1}(y_t - w_{t-1}c_{t-1})$$

or replacing $y_t$ by $\gamma w_{t-1} + \varepsilon_t$,

$$c_t = c_{t-1} + t^{-1}R_t^{-1}w_{t-1}(w_{t-1}(\gamma - c_{t-1}) + \varepsilon_t) \qquad \textbf{(A.1.2)}$$

(A.1.1) can also be written in recursive form

$$R_t = R_{t-1} + t^{-1}(w_{t-1}^2 - R_{t-1}) \qquad \textbf{(A.1.3)}$$

(the system **(A.2)** and **(A.3)** is the same as Evans and Honkapohja (2001) (hereafter EH) equation (2.9) pp. 33)

To put **(A.1.2)** and **(A.1.3)** in standard form we rather use $R_{t-1}$ instead of $R_t$ on the RHS of this equation. The appropriate way to handle it is to define another variable $S$ such that $S_{t-1} = R_t$ (see EH page 37). The system then becomes

$$c_t = c_{t-1} + t^{-1}S_{t-1}^{-1}w_{t-1}(w_{t-1}(\gamma - c_{t-1}) + \varepsilon_t) \qquad \textbf{(A.1.4)}$$

$$S_t = S_{t-1} + t^{-1}\left(\frac{t}{t+1}\right)(w_t^2 - S_{t-1}) \qquad \textbf{(A.1.5)}$$

Next rewrite equation **(A.1.4)** as

$$c_t = c_{t-1} + t^{-1}S_{t-1}^{-1}w_{t-1}[w_{t-1}(T(c_{t-1}) - c_{t-1}) + \varepsilon_t] \qquad \textbf{(A.1.4')}$$

where $T$ implicitly defines the mapping from the PLM to the ALM

$$T(c_{t-1}) = \gamma \qquad \textbf{(A.1.6)}$$

The interpretation of the ALM is that it describes the stochastic process followed by the economy if forecasts are made under the fixed rule given by the PLM. Here of course that stochastic process, the ALM, is the data generating process (DGP) (here the sample is generated by nature) – which is independent of the PLM - and the PLM is the recursive estimate of the ALM (or nature).

The system **(A.1.4')-(A.1.5)** is now implicitly in standard form with the following definition of variables:

$\theta_t = (c_t, S_t)'$; $X_t = (w_t, w_{t-1}, \varepsilon_t)'$ and $g_t = t^{-1}$

So the system (A.1.4')-(A.1.5) can now be written as

$$\theta_t = \theta_{t-1} + g_t Q(t, \quad \theta_{t-1}, \quad X_t) \qquad \textbf{(A.1.7)}$$

In the case above the state vector $X_t$ follows an exogenous stochastic process; i.e. the sample has been generated by nature. However, as pointed out by EH, p. 35 this is not at all essential. In particular, in the general framework, $X_t$ can be permitted to follow a VAR (vector autoregression) with parameters that depend on $\theta_{t-1} = (c_{t-1}, S_{t-1})'$. Evans and Honkapohja (2001) state that this issue is discussed fully in Chapters 6 and 7 of their book, and is relevant for the cases of passive and optimal learning.

The function $Q$ expresses the way in which the estimate $\theta_{t-1}$ (or rather a vector of parameter estimates or *beliefs*) is revised in line with last period's observations. Here, $\theta_{t-1}$ will include all components of $c_{t-1}$ and $S_{t-1} = R_t$, $X_t$ is the state vector that includes the effects of $w_t$, $w_{t-1}$ and $\varepsilon_t$, and $g_t$ is a deterministic sequence of 'gains' - i.e. a non-increasing sequence of positive numbers - satisfying $\lim_{t\to\infty} tg_t = 1$. We are interested in the conditions under which $\lim_{t\to\infty} \theta_t = \overline{\theta}$, where $\overline{\theta}$ solves either $EQ(X_t, \overline{\theta}) = 0$ in the case that $\{X_t\}$ is drawn from a distribution that is stationary or $\lim_{t\to\infty} EQ(X_t, \overline{\theta}) = 0$ in the case that $\{X_t\}$ is asymptotically stationary (for the latter case see Appendix A.2)).

As pointed out by Sargent (1993, pp. 39-41), it has been discovered that the limiting behavior of a sequence $\{\theta_t\}$ determined by stochastic difference equation **(A.1.7)** is described by an associated differential equation,

$$\frac{d\theta}{d\tau} = EQ(X, \theta_\tau) \qquad \textbf{(A.1.8)}$$

where $EQ(X, \theta)$ is the expected value of $Q(X, \theta)$, evaluated with respect to the asymptotic stationary distribution of $\{X_t\}$ and $\tau$ denotes "notional" or "artificial" time (see EH pp. 31).

Having shown that the system can be placed in standard SRA (stochastic recursive algorithm) form, the next step is to compute the associated ODE. Therefore, we have to compute $EQ(X, \theta)$.

The easiest way to do this is to look at the two components of $Q$ separately. The first component of $Q$, giving the revisions to $c_{t-1}$ is given by

$$Q_c(t,\theta_{t-1},X_t)=S_{t-1}^{-1}w_{t-1}\left[w_{t-1}(T(c_{t-1})-c_{t-1})+\varepsilon_t\right] \quad \textbf{(A.1.9)}$$

Hence, fixing the value of $c$ and $S$, and computing the expectation over $X_t$ we get

$$h_c(c,S)=\lim_{t\to\infty}EQ_c(t,\theta_{t-1},X_t)=\lim_{t\to\infty}ES^{-1}w_{t-1}\left[w_{t-1}(T(c)-c)+\varepsilon_t\right]= \\ \lim_{t\to\infty}ES^{-1}\left[w_{t-1}^2(T(c)-c)+w_{t-1}\varepsilon_t\right] \quad \textbf{(A.1.10)}$$

Similarly, the second component of $Q$ is given by

$$Q_S(t,\theta_{t-1},X_t)=\left(\frac{t}{t+1}\right)\left[(w_t^{\,2}-S_{t-1}\right] \quad \textbf{(A.1.11)}$$

Hence fixing the value of $c$ and $S$ and computing the expectation over $X_t$ we get

$$h_S(c,S)=\lim_{t\to\infty}EQ_S(t,\theta_{t-1},X_t)=\lim_{t\to\infty}\left(\frac{t}{t+1}\right)E\left[(w_t^{\,2}-S\right]$$

Since $Ew_t^2=Ew_{t-1}^2=Var(w_t)=\sigma_w^2$, $Ew_{t-1}\varepsilon_t=0$, and $\lim_{t\to\infty}t/(t+1)=1$ we obtain

$$h_c(c,S)=\frac{\sigma_w^2}{S}(T(c)-c) \quad \textbf{(A.1.12)}$$

$$h_S(c,S)=\sigma_w^2-S \quad \textbf{(A.1.13)}$$

where $Ew_t^2=Ew_{t-1}^2=\sigma_w^2$ is the unconditional second moment of $w_t$. In the case where the sample has been generated by nature, to make sure that the (asymptotic) variance $Ew_t^2=Ew_{t-1}^2=\sigma_w^2$ exists one can, for example, permit $w_t$ to follow a stationary exogenous AR (autoregressive) process, driven by a white noise shock with bounded moments.

The stochastic approximation approach associates an ordinary differential equation (ODE) with the stochastic recursive algorithm,

$$\frac{d\theta}{d\tau}=EQ(X,\theta_\tau)=h(\theta(\tau)) \quad \textbf{(A.1.14)}$$

We can write the differential equation component by component to obtain

$$\frac{dc}{d\tau}=EQ(X,c_\tau)=h(c(\tau))=\frac{\sigma_w^2}{S}(T(c)-c) \quad \textbf{(A.1.12')}$$

$$\frac{dS}{d\tau}=EQ(X,S_\tau)=h(S(\tau))=\sigma_w^2-S \quad \textbf{(A.1.13')}$$

As pointed out by EH, p. 38 this system is recursive (that is we 'first' compute the variance of the estimated parameter, or the 'Kalman gain' and then proceed in updating the estimate) and the second equation is globally stable[22] with $S \to \sigma_w^2$ from any starting point. It follows that $\frac{\sigma_w^2}{S} \to 1$ from any starting point, provided S is different from zero along the path, and hence that the stability of the differential equations **(A.1.12')** and **(A.1.13')** is determined entirely by the stability of the smaller dimension non-homogenous equation

$$\frac{dc}{d\tau} = (T(c) - c) = Ac = -c + \gamma \qquad \textbf{(A.1.15)}$$

where I have used **(A.1.6)**.

Clearly $c = \gamma$ is a stationary solution. The general solution [see e.g. Sargent (1993, p. 41)] is

$$c(t) = \gamma + (c(0) - \gamma)e^{-t}$$

which converges to $\gamma$ for any initial value $c(0)$.

## A.2 CONVERGENCE UNDER PASSIVE LEARNING

In the case where the central bank engages in estimation and control the data generating process for $w_t$ is not exogeneous anymore. Rather it follows an AR(1) process where the coefficient on the lagged term is a function of $c_t$. To see this, first solve for the long-term real interest rate. Substituting **(4.8)** into equation **(2.4)** and combining the result with **(3.8')** we obtain

$$R_t = \frac{1}{\beta_2}\pi_t + \frac{1+\beta_1}{\beta_2} z_t - k[c_t - \gamma) w_t] \qquad \textbf{(4.9)}$$

Then, we need the equilibrium equations for inflation and output gap[23]:

$$z_{t+1} = -[1 - k(c_t - \gamma)]z_t - [1 - k(c_t - \gamma)]\pi_t + d_{t+1} \qquad \textbf{(4.10)}$$

$$\pi_{t+1} = \pi_t + z_t - \eta_{t+1} \qquad \textbf{(2.2)}$$

Then adding **(4.10)** and **(2.2)** we get

$$\pi_{t+1} + z_{t+1} = k(c_t - \gamma)(\pi_t + z_t) + d_{t+1} + \eta_{t+1} \qquad \textbf{(A.2.1)}$$

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22 Clearly $S = \sigma_w^2$ is a stationary solution. The general solution [see e.g. Sargent (1993, p. 41)] is $S(t) = \sigma_w^2 + (S(0) - \sigma_w^2)e^{-t}$ which converges to $\sigma_w^2$ for any initial value $S(0)$.

23 The solution for the output gap $z$ can readily be obtained by substituting **(4.9)** into **(2.3)**.

or, in terms of $w_t$

$$w_{t+1} = k(c_t - \gamma)w_t - (d_{t+1} + \eta_{t+1}) \tag{A.2.2}$$

The difference with the case of an exogenous data sequence is that now we need to make sure that $\lim_{t\to\infty} Ew_t^2$ (which depends on the magnitude of $k(c_t - \gamma)$ in the AR(1) process) exits and is finite. The reason is that since we are interested in *local convergence*, it is necessary that in deriving the ODE, the process **(A.2.2)** be asymptotically stationary. For this we need to have $c_t = c$ sufficiently close to $\gamma$ ( the fixed point of interest) such that $|k(c-\gamma)| < 1$. Then the process

$$w_{t+1} = k(c - \gamma)w_t - (d_{t+1} + \eta_{t+1}) \tag{A.2.3}$$

will have a bounded second moment in the limit. Let $\lim_{t\to\infty} E[w_t(c)]^2 = Var_w(c)$. Denote the stationary points for $c$ and $S$ by $\bar{c}$ and $\bar{S}$ respectively. The ODE system is

$$\frac{dc}{d\tau} = h_c(c,S) = \frac{Var_w(c)}{S}(T(c) - c) \tag{A.2.4}$$

$$\frac{dS}{d\tau} = h_S(c,S) = Var_w(c) - S \tag{A.2.5}$$

Then as in Appendix (A.1) the stability of the ODE system **(A.2.4)** and **(A.2.5)** depends on the local stability of **(A.2.4)** at $\bar{c} = \gamma$. The general solution in real time will be

$$c(t) = \gamma + (c(0) - \gamma)e^{-t}$$

while the stationary solution will be $c(t) = \gamma$.

## APPENDIX B OPTIMAL MONETARY POLICY UNDER PERFECT KNOWLEDGE

The central bank chooses $\{r_\tau\}_{\tau=t}^{\infty}$ so as to maximize

$$E_t\left[\sum_{\tau=t}^{\infty} -\frac{\delta^{\tau-t}}{2}(\pi_\tau - \pi^*)^2\right] \tag{B.1}$$

subject to

$$\pi_{t+1} = \pi_t + \alpha_1 z_t - \eta_{t+1} \tag{2.2}$$

$$z_{t+1} = \beta_1 z_t - \beta_2 R_t + d_{t+1} \tag{2.3}$$

$$R_t = (1-k)r_t + k\hat{E}_t R_{t+1} \tag{2.4}$$

We can reformulate the problem above as choosing the indirect control variable $\{u_\tau\}_{\tau=t}^{\infty}$ to maximize

$$E_t\left[\sum_{\tau=t}^{\infty} -\frac{\delta^{\tau-t}}{2}(x_\tau - \pi^*)^2\right] \tag{B.2}$$

subject to

$$x_{t+1} = x_t + u_t + \xi_{t+1} \tag{B.3}$$

where $x_t = E_t\pi_{t+1}$ is the new state variable, $u_t = \alpha_1 E_t z_{t+1}$ is the new control variable and $\xi_{t+1} = -\eta_{t+1} + \alpha_1 d_{t+1}$.[24] We solve this problem by the method of Lagrange multipliers.[25]

The Lagrangian for this problem is

$$L = E_t\left[\sum_{\tau=t}^{\infty}\left\{-\frac{\delta^{\tau-t}}{2}(x_\tau - \pi^*)^2 - \delta^{\tau-t+1}\mu_{\tau+1}(x_{\tau+1} - x_\tau - u_\tau - \xi_{\tau+1})\right\}\right] \tag{B.4}$$

The first order conditions are

$$\frac{\partial L}{\partial u_t} = \delta E_t \mu_{t+1} = 0 \tag{B.5}$$

$$\frac{\partial L}{\partial x_t} = -(x_t - \pi^*) - \mu_t + \delta E_t \mu_{t+1} = 0 \tag{B.6}$$

From equation **(B.5)** we have $E_t\mu_{t+1} = 0$. Using this result in equation **(B.6)** gives

$$\mu_t = -(x_t - \pi^*) \tag{B.7}$$

Leading equation **(B.7)** by one period and taking expectations at time $t$ yields

$$E_t\mu_{t+1} = -(E_t x_{t+1} - \pi^*) \tag{B.8}$$

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[24] Equation **(B.3)** is derived by leading **(2.2)** by one period and taking expectations as of time t+1. This gives $E_{t+1}\pi_{t+2} = \pi_{t+1} + a_1 z_{t+1}$. The RHS variables can be decomposed as follows:
$\pi_{t+1} = E_t\pi_{t+1} - \eta_{t+1}$ and $\alpha_1 z_{t+1} = \alpha_1(E_t z_{t+1} + d_{t+1})$. Then we have
$E_{t+1}\pi_{t+2} = E_t\pi_{t+1} + \alpha_1 E_t z_{t+1} - \eta_{t+1} + \alpha_1 d_{t+1}$.

[25] For a discussion of the relative merits of the methods of dynamic programming and Lagrange, see Schaling (2001).

Since we have defined $x_{t+1} = E_{t+1}\pi_{t+2}$, using **(B.5)** and the law of iterated expectations the first order condition can be expressed as

$$E_t\pi_{t+2} = \pi^* \qquad \textbf{(B.9)}$$

Finally we combine **(B.9)**, **(2.2)**, **(2.3)** and **(2.4)** to get equation **(3.2)** in the main text.

**APPENDIX C.1 PASSIVE LEARNING**

In the case of passive learning at stage 1 the central bank first forms a sample estimate of the unknown parameter γ, and takes this estimate $c_t$ as given when it subsequently sets policy at stage 2 in the same period. Thus, the setup of the problem is similar to the case of perfect knowledge analyzed in Appendix B.1 - but with the additional simplifications that $\pi^* = 0$ and $\alpha_1 = 1$.

The model can now be written as follows. For the output gap, as before we decompose $z_{t+1}$ into the central bank's forecast $E_t z_{t+1}$ and its forecast error. To do this, first take expectations of **(2.3)** and **(2.4)** at time $t$

$$E_t z_{t+1} = \beta_1 z_t - \beta_2(1-k)r_t - \beta_2 k E_t[\hat{E}_t R_{t+1}] \qquad \textbf{(C.1.1)}$$

Combining **(C.1.1)** with equation **(4.6)** we get

$$\begin{aligned} E_t z_{t+1} &= \beta_1 z_t - \beta_2(1-k)r_t - \beta_2 k c_t w_t \\ &= \beta_1 z_t - \beta_2(1-k)r_t + \beta_2 k c_t \pi_t + \beta_2 k c_t z_t \end{aligned} \qquad \textbf{(C.1.1')}$$

The actual process for $z_{t+1}$ is given by

$$z_{t+1} = \beta_1 z_t - \beta_2(1-k)r_t + \beta_2 k\gamma\pi_t + \beta_2 k\gamma z_t + d_{t+1} \qquad \textbf{(C.1.2)}$$

Subtracting **(C.1.1')** from **(C.1.2)** yields

$$\begin{aligned} z_{t+1} &= (\beta_1 + \beta_2 k c_t)z_t - \beta_2(1-k)r_t + \beta_2 k c_t \pi_t + d'_{t+1} \\ &= E_t z_{t+1} + d'_{t+1} \end{aligned} \qquad \textbf{(C.1.3)}$$

Note that now the output equation becomes *nonlinear*. This can be seen from the presence of the 'product terms' $c_t z_t$ and $c_t \pi_t$ on the right-hand side of **(C.1.3)**.

The term $d'_{t+1} = -\beta_2 k(c_t - \gamma)(\pi_t + z_t) + d_{t+1}$ is the central bank's forecast error with respect to next period's level of the output gap. Compared to the case of perfect knowledge it can be seen that this error now consists of two terms: (i) the additive demand shock $d_{t+1}$, and (ii) a term that depends on its recursive forecast error of market expectations of next period's long real interest rate $-\beta_2 k(c_t - \gamma)(\pi_t + z_t)$. It is clear that if $c_t \rightarrow \gamma$, $d'_{t+1} \rightarrow d_{t+1}$ and results collapse to those under prefect knowledge (for the case that $\pi^* = 0$ and $\alpha_1 = 1$).

The algebra is now the same as in Appendix B.1, except that we have $d'_{t+1}$ instead of $d_{t+1}$. That is, we can reformulate the problem above as choosing the indirect control variable $\{u_\tau\}_{\tau=t}^{\infty}$ to maximize

$$E_t\left[\sum_{\tau=t}^{\infty} -\frac{\delta^{\tau-t}}{2}(x_\tau)^2\right] \qquad \textbf{(C.1.4)}$$

subject to

$$x_{t+1} = x_t + u_t + \xi'_{t+1} \qquad \textbf{(C.1.5)}$$

where all variables are defined as before and $\xi'_{t+1} = -\eta_{t+1} + d'_{t+1}$.

Following the same logic as in Appendix B.1, the first order condition can be expressed as

$$E_t \pi_{t+2} = 0 \qquad \textbf{(B.9)}$$

Finally we combine **(B.9)** with **(2.2)** and **(C.1.1')** to get equation **(4.8)** in the main text.

## APPENDIX C.2 OPTIMAL LEARNING WITH VARIABLE GAIN AND VARIABLE PREDICTION VARIANCE

The central bank chooses $\{r_\tau\}_{\tau=t}^{\infty}$ so as to maximize **(B.1')** subject to **(2.2)** and

$$z_{t+1} = (\beta_1 + \beta_2 k c_t) z_t - \beta_2 (1-k) r_t + \beta_2 k c_t \pi_t + d'_{t+1} \qquad \textbf{(C.1.3)}$$

$$y_{t+1} = \theta_t \qquad \textbf{(C.2.2)}$$

$$c_{t+1} = (1 + \varpi_t(\pi_t + z_t)) c_t + \varpi_t \theta_t \qquad \textbf{(C.3.1)}$$

$$\kappa_{t+1} = \varpi_t \qquad \textbf{(C.3.3)}$$

$$\kappa_{t+1} = -\psi_t (\pi_t + z_t)(\sigma_\varepsilon^2)^{-1} \qquad \textbf{(C.4.1)}$$

$$p_{t+1} = (1 + \omega_t(\pi_t + z_t)) p_t \qquad \textbf{(C.4.2)}$$

$$p_{t+1} = \psi_t \qquad \textbf{(C.4.3)}$$

The Lagrangian for this problem is

$$L = E_t \left[ \sum_{\tau=t}^{\infty} \left\{ \begin{aligned} & -\frac{\delta^{\tau-t}}{2}\pi_\tau^2 - \delta^{\tau-t+1}\mu_{\tau+1}^1(\pi_{\tau+1} - \pi_\tau - z_\tau + \eta_{\tau+1}) \\ & - \delta^{\tau-t+1}\mu_{\tau+1}^2(z_{\tau+1} - (\beta_1 + \beta_2 k c_\tau) z_\tau \\ & + \beta_2(1-k)r_\tau - \beta_2 k c_\tau \pi_\tau - d'_{\tau+1}) \\ & - \delta^{\tau-t+1}\mu_{\tau+1}^3(c_{\tau+1} - [1 + \varpi_\tau(\pi_\tau + z_\tau)]c_\tau - \varpi_\tau \theta_\tau) \\ & - \delta^{\tau-t+1}\mu_{\tau+1}^4(y_{\tau+1} - \theta_\tau) \\ & - \delta^{\tau-t+1}\mu_{\tau+1}^5\left(\kappa_{\tau+1} + \psi_\tau(\pi_\tau + z_\tau)\left(\sigma_\varepsilon^2\right)^{-1}\right) \\ & - \delta^{\tau-t+1}\mu_{\tau+1}^6\left(\kappa_{\tau+1} - \varpi_\tau\right) \\ & - \delta^{\tau-t+1}\mu_{\tau+1}^7\left(p_{\tau+1} - (1 + \varpi_\tau(\pi_\tau + z_\tau))p_\tau\right) \\ & - \delta^{\tau-t+1}\mu_{\tau+1}^8\left(p_{\tau+1} - \psi_\tau\right) \end{aligned} \right\} \right] \qquad \textbf{(C.4.4)}$$

The first order conditions are

$$\frac{\partial L}{\partial r_t} = -\beta_2(1-k)\delta E_t \mu_{t+1}^2 = 0 \qquad \textbf{(C.2.4)}$$

$$\frac{\partial L}{\partial c_t} = \delta\beta_2 k(\pi_t + z_t)E_t\mu_{t+1}^2 - \mu_t^3 + \delta[1 + \varpi_t(\pi_t + z_t)]E_t\mu_{t+1}^3 = 0 \qquad \textbf{(C.3.7)}$$

$$\frac{\partial L}{\partial \theta_t} = \delta\varpi_t E_t\mu_{t+1}^3 + \delta E_t\mu_{t+1}^4 = 0 \qquad \textbf{(C.3.8)}$$

$$\frac{\partial L}{\partial y_t} = -\mu_t^4 = 0 \qquad \textbf{(C.2.9)}$$

$$\frac{\partial L}{\partial \kappa_t} = -\mu_t^5 - \mu_t^6 = 0 \qquad \textbf{(C.3.10)}$$

$$\begin{aligned} & \frac{\partial L}{\partial \pi_t} = -\pi_t - \mu_t^1 + \delta E_t\mu_{t+1}^1 + \delta\beta_2 k c_t E_t\mu_{t+1}^2 + \delta\varpi_t c_t E_t\mu_{t+1}^3 \\ & - \delta\psi_t\left(\sigma_\varepsilon^2\right)^{-1} E_t\mu_{t+1}^5 + \delta\varpi_t p_t E_t\mu_{t+1}^7 = 0 \end{aligned} \qquad \textbf{(C.4.5)}$$

$$\begin{aligned} & \frac{\partial L}{\partial z_t} = -\mu_t^2 + \delta E_t\mu_{t+1}^1 + \delta(\beta_1 + \beta_2 k c_t)E_t\mu_{t+1}^2 + \delta\varpi_t c_t E_t\mu_{t+1}^3 \\ & - \delta\psi_t\left(\sigma_\varepsilon^2\right)^{-1} E_t\mu_{t+1}^5 + \delta\varpi_t p_t E_t\mu_{t+1}^7 = 0 \end{aligned} \qquad \textbf{(C.4.6)}$$

$$\frac{\partial L}{\partial \varpi_t} = \delta[c_t(\pi_t + z_t) + \theta_t]E_t\mu_{t+1}^3 + \delta E_t\mu_{t+1}^6 + \delta(\pi_t + z_t)p_t E_t\mu_{t+1}^7 = 0 \qquad \textbf{(C.4.7)}$$

$$\frac{\partial L}{\partial p_t} = -\mu_t^{\ 7} + \delta[1 + \varpi_t(\pi_t + z_t)]E_t\mu_{t+1}^7 - \mu_t^{\ 8} = 0 \qquad \textbf{(C.4.8)}$$

$$\frac{\partial L}{\partial \psi_t} = -\delta(\sigma_\varepsilon^2)^{-1}(\pi_t + z_t)E_t\mu_{t+1}^5 + \delta E_t\mu_{t+1}^8 = 0 \qquad \textbf{(C.4.9)}$$

From equation **(C.2.4)** and **(C.2.9)** we have $E_t\mu_{t+1}^2 = 0$ and $\mu_t^4 = E_t\mu_{t+1}^4 = 0$ respectively. Then, from **(C.3.8)** we see that $\varpi_t E_t\mu_{t+1}^3 = 0$. Using, this information **C.4.5)**, **(C.4.6)** and **(C.3.7)** simplify to

$$\begin{aligned} &-\pi_t - \mu_t^1 + \delta E_t\mu_{t+1}^1 \\ &-\delta\psi_t\left(\sigma_\varepsilon^2\right)^{-1}E_t\mu_{t+1}^5 + \delta\varpi_t p_t E_t\mu_{t+1}^7 = 0 \end{aligned} \qquad \textbf{(C.4.5')}$$

$$\begin{aligned} &-\mu_t^2 + \delta E_t\mu_{t+1}^1 \\ &-\delta\psi_t\left(\sigma_\varepsilon^2\right)^{-1}E_t\mu_{t+1}^5 + \delta\varpi_t p_t E_t\mu_{t+1}^7 = 0 \end{aligned} \qquad \textbf{(C.4.6')}$$

$$-\mu_t^3 + \delta E_t\mu_{t+1}^3 = 0 \qquad \textbf{(C.3.7'')}$$

Since we are dealing with the case of a non-zero gain, **(C.3.8)** implies $E_t\mu_{t+1}^3 = 0$. Then **(C.4.7)** simplifies to

$$E_t\mu_{t+1}^6 + (\pi_t + z_t)p_t E_t\mu_{t+1}^7 = 0 \qquad \textbf{(C.4.7')}$$

From equation **(C.3.10)** we have $\mu_t^5 = \mu_t^6$, so that $E_t\mu_{t+1}^5 = E_t\mu_{t+1}^6$. We can then rewrite **(C.4.7')** as

$$E_t\mu_{t+1}^5 = -(\pi_t + z_t)p_t E_t\mu_{t+1}^7 \qquad \textbf{(C.4.7'')}$$

Using **(C.4.7'')** in **(C.4.5')** and **(C.4.6')** we get

$$-\pi_t - \mu_t^1 + \delta E_t\mu_{t+1}^1 + \delta p_t[\psi_t\left(\sigma_\varepsilon^2\right)^{-1}(\pi_t + z_t) + \varpi_t]E_t\mu_{t+1}^7 = 0 \qquad \textbf{(C.4.5'')}$$

$$-\mu_t^2 + \delta E_t\mu_{t+1}^1 + \delta p_t[\psi_t\left(\sigma_\varepsilon^2\right)^{-1}(\pi_t + z_t) + \varpi_t]E_t\mu_{t+1}^7 = 0 \qquad \textbf{(C.4.6'')}$$

Now it can easily be seen that the term in square brackets appearing before $E_t\mu_{t+1}^7$ is zero by definition. To see this note that the constraints containing the Kalman gain $\kappa_{t+1}$ (i.e. **(C.4.1)** and **(C.3.3)**) imply that $\psi_t\left(\sigma_\varepsilon^2\right)^{-1}(\pi_t + z_t) = -\varpi_t$. With this result equations **(C.4.5'')** and **(C.4.6'')** will collapse to

$$-\pi_t - \mu_t^1 + \delta E_t \mu_{t+1}^1 = 0 \qquad \textbf{(C.4.5''')}$$

$$-\mu_t^2 + \delta E_t \mu_{t+1}^1 = 0 \qquad \textbf{(C.4.6''')}$$

These equations correspond with the first order conditions **(B.5)** and **(B.6)** respectively for the case of perfect knowledge (see Appendix B.1).[26]

Hence, the first order condition is

$$E_t \pi_{t+2} = 0 \qquad \textbf{(B.9)}$$

[26] There $\mu_t^2 = 0$ as we have only one constraint, formulated in terms of the inflation *forecast* $x_t = E_t \pi_{t+1}$ rather then in terms of the actual inflation rate, and the output *forecast* $E_t z_{t+1}$ becomes the (indirect) control .