Monetary Policy and the Transition to Rational Expectations

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(This version April 2004)

Abstract

Under the assumption of bounded rationality, economic agents learn from their past mistaken predictions by combining new and old information to form new beliefs. The purpose of this paper is to examine how the policy-maker, by affecting private agents' learning process, determines the speed at which the economy converges to the rational expectation equilibrium. I find that by reacting strongly to private agents' expected inflation, a central bank would increase the speed of convergence.

I assess the relevance of the transition period when looking at a criterion for evaluating monetary policy decisions and suggest that a fast convergence is not always suitable.

1 Introduction

There is wide consensus on the fact that monetary policy may affect real variables in the short run. One recent strand of research that has obtained this result explicitly incorporating frictions in a dynamic general equilibrium framework, gives a central role for forecasts of future courses of the economy (Clarida, Galí and Gertler, 1999 and Woodford, 2003). This litterature has been conducted under the rational expectations hypothesis, meaning that agents do not make systematic forecasting errors and their guesses about the future are on average correct.

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Many recent models in the analysis of economic policy emphasize, however, that the RE assumption should not be taken for granted, since expectations can be out of equilibrium at least for a period of time. It can be argued, for example, that in the presence of policy regime change the public needs to learn about the new regime: in the early stages of this process, previously held public beliefs could lead to biased predictions. For this reason there is now a substantial interest on the question of whether RE can be attained as the outcome of a learning process. Analyses on monetary policy and learning (Evans and Honkapohja, 2002a, 2002b, 2003 and Bullard and Mitra, 2002) suggest that economic policies should be designed to be conducive to long-run convergence of private expectations to rational expectations (*E-Stability*)[1], in order to avoid asymptotic instability in the economy. These works are extensively devoted to the study of the asymptotic properties of the equilibrium attainable under learning. A small but growing body of litterature is concerned with the dynamic properties along the convergence process. Papers centred on the analysis of the transition to the REE include Marcet and Sargent (1995), Timmerman (1996), Sargent (1999), Marcet and Nicolini (2003), Giannitsarou (2003), Aoki and Nikolov (2003) and Orphanides and Williams (2003).

The purpose of this work is to examine how the policy-maker, by affecting private agents' learning process, can influence the transition to the rational expectations equilibrium. I show that policies driving the economy to the same asymptotic REE may imply very different transitional dynamics. By reacting strongly to expected inflation, a central bank can shorten the transition and increase the speed of convergence to the REE. This is particularly relevant when policy decisions aim to influence social welfare: if policy-makers know that after a regime change private agents' expected inflation would be higher than the REE, they can substantially increase social welfare by choosing a policy that reacts strongly to expected inflation. If, instead, perceived inflation is initially lower than the REE, a weak response to expected inflation and a slow transition might be preferable.

In order to study the transition in the learning process to the REE I adapt arguments described by Marcet and Sargent (1995), which in turn are based on the theoretical results of Benveniste, Metivier and Priouret (1990).

The paper is organized as follows. Section 2 presents the monetary policy problem, describing the learning dynamics under two different policy rules. The section ends showing that under the optimal RE discretionary policy, transition to the REE is very slow. In section 3 I show how policies could be characterized by evaluating the speed of converges to the REE. In section 4 I study policies that allow the central bank to shorten (or extend) the transition without affecting the long-run equilibrium (i.e., the REE equilibrium) and in section 5 I analyze how these policies influence social welfare. Section 6 summarizes and concludes.

[1] An earlier paper by Howitt (1992) had already shown that under some interest rate rules the rational expectation equilibrium is not learnable.

2 The framework

2.1 The baseline model

Much of the recent theoretical analysis on monetary policy has been conducted under the "New Phillips curve" paradigm reviewed in Clarida, Galí and Gertler (1999) and Woodford (2003). The baseline framework is a dynamic general equilibrium model with money and temporary nominal price rigidities. I consider the linearized reduced form of the economy with competitive monopolistic firms, staggered prices and private agents that maximize intertemporal utility. From the private agents' point of view there is an intertemporal *IS curve*[2]

$$x_t = E_t^* x_{t+1} - \varphi \left(i_t - E_t^* \pi_{t+1} \right) + g_t \tag{2.1}$$

and an aggregate supply (AS) modeled by an expectations-augmented *Phillips curve*[3]:

$$\pi_t = \alpha x_t + \beta E_t^* \pi_{t+1}, \tag{2.2}$$

where x_t is the output gap, measured as the log deviation of actual output (y_t) from potential output (z_t) (i.e., the level of output that would arise if wages and prices were perfectly competitive and flexible), π_t is actual inflation at time t, $E_t^* \pi_{t+1}$ is the level of inflation expected by private agents for period $t+1$, given the information at time t. Similarly $E_t^* x_{t+1}$ is the level of the output gap that private agents expect for period $t+1$, given the information at time t. I write E_t^* to indicate that expectations need not be rational (E_t without $*$ denotes RE); i_t is the short-term nominal interest rate and is taken to be the instrument for monetary policy; g_t is a demand shock, $g_t = \rho_g g_{t-1} + \varepsilon_{gt}$ with $\varepsilon_{gt} \sim N\left(0, \sigma_g^2\right)$ and i.i.d.

In order to complete the model, it is necessary to specify how the interest rate is settled and how agents form beliefs. I consider the nominal interest rate as the policy instrument and model it by means of a reaction function, that is, a functional relationship between a dependent variable (the interest rate) and some endogenous (expected inflation and output gap) and exogenous (shocks) variables. I consider three cases. I start with a simple expectations-based policy rule that helps me to introduce in a very simple and intuitive way the concept of speed of convergence. Then, I describe the optimal RE policy under discretion derived in Evans and Honkapohja (2002)[4]. Finally, I

[2]The IS relationship approximates the Euler equation characterizing optimal aggregate consumption choices and the parameter φ can be interpreted as the rate of intertemporal substitution.

[3]The AS relation approximates aggregate pricing emerging from monopolistically competitive firms' optimal behaviour in Calvo's model of staggered price determination. Here I'm not considering cost-push shocks. Introducing cost-push shocks, would not change substantially the results on speed of convergence and the role of policy decisions along the transition. However, in section 5 I analyze briefly results in terms of welfare also in presence of cost-push shocks.

[4]I leave for future research a general study of the transition of learning process for monetary policy problem under commitment

introduce a set of expectations-based policy rules and show how to characterize the elements of this set, using a measure of the speed of convergence.

Concerning beliefs, I start each analysis by considering the rational expectations hypothesis in order to focus and discuss subsequently the implications of bounded rationality.

2.2 A simple expectations-based reaction function

It has long been recognized that monetary policy needs a forward-looking dimension. Let us assume that the central bank, in order to set the current interest rate, uses simple policy rules that feed back from expected values of future inflation and output gap

$$i_t = \gamma + \gamma_x E_t^* x_{t+1} + \gamma_\pi E_t^* \pi_{t+1}. \tag{2.3}$$

The class of *expectations-based reaction functions* that I first consider has $\gamma_x = \frac{1}{\varphi}$ in order to simplify the interaction between actual and expected variables. Under (2.3), in fact, the economy evolves according to the following system of equations:

$$Y_t = Q + F E_t^* Y_{t+1} + S g_t \tag{2.4}$$

where

$$Y_t = \begin{bmatrix} \pi_t \\ x_t \end{bmatrix},$$

$$Q = \begin{bmatrix} -\alpha\varphi\gamma \\ -\varphi\gamma \end{bmatrix}, \quad F = \begin{bmatrix} \beta + \alpha\varphi(1-\gamma_\pi) & 0 \\ \varphi(1-\gamma_\pi) & 0 \end{bmatrix}, \quad S = \begin{bmatrix} \alpha \\ 1 \end{bmatrix} \tag{2.5}$$

and neither the IS nor the AS are affected by expectations on output gap[5].

Under rational expectations (i.e. $E_t^* x_{t+1} = E_t x_{t+1}$ and $E_t^* \pi_{t+1} = E_t \pi_{t+1}$) it has been shown that the dynamic system defined by (2.4) has a unique non-explosive equilibrium (Bullard and Mitra, 2002). In particular, assuming for simplicity[6] that $\rho_g = 0$, the equilibrium can be written as a linear function of a constant and the shock

$$\pi_t = \overline{a}_\pi + \alpha g_t \qquad \text{and} \qquad x_t = \overline{a}_x + g_t, \tag{2.6}$$

while agents' forecasts are just constant

$$E_t \pi_{t+1} = \overline{a}_\pi \quad \text{and} \quad E_t x_{t+1} = \overline{a}_x. \tag{2.7}$$

[5] For a more general class of expectations-based policy rules without restrictions on γ_x I refer to section 3.

[6] Considering an i.i.d stochastic process instead of an AR(1) does not affect the results on speed of convergence. However, since the litterature usually consider AR(1) shocks, the welfare analysis in section 5 is obtained assuming a persistent demand shock.

2.2.1 Adaptive Learning

Let us assume now that private agents form expectations by learning from past experiences and update their forecasts through recursive least squares estimates[7].

Since, under the *simple expectations-based reaction function* (2.3), neither the IS nor the AS relations depend on expected output gap, the system under learning can be described by focusing directly on beliefs regarding expected inflation[8].

I assume that agents do not know the effective value of $\overline{a}_{\pi}$ in equation (2.5), but estimate it using past information. In this case, private agents' expected inflation is given by:

$$E_t^* \pi_{t+1} = a_{\pi,t}, \tag{2.8}$$

where $a_{\pi,t}$ is a statistic inferred recursively from past data according to

$$a_{\pi,t} = a_{\pi,t-1} + t^{-1} \left(\pi_{t-1} - a_{\pi,t-1} \right). \tag{2.9}$$

Forecasts are updated by a term that depends on the last prediction error[9] weighted by the *gain sequence* t^{-1}. It is well known that in this case the adaptive procedure is the result of a least squares regression of inflation on a constant, and perceived inflation is just equal to the sample mean of past inflations:

$$a_{\pi,t} = \frac{1}{t} \sum_{i=1}^{t} \pi_{i-1}. \tag{2.10}$$

An important aspect of recursive learning is that agents' beliefs may converge to RE, i.e., the estimated parameters $a_{\pi,t}$ may converge asymptotically to $\overline{a}_{\pi}$. The E-Stability principle (Evans and Honkapohja, 2001) provide conditions for asymptotic stability of the REE under least squares learning.

Before analyzing speed of convergence I describe briefly E-stability, since the building blocks of the two concepts are the same. The stability under learning (E-stability) of a particular equilibrium, is addressed by studying the mapping from the estimated parameters, i.e., the perceived law of motion (PLM), to the true data generating process, i.e., the actual law of motion (ALM).

When expectations in system (2.4) evolve according to expression (2.8), the inflation's ALM is

$$\pi_t = T\left(a_{\pi,t}\right) + \alpha g_t, \tag{2.11}$$

where

$$T\left(a_{\pi,t}\right) = -\alpha\varphi\gamma + \left[\beta + \alpha\varphi\left(1 - \gamma_{\pi}\right)\right] a_{\pi,t} \tag{2.12}$$

is the mapping from PLM to ALM of inflation.

[7] See Marcet and Sargent (1989 a, b) or Evans and Honkapohja (2001) for a detailed analysis of least squares learning.

[8] In the next section I show formally that this does not affect the results.

[9] This formula implies that private agents do not use today's inflation to formulate their forecasts. The assumption is made purely for convenience and it is often made in models of learning as it simplifies solving the model. The dynamics of the model are unlikely to change.

As shown in Marcet and Sargent (1999a,b) and Evans and Honkapohja (2001), it turns out that the dynamic system described by equations (2.9), (2.11) and (2.12) can be studied in terms of the associated *ordinary differential equation* (ODE)

$$\frac{da_\pi}{d\tau} = h(a_\pi) = T(a_\pi) - a_\pi, \tag{2.13}$$

where τ denotes "notional" or "artificial" time and $h(a_\pi)$ is the asymptotic mean prediction error, i.e. the mean distance between the ALM and the PLM:

$$h(a_\pi) = \lim_{t\to\infty} \left[T(a_{\pi,t}) - a_{\pi,t}\right]. \tag{2.14}$$

The REE is said to be E-stable if it is locally asymptotically stable under equation (2.14) and under some regularity conditions. In our example E-stability conditions are readly obtained by computing the derivative of the ODE with respect to a_π and checking wether it is smaller than zero[10].

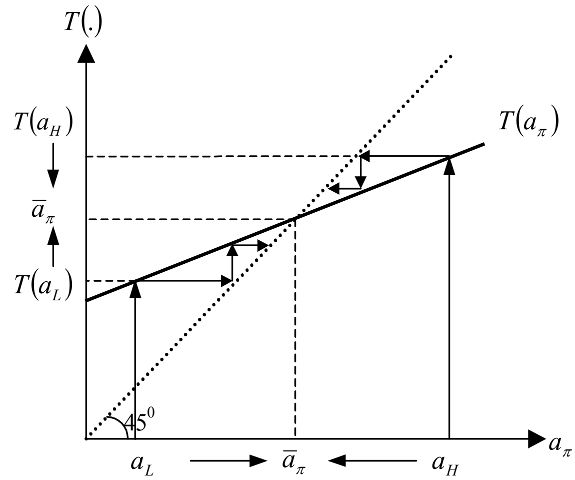
2.2.2 Speed of convergence to the REE

It turns out that policy decisions (i.e., choices concerning γ_π) are important not only to describe asymptotic properties of the equilibrium under learning, but also to determine the speed at which the distance between PLM and ALM shorten over time.

Figure 1 plots the mapping from PLM to ALM (2.12) and shows how private agents' estimates affect actual inflation along the transition to the REE.

Fig 1

Mapping from PLM to ALM



[10]This coincide with checking wether the derivative of the mapping from PLM to ALM is smaller than 1.

First of all note that if the slope of the mapping is smaller than 1, the REE is E-stable. In other words, if the economy starts from a perceived level of inflation $a_L < \overline{a}_\pi$ or $a_H > \overline{a}_\pi$, the mean of the prediction error (i.e., the mean distance between the ALM and the $T(.)$ mapping), $T(a_{\pi,t}) - a_{\pi,t}$, decreases over time and asymptotically converges to zero (i.e., it converges to the point $\overline{a}_\pi$).

Is there any difference between a policy that results in the slope of $T(.)$ equal to 0.01 and one with the slope equal to 0.99? The recent literature on monetary policy and learning (Evans and Honkapohja 2001, 2002a, 2002b, 2003 and Bullard and Mitra 2002), by focusing on asymptotic properties of the equilibrium, does not answer this question. Since in both cases the REE is unique and E-stable, both policies are "good"[11].

In the literature, the problem of the speed of convergence of recursive least square learning algorithms has been analyzed mainly through numerical procedures and simulations[12]. The few analytical results are obtained by using a theorem of Benveniste, Metiver and Priouret (1990) that relates the speed of convergence of the learning process to the eigenvalues of the associated ordinary differential equation (ODE) at the fixed point[13]. In the present case, the ODE to be analyzed is the one described in expression (2.14) and the associated eigenvalue is the slope of the mapping from PLM to ALM (2.12).

The following propositions, adapting arguments in Marcet and Sargent (1995), show that by choosing the γ_π, the policy-maker not only determines the level of inflation in the long run, but also the speed at which the distance between perceived and actual inflation narrows over time.

Proposition 1 *Let us define*

$$S_1 = \left\{ \gamma_\pi : \gamma_\pi > \frac{\alpha\varphi + \beta - 1/2}{\alpha\varphi} \right\}$$

Under the simple expectations-based reaction function (2.3), if $\gamma_\pi \in S_1$, then

$$\sqrt{t}\left(a_{\pi,t} - \overline{a}_\pi\right) \xrightarrow{D} N\left(0, \sigma_a^2\right)$$

with

$$\sigma_a^2 = \frac{\alpha^2 \sigma_g^2}{\left[1 - \beta - \alpha\varphi\left(1 - \gamma_\pi\right)\right]} \tag{2.15}$$

Proof. See Appendix A. ■

If the conditions of Proposition 1 are satisfied, the estimates $a_{\pi,t}$ converge to the REE, $\overline{a}_\pi$, at root-t speed. Root-t is the speed at which, in classical econometrics, the mean of the distribution of the *least square estimates* converges to the true value of the parameters estimated. Note that the formula for the

[11]Here, and thereafter, with "good" policy I refer to the criterion used by Bullard and Mitra (2002) to evaluate policy rules, based on determinacy and E-stability of the REE

[12]Orphanides and Williams (2003) have recently analyzed the role of monetary policy decisions along the transition to the REE by means of simulations.

[13]See see for example Marcet and Sargent (1995) for an interpretation of the ODE.

variance of the estimator a_π is modified with respect to the classical case where $\sigma_a^2 = \alpha^2\sigma_g^2$. Proposition 2 shows that for lower values of γ_π convergence is slower.

Proposition 2 *Under the simple expectations-based reaction function (2.3), if $\gamma_\pi \in S_1$, then the weaker the response to expected inflation (the smaller γ_π), the greater the asymptotic variance of the limiting distribution, σ_a^2.*

Proof. See Appendix B. ■

Looking at the formula for the asymptotic variance (2.15) it is easy to understand the role of policy decisions in determining the speed of convergence to the REE: for a weaker response to expected inflation, the convergence is slower in the sense that the asymptotic variance of the limiting distribution is greater.

What happen when the slope of the mapping (2.12) is smaller than 1, but bigger than 0.5? Let us define $S_2 = \left\{\gamma_\pi : \frac{\alpha\varphi+\beta-1}{\alpha\varphi} < \gamma_\pi < \frac{\alpha\varphi+\beta-1/2}{\alpha\varphi}\right\}$. If $\gamma_\pi \in S_2$, the estimates $a_{\pi,t}$ converge to the REE $\overline{a}_\pi$, but at a speed different from root-t. In this case, as Marcet and Sargent (1995) suggest, the importance of initial conditions fails to die out at an exponential rate (as it is needed for *root-t convergence*) and agents' beliefs converge to RE at a rate slower than root-t. In particular, even when $\gamma_\pi \in S_2$ it is possible to show by means of simulations that as the slope of the $T(.)$ mapping increases, the speed of convergence decreases[14]. Figure 2 shows examples for the two cases where $\gamma_\pi \in S_1$ and $\gamma_\pi \in S_2$.

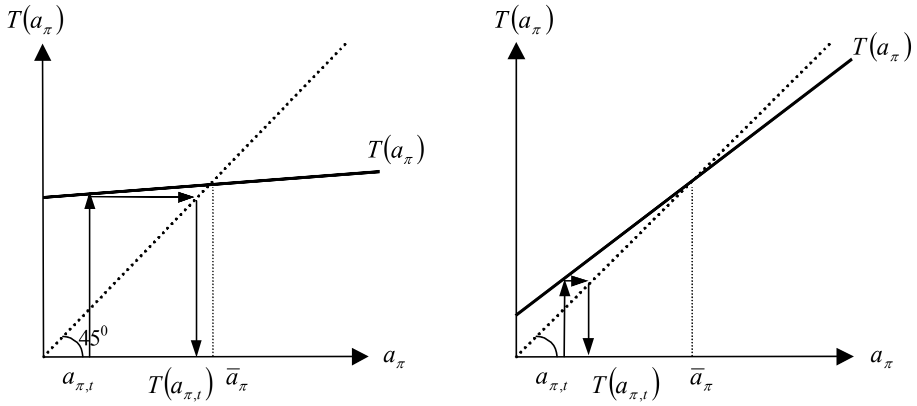
Since the least squares algorithm adjusts each parameter towards the truth when new information is received, the new belief $a_{\pi,t+1}$ will be an average of the previous beliefs $a_{\pi,t}$ and the actual value $T(a_{\pi,t})$ plus an error. When the reaction of the policy maker to expected inflation is *strong* ($\gamma_\pi \in S_1$), the derivative of $T(.)$ is smaller than (or equal to) 1/2 and $T(a_{\pi,t})$ is close to $\overline{a}_\pi$; when the reaction is *weak* ($\gamma_\pi \in S_2$), the derivative of $T(.)$ is larger than 1/2 and $T(a_{\pi,t})$ is close to $a_{\pi,t}$ instead of being close to $\overline{a}_\pi$, so the average can stay far from the REE for a long time.

[14] See section 4 for simulations that relate speed of convergence and the slope of the $T()$ mapping.

Fig. 2

Mapping from PLM to ALM and the speed of convergence

a) Low γ_π b) High γ_π



It is worth noting that even though the transition is quite different in the two cases analyzed here, the learning equilibrium could end up converging to the same REE and, according to policy-maker preferences, the speed of convergence could become a relevant variable in the policy decision problem.

2.3 Optimal monetary policy under discretion

The reason the analysis started with the *simple expectations-based reaction function* (2.3) was that it simplified the dynamics under learning. I now consider the optimal monetary policy problem without commitment (discretionary policies), where any promises made in the past by the policy-maker do not constrain current decisions. In deriving the optimal discretionary policy, I follow Evans and Honkapohja (2002), assuming that the policy-maker cannot manipulate private agent's beliefs. This assumption implies that the optimality conditions derived under learning are equivalent to the ones obtained under RE.

The policy problem consists in choosing the time path for the instrument i_t to engineer a contingent plan for target variables π_t and $(x_t - \overline{x})$ that maximizes the objective function

$$\underset{x_t,\pi_t}{Max} - E_0 \sum_{t=0}^{\infty} \beta^t L(\pi_t, x_t)$$

where

$$L(\pi_t, x_t) = \frac{1}{2}\left[\pi_t^2 + \lambda (x_t - \overline{x})^2\right]$$

subject to the constraints (2.1) and (2.2) and $E_t^* \pi_{t+1}$, $E_t^* x_{t+1}$ given.

The solution of this problem[15], as derived in Evans and Honkapohja (2002), yields to a reaction function that relates the policy instrument i_t to the current state of the economy and private agents' expectations:

$$i_t = \gamma^\star + \gamma_x^\star E_t^* x_{t+1} + \gamma_\pi^\star E_t^* \pi_{t+1} + \gamma_g^\star g_t \tag{2.16}$$

where $\gamma^\star = -\frac{\lambda}{(\lambda+\alpha^2)\varphi}\overline{x}$, $\gamma_x^\star = \gamma_g^\star = \frac{1}{\varphi}$ and $\gamma_\pi^\star = 1 + \frac{\alpha\beta}{(\lambda+\alpha^2)\varphi}$.

Since interest rate rule (2.16) states that the policy maker should react to the expected inflation and output gap, it is sometimes called the *optimal expectations-based reaction function (*Evans and Honkapohja, 2002). However, to stress the fact that this policy is optimal under rational expectations but is not necessarily optimal under learning, it would be worth to call it the *RE-optimal expectations-based reaction function*; in order to avoid notational flutter, I call it *Evans and Honkapohja (EH) policy.*

Under rational expectations (i.e. $E_t^* x_{t+1} = E_t x_{t+1}$ and $E_t^* \pi_{t+1} = E_t \pi_{t+1}$) the equilibrium is:

$$\pi_t = E_t \pi_{t+1} = \overline{a}_\pi \quad \text{and} \quad x_t = E_t x_{t+1} = \overline{a}_x, \tag{2.17}$$

Assuming that private agents do not know $\overline{a}_\pi$ and $\overline{a}_x$ but estimate them recursively, the expected inflation and output gap evolve as described in section 2.2, while the mapping from PLM to ALM is now given by

$$T\left(a_{\pi,t}, a_{x,t}\right) = \left(\Phi^* + \Gamma^* a_{\pi,t}, \Phi^* \alpha^{-1} - \left(\beta - \Gamma^*\right) \alpha^{-1} a_{\pi,t}\right). \tag{2.18}$$

where

$$\Gamma^* = \frac{\lambda\beta}{(\lambda+\alpha^2)}, \Phi^* = \frac{\lambda\alpha}{(\lambda+\alpha^2)}\overline{x}$$

Again, since the right-hand side of (2.18) does not depend on $a_{x,t}$, properties of the equilibrium under learning can be described simply by focusing on the mapping from perceived inflation to actual inflation.

Evans and Honkapohja (2002) showing that policy (2.16) results in a unique and E-stable REE conclude that the policy derived as the optimal solution of the problem under discretion and rational expectations is also "good" under learning.

Now, if we simulate the model under the *RE-optimal expectations-based reaction function*, and Clarida, Galì and Gertler (2000) calibration[16], it turns out that the distance between the actual inflation and the REE would be significatively different from zero for many periods[17].

[15] I consider λ as an exogenous policy parameter, as is often done in monetary policy literature. An alternative approach is to obtain λ as the result of the general equilibrium problem. In this case λ would depend on representative consumer preferences and firms' price setting rules.

[16] Clarida, Galí and Gertler (2000) derive from regressions on US data, $\varphi = 1$, $\alpha = 0.3$, $\beta = 0.99$, $\rho_u = 0.35$; Woodford (1999) finds $\varphi = 0.17$, $\alpha = 0.024$, $\beta = 0.99$, $\rho_u = 0.35$. In Figure 3 I assume $\lambda = 0.5$

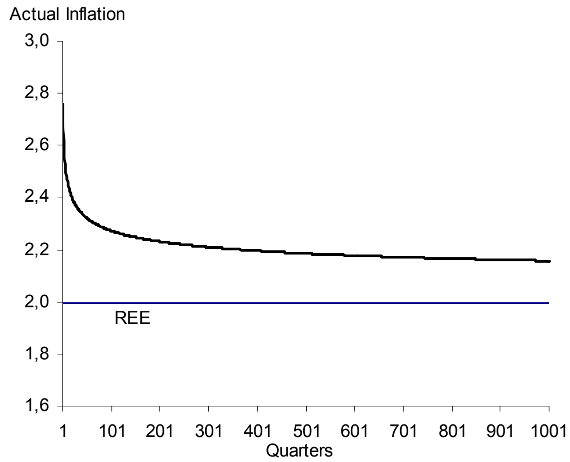
[17] In simulations, the weight that is attributed to the initial belief plays an important role. In equation (2.9), an initial t very small would imply a much higher weight to the present than to the past. In general, the bigger the t, the higher the weight given to previous belives. Here I consider $t_0 = 2$.

Figure 3 shows the evolution of perceived inflation under learning. Assuming that the policy-maker follows a flexible inflation targeting policy rule with $\lambda = 0.3$, the output gap target is $\overline{x} = 0.02$ and using CGG calibration, the REE for inflation is around 2 per cent. I consider an initial expected inflation 1 percentage point higher than the REE. After 1000 periods (quarters) perceived inflation is still 0.2 percentage points higher than the REE[18].

Fig 3

Deviation of actual inflation from the REE

$(\pi^{RE} = 2\%;\ \ E_0^*\pi_1 = 3\%)$



Applying a similar argument to that used in Propositions 1 and 2 it is possible to state the following proposition about the speed of convergence and the role of the weight to output gap in the welfare function, λ.

Proposition 3 *Under the EH policy, the speed of convergence of the learning process depends negatively on the weight that the policy-maker gives to output gap relative to inflation. In particular, under flexible inflation targeting policies* ($\lambda > 0$)*, the greater the weight to output gap, the slower the learning process.*

Proof. See Appendix C. ■

Proposition 3, by looking at the slope of the mapping from perceived inflation to actual inflation, relates the speed of convergence of the learning equilibrium to the importance of output gap in the objective function.

[18] With Woodford (2003) calibration the convergence is even slower.

Fig 4

Slope of the mapping from PLM to ALM

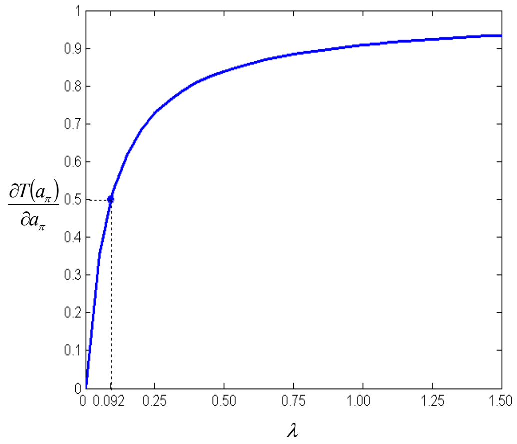


Figure 4 shows how the slope of the mapping from the PLM to ALM of inflation changes as the weight that the policy-maker gives to the output gap relatively to inflation increases[19]. When the policy-maker cares equally about output gap and inflation ($\lambda = 1$), the slope of the mapping is around 0.9; when he cares less about output gap than inflation the slope is smaller (for example if $\lambda = 0.5$, then the slope is 0.84) but, unless λ is smaller than 0.1, *root-t convergence* is never reached.

The fact that the learning speed could be very slow (or very fast) depending on policy decisions[20], suggests that when they consider the monetary policy problem under learning, policy-makers should take into account the transition to the REE. E-stability is not itself sufficient to characterize policies in a context of adaptive learning, and policy (2.16), which is optimal under rational expectations, may not be optimal under learning.

In the following sections I show that, in general, the analysis of the speed of convergence is helpful in evaluating policy rules.

3 Speed of convergence and policy design

Let us consider a third and more generic set of *expectations-based reaction functions*

$$i_t = \gamma + \gamma_x E_t^* x_{t+1} + \gamma_\pi E_t^* \pi_{t+1} + \gamma_g g_t \tag{3.1}$$

[19] I use the Clarida, Gali and Gertler (CGG) calibration for US. Similar results obtain with the Woodford (W) calibration.

[20] This result could be applied to the problem of "optimal delegation", justifing a conservative central bank when fast convergence is required.

and show how to characterize the elements of this set using a measure of the speed of convergence.

Under a generic *expectations-based reaction function*, the economy evolves according to the following expression:

$$Y_t = Q + F \times E_t^* Y_{t+1} + S g_t, \tag{3.2}$$

where

$$Q = \begin{bmatrix} -\alpha\varphi\gamma \\ -\varphi\gamma \end{bmatrix}, \quad S = \begin{bmatrix} \alpha\left(1-\varphi\gamma_g\right) \\ \left(1-\varphi\gamma_g\right) \end{bmatrix} \tag{3.3}$$

$$F = \begin{bmatrix} (\beta + \alpha\varphi(1-\gamma_\pi)) & \alpha(1-\varphi\gamma_x) \\ \varphi(1-\gamma_\pi) & (1-\varphi\gamma_x) \end{bmatrix}. \tag{3.4}$$

and the REE is of the form

$$Y_t = \overline{A} + S g_t, \tag{3.5}$$

If private agents do not know $\overline{A}$ but estimate it recursively, expected inflation and output gap evolve in a more complex way then described in section 2. As both the IS and the AS relations also depend on the expected output gap[21], the learning process cannot be described only by focusing on beliefs regarding expected inflation (see Appendix D for a complete description of the learning mechanism in this case).

Expectations are given by:

$$E_t^* Y_{t+1} = A_t, \tag{3.6}$$

where elements in A_t are estimated similarly to (2.9).

Lemma 4 is a slight generalization of a result obtained in Bullard and Mitra (2002) and describes the necessary and sufficient conditions under which the REE (3.5) is E-stable.

Lemma 4 *Under a generic expectations-based reaction function (3.1), the necessary and sufficient condition for a rational expectations equilibrium to be E-stable is*

$$\gamma_\pi > \max\left[1 - \frac{1-\beta}{\alpha}\gamma_x, 1 - \frac{1-\beta}{\alpha\varphi} - \frac{\gamma_x}{\alpha}\right]$$

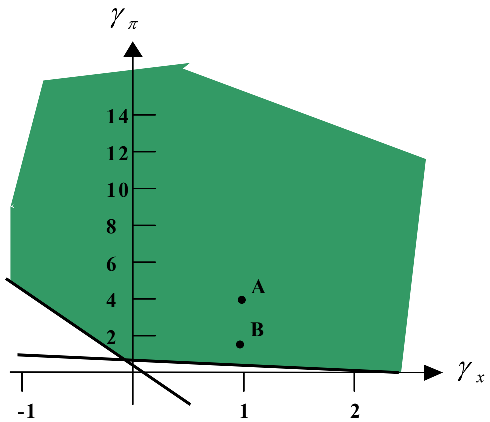
Proof. See Appendix D. ■

Figure 5 shows, with CGG calibration, all the combinations (γ_π, γ_x) under which the REE is E-stable.

[21] Under *generic expectations-based reaction functions* (3.1) the elements in the second column of the F matrix are not necessarily zero.

Fig 5

E-stable region under the expectations-based policy rule



Note that, since the *EH policy* (2.16) is an element of the set of *generic expectations-based policy rules* (3.1), points A and B represent the combination $\gamma_\pi^\star, \gamma_x^\star$ in the two extreme cases where policy-makers do not care about the output gap, $\lambda = 0$ (point A), and where they give equal weight to both inflation and the output gap, $\lambda = 1$ (point B). Figure 5 shows that in both cases the REE is E-Stable[22]. However, for $\lambda = 1$ this is very close to the bounds of the E-stability region; in this case, if the policy-maker chooses the *EH policy rule,* but improperly calibrates the model it can easily end up outside the stability region, enforcing a non-stationary policy.

Finally, the fact that the origin is not in the stable region is consistent with the non-convergence result of Evans and Honkapohja (2002): policies that react only to shocks, ignoring expectations, are unstable under learning.

3.1 The transition to the REE

In the previous sections, policy-makers settled the coefficients of matrix F, by means of reaction functions. This means that the evolution of estimated coefficients in private agents' forecasts (i.e., the speed at which private agents learn) strictly depends on policy decisions.

Proposition 5 provides conditions for root-t convergence.

Proposition 5 *Under expectations-based reaction functions (3.1), if*

$$\gamma_\pi > \max\left[1 + \frac{1-2\beta}{2\alpha\varphi} - \frac{1-2\beta}{\alpha}\gamma_x, 1 + \frac{\beta}{\alpha\varphi} - \frac{\gamma_x}{\alpha}\right], \tag{3.8}$$

[22]It is possible, moreover, to show that for any positive and finite value of λ, i.e., for all *flexible inflation targeting* policies under the *EH policy* (2.16) the rational expectation equilibrium is E-Stable (Evans and Honkapohja, 2002).

then

$$\sqrt{t}\,(A_t - A) \xrightarrow{D} N\,(0, \Omega)$$

where the matrix Ω *satisfies*

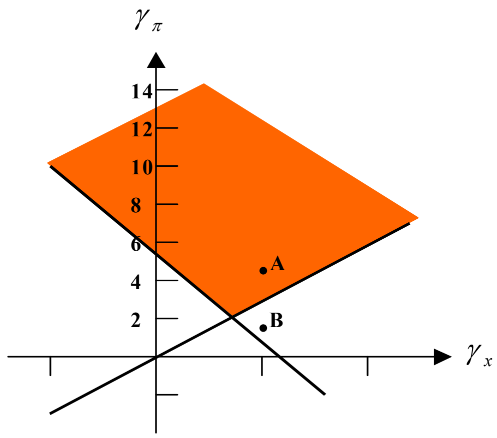
$$\left[\frac{I}{2}(F-I)\right]\Omega + \Omega\left[\frac{I}{2}(F-I)\right]' + SS'\sigma_g^2 = 0 \tag{3.9}$$

Proof. See Appendix E. ■

Under the *generic expectations-based reaction function* (3.1), if the REE is E-stable but conditions in Proposition 5 are not satisfied, then not all the eigenvalues of the matrix F have real part smaller than one half**.** In this case, as suggested in section 2, the learning equilibrium converges to the REE at a slower rate than root-t. Figure 6 shows all combinations of γ_π and γ_x for which there is root-t convergence.

Fig 6

Root-t convergence under the expectations-based policy rule



By comparing Figure 5 and Figure 6, it is apparent that the set of combinations (γ_x, γ_π) resulting in root-t convergence is much smaller than the one under which E-stability holds. Points A and B in Figure 6 show the combination $\gamma_\pi^\star, \gamma_x^\star$, i.e., the reaction to expected inflation and output gap under the *EH policy rule*, in the two extreme cases where policy-makers do not care about the output gap, $\lambda = 0$ (point A), and where they give equal weight to both inflation and the output gap, $\lambda = 1$ (point B). As derived in section 2, Figure 6 shows that when the policy-maker gives weight $\lambda = 1$ there is no *root-t* convergence.

In the previous sections, in order to characterize how policies determine the speed of converge to REE, I focused only on one policy parameter at a time (γ_π in section 2.2 and λ in section 2.3). Here, on the contrary, since the speed of convergence is determined by the eigenvalues of F and this matrix depends on both γ_π and γ_x, it is necessary to focus on two policy parameters at a time. For this reason I define the speed of convergence isoquants that map elements of the set of *expectations-based reaction functions* into a speed of convergence measure[23].

Definition 6 *A speed of convergence isoquant is a curve in R^2 along which all points (i.e., combinations (γ_π, γ_x) of an expectations-based reaction function (3.1)) result in the same real part of the largest eigenvalue z_1 of the matrix F.*

For simplicity I restrict the analysis to the set

$$\Gamma = \{\gamma_\pi, \gamma_x : \gamma_\pi > 0, \gamma_x > 0 \text{ and } 0 \leq z_1 < 1\}.$$

The following definition and proposition describe the main properties of the speed of convergence isoquants:

Definition 7 *The speed of convergence, represented by the speed of convergence isoquants, is monotonically increasing in the reaction to expected inflation (γ_π) if, given the reaction to the expected output gap (γ_x), the real part of the largest eigenvalue z_1 of the matrix F is decreasing in γ_π.*

A similar definition for monotonicity with respect to the expected output gap could be settled.

Proposition 8 *The speed of convergence relation, represented by the speed of convergence isoquants and defined over Γ is: (i) monotonically increasing in γ_π, (ii) not monotonic with respect to γ_x.*

Proof. See Appendix F. ■

Proposition 8 states that, for a given reaction to output gap expectations, the policy-maker, by increasing the reaction to expected inflation increases monotonically the speed at which private agents learn. On the contrary, for a given reaction to expected inflation, by increasing the reaction to the expected output gap, private agents could learn both faster or slower, depending on the value of γ_π.

Figure 7 shows the speed of convergence isoquants: the lower the isoquant, the slower the convergence. In fact, the larger the real part of z_1, the lower the isoquant and, from Marcet and Sargent (1995), the larger the real part of z_1, the slower the convergence.

[23]In the definition I relate speed of convergence to the eigenvalues of the matrix F. In general, as shown in previous sections, the speed of convergence is related to the eigenvalues of the derivatives of the mapping from PLM to ALM, $T(A)$. In this case, the derivative is equal to F.

Fig 7

The speed of learning isoquants

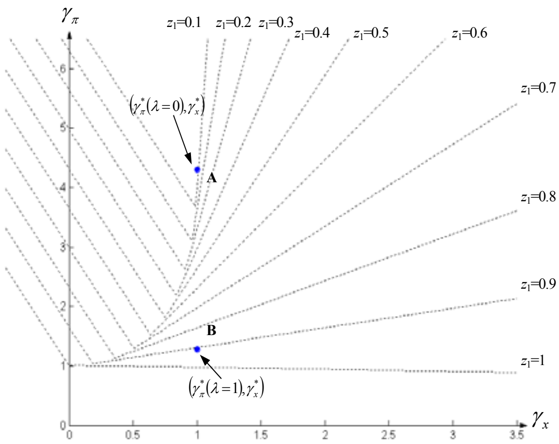


Figure 7 illustrates a practical way of using speed of convergence in order to characterize monetary policies. For example, a combination of γ_π and γ_x just above the isoquant $z_1 = 1$ (point B) determines an E-stable REE, but would imply very slow convergence. The combinations of γ_π and γ_x placed above the isoquant $z_1 = 0.1$ imply a very fast learning process. The combinations of γ_π and γ_x that stay above the isoquant $z_1 = 0.5$ imply a learning process that converges to the REE at a *root-t* speed.

Let us now see how to make active use of the speed of convergence in the study of optimal policies under discretion.

4 Discretionary policy and learning

Evans and Honkapohja (2002) say that the *expectations-based reaction function* (2.16) is not only a "good" policy because it determines an E-stable REE, but it also "*implements optimal discretionary policy in every period and for all values of private expectations*" in a context where "*private agents behave in a boundedly rational way*". In order to identify *EH policy* (2.16) as the optimal policy under discretion and learning, however, the crucial assumption is that "*the policy-maker does not make active use of learning behaviour on the part of agents*" (Evans and Honkapojha, 2002).

If, under rational expectations, the problem of optimal "discretionary policy" implies, by definition, that policy-makers cannot affect private agents' expectations, under the hypothesis of bounded rational private agents, since policy decisions affect the learning process, a rational policy-maker with full information should take transition into account. In fact, if private agents' expectations are the result of estimations that depend on past values of the monetary policy instrument, the policy-maker's decisions, will affect future estimates and, consequently, the private agents' learning process. *EH policy* (2.16) is not necessarily optimal under learning but could be defined as *asymptotically-optimal*. However, if private agents' perceived law of motion is well specified, once the learning process has converged to rational expectations, not only the policy rule (2.16) will be optimal, but the following Lemma says that there is a continuum of *expectations-based policy rules* that result in the same REE.

Lemma 9 *Under rational expectations, in the set of expectations-based reaction functions (3.1) there are infinitely many elements, i.e. combinations of* $\gamma, \gamma_x, \gamma_\pi, \gamma_g$ *that result in the optimal REE for* $\{\pi_t, x_t\}$ *defined in (2.17).*

Proof. See Appendix G. ■

Since all the policies that in the long run result in the same optimal allocation, could determine different transitions to the REE, a device for discriminating between them is required[24]. The speed of convergence isoquants derived in the previous section could be a useful starting point.

Let us consider a characterize set of *asymptotically-optimal expectations-based reaction functions* that allow to completely offset demand shocks, as under the *EH policy*.

Proposition 10 *The maximum speed of convergence of the learning process that could be reached under the restricted set of asymptotically-optimal expectations-based reaction functions,*

$$i_t = \gamma' + \gamma'_x E_t x_{t+1} + \gamma'_\pi E_t \pi_{t+1} + \gamma'_g g_t, \tag{4.1}$$

with $\gamma'_g = \gamma^*_g = \frac{1}{\varphi}$, $\gamma' = \gamma^* = -\frac{\lambda}{(\lambda+\alpha^2)\varphi}\overline{x}$ *and* $\gamma'_\pi = \frac{\lambda\left((1+\alpha\varphi)\left(\lambda+\alpha^2\right)-\lambda\beta\right)}{(\lambda+\alpha^2)-\lambda\beta}\overline{x} - \frac{1-\beta}{\alpha}\gamma'_x$, *depends negatively on the weight that the policy-maker gives to output gap relative to inflation.*

Proof. See Appendix H. ■

Proposition 10 states that under the set of reaction functions (4.1), the economy converges asymptotically to the optimal REE under discretion, but for a given λ the policy-maker can bring about a different speed of convergence. Note, instead, that under EH policy (2.16), each λ was associated with a given speed of convergence. In particular, under *asymptotically optimal expectations-based reaction functions* (4.1), the larger the relative weight on output gap, λ, the larger will be the real part of the biggest eigenvalue of the F matrix and the slower the fastest speed of convergence that a policy-maker can reach. Figure 8 shows, in the same picture, the speed of learning isoquants and, for given λ, combinations of γ_x and γ_x under which the economy will converge asymptotically to the optimal REE under discretion.

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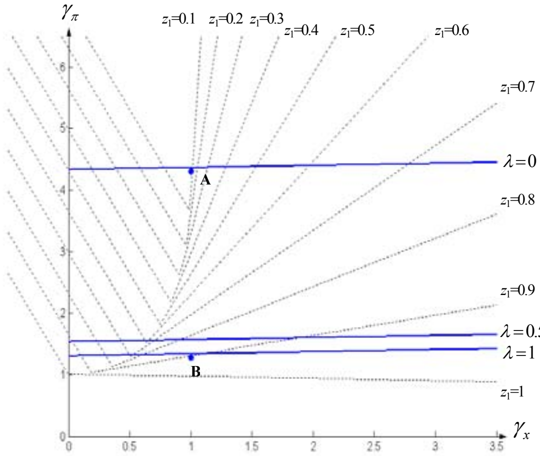
[24] *"There is no single policy rule that is uniquely consistent with the optimal equilibrium. Many rules may be consistent with the same equilibrium, even though they are not equivalent insofar as they imply a commitment to different sorts of out-of equilibrium behaviour" (Svensson and Woodford, 1999).*

Fig 8

Asymptotically-optimal expectations-based reaction functions



The line $\lambda = 0$ shows that if the policy-maker does not care about the output gap, by imposing $\gamma_\pi = \gamma'_\pi$, he can choose combinations of γ_π and γ_x such that the speed of convergence ranges from very slow to very fast: the line $\lambda = 0$, in fact, intersects isoquants $z_1 = 0.1$ (very fast speed), $z_1 = 0.7$ (speed of convergence slower than root-t) and many others. If, instead, the relative weight to output gap is one half, i.e., the line $\lambda = 0.5$, the policy-maker could choose only combinations of γ_π and γ_x such that the speed of convergence is slower than root-t: the line $\lambda = 0.5$ does not intersect any isoquant with $z_1 \leq 0.5$; if the policy-maker cares equally about inflation and output gap, i.e., the line $\lambda = 1$, he can choose only combinations of γ_π and γ_x such that the speed of convergence is very slow, slower than root-t, since the line $\lambda = 1$ does not intersect any isoquant with $z_1 \leq 0.7$.

Points A and B in Figure 8 also show another important result that will be analyzed further in the next section: for a given value of λ there are infinitely many expectations-based policies that determine asymptotically the same REE, but induce a faster (or slower) speed of convergence than the one determined by EH policy (2.16).

4.1 The mapping from PLM to ALM

In order to show how the central bank can make active use of private agents' learning behaviour in the monetary policy problem under discretion, I now consider more in detail the mapping from perceived to actual variables.

In section 2.3 I have shown that the analysis of the transition to the REE, under EH policy stands on the mapping from perceived inflation to actual inflation

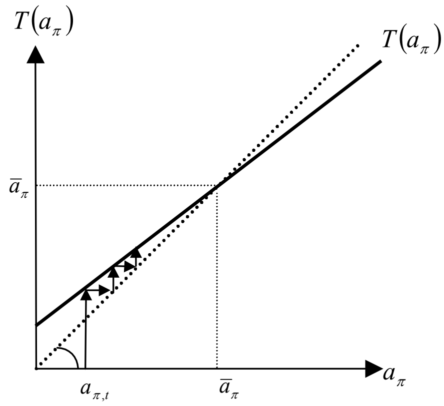
$$T\left(a_{\pi,t}\right) = \Phi^{*} + \Gamma^{*} a_{\pi,t} \tag{4.2}$$

and the necessary and sufficient condition for E-stability reduces to $\Gamma^{*} < 1$.

To give an example, since I consider λ to be an exogenous policy parameter, let us assume that the policy-maker gives a positive weight $\lambda = 0.3$ (note that with this weight I assume that the policy-maker cares more than three times more about inflation than about output gap). In this case the mapping $T\left(a_{\pi,t}\right)$ has a slope equal to 0.76 under CGG parametrization. Figure 9 shows the mapping from PLM to ALM. Even if initial perceived inflation is not too far from the REE, since the slope of the $T(.)$ mapping is close to 1, the transition from the learning to the RE equilibrium is very slow.

Fig 9

The mapping from PLM to ALM under the EH policy ($\boldsymbol{\lambda} = \mathbf{0.3}$)



4.2 Adjusting the learning speed

The question now is whether a policy-maker who wants to reach in the long run the same REE determined by the EH policy (2.16) can speed up or slow down the private agents' learning process. To answer to this question in a very intuitive way I consider a subset of asymptotically optimal policies that allow to offset not only demand shocks, but also expected output gap movements, as under the EH policy[25].

Definition 11 *The Adjusted Learning Speed-Γ' (ALS-Γ') policy rule, is an*

[25] At the beginning of this section I showed an asymptotically-optimal policy (4.1) that allowed a choice to be made among different speeds of convergence. However, under that policy, the analysis of the learning dynamics involved a mapping from PLM to ALM with both perceived inflation and output gap. Here, instead, I consider a policy that allows a choice between diffent speeds of convergence just by looking at a mapping from PLM to ALM involving only expected inflation, as under the *EH Policy*.

expectations-based reaction function

$$i_t^{ALS}(\Gamma') = \gamma^{ALS} + \gamma_x^{ALS} E_t x_{t+1} + \gamma_\pi^{ALS} E_t \pi_{t+1} + \gamma_g^{ALS} g_t \tag{4.3}$$

with coefficients $\gamma^{ALS} = -\frac{\Phi^*(1-\Gamma')}{(1-\Gamma^*)\alpha\varphi}$, $\gamma_x^{ALS} = \gamma_g^{ALS} = \frac{1}{\varphi}$, $\gamma_\pi^{ALS} = \left(1 + \frac{\beta-\Gamma'}{\alpha\varphi}\right)$, *where* $1 < \Gamma' < 0$ *is the slope of the new mapping from perceived inflation to actual inflation obtained under the ALS-*Γ' *policy:*

$$T'(a_{\pi,t}) = \frac{(1-\Gamma')}{(1-\Gamma^*)}\Phi^* + \Gamma' a_{\pi,t}. \tag{4.4}$$

Note that, under least square learning, the ALS-Γ' policy leads to a mapping from PLM to ALM

$$T'(a_{\pi,t}, a_{x,t}) = \left(\frac{(1-\Gamma')}{(1-\Gamma^*)}\Phi^* + \Gamma' a_\pi, \frac{\Phi^*(1-\Gamma')}{(1-\Gamma^*)\alpha} - \frac{(\beta-\Gamma')}{\alpha} a_{\pi,t}\right) \tag{4.5}$$

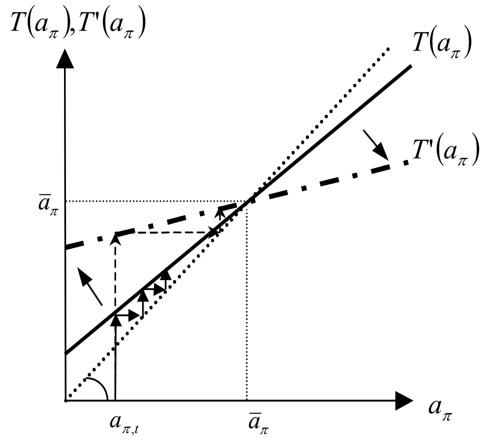
that does not depend on the perceived output gap. Therefore, in order to study convergence to the REE, as under EH policy, the analysis can focus on the mapping from perceived inflation to actual inflation (4.5).

Figure 10 shows the new mapping $T'(a_{\pi,t})$ under the *ALS-*Γ'*policy.*

Fig 10

The mapping from PLM to ALM under the ALS-Γ' policy

($\Gamma' < \Gamma^*$)



In particular, it can be observed that $T'(a_{\pi,t})$ has the same fixed point, $\bar{a}_\pi$, as under the *EH policy*, but the intercept and the slope are different. The policy-maker, in order to speed up (slow down) the transition to the REE can follow an expectations-based reaction function that induces a rotation of the mapping from PLM to ALM around the fixed-point (i.e., the REE), with a slope Γ' lower (higher) than under the EH policy.

The following proposition formalizes this result.

Proposition 12 *Under rational expectations, the ALS-Γ' policy results in the same REE for $\{\pi_t, x_t\}$ derived under the EH policy. Under least squares learning, the ALS-Γ' policy results asymptotically in the same REE for $\{\pi_t, x_t\}$ derived under the EH policy.*

Proof. See Appendix I. ■

Taking parameters α, φ, β as given, under the EH policy, the speed of convergence relies entirely on λ. By choosing a λ the policy-maker is also choosing the slope of the $T(.)$ mapping (in the previous example, with $\lambda = 0.3$, the slope was equal to 0.76) and, as shown in section 2, he determines the speed of convergence. However, under *ALS-Γ' policy,* the policy-maker could choose separately the relative weight on output gap and the speed at which agents learn without affecting the REE.

Lemma 13 *Under ALS-Γ' policy (4.3) the speed of convergence does not depend on the relative weight to outputg gap.*

Proof. See Appendix L. ■

Comparing now *EH* and *ALS-Γ'* policies, we have that

Lemma 14 *The response of interest rate to a rise in expected inflation is higher under the ALS-Γ' than under the EH policy if $\Gamma' < \Gamma^*$, is lower if $\Gamma' > \Gamma^*$.*

Proof. See Appendix M. ■

The following proposition and its corollary formally compare the transition under ALS-Γ' and under the EH policies.

Proposition 15 *Assume that private agents form expectations through recursive least squares learning and that initial perceived inflation is the same under both ALS-Γ' and EH policies but different from the REE. Now, if the reaction to expected inflation is stronger under ALS-Γ' than under EH policy, i.e. $\gamma_\pi^{ALS} > \gamma_\pi^*$, then perceived and actual inflation will be closer to the REE under ALS-Γ' than under the EH policy, along the transition. The opposite is true when $\gamma_\pi^{ALS} < \gamma_\pi^*$.*

Proof. See Appendix N. ■

Corollary 16 *Consider two ALS policies $i^{ALS}(\Gamma'_1)$ and $i^{ALS}(\Gamma'_2)$ with $0 < \Gamma'_1 < \Gamma'_2 < 1$ and $a_{\pi,0}\left(i^{ALS}(\Gamma'_1)\right) = a_{\pi,0}\left(i^{ALS}(\Gamma'_2)\right) \neq \overline{a}_\pi$. Along the transition, perceived and actual inflation will be closer to the REE under ALS-$\Gamma'(\Gamma'_1)$ than under ALS-$\Gamma'(\Gamma'_2)$ policy.*

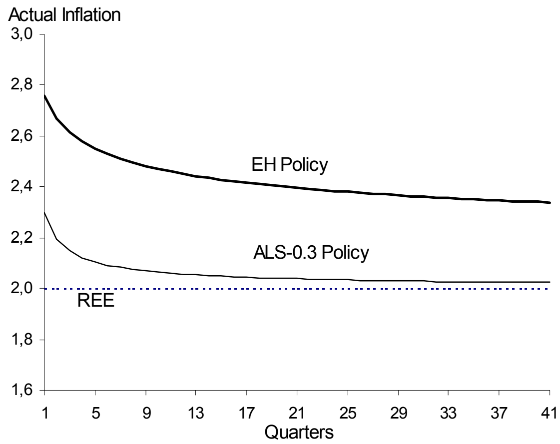
The intuition is the following: if the policy-maker reacts strongly to a change in expected inflation, the difference between private agents' expectations and actual inflation will be greater and prediction error will be initially bigger; if private agents make larger errors they will adapt their estimates faster and both expected and actual inflation will move closer to the REE. In other words, the stronger the policy-maker's response to a change in private agents' expectations, the faster private agents learn and the shorter the transition to the REE.

The fact that under the ALS policy for every $0 < t < \infty$ the distance from the REE could be smaller (greater) than under the EH policy brings to the following question: how long does it take under the two policies to get ε-close to the REE, i.e., starting from the same distance from the REE, $|a_{\pi,0} - \overline{a}_{\pi}| > \varepsilon$, how many periods are needed under the two policies in order to get $|\pi_t - \overline{a}_{\pi}| < \varepsilon$?

Assuming that the policy-maker follows a flexible inflation targeting policy rule with $\lambda = 0.3$, the output gap target is $\overline{x} = 0.02$ and using CGG calibration, figure 11 compares the results of a simulation under the EH policy and under an ALS-Γ' policy with $\Gamma' = 0.3$ (i.e., root-t convergence is imposed). Given that the REE for inflation is around 2 per cent, I consider an initial expected inflation 1 percentage point higher than the REE.

Fig 11

Deviation of actual inflation from the REE



Under the ALS-Γ' policy, after 1 quarter inflation is at $2,3$ per cent, after 1 year is below $2,2$ per cent and after 5 years its distance from the REE is smaller than $0,1$ percentage point. On the contrary, under EH policy, after 1 quarter inflation is at $2,8$ per cent, after 1 year is still close to $2,6$ per cent and after 20 years inflation is still $0,4$ percentage points higher than the REE.

Table 2 compares the transition to the REE for different ALS-Γ' policies.

Tab 2

Transition under the ALS-Γ' policy

(Quarters neede in order to have $(\pi_t - \overline{\pi}_{REE})$ smaller than)

γ_π^{ALS}	Γ'	0.7%	0.5%	0.3%	0.2%	0.1%
4.0	0.1	1	1	1	1	3
3.3	0.3	1	1	2	3	15
2.6	0.5	1	2	5	14	120
2.3	0.6	1	3	13	41	610
2	0.7	1	7	43	225	> 1000
1.8[1]	0.76	3	14	147	> 1000	> 1000
1.6	0.8	5	26	463	> 1000	> 1000
1.1	0.95	> 1000	> 1000	> 1000	> 1000	> 1000

[1]For $\gamma_\pi^{ALS} = 1.8$, the ALS and EH policies coincide, when $\lambda = 0.3$.

Let us consider, for example, the ALS-Γ' with $\gamma_\pi^{ALS} = 3.3$. Given that in equilibrium inflation is 2 per cent and assuming an initial expectation at 3 percent, inflation can be reduced by more than 0.5 percentage point, in:

- 1/2 of the time needed under the ALS-Γ' with $\gamma_\pi^{ALS} = 2.6$
- approximately 1/7 of the time needed under the ALS-Γ' with $\gamma_\pi^{ALS} = 2$
- 1/14 of the time needed under the EH policy!

Moreover, under the ALS-Γ' with $\gamma_\pi^{ALS} = 2.6$, that allows root-t convergence, inflation can be reduced by more than 0.7 percentage point, in approximately 1/30 of the time needed under EH policy.

This section looked at the role of policy decisions in determining the speed of convergence under learning, focusing on the mapping from perceived inflation to actual inflation. Before asking how the policy-maker can make use of his role to increase social welfare, the following lemma concerns the behaviour of the output gap along the transition.

Lemma 17 *Under EH and ALS policies, when initial perceived inflation is higher (lower) than the REE, the output gap converges to the REE from below (above).*

Proof. See Appendix N. ■

Now it is possible to return to the question addressed at the beginning of the paper: is the *EH policy* still optimal under learning? Are policies that speed up the learning process always better than policies that involve a slow transition to the REE?

5 Welfare analysis

In January 1999, with the start of stage 3 of the Economic and Monetary Union, monetary competencies were transferred from each country of the European

Union to the European Central Bank. Before that date people were accustomed to take into account the monetary policy of their own country when making economic decisions. After the start of stage 3, they faced a new policy-maker (and a new monetary policy) and inflation and output gap equilibria determined under the new policy regime were, in some cases, different from the ones implied by the previous policies. Let us consider, for example, countries like Italy or Spain, whose rates of inflation are historically higher than in other member states, and assume that in those two countries expected inflation at the start of the EMU was higher than the REE determined by the new monetary regime. Under the assumption that private agents need time to learn the new equilibrium, it is clear that the dynamics of the learning equilibrium along the transition to the REE play an important role in the analysis of monetary policy decisions based on welfare measures. Questions like the ones raised at the end of the previous section show up spontaneously.

To answer to those questions I consider separately the two cases where initial expected inflation is higher than the REE and where it is lower. The reason why I proceed in this way is twofold. First, under adaptive learning, when the policy-maker chooses the policy, he already knows private agents' expectations and he could infer wether the initial bias in agents' prediction is positive or negative. Second, the welfare implications differs in the two cases. In the literature it is well known that under the loss function described in section 2.3 the first best plan would be, for all t, to have inflation and output gap at their target levels, i.e., $\pi_t^{FB} = 0$ and $x_t^{FB} = \overline{x}$. As many works have shown, under no commitment, the first best solution is not feasible if $\overline{x} \neq 0$. The optimal (time-consistent) policy in this case leads to a REE with inflation higher than the first best and output gap lower[26]:

$$\pi_t^{REE} = \frac{\lambda\alpha}{(\lambda+\alpha^2)-\lambda\beta}\overline{x} \; > \pi_t^{FB} \quad \text{for all } t$$

$$x_t^{REE} = \frac{\lambda(1-\beta)}{(\lambda+\alpha^2)-\lambda\beta}\overline{x} < x_t^{FB} \quad \text{for all } t$$

Under learning, however, inflation and output gap could remain far from the REE for a long time. Therefore, if initial perceived inflation is higher than the REE, as in previous section, actual inflation will be higher and output gap lower than the REE along the transition. In this case, a policy-maker who bases decisions on the loss function described in section 2.3 would prefer policies that make inflation fall and output gap rise quickly to the REE. On the contrary, if initial perceived inflation is lower than the REE, the policy-maker would prefer policies that make inflation climbing and output gap landing slowly to the REE. Since *EH policy* is not taking into account the transition, I claim that there are *ALS-*Γ' *policies* that will make our economy better off.

[26]Under CGG parametrization, assuming $\lambda = 0.5$, the REE would be

$$\pi_t = 1.57 * \overline{x} \quad \text{and} \quad x_t = 0.05 * \overline{x}$$

In order to verify this claim, let us start by assuming that the *EH policy* (which is optimal under RE) is also optimal when private agents form expectations through adaptive learning. The aim is to compute the welfare cost of alternative monetary policies, i.e., ALS-Γ', that asymptotically result in the same REE as the *EH policy*, but along the transition result in different learning equilibria.

The social loss associated with *EH policy* is defined as:

$$L_0^{EH} = E_0 \sum_{t=0}^{\infty} \beta^t L\left(\pi_t\left(i^{EH}\right), x_t\left(i^{EH}\right)\right),$$

where $L\left(\pi_t\left(i^{EH}\right), x_t\left(i^{EH}\right)\right)$ is the period t loss function defined above and $\pi_t\left(i^{EH}\right), x_t\left(i^{EH}\right)$ denote the contingent plans for inflation and output gap under *EH policy*. Similarly, the social loss associated with *ALS*-Γ' *policies* is defined as

$$L_0^{ALS}\left(\Gamma'\right) = E_0 \sum_{t=0}^{\infty} \beta^t L\left(\pi_t\left(i^{ALS}\left(\Gamma'\right)\right), x_t\left(i^{ALS}\left(\Gamma'\right)\right)\right).$$

As a measure of the welfare loss (gain) I consider the percentage increase (decrease) in the social loss of moving from *EH* to *ALS*-Γ'*policy*:

$$\omega\left(L_0^{ALS}\left(\Gamma'\right)\right) = \left(\frac{L_0^{ALS}\left(\Gamma'\right) - L_0^{EH}}{L_0^{EH}}\right) * 100.$$

Note that for values of $\omega\left(L_0^{ALS}\left(\Gamma'\right)\right) < 0$ there is a welfare gain in adopting ALS-Γ' policy instead of EH, while for $\omega\left(L_0^{ALS}\left(\Gamma'\right)\right) > 0$, there is a welfare loss.

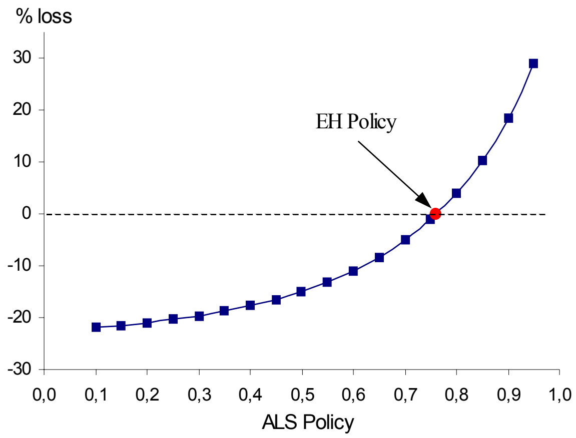
I run simulations[27] of the model for 10000 periods, assuming that the policymaker follows a flexible inflation targeting policy rule with $\lambda = 0.3$, the output gap target is $\overline{x} = 0.02$ and using CGG calibration. The REE for inflation is around 2 per cent. I consider an initial expected inflation 1 percentage point higher than the REE and I compute social losses under the EH and ALS-Γ' policies for different values of γ_π^{ALS} (i.e., different Γ').

Figure 13 shows that ALS-Γ' policies with $\Gamma' < \Gamma^*$, by inducing a fast convergence, reduce the social loss up to 20 per cent relative to EH policy. Policies with $\Gamma' > \Gamma^*$, on the contrary, increase the social loss by up to 30 per cent. In particular, a central bank that follows an ALS-Γ' policy with $\gamma_\pi^{ALS} = 2.6$ can, by increasing the speed of convergence to root-t, lower the value of the loss function by 15 per cent relative to the EH policy.

[27]In the following simulations I consider as AR(1) stochastic process for the demand shock, with $\rho_g = 0.95$ and $\varepsilon_{gt} \sim N(0, 0.05)$

Fig 13

Percentage loss in total welfare ($\pi_0 > \pi^{RE}$)



In order to analyze how the percentage increase (decrease) in the social loss evolves along the transition simulations are also run for $T < 10000$ periods. Table 3 shows the results, pointing out that most of the gain from using an ALS-Γ' policy with fast transition is concentrated in the first 20 quarters.

Tab 3

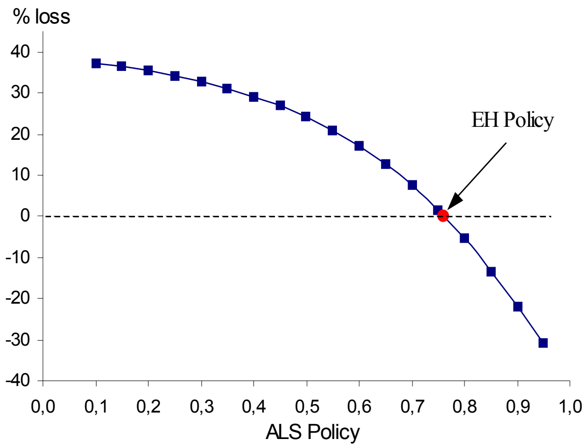
Percentage loss in total welfare after T quarters ($\pi_0 > \pi^{RE}$)

γ_π^{ALS}	Γ'	T=10	T=20	T=50	T=100	T=10000
4.0	0.1	$-18,3$	-22	$-23,6$	$-23,2$	$-21,9$
3.3	0.3	$-15,0$	$-18,5$	$-20,3$	$-20,4$	$-19,6$
2.6	0.5	$-10,2$	$-12,8$	$-14,4$	$-15,1$	$-15,0$
2.3	0.6	$-7,0$	$-8,9$	$-10,4$	$-10,9$	$-11,0$
1.6	0.8	$2,1$	$2,8$	$3,4$	$3,7$	$3,9$
1.1	0.95	$13,0$	$17,2$	$22,5$	$25,7$	$29,0$

Figure 14 and Table 4 show that under the assumption of an initial expected inflation 1 percentage point lower than the REE, by inducing a slower convergence, the policy-maker could greatly reduce the welfare loss.

Fig 14

Percentage loss in total welfare ($\pi_0 < \pi^{RE}$)



A central bank that follows an ALS-Γ' policy with $\gamma_{\pi}^{ALS} = 1.1$ can, by increasing the slope of the mapping from perceived inflation to actual inflation to $\Gamma' = 0.95$, slow down the transition and lower the value of the loss function by approximately 30 per cent relative to the EH policy. On the contrary, a policy-maker who speeds up the transition to root-t convergence, following an ALS-Γ' policy with $\gamma_{\pi}^{ALS} = 2.6$, would increase the value of the loss function by approximately 25 per cent relative to the EH policy. Again, Table 4 shows that most of the loss from using an *ALS*-Γ' *policy* with fast transition is concentrated in the first 20 quarters, while advantages from inducing a slow convergence are distributed along the transition.

Tab 4

Percentage loss in total welfare after T quarters ($\pi_0 < \pi^{RE}$)

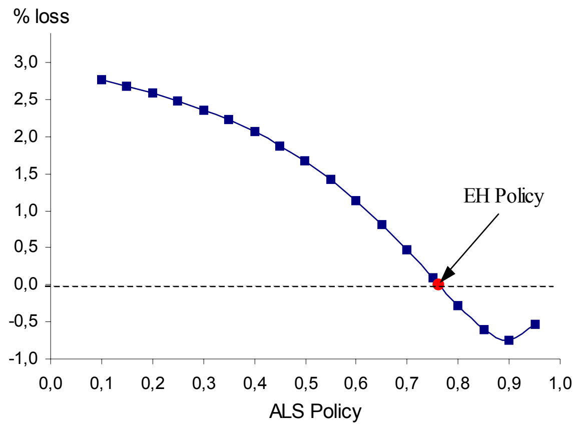
γ_{π}^{ALS}	Γ'	T=10	T=20	T=50	T=100	T=10000
4.0	0.1	$60,7$	$55,2$	$47,5$	$42,3$	$36,3$
3.3	0.3	$44,8$	$43,3$	$39,5$	$36,2$	$31,9$
2.6	0.5	$26,4$	$27,5$	$26,9$	$25,7$	$23,6$
2.3	0.6	$16,3$	$17,8$	$18,2$	$17,8$	$16,9$
1.6	0.8	$-3,6$	$-4,4$	$-5,1$	$-5,2$	$-5,3$
1.1	0.95	$-14,3$	$-19,7$	$-25,1$	$-27,7$	$-30,1$

Finally, under the assumption that initial expected inflation is random and distributed symmetrically[28] around the REE figure 15 shows that if the policy-maker does not take into account the initial bias in agents' prediction, *ALS*-Γ' *policies* that induce a slower convergence than under EH policy would be slightly preferable than policies that determine a fast convergence. In particular, the maximum gain in welfare can be obtained under the ALS-0.9 policy.

Fig 15

Percentage loss in total welfare

(π_0 symmetrically distributed around π^{RE})



Before concluding I wish to emphasize some aspects concerning the robustness of welfare results.

6 Robustness

In the previous section the speed of convergence and welfare were studied by running simulations with $\lambda = 0.3$ and $\overline{x} = 0.02$. Changing these parameters would not change the finding that *EH policy*, that is optimal under rational expectations, is not optimal under learning and that, when initial perceived inflation is higher than the REE, the central bank could increase welfare by inducing a faster transition. However, in the extreme case where $\overline{x} = 0$, if initial inflation is lower than the REE, the finding that a slower convergence to the REE increases welfare does not hold anymore. In fact, when $\overline{x} = 0$, in our model, the optimal policy under discretion results in a REE with inflation and output gap equal to the first best, and a faster transition will always be better

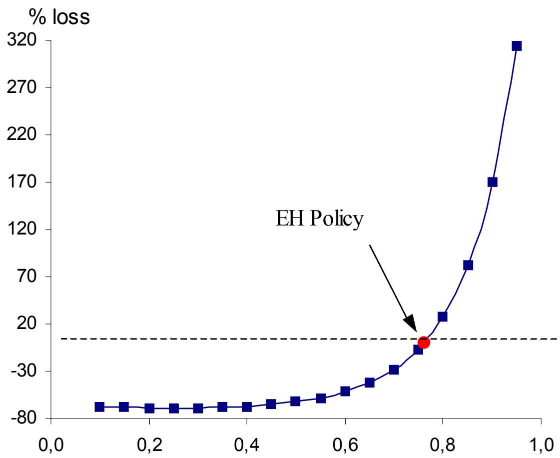
[28]Here I assume an uniform distribution between -1 and $+1$ percentage point around the REE.

(Figure 16).

Fig 15

Percentage loss in total welfare, when $\overline{x} = 0$

(π_0 symmetrically distributed around π^{RE})



6.1 An Economy with Cost-push Shocks

The new-Keynesian model analyzed in this paper is derived assuming that only one shock affects the economy. Under this assumption the policy-maker neutralizes real effects of the shock whether it follows the *EH policy* or an *ALS*-Γ' *policy*, i.e., $\gamma_g^* = \gamma_g^{ALS} = \frac{1}{\varphi}$. However, when an additional shock hits the economy (for example, a "cost-push shock", u_t) the policy-maker cannot, in general, neutralize both shocks at the same time. In this case, since the two policies along the transition to the REE would react differently to u_t, welfare analysis could be affected. Simulations show that the introduction of a cost-push shock affects the results only in the amount of the welfare gain (or loss).

Tab 6

Percentage loss in total welfare with cost-push shocks

$\left(\omega\left(L_0^{ALS}(\Gamma')\right);\ \mathbf{T} = \mathbf{10000}\right)$

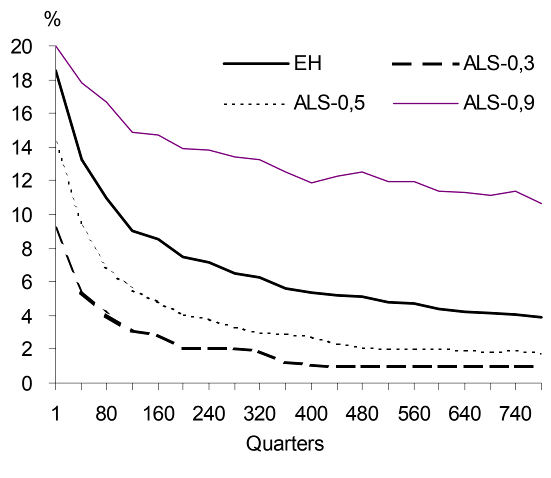
γ_π^{ALS}	Γ'	$\pi_0 > \pi^{RE}$	$\pi_0 < \pi^{RE}$
4.0	0.1	$-15,6$	$43,8$
2.6	0.5	$-11,4$	$28,1$
2.3	0.6	$-8,5$	$19,9$
1.6	0.8	$3,1$	$-6,2$
1.1	0.95	$23,5$	$-33,4$

Table 6 shows that adding an AR(1) shock u_t in the aggregate supply equation[29], when initial private agents' perceived inflation is higher than the REE, a central bank that follows an ALS-Γ' policy with $\gamma_{\pi}^{ALS} = 2.6$ can lower the value of the loss function by approximately 11 per cent relative to the EH policy (15 per cent without cost-push shocks); when initial private agents' perceived inflation is lower than the REE, an *ALS*-Γ' policy with $\gamma_{\pi}^{ALS} = 1.1$ can lower the value of the loss function by approximately 33 per cent (30 per cent without cost-push shocks).

In particular it is worth to notice that increasing the speed of convergence also reduces the variablity of inflation along the transition. Figure 17 shows that the faster the convergence, the smaller would be the difference between the standard deviation of actual inflation under learning and under RE. Under *ALS*-Γ' *policies* with $\Gamma' = 0,3$, at the beginning of the transition to the REE, inflation's standard deviation is 10 per cent higher than under RE, while under EH policy is approximately 20 per cent. After 10 years, under *ALS*-Γ' policy the standard deviation is only 5 per cent higher, while under EH policy it is 14 per cent higher.

Fig 17

Differences between standard deviation of inflation under learning and under RE along the transition



The results obtained in this section show that optimal policies derived under RE are not optimal under learning. Using results for the speed of convergence

[29] I assume $\lambda = 0.5$, $\overline{x} = 0.02$, $u_t = \rho_u u_{t-1} + \varepsilon_{u,t}$ with $\rho_u = 0.5$ and $\varepsilon_{u,t} \sim N(0, 0.05)$

could help to increase social welfare by taking into account the transition from learning equilibrium to the REE. Solving for the true optimal policy under discretion and learning would envolve taking into account that the policy-maker could make active use of private agents' learning behaviour. However, since the optimal monetary policy has to be derived by substituting the private agents' PLM into the objective function, it would be time-dependent. Further analysis in this direction is required and will be left for future research.

7 Conclusions

In this paper I have shown that considering learning in a model of monetary policy design is particularly important in order to describe not only the asymptotic properties of rational expectations equilibrium to which the economy could converge, but even to describe the dynamics that characterize the transition to this equilibrium.

The central message of the paper is that policy-makers should not only look at monetary policies that determine a stable equilibrium under learning, but also take into account how policy decisions affect the speed at which learning converges to rational expectations. In particular, under certain policies, the REE is E-stable, but the period needed to converge to this equilibrium could be incredibly long. Reacting strongly to expected inflation, a central bank would shorten the transition and increase the speed of convergence from the learning equilibrium to the REE.

A policy-maker who considers his role in determining the dynamics of the private agents' learning process could choose a policy rule that induces agents to learn at a given speed, affecting the welfare of society. In particular, if the policy-maker knows that after a regime change private agents' perceived inflation would be higher than the REE, by choosing a policy that reacts strongly to expected inflation he would determine a fast convergence and could increase social welfare. If, instead, perceived inflation is initially lower than the REE, a slow transition is preferred when the output gap target is greater than zero, a fast transition when the target is equal to zero.

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8 Appendix

A. Proof of Proposition 1

Given the recursive stochastic algorithm

$$a_{\pi,t} = a_{\pi,t-1} + t^{-1}\left(-\alpha\varphi\gamma + \left[\beta + \alpha\varphi\left(1-\gamma_\pi\right)\right]a_{\pi,t-1} + \alpha g_{t-1} - a_{\pi,t-1}\right)$$

let

$$h\left(a_\pi\right) = \left[-\alpha\varphi\gamma + \left[\beta + \alpha\varphi\left(1-\gamma_\pi\right)\right]a_\pi - a_\pi\right]$$

and let a_π be such that $h\left(a_\pi\right) = 0$. By the theorem of Benveniste et. al.(Theorem 3, page 110), if the derivative of $h\left(a_\pi\right)$ is smaller than $-1/2$, then

$$\sqrt{t}\left(a_{\pi,t} - \overline{a}_\pi\right) \xrightarrow{D} N\left(0, \sigma_a^2\right)$$

where σ_a^2 satisfies

$$\left[h'\left(\overline{a}_\pi\right)\right]\sigma_a^2 + E\left[-\alpha\varphi\gamma + \left[\beta + \alpha\varphi\left(1-\gamma_\pi\right)\right]\overline{a}_\pi - \overline{a}_\pi + \alpha g_t\right]^2 = 0$$

Note that the derivative of $E\left[-\alpha\varphi\gamma + \left[\beta + \alpha\varphi\left(1-\gamma_\pi\right)\right]a_\pi - a_\pi\right]$ being smaller than $-1/2$ coincides with $\left[\beta + \alpha\varphi\left(1-\gamma_\pi\right)\right]$ being smaller than $1/2$, i.e., γ_π being larger than $1 - \frac{1/2-\beta}{\alpha\varphi}$

B. Proof of Proposition 2

The formula for the asymptotic variance of the limiting distribution is

$$\sigma_a^2 = \frac{\alpha^2}{\left[1-\beta-\alpha\varphi\left(1-\gamma_\pi\right)\right]}\sigma_g^2$$

and the derivative,

$$\frac{\partial\sigma_a^2}{\partial\gamma_\pi} = -\frac{\alpha\varphi}{\left[1-\beta-\alpha\varphi\left(1-\gamma_\pi\right)\right]^2}\alpha^2\sigma_g^2 < 0$$

C. Proof of Proposition 3

The argument is similar to the one used in the proof of Propositions 1 and 2.

In order to have root-t convergence,

$$\lambda < \frac{\alpha^2}{2\beta - 1}$$

For values of $\lambda > \frac{\alpha^2}{2\beta-1}$ there is no root-t convergence and convergence will be slower.

D. Proof of Lemma 4

In the context of the present model, expected inflation and output gap are

$$E_t Y_{t+1} = \begin{pmatrix} a_{\pi,t} \\ a_{x,t} \end{pmatrix} = A_t$$

where $a_{\pi,t}$ and $a_{x,t}$ are estimated recursively

$$a_{\pi,t} = a_{\pi,t-1} + t^{-1} \left(\pi_{t-1} - a_{\pi,t-1} \right)$$

$$a_{x,t} = a_{x,t-1} + t^{-1} \left(x_{t-1} - a_{x,t-1} \right)$$

The ALM of inflation and output gap is

$$\begin{bmatrix} \pi_t \\ x_t \end{bmatrix} = Q + F A_t$$

Thus the mapping from PLM to ALM takes the form

$$T\left(A_t' \right) = Q + F A_t$$

Consider the stability under learning (E-stability) of the rational expectation solution $\overline{A}$ as the situation where the estimated parameters A_t converge to $\overline{A}$ over time.

From Evans and Honkapohja (2001), the E-stability is determined by the following differential equation

$$\frac{d}{d\tau} \left(A' \right) = T\left(A' \right) - A'$$

For this framework E-stability conditions are readily obtained by computing the derivative of $T\left(A'\right) - A'$ and imposing that the determinant of the matrix with the derivatives of the previous differential equation with respect to A is greater than zero and the trace of the matrix with the derivative is greater than zero. In particular, the eigenvalues of F, z_1 and z_2, must have real parts less than one (let us define the biggest eigenvalue of the F matrix as z_1).

Then, let us distinguish between the two cases:

1. The "real" case.

   In this case two conditions must be satisfied in order to have convergence to the REE:

   (a) For reality

   $$\left(\alpha\varphi \left(1 - \gamma_\pi \right) + \beta + \left(1 - \varphi\gamma_x \right) \right)^2 - 4\beta \left(1 - \varphi\gamma_x \right) > 0$$

   (b) $\boxed{z_1 < 1}$ implies

   $$\gamma_\pi > 1 - \frac{(1-\beta)}{\alpha} \gamma_x$$

   Since by hypothesis $z_1 \geq z_2$, if $z_1 < 1$ then also $z_2 < 1$.

2. The "complex" case.

   In this case two conditions must be satisfied in order to have convergence to the REE:

   (a) For the solution to be imaginary,

   $$(\alpha\varphi(1-\gamma_\pi)+\beta+(1-\varphi\gamma_x))^2-4\beta(1-\varphi\gamma_x)<0$$

   (b) $\boxed{\text{Real part of } z_1 < 1}$ implies

   $$\frac{(\alpha\varphi(1-\gamma_\pi)+\beta+(1-\varphi\gamma_x))}{2}<1$$

   That is

   $$\gamma_\pi > 1-\frac{1-\beta}{\alpha\varphi}-\frac{\gamma_x}{\alpha}$$

   Since by hypothesis $z_1 \geq z_2$, if $z_1 < 1$ then also $z_2 < 1$.

From case 1 and case 2, we obtain the necessary and sufficietn condition for E-stability,

$$\gamma_\pi > \max\left[1-\frac{1-\beta}{\alpha}\gamma_x, 1-\frac{1-\beta}{\alpha\varphi}-\frac{\gamma_x}{\alpha}\right]$$

E. Proof of Proposition 5

Consider again the mapping from PLM to ACL under the least square learning hypothesis:

$$T(A_t')=Q+FA_t$$

From Marcet and Sargent (1992) it follows that in order to have root-t convergence the eigenvalues of F must have the real part smaller than $\frac{1}{2}$.

Then, let us distinguish between the two cases:

1. The "real" case.

   In this case two conditions must be satisfied in order to have convergence to the REE:

   (a) For reality

   $$(\alpha\varphi(1-\gamma_\pi)+\beta+(1-\varphi\gamma_x))^2-4\beta(1-\varphi\gamma_x)>0$$

   (b) $\boxed{z_1 < 0.5}$ implies

   $$\gamma_\pi > 1+\frac{1-2\beta}{2\alpha\varphi}-\frac{1-2\beta}{\alpha}\gamma_x$$

   Note that if $z_1$is smaller than $\frac{1}{2}$ then even z_2 is smaller than $\frac{1}{2}$.

2. The "complex" case.

   In this case two conditions to be satisfied in order to have root-t convergence:

   (a) For the solution to be imaginary,

   $$(\alpha\varphi(1-\gamma_\pi)+\beta+(1-\varphi\gamma_x))^2-4\beta(1-\varphi\gamma_x)<0$$

   (b) $\boxed{\text{Real part of } z_1<\frac{1}{2}}$ implies

   $$\frac{(\alpha\varphi(1-\gamma_\pi)+\beta+(1-\varphi\gamma_x))}{2}<\frac{1}{2}$$

   That is

   $$\gamma_\pi>1+\frac{\beta}{\alpha\varphi}-\frac{\gamma_x}{\alpha}$$

   Note that if $z_1$is smaller than $\frac{1}{2}$ then even z_2 is smaller than $\frac{1}{2}$.

From case 1 and case 2, we obtain the necessary and sufficietn condition for root-t convergence,

$$\gamma_\pi>\max\left[1+\frac{1-2\beta}{2\alpha\varphi}-\frac{1-2\beta}{\alpha}\gamma_x,1+\frac{\beta}{\alpha\varphi}-\frac{\gamma_x}{\alpha}\right]$$

F. Proof of Proposition 8

Consider the set $\Gamma=\{\gamma_\pi,\gamma_x:\gamma_\pi>0,\gamma_x>0 \text{ and } 0\leq z_1<1\}$.

Monotonically increasing with respect to γ_π: for every $h=\left(\gamma_\pi^1,\gamma_x^1\right)\in\Gamma$ and $w=\left(\gamma_\pi^2,\gamma_x^1\right)\in\Gamma$, with $\gamma_\pi^2\geq\gamma_\pi^1$, w implies a value for the real part of z_1 smaller or equal to the one with h.

Proof. z_1 is the biggest eigenvalue of F:

$$\begin{aligned} z_1 &= \frac{(\alpha\varphi(1-\gamma_\pi)+\beta+(1-\varphi\gamma_x))}{2}+ \\ &+\frac{\sqrt{(\alpha\varphi(1-\gamma_\pi)+\beta+(1-\varphi\gamma_x))^2-4\beta(1-\varphi\gamma_x)}}{2} \end{aligned}$$

Consider a $h=\left(\gamma_\pi^1,\gamma_x^1\right)\in\Gamma$ such that $z_1=z_1^1$ is real. In this case

$$(\alpha\varphi(1-\gamma_\pi)+\beta+(1-\varphi\gamma_x))^2-4\beta(1-\varphi\gamma_x)>0$$

For every $\varepsilon\geq0$ there is a $w=\left(\gamma_\pi^2,\gamma_x^2\right)=\left(\gamma_\pi^1+\varepsilon,\gamma_x^1\right)\in\Gamma$ with $\gamma_\pi^2\geq\gamma_\pi^1$.

For the combination $(\gamma_\pi, \gamma_x) = w$, the biggest eigenvalue of F, z_1^2 is equal to

$$
\begin{aligned}
z_1^2 &= \frac{\left(\alpha\varphi\left(1-\left(\gamma_\pi^1+\varepsilon\right)\right)+\beta+\left(1-\varphi\left(\gamma_x^1\right)\right)\right)}{2}+ \\
&+\frac{\sqrt{\left(\alpha\varphi\left(1-\left(\gamma_\pi^1+\varepsilon\right)\right)+\beta+\left(1-\varphi\left(\gamma_x^1\right)\right)\right)^2-4\beta\left(\gamma_x^1\right)}}{2}
\end{aligned}
$$

There could be two cases:

1. w is such that z_1^2 is real. In this case

   $$\left(\alpha\varphi\left(1-\left(\gamma_\pi^1+\varepsilon\right)\right)+\beta+\left(1-\varphi\gamma_x^1\right)\right)^2-4\beta\gamma_x^1>0$$

   Now, it is obvious that $z_1^2 - z_1^1 < 0$ and monotonicity with respect to γ_π is satisfied.

2. w is such that z_1^2 is complex. In this case z_1^1 should be compared with the real part of z_1^2

   Since

   $$\frac{\left(\alpha\varphi\left(1-\left(\gamma_\pi^1+\varepsilon\right)\right)+\beta+\left(1-\varphi\gamma_x^1\right)\right)}{2}-\frac{\left(\alpha\varphi\left(1-\gamma_\pi\right)+\beta+\left(1-\varphi\gamma_x\right)\right)}{2}<0$$

   monotonicity with respect to γ_π is satisfied.

Consider an $h = \left(\gamma_\pi^1, \gamma_x^1\right)$ such that z_1^1 is complex. In this case only the real part of z_1^1 is of interest.

Take a $w = \left(\gamma_\pi^1 + \varepsilon, \gamma_x^1\right)$, in this case $\|w - h\| = \left[\left(\gamma_\pi^1 + \varepsilon - \gamma_\pi^1\right)^2\right]^{\frac{1}{2}} = \varepsilon$. In the point $(\gamma_\pi, \gamma_x) = w$, the biggest eigenvalue of F, z_1^2 is equal to

$$
\begin{aligned}
z_1^2 &= \frac{\left(\alpha\varphi\left(1-\left(\gamma_\pi^1+\varepsilon\right)\right)+\beta+\left(1-\varphi\gamma_x^1\right)\right)}{2}+ \\
&+\frac{\sqrt{\left(\alpha\varphi\left(1-\left(\gamma_\pi^1+\varepsilon\right)\right)+\beta+\left(1-\varphi\gamma_x^1\right)\right)^2-4\beta\gamma_x^1}}{2}
\end{aligned}
$$

Note now, that if z_1^1 is complex, z_1^2 cannot be real: if z_1^1 is complex $4\beta\gamma_x^1 > \left(\alpha\varphi\left(1-\gamma_\pi^1\right)+\beta+\left(1-\varphi\gamma_x^1\right)\right)^2$.

Now, since

$$\left(\alpha\varphi\left(1-\gamma_\pi^1\right)+\beta+\left(1-\varphi\gamma_x^1\right)\right)^2>\left(\alpha\varphi\left(1-\left(\gamma_\pi^1+\varepsilon\right)\right)+\beta+\left(1-\varphi\gamma_x^1\right)\right)^2$$

then $4\beta\gamma_x^1 > \left(\alpha\varphi\left(1-\left(\gamma_\pi^1+\varepsilon\right)\right)+\beta+\left(1-\varphi\gamma_x^1\right)\right)^2$, i.e., z_1^2 is complex. In this case it is obvious that monotonicity with respect to γ_π is satisfied. ■

No Monotonicity with respect to γ_x: Consider an $h = \left(\gamma_\pi^1, \gamma_x^1\right) \in \Gamma$ and a $w = \left(\gamma_\pi^1, \gamma_x^2\right) = \left(\gamma_\pi^1, \gamma_x^1 + \varepsilon\right) \in \Gamma$ such that z_1^1 and z_1^2 are complex. In this case it is easy to see (using a similar argument to the previous proof) that $z_1^1 \le z_1^2$; take now $h = \left(\gamma_\pi^1, \gamma_x^1\right) \in \Gamma$ and a $w = \left(\gamma_\pi^1, \gamma_x^2\right) = \left(\gamma_\pi^1, \gamma_x^1 + \varepsilon\right) \in \Gamma$ such that z_1^1 and z_1^2 are real and it is easy to see that $z_1^2 \le z_1^1$.

G. Proof of Lemma 9

Substituting the value of the conditional expectations into (2.21), the optimal policy rule could be written as:

$$i_t = \gamma^R + \gamma_g^R g_t$$

$$\begin{aligned} \gamma^R &= \frac{\lambda\alpha}{(\lambda+\alpha^2) - \lambda\beta}\overline{x} \\ \gamma_g^R &= \frac{1}{\varphi} \end{aligned}$$

This expression says that the policy-maker should offset demand shocks (g_t) by adjusting the nominal interest rate in order to neutralize any shock to the IS curve. Since this optimal policy rule involves only the fundamentals of the economy (demand and supply shocks), it could be defined as the *optimal fundamentals-based reaction function* under rational expectations (Evans and Honkapohja (2002))[30].

Now, consider a generic *expectations-based* policy rule of the form:

$$i_t = \gamma + \gamma_x E_t x_{t+1} + \gamma_\pi E_t \pi_{t+1} + \gamma_g g_t$$

Assuming rational expectations, expected values could be substituted in the previous expression to obtain the following policy rule:

$$i_t = (\gamma + \gamma_x a_x + \gamma_\pi a_\pi) + \gamma_g g_t$$

By comparing this equation with the *optimal fundamentals-based policy rule,* a system of two equations on four unknowns $(\gamma, \gamma_x, \gamma_\pi, \gamma_g)$ is obtained:

$$\begin{aligned} \gamma^R &= (\gamma + \gamma_x a_x + \gamma_\pi a_\pi) \\ \gamma_g^R &= \gamma_g \end{aligned}$$

Obviously, this system has multiple solutions.

H. Proof of proposition 10

By considering the values of the coefficients of the reaction function γ_g^*, γ^*,γ^R and the rational expectations values a_x, a_π given, the combinations of γ_x and γ_π are obtained that determine asymptotically the same equilibrium derived under the *optimal expectations-based reaction function* (2.21):

$$\gamma_\pi = \frac{\left(\lambda+\alpha^2\right)(1+\alpha\varphi) - \lambda\beta}{\alpha\left(\lambda+\alpha^2\right)\varphi} - \frac{(1-\beta)}{\alpha}\gamma_x$$

[30]Many autors (see for example Woodford (1999)) have shown that this interest rate rule leads to indeterminacy, i.e., a multiplicity of rational expectations equilibria.

Consider the isoquants of Figure 8:

$$\gamma_\pi = 1 - \frac{(1-z_1)(\beta - z_1)}{z_1 \alpha \varphi} + \frac{(\beta - z_1)}{z_1 \alpha}\gamma_x \qquad \text{for } \gamma_x < \widehat{\gamma}_x$$

$$\gamma_\pi = 1 + \frac{\beta + 1 - 2z_1}{\alpha\varphi} - \frac{1}{\alpha}\gamma_x \qquad \text{for } \gamma_x \geq \widehat{\gamma}_x$$

with a kink on

$$\left(\widehat{\gamma}_x = \frac{-z_1^2 + \beta}{\varphi\beta}, \widehat{\gamma}_\pi = 1 + \frac{(\beta - z_1)^2}{\alpha\varphi\beta} \right)$$

Restricting the analysis to the set $\Gamma_\beta = \{\gamma_\pi, \gamma_x : 0 < z_1 < \beta, \gamma_\pi > 0, \gamma_x > 0\}$, now the maximum speed of convergence problem defined for $0 < z_1 < \beta$:

$$\begin{aligned} &\min_{\gamma_\pi, \gamma_x} Z(\gamma_\pi, \gamma_x) \\ s.t. \quad \gamma_\pi \quad &= \quad \frac{(\lambda + \alpha^2)(1 + \alpha\varphi) - \lambda\beta}{\alpha(\lambda + \alpha^2)\varphi} - \frac{(1-\beta)}{\alpha}\gamma_x \end{aligned}$$

has a solution (use proposition 3.D.1 in Mas-Colell et al., 1995), and there is also an indirect speed of convergence function $v(\lambda)$ that is strictly decreasing on λ (use proposition 3.D.3 in Mas-Colell et al., 1995). The maximum speed of convergence that could be induced by a combination (γ_π, γ_x) for a given λ will always coincide with the kink. Note that

$$\frac{\partial \widehat{\gamma}_x}{\partial z_1} = \frac{-2z_1}{\varphi\beta} < 0$$

$$\frac{\partial \widehat{\gamma}_\pi}{\partial z_1} = \frac{-2(\beta - z_1)}{\alpha\varphi\beta} < 0 \quad \text{for} \quad z_1 < \beta$$

Now, since the higher the level curve, the faster the convergence, it must be shown that as λ increases, the line

$$\gamma_\pi = \frac{(\lambda + \alpha^2)(1 + \alpha\varphi) - \lambda\beta}{\alpha(\lambda + \alpha^2)\varphi} - \frac{(1-\beta)}{\alpha}\gamma_x$$

moves downward and the fastest speed of convergence that is feasible is lower, or in other words the smallest z_1 that can be reached is larger.

I. Proof of Proposition 12

Under the EH policy, the economy evolves according to the following dynamic system:

$$\begin{bmatrix} \pi_t \\ x_t \end{bmatrix} = \begin{bmatrix} \Phi^* \\ \frac{\Phi^*}{\alpha} \end{bmatrix} + \begin{bmatrix} \Gamma^* & 0 \\ -\frac{(\beta - \Gamma^*)}{\alpha} & 0 \end{bmatrix} \begin{bmatrix} E_t \pi_{t+1} \\ E_t x_{t+1} \end{bmatrix}$$

Under the ALS-Γ' policy, the economy evolves according to the following dynamic system:

$$\begin{bmatrix} \pi_t \\ x_t \end{bmatrix} = \begin{bmatrix} \frac{\Phi^*(1-\Gamma')}{(1-\Gamma^*)} \\ \frac{\Phi^*(1-\Gamma')}{(1-\Gamma^*)\alpha} \end{bmatrix} + \begin{bmatrix} \Gamma' & 0 \\ -\frac{(\beta-\Gamma')}{\alpha} & 0 \end{bmatrix} \begin{bmatrix} E_t\pi_{t+1} \\ E_t x_{t+1} \end{bmatrix}$$

The REE under both policies is

$$\pi_t = \frac{\Phi^*}{(1-\Gamma^*)} \text{ and } x_t = \frac{\Phi^*(1-\beta)}{(1-\Gamma^*)\alpha}$$

and under learning, when both Γ' and Γ^* are smaller than one the REE is E-stable.

L. Proof of Lemma 13

Since under ALS-Γ' policy,

$$\Gamma' = \left(1-\gamma_\pi^{ALS}\right)\alpha\varphi + \beta,$$

given the parameters α, φ, β, each value of the policy reaction parameter γ_π^{ALS} has a corresponding slope of the $T(.)$ mapping, Γ', independently from λ.

M. Proof of Lemma 14

If

$$1 > \Gamma^* > \Gamma'$$

then

$$\gamma'_\pi = \left(1 + \frac{(\beta-\Gamma')}{\alpha\varphi}\right) > \left(1 + \frac{\beta-\Gamma^*}{\alpha\varphi}\right) = \gamma^*_\pi$$

Similarly if

$$1 > \Gamma' > \Gamma^*$$

N. Proof of Proposition 15

Define $a_{\pi,t}\left(i^{ALS}(\Gamma')\right)$ and $a_{\pi,t}\left(i^{EH}\right)$ the perceived inflation under ALS-Γ' and EH policies, $\pi_t\left(i^{ALS}(\Gamma')\right)$ and $\pi_t\left(i^{EH}\right)$ actual inflation under ALS-Γ' and EH policies. Assume that the economy starts from a point where the learning equilibrium and the REE do not coincide,

$$a_{\pi,0}\left(i^{ALS}(\Gamma')\right) = a_{\pi,0}\left(i^{EH}\right) \neq \overline{a}_\pi$$

I have to show that if $\gamma_\pi^{ALS} > \gamma_\pi^*$, then for every $0 < t < \infty$

$$\left|a_{\pi,t}\left(i^{ALS}(\Gamma')\right) - \overline{a}_\pi\right| < \left|a_{\pi,t}\left(i^{EH}\right) - \overline{a}_\pi\right|$$

and

$$\left|\pi_t\left(i^{ALS}\left(\Gamma'\right)\right)-\overline{a}_\pi\right|<\left|\pi_t\left(i^{EH}\right)-\overline{a}_\pi\right|$$

while, if $\gamma_\pi^{ALS}<\gamma_\pi^*$, then for every $0<t<\infty$

$$\left|a_{\pi,t}\left(i^{ALS}\left(\Gamma'\right)\right)-\overline{a}_\pi\right|>\left|a_{\pi,t}\left(i^{EH}\right)-\overline{a}_\pi\right|$$

and

$$\left|\pi_t\left(i^{ALS}\left(\Gamma'\right)\right)-\overline{a}_\pi\right|>\left|\pi_t\left(i^{EH}\right)-\overline{a}_\pi\right|$$

I will proove the proposition for $\gamma_\pi^{ALS}>\gamma_\pi^*$. A similar procedure could be used for $\gamma_\pi^{ALS}<\gamma_\pi^*$.

Let $\gamma_\pi^{ALS}>\gamma_\pi^*$, then

$$\Gamma'<\Gamma^*$$

Now, for $t=0$, since

$$\Gamma^*\left(a_{\pi,0}-\frac{\Phi^*}{(1-\Gamma^*)}\right)>\Gamma'\left(a_{\pi,0}-\frac{\Phi^*}{(1-\Gamma^*)}\right)$$

then

$$\left|\pi_0\left(i^{ALS}\left(\Gamma'\right)\right)-\overline{a}_\pi\right|<\left|\pi_0\left(i^{EH}\right)-\overline{a}_\pi\right|$$

For $t=1$, since

$$a_{\pi,1}\left(i^{EH}\right)-\overline{a}_\pi=\pi_0\left(i^{EH}\right)-\overline{a}_\pi$$
$$a_{\pi,t}\left(i^{ALS}\left(\Gamma'\right)\right)-\overline{a}_\pi=\pi_0\left(i^{ALS}\left(\Gamma'\right)\right)-\overline{a}_\pi$$

then

$$\left|a_{\pi,t}\left(i^{ALS}\left(\Gamma'\right)\right)-\overline{a}_\pi\right|<\left|a_{\pi,1}\left(i^{EH}\right)-\overline{a}_\pi\right|$$

Moreover, since

$$\pi_1\left(i^{EH}\right)-\overline{a}_\pi=\Gamma^{*2}\left(a_{\pi,0}-\frac{\Phi^*}{(1-\Gamma^*)}\right)$$

$$\pi_1\left(i^{ALS}\left(\Gamma'\right)\right)-\overline{a}_\pi=\Gamma'^2\left(a_{\pi,0}-\frac{\Phi^*}{(1-\Gamma^*)}\right)$$

and since $\Gamma^{*2}>\Gamma'^2$, then

$$\left|\pi_1\left(i^{ALS}\left(\Gamma'\right)\right)-\overline{a}_\pi\right|<\left|\pi_1\left(i^{EH}\right)-\overline{a}_\pi\right|$$

Similarly for $t>1$.

N. Proof of Lemma 17

Given that

$$x^{REE}=\frac{\Phi^{*}\left(1-\beta\right)}{\left(1-\Gamma^{*}\right)\alpha}$$

it must be shown that if $a_{\pi,0}\left(i^{ALS}\left(\Gamma'\right)\right)=a_{\pi,0}\left(i^{EH}\right)>\overline{a}_{\pi}$, then for every $0\leq t<\infty$,

$$x_{t}\left(i^{ALS}\left(\Gamma'\right)\right),x_{t}\left(i^{EH}\right)<x^{REE}$$

and for all $0\leq t',t<\infty$, with $t'>t$

$$x_{t}\left(i^{ALS}\left(\Gamma'\right)\right)<x_{t'}\left(i^{ALS}\left(\Gamma'\right)\right)<x^{REE}\quad\text{and}\quad x_{t}\left(i^{EH}\right)<x_{t'}\left(i^{EH}\right)<x^{REE}$$

If $a_{\pi,0}\left(i^{ALS}\left(\Gamma'\right)\right)=a_{\pi,0}\left(i^{EH}\right)<\overline{a}_{\pi}$, then for every $0\leq t<\infty$,

$$x_{t}\left(i^{ALS}\left(\Gamma'\right)\right),x_{t}\left(i^{EH}\right)>x^{REE}$$

and for all $0\leq t',t<\infty$, with $t'>t$

$$x_{t}\left(i^{ALS}\left(\Gamma'\right)\right)>x_{t'}\left(i^{ALS}\left(\Gamma'\right)\right)>x^{REE}\quad\text{and}\quad x_{t}\left(i^{EH}\right)>x_{t'}\left(i^{EH}\right)>x^{REE}$$

$$x_{t}\left(i^{EH}\right)=\frac{\Phi^{*}}{\alpha}-\frac{\left(\beta-\Gamma^{*}\right)}{\alpha}a_{\pi,t}\left(i^{EH}\right)$$

$$x_{t}\left(i^{ALS}\left(\Gamma'\right)\right)=\frac{\Phi^{*}\left(1-\Gamma'\right)}{\left(1-\Gamma^{*}\right)\alpha}-\frac{\left(\beta-\Gamma'\right)}{\alpha}a_{\pi,t}\left(i^{ALS}\left(\Gamma'\right)\right)$$

Let $a_{\pi,0}<\overline{a}$. Since

$$\frac{\left(\Gamma'-\beta\right)}{\alpha},\frac{\left(\Gamma^{*}-\beta\right)}{\alpha}<0$$

and

$$x_{0}\left(i^{EH}\right)-x^{REE}=\frac{\left(\Gamma^{*}-\beta\right)}{\alpha}\left(a_{\pi,0}-\frac{\Phi^{*}}{\left(1-\Gamma^{*}\right)}\right)$$

$$x_{0}\left(i^{ALS}\left(\Gamma'\right)\right)-x^{REE}=\frac{\left(\Gamma'-\beta\right)}{\alpha}\left(a_{\pi,0}-\frac{\Phi^{*}}{\left(1-\Gamma^{*}\right)}\right)$$

then $x_{0}\left(i^{EH}\right),x_{0}\left(i^{ALS}\left(\Gamma'\right)\right)<0$.

For $t=1$, we have

$$x_{1}\left(i^{EH}\right)-x^{REE}=\Gamma^{*}\frac{\left(\Gamma^{*}-\beta\right)}{\alpha}\left(a_{\pi,0}-\frac{\Phi^{*}}{\left(1-\Gamma^{*}\right)}\right)$$

$$x_{1}\left(i^{ALS}\left(\Gamma'\right)\right)-\overline{a}_{\pi}=\Gamma'\frac{\left(\Gamma'-\beta\right)}{\alpha}\left(a_{\pi,0}-\frac{\Phi^{*}}{\left(1-\Gamma^{*}\right)}\right)$$

and again it is obvious that $x_{1}\left(i^{EH}\right),x_{1}\left(i^{ALS}\left(\Gamma'\right)\right)<0$ and

$$x_{1}\left(i^{ALS}\left(\Gamma'\right)\right)>x_{0}\left(i^{ALS}\left(\Gamma'\right)\right)\quad\text{and}\quad x_{1}\left(i^{EH}\right)>x_{0}\left(i^{EH}\right)$$

similarly for t=2,3,... and for the case $a_{\pi,0}\left(i^{ALS}\left(\Gamma'\right)\right)=a_{\pi,0}\left(i^{EH}\right)>\overline{a}_{\pi}$.