Optimal Allocation Mechanisms When Bidders Ranking for the Objects is Common

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Abstract

Search engines commonly use "sponsored links", where certain advertisers' links are promoted to be placed above others in return for monetary payment. It is natural to assume that all providers value a higher ranked placement more than lower ranked ones. Then how should the seller optimally sell these ranked slots is critical for the search engines. In this paper we study the seller's (search engine) optimal selling mechanism in the following setting: buyers (advertisers), each of whom has unit demand, compete for positions offered by the seller. While each buyer's valuation for each position is private and independent, the ranking for these positions is common among all the buyers. However the rate at which these valuations change might be different. We begin with 4 simplified scenarios specifying how buyers valuations change for different positions, namely,"parallel", "convergent", "divergent", and "convergent then divergent". We find that the optimal incentive compatible allocation mechanism is quite different in determining the "pivot" types and the order to fill in the positions. Under some conditions, these mechanisms are even efficient in terms of maximizing the total welfare of the auctioneer and bidders. When the buyers' valuations for lower positions decrease at different rates, the seller earns more than the case of simple second-price sequential auction.

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1 Introduction

The rise of "e-business" over the last few years has led to a revival of interest in well-known economic and strategic problems, such as the design of on line auctions [19], price dispersion of homogenous goods, [6], online reputation [5], just to give a few examples. Arguably, it has also generated some qualitatively new problems to which the methods of economic theory could be brought to bear. This paper addresses one new phenomenon arising in question with the widespread popularity of search engines such as Google.com. More specifically, this is one arising from the presence of "sponsored links" on the output of a key-word search through internet search engines. In contrast to the results generated through the search engine's retrieving algorithm, these links are explicitly sold to firms that have an interest in advertising their products to the search engine users. The total paid placement market is now worth some $2 billion a year (2003), and is widely credited for the revitalization of the search engine business [1].

In allocating the paid slots to advertisers, the order of these links matters – because a higher placement on a search page leads to higher traffic, and eventually an increased financial payoff [8, 7]. Therefore the earlier link is more valuable than its successors. However, it is not clear how much more valuable it is and the difference of valuations could quite evidently depend on the identities of the advertisers concerned. For example, Walmart might not see much difference between obtaining the first slot and the second; however a small firm seeking to use these links to catch the consumer's eye might find much more value in a higher link than in a lower one, even though it values each link less than Walmart might do. This gives rise to a mechanism design problem from the point of view of the seller, where buyers only need one object, they value objects differently, their rankings of valuations are common, but the rates

[1] http://www.economist.com/displaystory.cfm?story_id=1932434

at which these valuations change may be different for different buyers.

In practice, most of the search engines use auctions to sell these paid slots[10]. For example, Overture sells their paid positions mainly through real time pay-your-bid auctions. Goto.com even showed the current winning prices next to the paid links, like an open auction. Google uses some variant of a second-price auction through their so called "AdWord Program". However, no theoretical research has been done to show the optimal mechanism of selling these ranked objects. Will a sequential auction that sells one position a time be better than a simultaneous auction? Should the highest position be auctioned off earlier than a lower position? Is the highest rejected bid auction optimal? Should the auctioneer leave some positions unfilled? This paper seeks to design an optimal selling mechanism from the perspective of the seller (search engine).

This kind of allocation can also be applied to other contexts. An example could be a scheduling problem where several tasks are waiting in a line for processing, each with a certain approaching deadline, and tasks bid for the position in the waiting queue. Or the selling of a set of condos where buyers' preferences towards the locations of these condos are roughly ranked in the same way.

The problem of allocating multiple objects to individuals who have different preferences for the objects has been studied in the matching literature (Roth and Sotomayor (90))[20]. However, the problem we consider here is different in that we focus our attention on Bayesian mechanism design.

The closest literature related to this paper is optimal (multiple unit) auction design. We follow very closely Myerson(81) [15], where he studied the optimal mechanism to sell a single object, and found that it is optimal to sell the object to the bidder with the highest valuation, given his virtual value [2] is non-negative. As a generalization, Maskin and Riley (1990) [13] studied the optimal auction of selling multiple identical objects, using a similar approach. For optimal auctions with heterogeneous objects, many papers consider multi-unit demand. Then

[2] In Myerson (81), the virtual value is defined as $t_i - \frac{1-F(t_i)}{f(t_i)}$

the question of demand reduction arises [3]; when the objects are complements or substitutes, or when the number of buyers is small or large, whether or not to sell the items separately or in bundles, [17], [4], [2]. Menezes (1998) [14] studied a pooled auction in the environment of identical or perfectly correlated objects where every bidder submits a single bid, and bidders with higher bids are given the rights to choose their ideal objects earlier. The environment in [14] is similar to one of the four distinct cases we study in this paper. For efficiency in multi-unit auctions, Dasgupta and Maskin (2000) [9] and Perry and Reny (1999) [18] show that when bidders' values are interdependent, the Vickrey mechanism can be generalized to achieve efficiency, as long as each bidder's signal is one dimensional. Jehiel and Moldovanu (2001) [12] shows that when bidders have multi-dimensional signals, efficiency is usually not obtained.

The problem we address in this paper is different from the above two streams. We consider the *optimal* allocation of non-identical objects, where buyers' values for these objects are ranked in the same order, and each buyer only needs one object (unit demand). Buyers have independent private values. We categorize the environment into four cases based on how bidders' valuation drops with the rank of the positions, relative to other bidders, namely, "parallel", "convergent", "divergent", and "convergent then divergent". In each case, the optimal allocation rule and payment scheme is characterized. We find that the optimal allocation is quite different under these different cases. Thus understanding the buyers' preference characteristics is vital in determining the optimal mechanism. More specifically, as long as the buyers' valuations for lower positions decrease at different rates, the seller can extract more buyers surplus than the standard second-price auction can, because the optimal expected payment in our mechanism is at least as high as the next highest buyer's valuation for that particular position; under some conditions these optimal allocation rules maximize the total welfare of the seller and the buyers, thus they are also socially efficient. We also find that this optimal mechanism cannot usually be implemented by simple sequential "highest rejected bid" auctions.

This paper is organized as follows. In section 2 we introduce the model and notation. We then discuss the optimal mechanism under four different specifications of buyers preferences in Section 3. Some issues related to the implementation of the optimal mechanism are discussed in Section 4, and we conclude in Section 5.

2 Model

Assume that a set of risk neutral buyers $N = \{1, 2, ...n\}$ compete for $K < N$ positions. Buyers have independent private types. Buyer i's type t_i is distributed over the interval $T_i = [a, b]$ $(a \geq 0)$ according to the distribution function F_i with associated density function f_i. Let $T = \times_{j=1}^{n} T_j$ denote the product of the set of buyers' types, and for all i, let $T_{-i} = \times_{j \neq i} T_j$. Define $f(t)$ to be the joint density of $t = (t_1, t_2, ..., t_n)$. Similarly, define $f_{-i}(t_{-i})$ to be the joint density of $t_{-i} = (t_1, ..., t_{i-1}, t_{i+1}, ..., t_n)$. Since the types are independent, $f(t) = f_1(x_1) \times f_2(x_2) \times ... \times f_n(x_n)$, and $f_{-i}(t_{-i}) = f_1(x_1) \times ... \times f_{i-1}(x_{i-1}) \times f_{i+1}(x_{i+1}) \times ... \times f_n(x_n)$.

Let $v_i^k(t_i)$ represent buyer i's valuation for the k'th position, which is non-increasing in k. For simplicity we write $v_i^k(t_i)$ and v_i^k interchangeably. Let t_0 represent the seller's type, so his valuation for position k is given by v_0^k. Assume that all buyers' valuation functions are either parallel, or there exists a position $\mu \in (-\infty, \infty)$ for which all the bidders have the same valuation and this is common knowledge. Assume for a certain position, we can separate the expression of the difference between types from the expression of the difference between any two values. More specifically, assume $v_i^k(t_i) - v_j^k(t_j) = (t_i - t_j)S(k)$, $i, j = 0, 1, ...n$, where $S(k)$ is independent of t. For simplicity, in this paper we assume buyers' valuations are linear in the rankings.

By the "Revelation Principle" ([1], [11], [16]), without loss of generality, we restrict our attention to direct mechanisms. Let $P : T \rightarrow \triangle$ represent the allocation rule, where $\triangle$ is the set of positions, and $X : T \rightarrow R^N$ represent the payment rule. Our goal is to identify the optimal mechanism (P, X) which is incentive compatible and individually rational. Following Myerson, let $p_i(t)$ represent the probability for buyer i to win one position and $x_i(t)$ be the buyer i's expected payment for his winning position. More specifically, let $p_i^k(t)$ represent the probability that buyer i wins the kth position, $x_i^k(t)$ be buyer i's expected payment for the kth position. Then we have $p_i(t) = \sum_K p_i^k(t)$, and $x_i(t) = \sum_K x_i^k(t)$.

Suppose the seller uses the direct mechanism (P, X). Then buyer i's expected utility is:

$$U_i(p, x, t_i) = \sum_K \int_{T_{-i}} \left[v_i^k(t_i) p_i^k(t_i, t_{-i}) - x_i^k(t_i, t_{-i}) \right] f_{-i}(t_{-i}) dt_{-i} \tag{1}$$

The seller's expected utility is:

$$U_0(p, x) = \sum_K \int_T \left[v_0^k(t) \left(1 - \sum_N p_i^k(t) \right) + \sum_N x_i^k(t) \right] f(t) dt \tag{2}$$

where

$$p_i^k(t) \geq 0 \ \ \forall i, \ \ \forall k, \ \ \forall t \in T \tag{3}$$

$$\sum_N p_i^k(t) \leq 1 \ \ \forall k, \ \ \forall t \in T \tag{4}$$

$$\sum_K p_i^k(t) \leq 1 \ \ \forall i, \ \ \forall t \in T \tag{5}$$

For the buyers, the "Individual Rationality" condition ensures that by not participating, a buyer can guarantee himself a payment of zero:

$$U_i(p, x, t_i) \geq 0 \ \ \forall i, \ \ \forall t_i \tag{6}$$

The "Incentive Compatibility" condition ensures that every buyer reports his true type. This is written as:

$$U_i(p, x, t_i; t_i) \geq U_i(p, x, s_i; t_i) = \sum_K \int_{T_{-i}} \left[v_i^k(t_i) p_i^k(t_{-i}, s_i) - x_i^k(t_{-i}, s_i) \right] f_{-i}(t_{-i}) dt_{-i} \ \ \forall i, \ \ \forall t_i, \ \ \forall s_i \neq t_i \tag{7}$$

Thus our goal is to identify the optimal $p_i^k(t)$ and $x_i^k(t)$ to maximize the expected payoff of the seller. That is,

$$\begin{aligned} &\max \left(\, 2 \right) \\ \text{s.t.} \quad &(3), (\, 4), (\, 5), (6), and (\, 7) \end{aligned}$$

Let $Q_i(p, t_i) = \sum_K \int_{T_{-i}} S(k) p_i^k(t_i, t_{-i}) f_{-i}(t_{-i}) dt_{-i}$.

Proposition 1 *When $S(k) \geq 0$, an allocation mechanism is feasible if and only if:*

$$\text{if } s_i \leq t_i, \quad \text{then } Q(p, s_i) \leq Q(p, t_i) \tag{8}$$

$$U_i(p, x, t_i) = U_i(p, x, a) + \int_a^{t_i} Q_i(p, s_i) ds_i \tag{9}$$

$$U_i(p, x, a) \geq 0 \tag{10}$$

(3), (4), and (5)

Please refer to appendix A.1 for proof. When $S(k) < 0$, everything follows through except (9) now becomes:

$$U_i(p, x, t_i) = U_i(p, x, b) - \int_{t_i}^{b} Q_i(p, s_i) ds_i \tag{11}$$

and (10) becomes

$$U_i(p, x, b) \geq 0 \tag{12}$$

.

We will discuss this further in section 3.3.

Re-arrange the objective function (2):

$$\begin{aligned} U_0(p, x) &= \textstyle\sum_K \int_T \left[v_0^k(t) \left(1 - \sum_N p_i^k(t)\right) + \sum_N x_i^k(t) \right] f(t) dt \\ &= \textstyle\sum_K \int_T v_0^k(t) f(t) dt + \sum_K \int_T \sum_N p_i^k(t) \left(v_i^k(t) - v_0^k(t)\right) f(t) dt \\ &\quad \textstyle + \sum_K \int_T \sum_N \left(x_i^k(t) - p_i^k(t) v_i^k(t)\right) f(t) dt \end{aligned} \tag{13}$$

For the last term of Eq.(13), using Eq.(9),

$$
\begin{aligned}
& \textstyle\sum_K \int_T \sum_N \left(x_j^k(t) - p_i^k(t) v_j^k(t) \right) f(t) dt \\
= \quad & -\textstyle\sum_K \sum_N \int_a^b U_i^k(p_i^k, x_i^k, t_i) f(t_i) dt_i \\
= \quad & -\textstyle\sum_K \sum_N \int_a^b \left(U_i^k(p_i^k, x_i^k, a^k) + \int_a^{t_i} Q_i^k(p_i^k, s_i) ds_i \right) f(t_i) dt_i \\
= \quad & -N \cdot U_i(p, x, a) - \textstyle\sum_N \int_a^b \left(\int_a^{t_i} Q_i(p, s_i) ds_i \right) f(t_i) dt_i \\
= \quad & -N \cdot U_i(p, x, a) - \textstyle\sum_N \int_a^b \left(\int_{s_i}^b Q_i(p, s_i) \right) f(t_i) dt_i ds_i \\
= \quad & -N \cdot U_i(p, x, a) - \textstyle\sum_N \int_a^b \left(1 - F_i(s_i)\right) Q_i(p, s_i) ds_i \\
= \quad & -N \cdot U_i(p, x, a) - \textstyle\sum_N \int_a^b \left((1 - F_i(t_i)) \sum_K \int_{T_{-i}} S(k) p_i^k(t) f_{-i}(t_{-i}) dt_{-i} \right) dt_i \\
= \quad & -N \cdot U_i(p, x, a) - \textstyle\sum_N \int_T (1 - F_i(t_i)) \sum_K S(k) p_i^k(t) f_{-i}(t_{-i}) dt_{-i} \\
= \quad & -N \cdot U_i(p, x, a) - \textstyle\sum_K \int_T \sum_N \left(S(k) \frac{1-F(t_i)}{f(t_i)} \right) p_i^k(t_i, t_{-i}) f(t) dt
\end{aligned}
$$

Plug this back to Eq.(13):

$$
\begin{aligned}
& U(p_i, x_i, t_0) \\
= \quad & \textstyle\sum_K \int_T v_0^k f(t) dt - N \cdot U_i(p, x, a) + \sum_K \int_T \sum_N \left[(v_i^k(t_i) - v_0^k) - S(k) \frac{1-F(t_i)}{f(t_i)} \right] p_i^k(t_i, t_{-i}) f(t) dt
\end{aligned} \tag{14}
$$

Maximizing(14) is equivalent to:

$$
\max \sum_K \int_T \sum_N \left[(v_i^k(t_i) - v_0^k) - S(k) \frac{1 - F_i(t_i)}{f_i(t_i)} \right] p_i^k(t_i, t_{-i}) f(t) dt - N \cdot U_i(p, x, a) \tag{15}
$$

such that

$$
U_i(p, x, a) \geq 0
$$

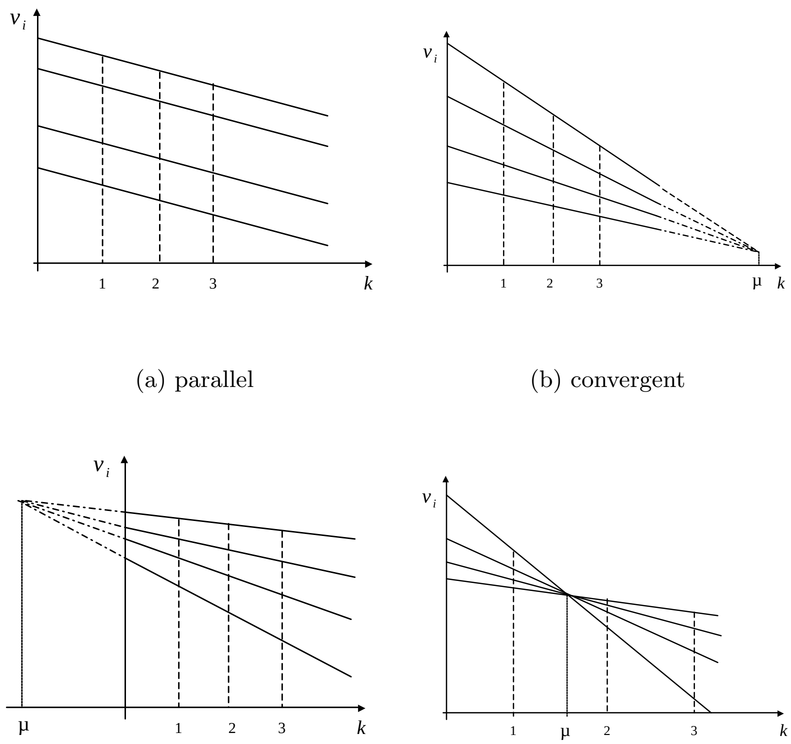
$$
Q_i(p, x, s_i) \leq Q_i(p, x, t_i), \quad \text{if} \;\; s_i \leq t_i
$$

$$
p_i^k(t) \geq 0 \;\; \forall i, \;\; \forall k, \;\; \forall t \in T
$$

$$
\sum_N p_i^k(t) \leq 1 \;\; \forall k, \;\; \forall t \in T
$$

$$
\sum_K p_i^k(t) \leq 1 \;\; \forall i, \;\; \forall t \in T
$$

Since buyers' valuations drop for a lower ranked position, based on how their valuations drop relative to other's (they may drop in the same rate, or some may drop faster/slower than the others), we categorize the situations into four cases, namely, parallel, convergent, divergent, and convergent then divergent. We are going to discuss the optimal mechanism for these different cases in Section 2. Figure 1 shows these 4 cases, where μ represents the position for which each buyer has the same valuation.



(a) parallel

(b) convergent

(c) divergent

(d) convergent and divergent

Figure 1: Different cases of buyers preferences with respect to the ranking of the positions

3 Four Different Cases

3.1 The Parallel Case

First consider the case where every buyer's valuation for a lower position drops at the same rate. In the example of search engine advertising (paid placement), if the competing advertisers have relatively the same taste or budget, the change of their valuation for positions may be relatively stable, thus we may approximate their valuation function by assuming that their valuations drop at the same rate. Let $v_i^k(t_i) = t_i - \alpha k$, where $\alpha > 0$ is a constant. Thus the type of the buyers are characterized by the intersections of their utility functions with the value axis (vertical axis in the figure). From now on assume that the seller's valuations for all the items are 0. Then $v_i^k(t_i) - v_0^k = t_i - \alpha k$ (here $S(k) = 1$). So Eq.(15) becomes:

$$\max \sum_K \int_T \left(\sum_N \left(t_i - \frac{1 - F_i(t_i)}{f_i(t_i)} - \alpha k \right) p_i^k(t) \right) f(t)dt - N \cdot U_i(p, x, a) \tag{16}$$

This is very similar to Myerson(1981)'s optimal auction design problem. Notice that in the objective function, the expression of t_i and k can be separated. Define $c(t_i)$ as the modified virtual value, which is represented by the t_i term in the objective function (in this case, $c(t_i) = t_i - \frac{1-F_i(t_i)}{f_i(t_i)}$. If the regularity condition is satisfied that $t_i - \frac{1-F_i(t_i)}{f_i(t_i)}$ is strictly increasing in t_i, and if we can impose that $U_i(p, x, a) = 0$, then Eq.(16) is maximized when the objects are assigned to the K buyers with the highest types (t_i), given that their virtual values are higher than αK. More importantly, as long as the winners are determined, it doesn't matter which buyer gets which object. This is because in the objective function, the part containing t_i and the part containing k can be fully separated. Thus the total contribution of the winning types to the objective function remains the same no matter which position k they are assigned. Notice the reserve price is a constant for each position according to this allocation rule ($r(k) = solves\{c(t_i) - \alpha K = 0\}$). That is to say, as long as a buyer is eligible to win the last position(i.e., if the buyer has the K'th highest non-negative virtual value), he is eligible to win every other position.

To formally state the allocation rule, let $C_j(t_{-i})$ as the j'th highest virtual value among all the buyers except i. Then define

$$z_k(t_{-i}) = inf\{s_i | c(s_i) - \alpha K \geq 0 \; and \; c(s_i) \geq C_k(t_{-i}), \;\; k = 1, 2, ..., K\} \tag{17}$$

Proposition 2 *In the parallel case, the optimal incentive compatible allocation rule is to allocate one position to each of the K bidders with the highest modified virtual values, given that their types ($t_i s$) satisfy that $c(t_i) - \alpha K \geq 0$. The allocation of the positions among the winners* *is inconsequential. In other words,*

$$p_i(t_{-i}, s_i) = \begin{cases} 1 & if \quad s_i \geq z_K(t_{-i}) \\ 0 & if \quad s_i < z_K(t_{-i}) \end{cases} \tag{18}$$

Please refer to Appendix A.2 for proof.

Now consider the payment scheme x. According to Eq.(9) and Eq.(1), $U_i(p,x,a)+\int_a^{t_i} Q_i(p,s_i)ds_i = \sum_K \int_{T_{-i}} \left[v_i^k(t_i)p_i^k(t_i,t_{-i}) - x_i^k(t_i,t_{-i})\right] f_{-i}(t_{-i})dt_{-i}$, the optimal expected payment function is determined by:

$$\begin{aligned} \sum_K x_i^k(t) &= \sum_K v_i^k p_i^k(t) - \sum_K \int_a^{t_i} S(k)p_i^k(s)f(s)ds \\ &= \sum_K v_i^k p_i^k(t) - S(k)\int_a^{t_i} S(k)p_i^k(s)f(s)ds \\ &= \sum_K v_i^k p_i^k(t) - S(k)(t_i - z_K(t_{-i})) \end{aligned} \tag{19}$$

So if buyer i wins position k, his payment will be $x_i^k = v_i^k - v_i^k + Z_K^k$, where Z_K^k is defined as the K'th highest type (other than i) buyer's valuation for the k'th position, or buyer $z_K(t_{-i})'$s valuation for position k. This means, it doesn't matter for a winning buyer which position he is allocated, as long as he is paying the $K+1$'s buyer's valuation for that particular position. His utility is the same because of the parallel characteristics of the value functions.

This mechanism can be implemented as a "pseudo-second-price auction", where every bidder bids their type, and the highest K bidders win. The higher their types, the higher the positions they are allocated. And each one pays the highest rejected bidder's valuation for his winning position. To better understand this mechanism, assume that buyers types follow a uniform distribution between $[0,1]$. Then the reserve price for each position is the same, that is, $\frac{1+\alpha K}{2}$. If there are 3 positions for sale and α is 0.05, the reserve price is 0.575 for each position . If the realized types are $t_1 = 0.9$, $t_2 = 0.8$, $t_3 = 0.75$, $t_4 = 0.7$, then the positions 1, 2, 3 will be allocated to buyer 1, 2, 3, respectively, with the expected payment for those position $v^1(0.7)$, $v^2(0.7)$, $v^3(0.7)$, which are, 0.65, 0.6, 0.55. On the other hand, if the realized types are: $t_1 = 0.9$, $t_2 = 0.7$, $t_3 = 0.5$, $t_4 = 0.2$. Then t_1 is allocated to position 1, t_2 is allocated to position 2, while position 3 is not allocated.

3.2 The Convergent Case

This describes the case that the higher type buyer's valuation drops faster, while still remain higher, than a lower type buyer's valuation for a lower position. In other words, the lower position means less to a high type buyer than to a low type buyer. For example, if a relatively unknown company has a big marketing budget to attract search engine user traffic, it may have a strong incentive to win a higher position, but its valuation for a lower position may drop much more quickly than its competitors, because the reduced attention from those lower positions does not serve the company's strategic goal.

Let $v_i^k = \beta - t_i(k - \mu)$, where $\mu > K$ is the position for which each buyer has the same valuation. In this example, μ is the horizontal value of the point where each utility function crosses. Here t_i is no longer the utility function's intersection on the y axis. It not only affects the intersection ($t_i\mu$ part), but also represents the slope of the utility function. This function guarantees that the higher type buyer (larger t_i) has a higher valuation for each position than a lower valued buyer. Thus $v_i^k - v_j^k = (t_i - t_j)(\mu - k)$, and $S(k) = (\mu - k) > 0$.

So Eq.(15) becomes:

$$\max \sum_K \int_T \left[\sum_N \left(\beta + \left(t_i - \frac{1 - F_i(t_i)}{f_i(t_i)} \right) (\mu - k) \right) p_i^k(t) \right] f(t)dt - N \cdot U_i(p, x, a) \tag{20}$$

Define the modified virtual value $c(t_i)$ the same way as in section 3.1. In this case, $c(t_i)$ again is represented by $t_i - \frac{1-F_i(t_i)}{f_i(t_i)}$. To maximize this expression, if the distribution function satisfies the regularity condition, again the positions should be allocated to those buyers with the largest virtual values, that is, the K buyers with highest t_is, whose types also satisfy $\beta + c(t_i)(\mu - k) \geq 0$. Notice that the reserve price $r(k) = solve\{c(t_i) \cdot (\mu - k) + \beta = 0\}$ is decreasing in k. However, different from Myerson (81), the winning modified virtual values can be *negative* because of the presence of a positive constant β in the objective function.

How to allocate the positions among the buyers? Similar to Sec. 3.1 (Eq.17), define

$$z_k(t_{-i}) = inf\{s_i | \beta + c(s_i) \cdot (\mu - k) \geq 0 \; and \; c(s_i) \geq C_k(t_{-i}), \;\; k = 1, 2, ..., K\} \tag{21}$$

and $z_0(t_{-i})$ equal to b, the upper bound of the buyers' value distribution. Then

Proposition 3 *The optimal incentive compatible allocation rule when the modified virtual value* $t_i - \frac{1-F_i(t_i)}{f_i(t_i)}$ *is non-decreasing is to allocate the higher positions to the buyers with higher modified virtual values, as long as* $\beta + \left(t_i - \frac{1-F_i(t_i)}{f_i(t_i)}\right)(\mu - k) \geq 0$. *In other words,*

$$p_i^k(t_{-i}, s_i) = \begin{cases} 1 & \text{if} \quad z_k(t_{-i}) \leq s_i \leq z_{k-1}(t_{-i}) \;\; \forall k \\ 0 & \text{otherwise} \end{cases} \tag{22}$$

Please refer to Appendix A.3 for proof of this proposition.

Thus, in this case the allocation rule satisfies the three criteria that Menezes (98) [14] mentioned: zero expected payoff for the lowest value; the K highest valued bidders win; and the higher the value, the higher the position allocated. According to this allocation rule, each position has a reserve price and the reserve prices are decreasing in the ranking of the positions; at the same time, the $t_i s$ receiving these positions are also decreasing in the rank of positions. That means, for a certain position k, if there is no t_i that satisfies the reserve price condition, then that particular position will not be allocated, but a position lower than that may still be allocated. This unallocated position can occur on the top, in the middle, or the bottom of the ranking. In practice there are some measures that the seller can take to make a specific position unavailable. For example, if there are K top paid links in a search engine and the k'th position is not sold, then the search engine can insert one of its own ads (an ad about the search engine itself) into that slot or insert a fake web link there; if these K positions represent the order in a queue where all the jobs are waiting for processing, and the k'th slot is unallocated, then the seller can deliberately delay the processing for all the jobs after the k'th. [3]

The payment function is determined according to Eq.(9):

$$\sum_K x_i^k = \sum_K v_i^k p_i^k - \sum_K \int_a^{t_i} S(k) p_i^k f(s) ds$$

[3] The case when there is a constraint that no lower position can be allocated before a higher ranked one is filled (that is, $\sum_{i \in N} p_i^k \geq \sum_{i \in N} p_i^{k+1}, \;\; \forall k = 1, 2, ..., K-1$) is not considered here. Please see Sec 4.

Notice that if buyer i is allocated the first object, then Eq.(19) becomes that:

$$x_i^1 = v_i^1 - \int_a^{t_i} S(1) p_i^1 f(s) ds$$

Define $Z_j^k(t_{-i})$ as in Sec.3.1 (buyer z_j's valuation for the k'th position), then $\int_a^{t_i} S(1)p_i^1 f(s)ds = S(1)(t_i - z_1(t_{-i}) = v_i^1 - Z_1^1(t_{-i})$. Thus the optimal payment for the first object is $x_i^1 = Z_1^1(t_{-i})$.

Now consider the second object. The buyer i can win the second object only if his type is between the first highest and second highest buyers' type other than his own. Thus we get $\int_a^{t_i} S(2)p_i^2 f(s)ds = S(2)(t_i - z_2(t_{-i}|z_2 \leq t_i \leq z_1) prob(z_2 \leq t_i \leq z_1) = (v_i^2 - Z_2^2(t_{-i}|z_2 \leq t_i \leq z_1)) \cdot prob(z_2 \leq t_i \leq z_1)$. Thus the optimal payment for the second object is:

$$\begin{aligned} x_i^2 &= v_i^2(1 - prob(z_2 \leq t_i \leq z_1) + Z_2^2(t_{-i}|z_2 \leq t_i \leq z_1) \cdot prob(z_2 \leq t_i \leq z_1) \\ &= v_i^2 \left(1 - \frac{\int_0^{t_i} \frac{(n-1)!}{(n-3)!(1)!} F(y)^{n-3}(1-F(t_i))f(y)dy}{\int_0^{t_i} \frac{(n-1)!}{(n-3)!(1)!} F(y)^{n-3}(1-F(y))f(y)dy}\right) + \frac{\int_0^{t_i} y \frac{(n-1)!}{(n-3)!(1)!} F(y)^{n-3}(1-F(t_i))f(y)dy}{\int_0^{t_i} \frac{(n-1)!}{(n-3)!(1)!} F(y)^{n-3}(1-F(y))^1 f(y)dy} \end{aligned}$$

The optimal payment for the rest of the positions can be obtained in the same way.

In general,

Proposition 4 *The optimal payment for the first position is $x_i^1(t) = Z_1^1(t_{-i})$ if $p_i^1(t) = 1$ and for the k'th position ($k > 1$) is:*

$$\begin{aligned} x_i^k(t_i) &= v_i^k(1 - prob(z_k \leq t_i \leq z_{k-1})) + Z_k^k(t_{-i}|z_k \leq t_i \leq z_{k-1}) \cdot prob(z_k \leq t_i \leq z_{k-1}) \\ &= v_i^k \left(1 - \frac{\int_0^{t_i} \frac{(n-1)!}{(n-1-k)!(k-1)!} F(y)^{n-1-k}(1-F(t_i))^{k-1}f(y)dy}{\int_0^{t_i} \frac{(n-1)!}{(n-1-k)!(k-1)!} F(y)^{n-1-k}(1-F(y))^{k-1}f(y)dy}\right) + \frac{\int_0^{t_i} y \frac{(n-1)!}{(n-1-k)!(k-1)!} F(y)^{n-1-k}(1-F(t_i))^{k-1}f(y)dy}{\int_0^{t_i} \frac{(n-1)!}{(n-1-k)!(k-1)!} F(y)^{n-1-k}(1-F(y))^{k-1}f(y)dy} \end{aligned} \tag{23}$$

if $p_i^1(k) = 1, k = 2, ...K$

Lemma 1 *In the convergent case, in expectation, if a buyer wins a position, he will pay at least as much as the next highest type bidder's valuation for that position.*

Please refer to Appendix A.3 for proof.

Thus it is obvious that the seller can extract much more surplus from the buyers than in the parallel case, where every winner pays $K+1'$th highest buyer's valuation for his winning position. More importantly, this mechanism performs better than the simple second-price sequential auction, where in the best scenario the buyers pay the next highest valuation for the winning position. Intuitively, when different buyers' valuations for lower positions fall at different rates, the seller has an incentive to optimally match the position to the buyers to maximize his expected payoff. To charge a higher price for a lower position can prevent a higher type buyer from shading his bid to win a lower position. This increases the seller's expected payoff, comparing to the sequential second-price auction.

To better understand this mechanism, let's assume that buyers' types follow a uniform distribution between $[0,1]$. Then for the k'th position, the reserve price is $\frac{1}{2} - \frac{\beta}{2(\mu-k)}$, which is decreasing in k. For example, if there are 3 positions available, and $\mu = 6$, $\beta = 1$, then the reserve price for positions 1, 2, 3 are $\frac{2}{5}$, $\frac{3}{8}$, $\frac{1}{3}$, respectively. If there are 4 buyers with realized types $t_1 = 0.8$, $t_2 = 0.6$, $t_3 = 0.4$, $t_4 = 0.2$, then the position 1, 2, 3 will be allocated to buyer 1, 2, 3, respectively, with the expected payment for those position $v_1^1(0.6)$, $v_2^2(0.467)$, $v_3^3(0.333)$. On the other hand, if the realized types are: $t_1 = 0.8$, $t_2 = 0.35$, $t_3 = 0.32$, $t_4 = 0.2$. Then t_1 is allocated to position 1, t_2 is allocated to position 3, while position 2 is not allocated.

3.3 The Divergent Case

This describes the case that the higher type buyer's valuation drops slower for a lower position than a lower type buyer's. For example, Amazon.com is a big player in the paid placement market. It spends a big amount of marketing money in attracting customers in every search engine, but it probably doesn't care which positions it wins. While a small company's valuation for a lower position may drop much faster.

Let $v_i^k = \beta - t_i(k-\mu)$, where $\mu < 1$. Then $v_i^k - v_j^k = (t_i - t_j)(\mu - k)$, and $S(k) = (\mu - k) < 0$. We can repeat the analysis of the last section, except that because $S(k) < 0$, some of the incentive compatibility conditions ((9) and (10)) should be rewritten as (11) and (12):

$$U_i(p, x, t_i) = U_i(p, x, b) - \int_{t_i}^{b} Q_i(p, s_i) ds_i$$

$$U_i(p, x, b) \geq 0$$

Re-arrange the last term of Eq.(13),

$$
\begin{aligned}
& \textstyle\sum_K \int_T \left(x_j^k(t) - p_i^k v_j^k(t)\right) f(t)dt \\
= & \ -\textstyle\sum_K \int_a^b U_i^k(p_i^k, x_i^k, t_i) f(t_i)dt_i \\
= & \ -\textstyle\sum_K \int_a^b \left(U_i^k(p_i^k, x_i^k, b) - \int_{t_i}^b Q_i^k(p_i^k, s_i)ds_i\right) f(t_i)dt_i \\
= & \ -U_i(p, x, b) + \textstyle\int_a^b \left(\int_a^{t_i} Q_i(p_i, s_i)ds_i\right) f(t_i)dt_i \\
= & \ -U_i(p, x, b) + \textstyle\int_a^b \left(\int_a^{s_i} Q_i(p, s_i)\right) f(t_i)dt_i ds_i \\
= & \ -U_i(p, x, b) + \textstyle\int_a^b \left(F_i(s_i)\right) Q_i(p, s_i)ds_i \\
= & \ -U_i(p, x, b) + \textstyle\int_a^b \left(F_i(t_i) \sum_K \int_{T_{-i}} S(k)p_i^k(t)f_{-i}(t_{-i})dt_{-i}\right) dt_i \\
= & \ -U_i(p, x, b) + \textstyle\int_T F_i(t_i) \sum_K S(k)p_i^k(t)f_{-i}(t_{-i})dt_{-i}
\end{aligned}
$$

Then the objective function becomes:

$$\max \sum_K \int_T \left(\sum_N \left(\beta + \left(t_i + \frac{F_i(t_i)}{f_i(t_i)} \right) S(k) \right) p_i^k(t) \right) f(t)dt - N \cdot U_i(p, x, b_i) \tag{24}$$

Again define $c(t_i) = t_i + \frac{F_i(t_i)}{f_i(t_i)}$ as the modified virtual value. If the modified virtual value is non-decreasing in t_i, (for example, uniform distribution, exponential distribution satisfy this condition), since $S(k)$ is negative, this objective function will be maximized if the K lowest types (t_is) have been selected, given that $\beta + \left(t_i + \frac{F_i(t_i)}{f_i(t_i)}\right) S(k)$ is non-negative. Notice that the "reserve price" $r(k) = solve\{c(t_i) \cdot (\mu - k) + \beta = 0\}$ is again decreasing in k, thus the lower the position, the tighter the reserve price condition. Further more, according to this allocation rule, the lower t_is are allocated the lower positions with tighter reserve price conditions. Thus we can allocate the positions from the bottom to the top. If for a certain position k we can not find any buyer's type lower than the reserve price, then this mechanism just automatically shifts all the allocated positions up for 1 rank. In other words, after the buyers' type are realized, we can identify the number of positions available ($\tilde{K}$) by calculating how many buyer's types are lower than the modified virtual value. Then identify the $\tilde{K}$ winners, and allocate the highest position to the highest t_i, and so on. Thus all the unavailable slots (if any) occurs neither in the top, nor in the middle, but in the bottom.

More specifically, define

$$d_k(t_{-i}) = sup\{s_i | \beta + c(s_i) \cdot (\mu - k) \geq 0 \ and \ c(s_i) \leq C_k(t_{-i}), \ \ k = 1, 2, ..., K\} \tag{25}$$

and $d_0(t_{-i})$ equal to a, the lower bound of the buyers' value distribution. Then

Proposition 5 *The optimal incentive compatible allocation mechanism in the diverging case is to allocate the lower position to the buyers with lower modified virtual values, given that buyer's type satisfies the reserve price condition. In other words,*

$$p_i^k(t_{-i}, s_i) = \begin{cases} 1 & \text{if} \;\; d_{\tilde{K}-k}(t_{-i}) \leq s_i \leq d_{\tilde{K}-k+1}(t_{-i}) \;\; \forall k \\ 0 & \text{otherwise} \end{cases} \tag{26}$$

where $\tilde{K}$ is the number of total available slots.

Proof is in appendix A.4. One thing needs to be noted is, if $t_i + \frac{F_i(t_i)}{f_i(t_i)}$ is increasing, when β is large enough, more specifically, when $\beta > -\left(b + \frac{1}{f(b)}\right) S(K)$, which means if every buyers' valuation for the last position is high enough, since the reserve price condition is the tightest for the last position, each buyer's type satisfies the reserve price condition for the rest of the positions (in other words, the reserve price condition becomes $t_i \leq b$). Thus this mechanism is automatically efficient and maximize the total payoff of the buyers and the seller.

The payment scheme can be worked out accordingly as in section 3.2. That is, the optimal payment for the lowest position (K) is $D_1^K(t_{-i})$; and the optimal payment for the position $1 \leq k < K$ is:

$$\begin{aligned} x_i^k(t_i) &= v_i^k(1 - prob(d_{K-k} \leq t_i \leq d_{K-k+1}) + D_k^k(t_i | d_{K-k} \leq t_i \leq d_{K-k+1}) \\ &\cdot prob(d_{K-k} \leq t_i \leq d_{K-k+1}) \\ &= v_i^k \left(1 - \frac{\int_{t_i}^1 \frac{(n-1)!}{(n-1-k)!(k-1)!} F(t_i)^{k-1}(1-F(y))^{n-1-k} f(y)dy}{\int_{t_i}^1 \frac{(n-1)!}{(n-1-k)!(k-1)!} F(y)^{k-1}(1-F(y))^{n-1-k} f(y)dy}\right) + \frac{\int_{t_i}^1 y \frac{(n-1)!}{(n-1-k)!(k-1)!} F(t_i)^{k-1}(1-F(y))^{n-1-k} f(y)dy}{\int_{t_i}^1 \frac{(n-1)!}{(n-1-k)!(k-1)!} F(y)^{k-1}(1-F(y))^{n-1-k} f(y)dy} \end{aligned} \tag{27}$$

Thus other than the buyer with the lowest modified virtual value (the buyer with the least steep slope), all the other winners are paying higher than the next buyer's valuation for that winning position. It is obvious that this mechanism works better than an simple second-price sequential auction, in which the lower positions are up for sale first, where the best scenario is to earn the next highest buyer's valuation for a winning position.

On the other hand, if $t_i + \frac{F_i(t_i)}{f_i(t_i)}$ is decreasing in t_i, then to allocate the lower position to the buyers with lower modified virtual value actually means the higher the t_i, the greater probability to win an item. This violates the incentive compatible constraint (8) that $Q(p_i, x_i, s_i) \leq Q(p_i, x_i, t_i)$, if $s_i \leq t_i$. In this case, randomization is one way to allocate the position. This is incentive compatible, but so far we do not have any results for optimality.

Again, if we assume that buyers' types follow uniform distribution between $[0, 1]$, then the reserve price condition for position k is: $t_i \leq \frac{\beta}{2(k-\mu)}$, where the reserve price is decreasing in k again, and the reserve price condition is tighter with the increase in the ranking. Thus the unallocated position will only be in the bottom. But if β is large enough such that $\beta \geq 2(K - \mu)$, then every type satisfies the reserve price condition and this mechanism is automatically efficient.

For example, if there are 3 positions available, and $\mu = 0$, $\beta = 3$, then the reserve price for positions 1, 2, 3 are 1, 0.75, 0.6, respectively. If there are 4 buyers with realized types $t_1 = 0.8$, $t_2 = 0.6$, $t_3 = 0.4$, $t_4 = 0.2$, then the position 1, 2, 3 will be allocated to buyer 2, 3, 4, respectively, with the expected payment for those position $v_2^1(0.7102)$, $v_3^2(0.5333)$, $v_4^3(0.4)$, which are 0.8694, 0.8668, 1, respectively, while their values for their winning positions are 1.2, 1.4, 2, respectively. On the other hand, if the realized types are: $t_1 = 0.9$, $t_2 = 0.85$, $t_3 = 0.8$, $t_4 = 0.2$. Then t_3 is allocated to position 1, and t_4 is allocated to position 2, while position 3 is not allocated. As we showed, the unallocated positions are always be in the bottom.

3.4 Convergent then Divergent

This is the extreme case of the convergent case, where if a buyer has a higher valuation for a top position, his valuation for a lower position may be lower than his competitors. This means different buyers utility functions are allowed to cross in the middle. Again in the example of search engine advertisers, a small company's willingness to pay for a top position may be higher than an established big company like Amazon.com. But because a small company often has a tight budget, its valuation for a bottom position may be much lower than Amazon. Write the utility function as $v_i^k = \beta - t_i(k - \mu)$, where $(k - \mu) > 0$ before a certain $\widetilde{k} \in [1, K]$ and after that $\widetilde{k}$, $(k - \mu) < 0$ is the point where each utility function crosses. Analyzing the utility function in the same way, $v_i^k - v_j^k = (t_i - t_j)(\mu - k)$, and $S(k) = (\mu - k) > 0$ when $k \leq \widetilde{k}$ and $S(k) < 0$ when $k > \widetilde{k}$.

Again one of the incentive compatibility conditions (9) should be rechecked because the sign of $S(k)$ changes before and after $k = \widetilde{k}$.

More specifically, there exists a $w \in (a, b)$ such that we can rewrite the expression of Eq. (9) into

$$\begin{array}{c} U_i(p,x,t_i) = U_i(p,x,w) + \int_w^{t_i} Q_i(p,s_i)ds_i \quad \text{if} \quad t_i \geq w \\ \text{and} \\ U_i(p,x,t_i) = U_i(p,x,w) - \int_{t_i}^{w} Q_i(p,s_i)ds_i \quad \text{if} \quad t_i < w \end{array} \tag{28}$$

and we have

$$U_i(p,x,w) \geq 0$$

The objective function becomes:

$$\begin{aligned} & \textstyle\sum_K \int_T \left(x_j^k(t) - p_i^k(t) v_j^k(t) \right) f(t)dt \\ = \quad & -\textstyle\sum_K \int_a^w U_i^k(p_i^k, x_i^k, t_i) f(t_i)dt_i - \sum_K \int_w^a U_i^k(p_i^k, x_i^k, t_i) f(t_i)dt_i \\ = \quad & -\textstyle\sum_K \int_a^w \left(U_i^k(p_i^k, x_i^k, w) - \int_a^w Q_i^k(p_i^k, s_i)ds_i \right) f(t_i)dt_i \\ & -\textstyle\sum_K \int_w^b \left(U_i^k(p_i^k, x_i^k, w) + \int_w^b Q_i^k(p_i^k, s_i)ds_i \right) f(t_i)dt_i \\ = \quad & -U_i(p,x,w) + \textstyle\int_a^w Q_i^k(p_i^k, s_i)ds_i f(t_i)dt_i - \int_w^b Q_i^k(p_i^k, s_i)ds_i f(t_i)dt_i \\ = \quad & -U_i(p,x,w) + \textstyle\int_a^w \left(\int_w^{s_i} Q_i(p,s_i) \right) f(t_i)dt_ids_i - \int_w^a \left(\int_w^{s_i} Q_i(p,s_i) \right) f(t_i)dt_ids_i \\ = \quad & -U_i(p,x,w) + \textstyle\int_a^w (F(s_i)) Q_i(p,s_i)ds_i - \int_w^a (1 - F(s_i)) Q_i(p,s_i)ds_i \\ = \quad & -U_i(p,x,w) + \textstyle\int_a^w \left(F(t_i) \sum_K \int_{T_{-i}} S(k) p_i^k(t_i, t_{-i}) f_{-i}(t_{-i})dt_{-i} \right) dt_i \\ & -\textstyle\int_w^a \left(1 - F(t_i) \sum_K \int_{T_{-i}} S(k) p_i^k(t_i, t_{-i}) f_{-i}(t_{-i})dt_{-i} \right) dt_i \end{aligned}$$

where $\sum_K \int_a^w U_i^k(p_i^k, x_i^k, t_i) f(t_i)dt_i$ can be written as $\sum_K \int_a^w \left(U_i^k(p_i^k, x_i^k, w) - \int_a^w Q_i^k(p_i^k, s_i)ds_i \right) f(t_i)dt_i$, which implies that $\int_a^w Q_i^k(p_i^k, s_i)ds_i f(t_i)dt_i$ is negative.

Thus the objective is:

$$\begin{aligned} & \max \textstyle\sum_K \int_a^w \left[\sum_N \left(\beta + \left(t_i + \frac{F_i(t_i)}{f_i(t_i)} \right) (\mu - k) \right) p_i^k(t) \right] f(t)dt \\ & + \textstyle\int_w^b \left[\sum_N \left(\beta + \left(t_i - \frac{1 - F_i(t_i)}{f_i(t_i)} \right) (\mu - k) \right) p_i^k(t) \right] f(t)dt - N \cdot U_i(p,x,w) \end{aligned} \tag{29}$$

Assume that both the virtual value $t_i - \frac{1-F_i(t_i)}{f_i(t_i)}$ and the modified virtual value $t_i + \frac{F_i(t_i)}{f_i(t_i)}$ are non-decreasing (for example, the uniform distribution satisfies these two conditions). To maximize this objective function, if we make $U_i(p,x,w)$ equal to 0, notice that when $t_i \geq w$, $\mu - k \geq 0$, thus it is optimal to allocate the highest $\lfloor \widetilde{k} \rfloor - k$ positions to the buyers with the highest virtual values $t_i - \frac{1-F_i(t_i)}{f_i(t_i)}$; when $t_i < w$, $\mu - k < 0$, thus it is optimal to allocate the lowest $K - \lceil \widetilde{k} \rceil$ positions to the buyers with the lowest modified virtual value of $t_i + \frac{F_i(t_i)}{f_i(t_i)}$.

More specifically, proposition 6 describes this allocation rule.

Proposition 6 *If the distribution of t_i satisfies the regularity conditions and $t_i + \frac{F_i(t_i)}{f_i(t_i)}$ is non-decreasing, the optimal allocation mechanism is: for each $k < \mu$, allocated the highest remaining position to the buyers with the highest remaining t_i, as long as $t_i \geq w$ and $\beta + \left(t_i - \frac{1-F_i(t_i)}{f_i(t_i)}\right)(\mu - k) \geq 0$, until $k = \lfloor \mu \rfloor$, otherwise leave that particular position unassigned; for each $k > \mu$, allocate the remaining lowest position to the buyers with the remaining lowest t_i, as long as $t_i < w$ and $\beta + \left(t_i + \frac{F_i(t_i)}{f_i(t_i)}\right)(\mu - k) \geq 0$; otherwise shift the allocation up for 1 rank.*

Proof. Follow the cases when $\mu - k \geq 0$ and $\mu - k < 0$, which follow the proof in section 3.2 and 3.3. ◇

The above gives the optimal mechanism given a specific $w \in (a,b)$. Let this value be $V(w)$. Now the problem is how to identify that optimal w? Our objective function now becomes:

$$\max_{w} V(w) \tag{30}$$

Obviously w^* is a function of μ, n, and K. For example, the ideal w^* should have the property that there are at least $\lceil K - \mu \rceil$ buyers whose types are below w, and at least $\lfloor \mu \rfloor$ buyers whose types are above w. And this indicates that, given K, the optimal w should be non-increasing in μ. But to complete this mechanism, w should be preannounced. So there is positive probability that the above condition can not be satisfied, thus this mechanism is *not efficient* in addition to the existence of reserve price for each position, because there is positive probability that a certain buyer whose type satisfies the reserve price, will give the seller higher profit if he wins, but can not win because its type falls on the a "wrong" side of w. Figure 2 shows one example of how the optimal w changes with μ, assuming that bidders' types are uniformly distributed between [0,1], and β is large enough so the reserve price condition is always satisfied. (In this example $n = 7$, $k = 4$, and μ can be anywhere between 1 and 4.)

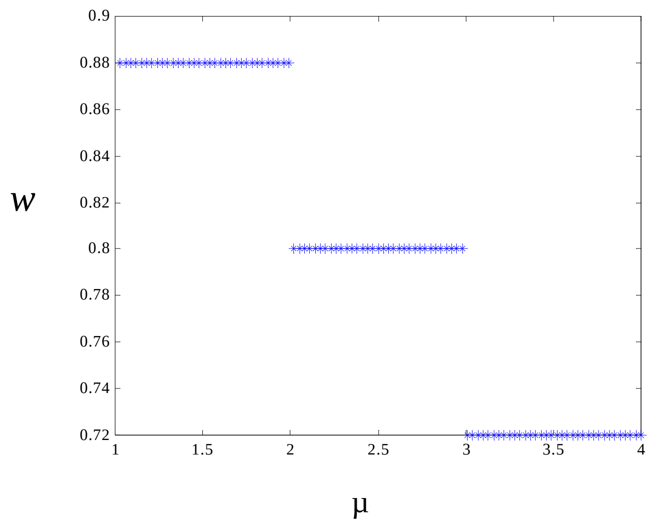


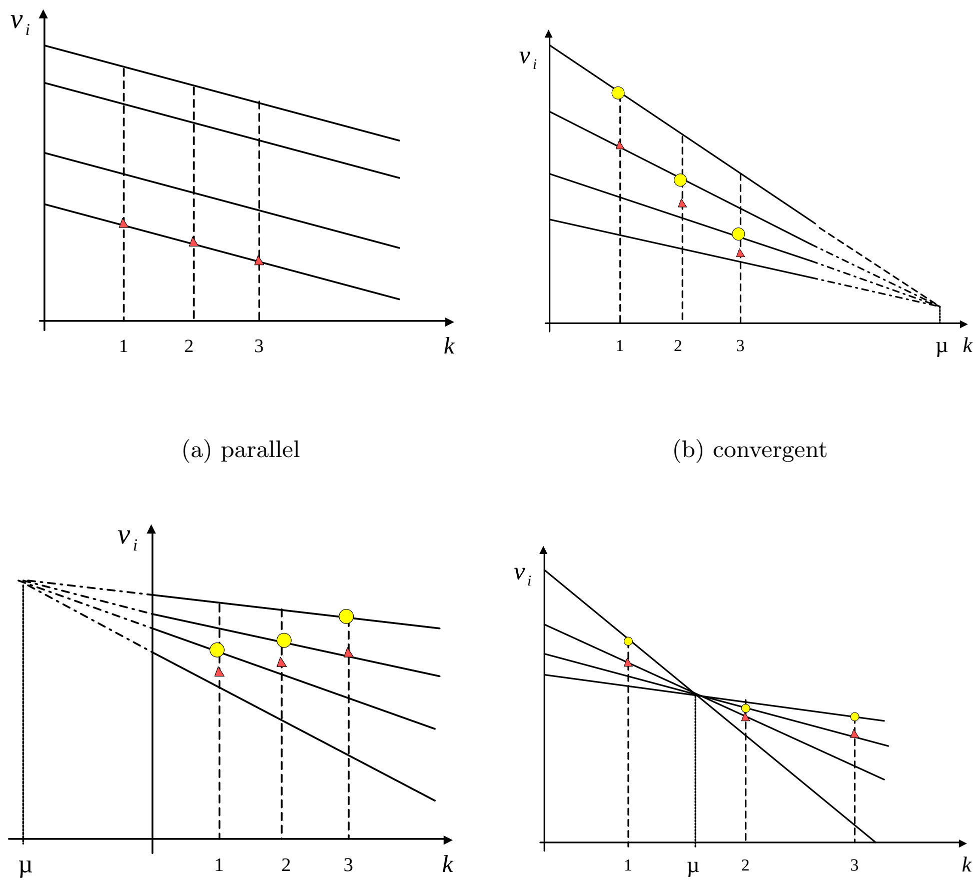
Figure 2: How the optimal w changes with μ

From Figure 2 we can see that when the reserve price condition is satisfied, w is decreasing with μ. Intuitively, the larger the μ, the more types should be above w, thus the smaller the w. But one thing needs to be noted is that once k and n are fixed, the only determinant of w is between which two positions μ is located, while the exact position of μ between those two positions does not matter.

Also in the example of uniform distribution between $[0, 1]$, the reserve price condition for position $k < \mu$ is $t_i \geq \frac{1}{2} - \frac{\beta}{2(\mu-k)}$; for position $k \geq \mu$ is $t_i \leq \frac{\beta}{2(k-\mu)}$. Combining with the choice of w, then the necessary condition to allocate a position k before μ is $t_i \geq \max\{w, \frac{1}{2} - \frac{\beta}{2(\mu-k)}\}$ and for a position after μ is $t_i < \min\{w, \frac{\beta}{2(k-\mu)}\}$. For example, let $K = 3$, $\mu = 1.5$, $\beta = 0.5$, $w = 0.5$. Then we have the actual reserve price for position 1 before $\tilde{k}$ is $\max\{0.5, 0\} = 0.5$, and the actual reserve price for position 2 and 3 (after $\tilde{k}$) are: $\min\{0.5, 1\} = 0.5$, and $\min\{0.5, 0.333\} = 0.333$. If the realized types are: $t_1 = 0.8$, $t_2 = 0.6$, $t_3 = 0.4$, $t_4 = 0.2$, then the position 1, 2, 3 will be allocated to buyer 1, 3, 4. On the other hand, if the realized types are: $t_1 = 0.45$, $t_2 = 0.4$, $t_3 = 0.3$, $t_4 = 0.2$. Then the first position is not allocated, while the second position is allocated to buyer 3, and the third position is allocated to buyer 4.

4 Implementation

Here we present an example of the optimal allocation and payment mechanism under the four different cases in figure 3.



(a) parallel

(b) convergent

(c) divergent

(d) convergent and divergent

Figure 3: Optimal allocation and payment schemes under the four different cases. (where ∘ represents the allocation, and △ represents the payment.)

Notice that in the mechanisms we discussed above, we assumed that the seller can commit to leave a position unfilled if no buyer's type satisfies the reserve price condition. We've discussed such examples in Sec 3.2. In reality this practice of leaving a position unfilled is commonly observed . For example, in the airline industry, passengers in the coach class are not allowed to sit in the first class without paying extra, even when the first class is not full. It's also observed in some competitions, sometimes the highest award given is the second prize, while the first prize remains un-assigned. This guarantees that those who can pay for first class don't understate their values, or high competition standard. In our model, this makes sure that buyers do not reduce their bids, hoping to win a more desirable position when there is lack of competition.

On the other hand, if the seller cannot commit to leave a position unfilled, that is, the k'th position has to be filled first in order to fill the $k+1'$th position, then we have an extra constraint,

$$\sum_{i \in N} p_i^k \leq \sum_{i \in N} p_i^{k-1} \quad k = 2, ...K \tag{31}$$

We find that this constraint is binding only in two cases: convergent, and the converging portion of the "convergent then divergent" case, because only in these two cases will a higher t_i be assigned to a higher position, with a higher reserve price. However this constraint changes the maximization problem. Now the reserve price should guarantee the summation of each term in the $\sum$ to be non-negative, instead of making sure each term in the $\sum$ to be non-negative. However we will not discuss this problem in detail in this paper.

5 Conclusion

In this paper we show how the earlier work about optimal auctions ([15], [13]) can be extended and applied to the allocation of non-identical objects where every buyer only has unit demand, and their preferences for these objects are ranked in the same order. We find that the optimal way to sell these non-identical objects is quite different when buyers preferences for different objects change in different way. Thus to understand the buyers preference characteristics is vital in determining the optimal mechanism. We find that our mechanism works better than simple second-price sequential auction. More specifically, when buyers' valuations for a lower position drop at different rates, the seller can extract more surplus from the buyers than when they drop at the same rate. Compare to the single unit or multiple identical unit case, besides the inefficiency created by the reserve price (under the assumption of symmetric buyers), this optimal allocation mechanism can be inefficient because of the choice of the "pivot" type in the fourth case.

We use linear value function in this paper, and assume that there exists some position for which all utility functions give the identical value. This assumption may seem special but it only says that there exists a position (probably very far away) such that every buyer's valuation for that position is the same (like a position in the very bottom of the result page). Many commonly used utility functions when dealing with heterogeneous consumers have this property (for example, $U(\theta) = \theta q$ where q is the quality of a product). In future research, we hope to study more general settings of bidders preferences than the one discussed in this paper. The case when the auctioneer can not commit to leave a position open (the addition of constraint (31) is another interesting extension.

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A Appendix

A.1 Feasibility of an Allocation Mechanism

Proposition 1. To show the "only if" part,

$$\begin{aligned}
& U(p, x, s_i; t_i) \\
= \; & \sum_K \int_{T_{-i}} \left[v_i^k(t_i) p_i^k(t_{-i}, s_i) - x_i^k(t_{-i}, s_i) \right] f_{-i}(t_{-i}) dt_{-i} \\
= \; & \sum_K \int_{T_{-i}} \left[(s_i + (t_i - s_i) S(k)) \, p_i^k(t_{-i}, s_i) - x_i^k(t_{-i}, s_i) \right] f_{-i}(t_{-i}) dt_{-i} \\
= \; & U_i(p, x, s_i) + \sum_K \int_{T_{-i}} ((t_i - s_i) S(k)) \, p_i^k(t_{-i}, s_i) f_{-i}(t_{-i}) dt_{-i} \\
= \; & U_i(p, x, s_i) + (t_i - s_i) Q_i(p, s_i)
\end{aligned}$$

The incentive compatibility constraint implies that:

$$U_i(p, x, t_i; t_i) \geq U_i(p, x, s_i; t_i) + (t_i - s_i) Q_i(p, s_i) \;\; \forall s_i \tag{32}$$

Use (32) twice we get:

$$(t_i - s_i) Q_i(p, s_i) \leq U_i(p, x, t_i) - U_i(p, x, s_i) \leq (t_i - s_i) Q_i(p, t_i) \tag{33}$$

So

$$Q_i(p, s_i) \leq Q_i(p, t_i) \tag{34}$$

when $s_i \leq t_i$

Let $t_i - s_i = \delta$, then (33) can also be written as:

$$\delta Q_i(p, s_i) \leq U_i(p, x, s_i + \delta) - U_i(p, x, s_i) \leq \delta Q_i(p, s_i + \delta) \tag{35}$$

Since $Q_i(p, s_i)$ is increasing in s_i, thus this equation is integrable and can be written as: $\int_a^{t_i} Q_i(p, s_i) ds_i = U_i(p, x, t_i) - U_i(p, x, a)$, so

$$U_i(p, x, t_i) = U_i(p, x, a) + \int_a^{t_i} Q_i(p, s_i) ds_i \tag{36}$$

From the other direction (the "if" part), to show (32), assume $s_i \leq t_i$, then using (8) and (9) we get:

$$\begin{aligned}
U_i(p, x, t_i) \; &= \; U_i(p, x, s_i) + \int_{s_i}^{t_i} Q_i(p, r_i) dr_i \\
&\geq \; U_i(p, x, s_i) + \int_{s_i}^{t_i} Q_i(p, s_i) dr_i \\
&= \; U_i(p, x, s_i) + (t_i - s_i) Q_i(p, s_i)
\end{aligned}$$

If $s_i > t_i$, then

$$\begin{aligned}
U_i(p,x,t_i) &= U_i(p,x,s_i) - \int_{t_i}^{s_i} Q_i(p,r_i)dr_i \\
&\geq U_i(p,x,s_i) - \int_{t_i}^{s_i} Q_i(p,s_i)dr_i \\
&= U_i(p,x,s_i) + (t_i - s_i)Q_i(p,s_i)
\end{aligned}$$

So when $S(k) \geq 0$, (p_i, x_i) is an optimal mechanism if it satisfies (8), (9), (10), (3), (4), and (5) and maximizes (2). $\diamond$

A.2 The Parallel Case

Proposition 2.

The optimality of this allocation rule is obvious because (16) will be maximized if we pick the buyers with the K highest virtual values $(c(t_i))$, given $t_i - \frac{1-F_i(t_i)}{f_i(t_i}$ is non-negative.

To check for the incentive compatibility constraint that $Q_i(p,x,s_i) \leq Q_i(p,x,t_i)$ when $s_i \leq t_i$, notice that if $c(t_i) \geq \max\{C_K(t_{-i}), \alpha k\}$, $j \in n$, where $C_K(t_{-i})$ is the K'th highest virtual value among all the other buyers, then he wins. Since $S(k) = 1$, this equation purely means the conditional probability of winning 1 item given type t_i is higher than type s_i $(s_i \leq t_i)$. Since the highest K buyers win, and $c(t_i)$ is increasing, we know that the probability that $c(s_i) \geq C_k(t_{-i})$ is increasing in s_i. So whenever buyer i could win by submitting s_i, he could also win by submitting t_i where $t_i > s_i$. So $Q(p,x,t_i)$ is indeed increasing in t_i. $\diamond$

A.3 The Convergent Case

Proposition 3. Since $S(k) = \mu - k$ is positive, the objective function will be maximized if the seller allocates the higher position (larger $\mu - k$) to the buyers of higher type (larger $t_i - \frac{1-F_i(t_i)}{f_i(t_i)}$. To see this, let y_j be the $j^{'}$th highest value of $t_i - \frac{1-F_i(t_i)}{f_i(t_i)}$, comparing $y_j(\mu-k) + y_{j+1}(\mu-k-1)$ and $y_{j+1}(\mu-k) + y_j(\mu-k-1)$. Note that the difference between these two expression is that:$y_j - y_{j+1} > 0$. This can be generalized to the case where there are more than 2 values. $\diamond$

Lemma 1. We only need to show for a specific position k, $v_i^k - Z_k^k(t_i)$ is no less than

$(v_i^k - Z_k^k(t_i|z_k \leq t_i \leq z_{k-1})) \cdot prob(z_k \leq t_i \leq z_{k-1})$. This is obvious because:

$$
\begin{aligned}
& v_i^k - Z_k^k(t_i) \\
= \; & (v_i^k - Z_k^k(t_i|z_k \leq t_i \leq z_{k-1})) \cdot prob(z_k \leq t_i \leq z_{k-1}) \\
& + (v_i^k - Z_k^k(t_i|z_k \leq t_i \;\&\; z_{k-1} \leq t_i)) \cdot prob(z_k \leq t_i \;\&\; z_{k-1} \leq t_i)
\end{aligned}
$$

and both of the terms to the right of "=" are non-negative. $\diamond$

A.4 The Divergent Case

Proposition 5. First we want to show that to allocate a lower position to a lower modified virtual value (lower type) is optimal. Let $0 < A_1 < A_2 < A_3$ and $0 < B_1 < B_2 < B_3$. The objective is to minimize $\sum_{i,j} A_i B_j$. And $A_1B_3 + A_2B_2 + A_3B_1 < A_1B_2 + A_2B_3 + A_3B_1$ because it is equivalent to $A_1(B_3 - B_2) + A_2(B_2 - B_3) < 0$; $A_1B_3 + A_2B_2 + A_3B_1 < A_1B_1 + A_2B_2 + A_3B_3$ because it is equivalent to $A_1(B_3 - B_1) + A_3(B_1 - B_3) = (A_1 - A_1)(B_3 - B_1) < 0$. This result can be generalized to the case where $i \geq 3$.

To check whether this allocation rule is incentive compatible, revisit the constraint (8) that $Q(p_i, x_i, s_i) \leq Q(p_i, x_i, t_i), \text{if} s_i \leq t_i$. Notice that here $Q(p_i, x_i, t_i) = \sum_K \int_{T_{-i}} S(k) p_i^k f_{-i}(t_{-i}) dt_{-i} \leq 0$. Thus the higher t_i, the less likely that the buyer is going to win, and the less negative $S(k)$, thus constraint (8) is indeed satisfied. $\diamond$