Optimal Auction Design for Heterogeneous Objects

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Abstract

This paper studies optimal auction for heterogenous objects when bidders have multidimensional independent types from continuous distributions. A new convergence result shows that the optimal auction mechanism can be obtained as a limit of multiproduct nonlinear pricing mechanisms of Rochet and Choné (1998). The reservation price excludes bidders with positive probabilities. Bunching is robust because of the conflict between participation constraints and incentive compatibility conditions and also from the quantity constraints. Numerical examples are presented.

1 Introduction

This paper studies the optimal auction mechanism when the seller has heterogeneous objects and each buyer has a multidimensional type drawn independently from a continuous distribution.

In case of a single good and a single dimensional type, Myerson (1981) presented a systematic study of the optimal auction mechanism. Buyer's incentive constraints are equivalent to an envelope condition and monotonicity of the expected allocation. Using integration by parts, the seller's expected

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profit is represented in terms of allocation and the marginal revenue. If the distribution of types satisfies the regularity condition, then the optimal auction mechanism allocates the object to the bidder with the highest marginal revenue if the marginal revenue is positive. A second price auction with a reserve price implements the optimal auction. If the regularity condition is not satisfied, then the seller uses the ironing procedure to restore monotonicity of the allocation.

But when I try to apply this approach to my auction problem, I encounter two difficulties. First, since types are multidimensional, incentive compatibility conditions are more complex. The second order condition is equivalent to convexity of expected surplus. In contrast to the single dimensional case, it is very difficult to find a regularity condition. Indeed, in the context of multidimensional nonlinear pricing, Rochet and Choné (1998) showed that this second order condition is generally binding. The second difficulty is that the seller may have an incentive to bundle the objects. McAfee, McMillan, and Whinston (1986) showed that, in the context of the monopoly selling, under a general condition on the distribution of types, the seller bundles the objects in order to increase the revenue from the buyer with a high type for one object and a low type for the other objects. As a result, an auction mechanism which allocates each object to the buyer with the highest marginal revenue for each object is not likely to be optimal.

In order to explain these two points, let me consider a very simple example where the seller has two objects. Each buyer has two dimensional types. Each type is distributed independently according to a uniform distribution on [0,1]. A buyer's payoff is the sum of values from each object minus the payment. In this example, since there are no complementarity in the payoffs and the distribution of types is independent, running an optimal auction for each object seems to be a reasonable candidate for the optimal auction mechanism. An optimal mechanism for each object in this case is the second price auction with the reserve price of 0.5. If there is only one bidder, the expected payoff is given by $2 \times 0.5 \times 0.5 = 0.5$. Now, consider a following pure bundling auction where the seller bundles object 1 and object 2, and sets the reserve price of 0.9 for the bundle. In this case, with one bidder, the expected payoff is $0.9 * (1 - 0.9^2 \times 0.5) = 0.5355$, which is strictly higher than the combination of the optimal auctions for each objects.

In this paper, I deal with these two problems as follows. I start with a new convergence result which derives an optimal auction mechanism as a limit of nonlinear pricing mechanisms by constructing the cost functions of the nonlinear pricing problem which converges to that of the optimal auction problem.

Since this is a key idea of the paper, let me explain the method in the simplest setting of a seller selling a single object to a single buyer. In the optimal auction setting, the seller has a zero marginal cost up to one unit, and faces an infinite marginal cost for more than one unit. I can represent this cost structure as a limit of nonlinear pricing cost structures $c(q) = q^n$. In this nonlinear pricing problem, the objective function is

$$\int [MR(t_1)q(t_1) - q(t_1)^n] f(t_1) dt_1$$

where $MR(t_1) = t_1 - (1 - F(t_1))/f(t_1)$. Assuming the regularity condition on the distribution of types, the optimal allocation is to allocate zero unit if the marginal revenue is negative, and the quantity which satisfies the first order condition $MR(t_1) = MC(q(t_1))$ if the marginal revenue is positive. As I let $n \to \infty$, I get the optimal auction mechanism where the seller allocates one unit to the buyer with a nonnegative marginal revenue.

Now suppose there are two buyers. In the optimal auction problem, the quantity constraints is $0 \leq q(t_1, t_2) + q(t_2, t_1) \leq 1$ for each t_1 and t_2. Let me consider nonlinear pricing problems with cost functions $(q(t_1, t_2) + q(t_2, t_1))^n$. The seller's objective function is

$$\int (MR(t_1)q(t_1, t_2) + MR(t_2)q(t_2, t_1) - (q(t_1, t_2) + q(t_{2,}t_1))^n) f(t_1, t_2) dt_1 dt_2$$

Since the cost function is symmetric in $q(t_1, t_2)$ and $q(t_2, t_1)$, the marginal cost is the same whether the seller sells an additional unit to either of buyer 1 or buyer 2. The seller chooses buyer i with the highest marginal revenue (assuming, for simplicity, it is positive) and supplies the quantity determined by $MR(t_i) = MC(q(t_i))$. At the limit of $n \to \infty$, the seller sells one unit of the object to the buyer with the highest marginal revenue.

Then I apply this convergence result to study an optimal auction mechanism of multidimensional types and heterogeneous objects. An important advantage in working from the nonlinear pricing problem is that, since the objective of the nonlinear pricing problem is smooth and the set of feasible mechanisms is a closed convex cone, I can apply first order conditions to characterize the optimal mechanism. I first construct a multi-product and multi-dimensional nonlinear pricing problem among multiple buyers corresponding to the optimal auction problem. This nonlinear pricing problem can be considered as a multiple buyer version of the nonlinear pricing problem by Rochet and Choné(1998). Then I apply the first result to derive the optimal auction mechanism as a limit of nonlinear pricing mechanisms.

I find three features in the optimal auction mechanism for heterogenous objects: exclusion, efficiency at the boundary, and bunching.

First, the seller is going to set a nontrivial reserve price. In nonlinear pricing problems, Armstrong (1996) showed the exclusion region has a strictly positive measure. An intuition is that, in multidimensional problems, the measure of buyers excluded is of a higher order (for example, if the buyer has a two dimensional type distributed on a rectangle $[0,1]^2$ and the seller sets a reserve price in the form of $t_1 + t_2 = \epsilon$, then the measure of the set of buyers who will be excluded is $\varepsilon^2/2$) than the first order benefit from charging higher prices to remaining buyers. I find that this property carries over to an optimal auction mechanism, and, moreover, the reserve price (the exclusion region) is the same, independent of the number of buyers and of the difference between nonlinear pricing mechanisms and optimal auction mechanisms. Intuitively, using an analogy from a single dimensional problem, a buyer's marginal revenue only depends on the realization of a type, and the seller excludes buyers with a negative marginal revenue. There the cost structure does not play any role in the determination of exclusion. This finding generalizes an analogous result in one dimensional single object case.

In nonlinear pricing problems, buyers at the upper boundary of the set of feasible types are not subject to distortion. An intuition is that there are no higher (in terms of types) buyers to worry about deviation. Since this property holds for every nonlinear pricing mechanism, it also holds in the limit case of optimal auction mechanisms.

The third result concerns bunching. Rochet and Choné(1998) found that bunching is robust in multidimensional nonlinear pricing models. In nonlinear pricing problem, bunching takes place because of the conflict among incentive constraints, exclusion, and continuity of allocations. The seller wishes to set nontrivial reserve prices. On the other hand, the seller wishes to satisfy the first order conditions. It will create a conflict at the boundary of the exclusion region, and can violate continuity of allocation. As a result, bunching will take place. In optimal auction problem, bunching is still robust, but with some differences from that of nonlinear pricing problems. In auctions, the allocation does not necessarily have to be continuous, but on the other hand, the allocation is exogenously bounded from above due to quantity constraints. This will create a new source of bunching. I find an example of the optimal auction where bunching region merges with differentiated region.

Finally I numerically compute optimal auction mechanisms for some examples. The result suggests a complex allocation schedule. At the boundary of the reserve price, the allocation pattern exhibits bundling. But at the upper boundary of the exclusion region, efficiency at the boundary implies that the allocation for an object solely depends on the type for this object. This complexity suggests that an explicit full characterization of an optimal auction mechanism may not be analytically easy.

For contributions, first, this paper provides a new linkage between optimal auction mechanisms and nonlinear pricing mechanisms. Improving a result in Bulow and Roberts (1989), my result allows an explicit derivation of an optimal auction mechanism from standard nonlinear pricing mechanisms. Second, this paper intends to make a progress, building on previous contributions, in a study of multidimensional mechanism design. This paper provides first characterization results of a general optimal auction mechanism for multidimensional types and heterogenous objects without assuming an exogenous reduction to one dimensional type, some restrictions of possible mechanisms, nor a discrete distribution of types. In addition, I provide computational examples of the auction mechanism for heterogeneous objects.

The plan of the paper is as follows. In section 2, I define the optimal auction problem. Section 3 presents the convergence result. In section 4, I apply this result to consider an auction for heterogeneous objects. Section 5 presents computational examples. Section 6 concludes.

1.1 Related Literature

First this paper relates to Bulow and Roberts (1989) which examined the relation between optimal auction problems and a nonlinear pricing problem with the seller having zero marginal cost up to a capacity constraint. In this paper, I tighten the connection further between optimal auctions by deriving the optimal auction mechanism from nonlinear pricing problems with standard cost functions (e.g. quadratic). This technique is valid for an arbitrary finite number of objects, bidders, and when the objective function is not restricted to revenue maximization (e.g. welfare maximization).

Second this paper builds on the advances made in the study of multidimensional nonlinear pricing problems[1].

Armstrong (1996) considered multiproduct and multidimensional nonlinear pricing problem. Armstrong (1996) proved the key exclusion result in multidimensional mechanisms and characterized sufficient conditions to derive the optimal nonlinear pricing mechanism based on a cost-based tariff.

Rochet and Choné(1998) provided a general characterization result of nonlinear pricing mechanisms with a multidimensional type and heterogeneous objects. This paper builds on their new methods and extends their insights to optimal auction mechanisms.

Third, let me explain the relation to the previous contributions to the study of optimal auction for heterogeneous objects.

Palfrey (1983)'s elegant analysis concerns the seller bundling decision in Vickrey auctions. Palfrey (1983) showed that the bundling decision can depends on the number of bidders. In contrast, this paper studies a general characterizations of the optimal auction mechanism without exogenous restrictions on the possible mechanisms.

Levin (1997) succeeded in the full characterization of an optimal auction problem for heterogenous objects when the bidder has one dimensional type so that the techniques developed in Myerson (1981) can be applied. This paper considers an optimal auction problem when bidders have multidimensional types.

Jehiel, Moldovanu, and Stachetti (1999) considered an auction design problem when the seller has a single object and the seller and the buyers have multidimensional types which represents externalities on the identity of the winner of the auction. They provided a characterization result for incentive compatibility for multidimensional types. Furthermore, they showed that a second price auction with entry fee is optimal among the standard bidding mechanisms where the bidder is restricted for a one dimensional bid and the seller always sell the object to a bidder. This paper considers an optimal mechanism where the seller can use full multidimensional information from bidders.

Avery and Hendershott (2000) analyzed the optimal auction mechanism of two objects when there is a buyer who is interested in both objects and other bidders are interested in one object. Avery and Hendershott (2000) showed that the optimal auction mechanism involves bundling. This paper considers a general case where each bidder is interested in both objects and

[1] Rochet and Stole (2000) provided a survey on the literature.

the type comes from a continuous distribution. The computational results suggests a bundling persists in optimal auction mechanisms.

Armstrong (2000) provided a detailed characterization of the optimal auction mechanism when the bidder's type is from a binary distribution. Armstrong (2000) showed that the optimal auction mechanism depends on the correlation among types, and a bundling auction is optimal with independently distributed types, the insight this paper confirms in a case of a continuous distribution.

2 The Model

In this section I explain the model of the optimal auction with heterogeneous objects and multidimensional types.

The Supply Side. The single seller has K different objects. The seller has zero value for each object. There are N buyers. Let q_i^k be an allocation of object k to buyer i. The seller faces quantity constraints: $0 \leq \sum_{i=1}^N q_i^k \leq 1$ for each k and $0 \leq q_i^k \leq 1$, for each of i and k. The seller is risk-neutral.

The Demand Side. Buyer $i = 1, ..., N$ has a $K-$dimensional type t_i. The distribution of types among buyers is iid with a continuous and bounded density [2] f. The support of the distribution of types is $\Omega = [0, 1]^K$. Buyer i has quasilinear preferences: given type t_i, an allocation $q_i = (q_i^1,..., q_i^K)$, and a payment $p_i \in \mathbb{R}_+$, the payoff is given by $t_i \cdot q_i - p_i$. Each buyer has an outside option with a payoff of 0 and is risk neutral.

The Seller's Problem. The seller chooses a mechanism to maximize the expected revenue. By the revelation principle, it is without loss of generality to restrict attentions to a class of direct revelation mechanisms, which are maps from the set of reports from buyers to an allocation and a payment. Let $q^k : [0, 1]^{KN} \to \mathbb{R}_+, k = 1,..., K$ and $p : [0, 1]^{KN} \to \mathbb{R}_+$ be the allocation and the payment function. The seller restrict attentions to symmetric mechanisms, that is, for each k, $q^k(t_i, t_{-i}) = q^k(t_j, t_{-j})$ and $p(t_i, t_{-i}) = p(t_j, t_{-j})$ if $t_i = t_j$ and $t_{-i} = t_{-j}$. I assume that q^k and p are from $L^1([0, 1]^{KN})$ endowed with a usual (norm) topology. The seller's expected revenue from the

[2] For simplicity of notations, I use f for the density of the type t_i, or a vector of types $t = (t_1, ..., t_N)$ and others while avoiding confusion.

mechanism (p, q) is

$$\Phi(p,q) = \int \sum_{i=1}^{N} p(t_i, t_{-i}) f(t_i, t_{-i}) dt.$$

Now consider constraints for a buyer. Let $Q^k(t_i) = \int q^k(t_i, t_{-i}) f(t_{-i}) dt_{-i}$ and $Q(t_i) = (Q^1(t_i), ..., Q^K(t_i))$ be the expected allocation for a buyer with type t_i,assuming truthful reporting. Let $P(t_i) = \int p(t_i, t_{-i}) f(t_{-i}) dt_{-i}$ be the expected payment, assuming truthful reporting. Let $U(t_i) = t_i \cdot Q(t_i) - P(t_i)$ be the expected surplus from the mechanism. The incentive constraints are $U(t_i) \geq t_i \cdot Q(t_i') - P(t_i')$, $\forall t_i, t_i'$. Alternatively, $U(t_i) = \sup_{t_i'}(t_i \cdot Q(t_i') - P(t_i'))$, $\forall t_i$. The individual rationality constraints are $U(t_i) \geq 0$, $\forall t_i$.

I summarize the seller's maximization problem:

$$\sup_{p,q \in L^1([0,1]^{NK})} \Phi(p,q) = \int \sum_{i=1}^{N} p(t_i, t_{-i}) f(t_i, t_{-i}) dt.$$

subject to $U(t_i) \geq t_i \cdot Q(t_i') - P(t_i')$, $\forall t_i, t_i'$,$U(t_i) \geq 0$, $\forall t_i$,and $0 \leq \sum_{i=1}^{N} q^k(t_i, t_{-i}) \leq 1$, $0 \leq q^k(t_i, t_{-i}) \leq 1$,for each $k = 1, ..., K$ and (t_i, t_{-i}).

Finally I recall a characterization of the incentive constraints:

Lemma 1 (Rochet (1987)) *Given a surplus function U, there exists an expected allocation Q and a payment function P which satisfies the incentive constraints if and only if $Q(t_i) = \nabla U(t_i)$ for t_i almost everywhere and $U(t_i)$ is convex continuous.*

This lemma extends a standard constraint simplification theorem for one dimensional type, since convexity of $U(t_i)$ in one dimensional t_i is equivalent to a monotonicity of its subgradient $Q(t_i)$.

3 Approaches

In the previous section, I defined an optimal auction problem. In this section, I review two possible previous approaches to the problem. The first one is a direct approach, taken by Myerson (1981) in the study of a single object one dimensional problem. I explain that complexity of the second order condition of incentive constraints makes the application of this approach difficult in my problem. The second approach is an indirect approach, taken by Rochet and Choné (1998) in the study of a multiproduct multidimensional problem. This approach has an advantage in dealing with the second order condition of incentive compatibility conditions as a constraint in calculus of variation problem. But in the auction problem, the set of possible mechanisms is not a closed convex cone because of quantity constraints, which makes an immediate application of their methodology difficult. Thus I will present a new approach to obtain an optimal auction mechanism as a limit of nonlinear pricing mechanisms, which will be detailed in the next section.

3.1 A Direct Approach

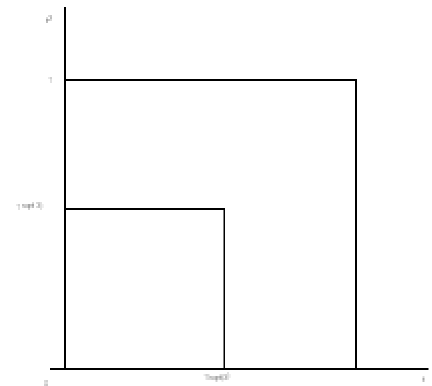
A first approach to a multidimensional problem would be to extend an integration by parts approach.

The seller's expected revenue is, by applying integration along the ray (Armstrong (1996)),

$$\begin{aligned}\Phi(p,q) &= \int (\sum_{i=1}^{N} p(t_i,t_{-i}))f(t_i,t_{-i})dt \\ &= \int \sum_{i=1}^{N}(1-\frac{g(t_i)}{f(t_i)})\sum_{k=1}^{K} t_i^k q_i^k(t_i,t_{-i})f(t)dt \\ &= \int \sum_{i=1}^{N}\sum_{k=1}^{K}(1-\frac{g(t_i)}{f(t_i)})t_i^k q_i^k(t_i,t_{-i})f(t)dt\end{aligned}$$

with $g(t_i) = \int_1^\infty \alpha f(t_i\alpha)d\alpha$. Thus I can derive a similar expression for the marginal revenue for heterogeneous object case $\mathrm{MR}_i^k(t_i) = (1-\frac{g(t_i)}{f(t_i)})t_i^k$.

Let me consider whether the solution of this first order condition can be an optimal solution. For example, consider the case of two objects and two dimensional types, with the distribution being rectangle $[0,1]^{2.}$ Then $g(t) = \int_1^{\min\{1/t_i^1, 1/t_i^2\}} \alpha d\alpha = \frac{1}{2}([\min(\frac{1}{t_i^1},\frac{1}{t_i^2})]^2-1)$. Thus $\mathrm{MR}_i^k(t_i) = (\frac{3}{2}-\frac{1}{2}[\min(\frac{1}{t_i^1},\frac{1}{t_i^2})]^2)t_i^k$.Thus the pointwise maximization implies that the seller does not sell to the buyer if $\min(\frac{1}{t_i^1},\frac{1}{t_i^2}) > \sqrt{3} \Leftrightarrow \max(t_i^1,t_i^2) < \frac{1}{\sqrt{3}} = 0.57735$. Thus the exclusion region for each object is given by $\Omega_0 = \{(t_i^1,t_i^2) : \max(t_i^1,t_i^2) < \frac{1}{\sqrt{3}}\}$. But this allocation is not optimal, as will be seen from computations. Intuitively, the seller is going to set a mixed reserve price discounted for a buyer with a higher type for each object (see Section 6 for more detail).



An Allocation According to the First Order Condition for Two Objects when the Distribution is from Uniform [0,1].

Let me explain the difficulty in satisfying incentive constraints in multidimensional problems[3]. The incentive constraints, expressed in Lemma 1, can be expressed in terms of cyclical monotonicity of expected allocation:

Definition (Rockerfeller (1967)). *A function Q: $\Omega \to \mathbf{R}$ is cyclically monotone if and only if for every $n>0$, $\{t_0,..., t_n\}$,*

$$(t_0 - t_n) \cdot Q(t_n) + ... + (t_2 - t_1) \cdot Q(t_1) + (t_1 - t_0) \cdot Q(t_0) \leq 0.$$

Lemma 2 (Rockerfeller (1967)) *In order for the existence of a proper convex function U such that its subgradient ∂U contains a function Q, it is necessary and sufficient that Q is cyclically monotone.*

Given these lemma, what conditions for Q will imply cyclical monotonicity ? Let me consider a weaker condition for monotonicity $(t_2 - t_1) \cdot (Q(t_2) - Q(t_1))$. Consider two objects case. The condition implies,

[3] See Jehiel, Moldovanu and Stacchetti (1999) for a characterization result.

$$
\begin{aligned}
& (t_2^1 - t_1^1, t_2^2 - t_1^2) \cdot (Q^1(t_2^1, t_2^2) - Q^1(t_1^1, t_1^2), Q^2(t_2^1, t_2^2) - Q^2(t_1^1, t_1^2)) \\
= \; & (t_2^1 - t_1^1)(Q^1(t_2^1, t_2^2) - Q^1(t_2^1, t_1^2)) + (t_2^2 - t_1^2)(Q^2(t_2^1, t_2^2) - Q^2(t_1^1, t_1^2)) \\
= \; & (t_2^1 - t_1^1)(Q^1(t_2^1, t_2^2) - Q^1(t_1^1, t_2^2)) + (t_2^1 - t_1^1)(Q^1(t_1^1, t_2^2) - Q^1(t_1^1, t_1^2)) \\
& + (t_2^2 - t_1^2)(Q^2(t_2^1, t_2^2) - Q^2(t_1^1, t_2^2)) + (t_2^2 - t_1^2)(Q^2(t_1^1, t_2^2) - Q^2(t_1^1, t_1^2))
\end{aligned}
$$

Even if the distribution satisfies monotonicity in each dimension, i.e., $(t_2^1 - t_1^1)(Q^1(t_2^1, t_2^2) - Q^1(t_1^1, t_1^2)) \geq 0$ and $(t_2^2 - t_1^2)(Q^2(t_1^1, t_2^2) - Q^2(t_1^1, t_1^2)) \geq 0$, it is still necessary to show that

$$(t_2^1 - t_1^1)(Q^1(t_1^1, t_2^2) - Q^1(t_1^1, t_1^2)) + (t_2^2 - t_1^2)(Q^2(t_2^1, t_2^2) - Q^2(t_1^1, t_2^2)) \geq 0,$$

which cannot be derived from the standard ordinal conditions.

3.2 A Dual Approach

An alternative will be to apply the calculus of variation analysis by Rochet and Choné (1998) to the optimal auction problem. By representing the objective function in terms of the surplus, I get

$$\Phi(U) = \int \sum_{i=1}^{N} (t_i \cdot \nabla U(t_i) - U(t_i)) f(t) dt.$$

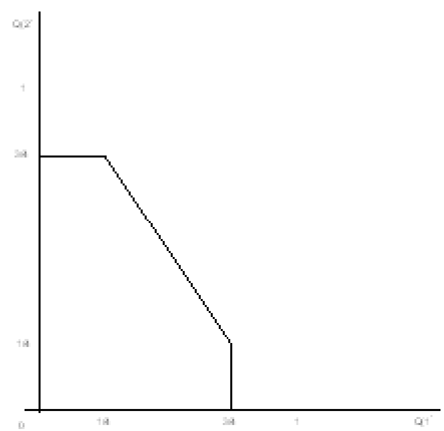
With multiple buyers, the seller's choice of expected surplus will yield the expected allocation as its subgradient. Since the quantity constraints is expressed in terms of unconditional quantities, I apply Border (1991)'s implementability conditions. Armstrong (2000) is the first to apply Border (1991)'s result to the analysis of the optimal auction mechanism.

Lemma 3 (Border (1991), Proposition 3.1) *Given $Q^k\!:\Omega \to [0,1]$, there exists $q^k(t_i, t_{-i})$ such that $Q^k(t_i) = \int q^k(t_i, t_{-i}) f(t_{-i}) dt_{-i}$ if and only if for each closed set $B \subseteq \Omega$,*

$$
\begin{aligned}
\int_B Q^k(t_i) f(t_i) dt_i \;&\leq\; \overline{Q}(B) \\
\textit{where } \overline{Q}(B) \;&\equiv\; \frac{1 - P(\Omega/B)^N}{N}.
\end{aligned}
$$

Let me take a moment to understand the condition. Intuitively, the expected quantity is computed by taking an integral of the unconditional allocation. The unconditional allocation satisfies quantity constraints. By inverting these two conditions, there will be inequalities for Q.

For an example (Border (1991)), consider a case of two buyers $N = 2$ and binary types $T = \{1, 2\}$ with equal probabilities $P(\{1\}) = P(\{2\}) = 1/2$. In this case, the allocation is given by $\{q(1,1), q(1,2), q(2,1), q(2,2)\}$. The expected allocation is $Q(1) = (1/2)q(1,1) + (1/2)q(1,2)$ and $Q(2) = (1/2)q(2,1)+(1/2)q(2,2)$. The quantity constraints are $q(1,1) \leq 1/2, q(2,2) \leq 1/2$ and $q(1,2) + q(2,1) \leq 1$. The set of $(Q(1), Q(2))$ which satisfy these conditions are graphed below (see Figure 4). Alternatively, I can draw the picture from corresponding conditions: $Q(1) \leq 3/4, Q(2) \leq 3/4$, and $(1/2)(Q(1) + Q(2)) \leq 1/2$.



The Set of Possible (Q(1),Q(2))

The optimal auction problem is

$$\max \Phi(U) = \int \sum_{i=1}^{N} (t_i \cdot \nabla U(t_i) - U(t_i)) f(t) dt.$$

subject to $U(t_i)$ convex and continuous and $\int_B \nabla U^k(t_i) f(t_i) dt_i \leq \overline{Q}(B)$ for each k and closed set $B \subseteq [0,1]^K$.

Still, it is difficult to directly apply the Rochet and Choné (1998) approach to my optimal auction problem. It is because the set of feasible mechanisms does not form a cone because of quantity constraints. As a result, the analysis would require additional Lagrangian multipliers for the quantity constraints. Instead, in the next section, I am going to show that the optimal auction mechanism can be obtained as a limit of nonlinear pricing mechanisms.

4 The Optimal Auction Mechanism as a Limit of Multiproduct Nonlinear Pricing Mechanisms

In this section, I will explain that an optimal auction mechanism can be derived as a limit of nonlinear pricing mechanisms whose cost structures converge to the cost structure of the optimal auction problem. In this section, I first present an example with a single object and one dimensional type to explain the intuition. Then I will explain the construction of the cost structure in the nonlinear pricing mechanisms. Finally I present the convergence result.

4.1 An Example with a Single Object and One Dimensional Type

In this subsection, I take a simple example of a single object and one dimensional type and show that an optimal auction mechanism of Myerson (1981) can be obtained as a limit of nonlinear pricing mechanisms in Mussa and Rosen (1978). I start with the simplest case of a single buyer, and then later move to the case of multiple buyers.

4.1.1 The Single Buyer Case

Consider a seller with a single good. The distribution of the value by a buyer is uniform [0,1].

Let me begin with the optimal auction problem. The seller has a zero marginal cost up to 1 unit and then faces an infinitely high marginal cost beyond one unit. Thus the cost structure is, by denoting q to be the quantity sold,

$$\begin{aligned} c^A(q) &= 0 \text{ if } 0 \leq q \leq 1 \\ &\quad \infty \text{ otherwise.} \end{aligned}$$

The buyer's incentive constraints is equal to the envelope condition and monotonicity of the allocation. Using integration by parts, I can compute the seller's expected revenue

$$\int_0^1 (MR(t_i)q(t_i) - c^A(q(t_i)))f(t_i)dt_i$$

Given the uniform distribution assumption, the marginal revenue is $t_i - \frac{1-F(t_i)}{f(t_i)} = 2t_i - 1$. The optimal mechanism is that the seller sells one unit if the buyer has a positive marginal revenue. That is,

$$\begin{aligned} q^A(t_i) &= 1 \text{ if } t_i \geq 1/2 \\ & \quad 0 \text{ else.} \end{aligned}$$

This allocation satisfies the second order condition of incentive constraints.

Next let me study nonlinear pricing problems. Consider a nonlinear pricing problem with a cost function $c^n(q) = q^n, n \geq 1$. The seller's expected revenue is

$$\int_0^1 (MR(t_i)q(t_i) - c^n(q(t_i)))f(t_i)dt_i$$

The first order condition is, $2t_i - 1 = n(q^n(t_i))^{n-1}$.Then

$$\begin{aligned} q^n(t_i) &= (\frac{2t_i - 1}{n})^{\frac{1}{n-1}} \text{ if } t \geq 1/2 \\ & \quad 0 \text{ otherwise.} \end{aligned}$$

The cost functions of nonlinear pricing problem converge to that of the optimal auction problem. That is, $c^n(q) \to c^A(q)$ pointwise except for $q = 1$. That is, as I increase n, the cost of providing $q < 1$ unit will converge to 0 and will diverge if $q > 1$, and this is exactly the cost structure of the optimal auction problem (see Figure). Since the only difference between the optimal auction problem and the nonlinear pricing problem is the costs, the nonlinear pricing mechanisms will converge to an optimal auction mechanism as $n \to \infty$ (see Figure).

4.1.2 Multiple Buyers

With multiple buyers, the seller's quantity constraints are $0 \leq \sum_{i=1}^N q(t_i, t_{-i}) \leq 1$. I begin by extending the nonlinear pricing model to allow multiple buyers with a cost function $(\sum_{i=1}^N q(t_i, t_{-i}))^n$. The objective function is

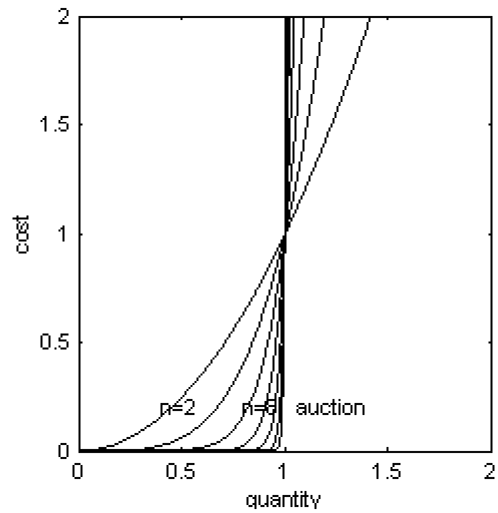


Figure 1: Convergence of Costs

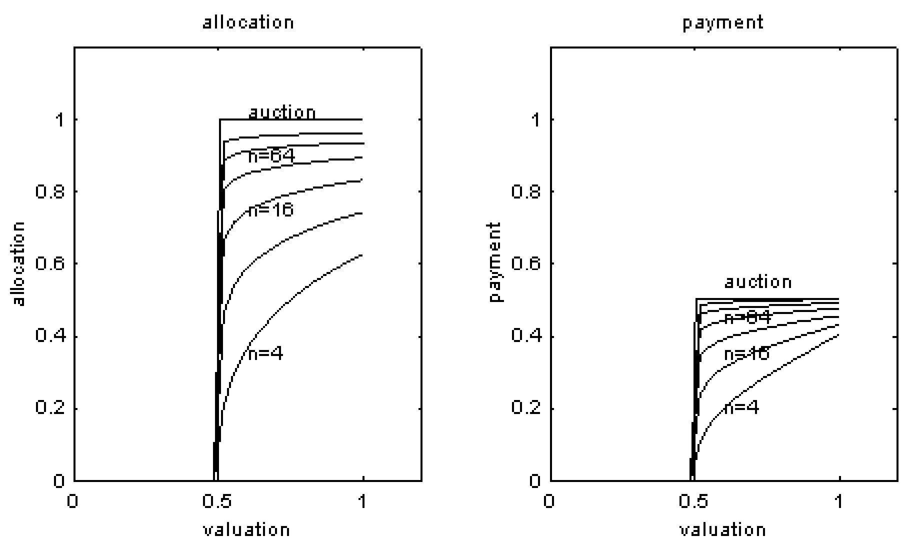


Figure 2: Convergence of Mechanisms

$$\int (\sum_{i=1}^{N} MR(t_i)q(t_i,t_{-i}) - (\sum_{i=1}^{N} q(t_i,t_{-i}))^n)f(t)dt$$

The marginal cost is identical for every buyer. The seller only wants to sell to the buyer with the highest marginal revenue. The optimal allocation is

$$\begin{aligned} q(t_i,t_{-i}) &= q \text{ such that } MR(t_i) = MC(q) \text{ if } t_i > t_j \text{ for all } j \neq i \text{ and } MR(t_i) \geq 0 \\ &= 0 \text{ else.} \end{aligned}$$

At the optimal the seller sells the object only to the buyer with the highest marginal revenue.

By sending $n \to \infty$, the optimal allocation[4] is

$$\begin{aligned} q(t_i,t_{-i}) &= 1 \text{ if } t_i > t_j \text{ for all } j \neq i \text{ and } MR(t_i) \geq 0 \\ &= 0 \text{ else.} \end{aligned}$$

Of course, it is exactly the allocation derived in Myerson (1981).

4.2 Statement of the Formal Result

In the previous subsection, I explained an example to show the idea of obtaining the optimal auction mechanism as a limit of nonlinear pricing mechanisms. In this subsection, I make the idea precise by first constructing nonlinear pricing mechanism correspondings to that optimal auction problem and then showing that its limit is indeed an optimal auction mechanism. This procedure decomposes an analysis of the optimal auction mechanism into two steps: (1) the analysis of the corresponding nonlinear pricing mechanisms and (2) taking the limit of the nonlinear pricing mechanism to derive the optimal auction mechanism.

[4] For simplicity, I do not consider a case of a tie in this example, but it is straightforward to handle.

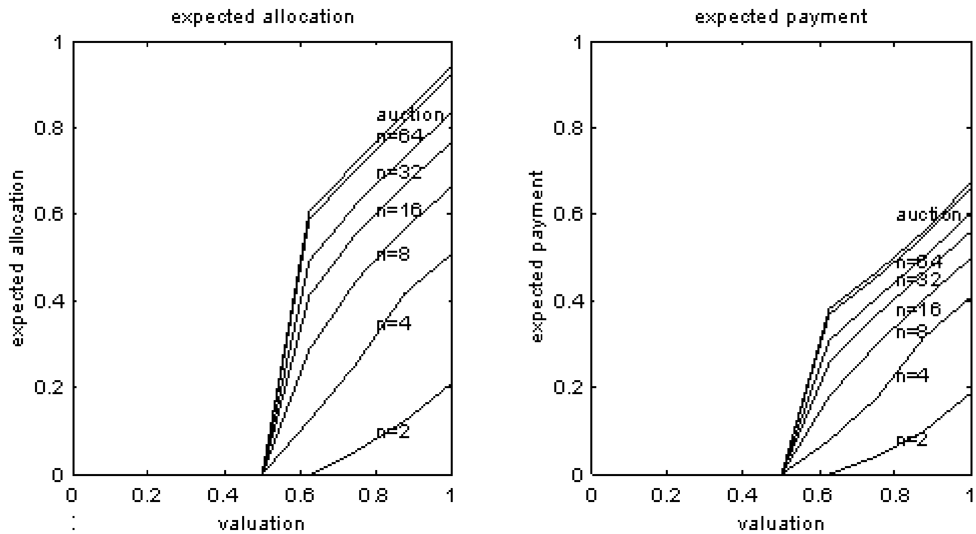


Figure 3: Convergence to Myerson (1981) Auctions

4.2.1 Reformulation of the Cost Structure

Let me begin with the construction of a measure on a space of closed sets of $[0,1]^K$. Let $\mathbf{F}$ be the family of closed subsets in $[0,1]^K$. Define a Hausdorff metric for two sets F_1 and F_2 by $d(F_1, F_2) = \sup_x |d_{F_1}(x) - d_{F_2}(x)|$ with $d_{F_1}(x) = \inf_{y \in F_1} ||x - y||$ and $d_{F_2}(x) = \inf_{y \in F_2} ||x - y||$. Then $\mathbf{F}$ is a compact metric space with this Hansdorff metric. Let $\mathcal{F}$ be the Borel σ-algebra of $\mathbf{F}$. Then ($\mathbf{F}$,$\mathcal{F}$) is universally measurable. Define a finite dimensional measure μ_1 by setting, for a finite B in $\mathbf{F}$, $\mu_1(B) = \sum_{x \in B} \mu^{LEB}(x)$. By a Kolmogorov extension theorem (Dudley (1989), theorem 12.1.2), there exists a measure μ which extends μ_1 to $\mathcal{F}$.

Next I construct a cost function. First, for each object k and for each closed set B,

$$\begin{aligned} c^{A,k,B}(\nabla U) &= 0 \\ (\text{if } \int_B \nabla U^k(t_i) f(t_i) dt_i &\leq \overline{Q}(B)) \\ &= 10NK \text{ else.} \end{aligned}$$

For each closed set B in Ω, the cost function takes a value of 0 if it satisfies the quantity constraints, and a very high value otherwise. Let me take a previous example of binary distributions. The cost functional is $c^{A,1,\{(1)\}}(Q) = 0$ if $Q(1) \leq 3/4, c^{A1,1,\{2\}}(Q) = 0$ if $Q(2) \leq 3/4$, and $c^{A1,1,\{1,2\}}(Q) = 0$ if $Q(1) + Q(2) \leq 1$.

Then I integrate over the set of possible closed sets:

$$c^{A,k}(\nabla U(t_i)) = \int_{\mathbf{F}} c^{A,k,B}(\nabla U) d\mu(B).$$

Finally sum up for all the objects:

$$c^{A}(\nabla U(t_i)) = \sum_k c^{A,k}(\nabla U(t_i))$$

Although this cost structure looks quite complicated, in the argument, I work from a discretized type space where the number of closed sets is finite, so it is going to be manageable.

The optimal auction problem is

$$\max \Phi(U : c^A) = \int \sum_{i=1}^{N} (t_i \cdot \nabla U(t_i) - U(t_i) - c^A(\nabla U(t_i))) f(t) dt.$$

subject to $U(t_i)$ convex and continuous.

4.2.2 Nonlinear Pricing Problems

I construct a cost function following the idea in section 2. Let me construct a cost function

$$\begin{aligned} c^{m,k,B}(\nabla U) &= \left(\int_B \nabla U^k(t_i) f(t_i) dt_i / \overline{Q}(B) \right)^m \\ \text{if} \left(\int_B \nabla U^k(t_i) f(t_i) dt_i / \overline{Q}(B) \right)^m &\leq 10NK) \\ &= 10NK \text{else} \end{aligned}$$

Then following the same steps,

$$c^{m,k}(\nabla U) = \int_{\mathbf{F}} c^{m,k,B}(\nabla U(t_i)) d\mu(B)$$

and

$$c^m(\nabla U(t_i)) = \sum_k c^{m,k}(\nabla U(t_i)).$$

Note ∇U is bounded pointwise, c^m is differentiable and convex in ∇U. $c^{m,k,B}(\nabla U) \to c^{A,k,B}(\nabla U)$ except for B that $\int_B \nabla U^k(t_i) f(t_i) dt_i = \overline{Q}(B)$. It is an analogous situation at Section 2 where the nonlinear pricing cost structures are convergent to that of the optimal auction except for (t_i, t_{-i}) for the case $\sum_{i=1}^N q_i(t_i, t_{-i}) = 1$.

The nonlinear pricing problems are

$$\max_{U \in L^1} \Phi(U; c^m) = \sum_{i=1}^N \int (t_i \cdot \nabla U(t_i) - U(t_i) - c^m(\nabla U(t_i)) f(t_i) dt$$

subject to $U(t_i)$ is convex continuous.

4.2.3 Convergence of Nonlinear Pricing Mechanisms to an Optimal Auction Mechanism

I formulate the idea of convergence in the next proposition.

Proposition 1. *For each m, there exist a solution to a nonlinear problem U^m with a cost function c^m. There exists a limit U^A for a sequence of nonlinear pricing mechanisms U^m, and it is an optimal auction mechanism.*

This proposition will be proved in the appendix. The argument is as follows. In order for the clarity of the argument, I first work in terms of P and Q, and then use the envelope theorem to restore U.

First I show existence of the solution for a nonlinear pricing problem with a cost c^m. I first consider existence in a discretized problem by discretizing the type space. With a discretization of the type space, the set of closed subsets is finite, so the cost function is easier to handle. Since the objective function is continuous and the set of mechanisms which satisfy incentive and individual rational constraints is compact, there exists a solution to the discretized problem. Since the marginal cost diverges in the problem, the mechanism is bounded. By applying a version of Helly's selection theorem on multidimensional functions, there exists a subsequence limit of solutions as the grid size goes to 0. Since the objective function is continuous, the subsequence limit is a solution of the original nonlinear pricing problems.

The next step is to show that the sequence of nonlinear pricing mechanisms converges by showing that these sequences are Cauchy. Intuitively, for a large M, the seller does not sell more than $\overline{Q}(B)$ for each B, and for this region, the difference between cost function c^r and c^s will be very small for *every* $r, s \geq M$ from the property of the power function. If the mechanisms do not converge, then there will be a contradiction.

I then show that the limit is an optimal auction mechanism. The only difficulty lies in the fact that the nonlinear pricing cost structures do not converge at B with $\int_B \nabla U^k(t_i) f(t_i) dt_i = \overline{Q}(B)$. That is, in the auction cost structure, it is fine to satisfy the quantity constraint with equality, but in the nonlinear pricing cost structure, it takes a cost of 1. In order to work around this problem, I first consider a modified auction cost structure with cost 1 for binding constraints. Then, since the nonlinear cost structures converge pointwise to that modified cost structure, the limit of the sequence of nonlinear pricing mechanisms is a solution of the auction problem with the modified cost structure. Then, if it is not a solution of the original optimal auction problem, then, starting from this deviation, I can construct a mechanism by reducing the supply by removing the binding constraints while satisfying the incentive constraints by adjusting the payments. It will give a valid deviation for the auction problem with the modified cost structure. This gives me the contradiction with the assumption that the limit is the solution of the auction problem with the modified cost structure. Thus, the limit is a solution to the original auction problem.

Finally, by applying the envelope theorem, I can go back to the surplus U from the allocation Q.

5 Characterization of the Optimal Auction Mechanism

In the previous section, I showed that an optimal auction mechanism can be derived as a limit of nonlinear pricing mechanisms. In this section, based on this result, I consider the properties of the optimal auction mechanism.

5.1 Characterization of the Nonlinear Pricing Mechanisms

Now in order to understand the properties of the optimal auction mechanism, let me begin with the analysis of the nonlinear pricing mechanisms. Let me look an objective function in a little bit more detail:

$$\begin{aligned}\max_{U\in L^1}\Phi(U;c^m) &= \int\sum_{i=N}(t_i\cdot\nabla U(t_i)-U(t_i)-c^m(\nabla U))f(t)dt \\ &= N\int(t_i\cdot\nabla U(t_i)-U(t_i)-c^m(\nabla U(t_i)))f(t_i)dt_i\end{aligned}$$

since the buyers are identical ex ante. The seller's revenue is the sum of revenue from each buyers. In this way, the problem closely resembles to a multiple buyer version buyer of Rochet and Choné(1998).

I first recall the definition of Gateaux derivatives, which is a generalization of derivatives in a finite dimensional optimization problem:

Definition (Luenberger (1969)). *A Gateaux differential of Φ at U with increment h is*

$$L^m(U;h)=-\Phi'(U^*)h=\lim_{\alpha\to 0}\frac{1}{\alpha}[\Phi(U+\alpha h)-\Phi(U)]$$

if the limit exists.

In economic terms, a Gateaux differential is a marginal loss, which measures the change in the seller's expected profit when the seller provides some additional surplus $h(t_i)$ to the buyer with type t_i.

By applying multivariate analogue of integration by parts formula, let me calculate

$$\begin{aligned}L^m(U;h) &= N[\int_{\partial\Omega}(-t_i+\nabla c^m(\nabla U(t_i)))\cdot h(t_i)f(t_i)dt_i \\ &\quad +\int_{\Omega}div[(t_i-\nabla c^m(\nabla U(t_i)))f(t_i)]h(t_i)dt_i+\int_{\Omega}h(t_i)f(t_i)dt_i]\end{aligned}$$

This gives rise to the expression of the marginal loss:

$$L^m(U;h) = \int_{\Omega} \alpha(t_i)h(t_i)dt_i + \int_{\partial\Omega} \beta(t_i)h(t_i)d\sigma(t_i).$$

with

$$\begin{aligned}
\alpha^m(t_i) &= N\{f(t_i) + div[(t_i - \nabla c^m(\nabla U(t_i)))f(t_i)]\} \\
\beta^m(t_i) &= N(t_i - c^m(\nabla U(t_i))) \cdot \overrightarrow{n}(t_i)
\end{aligned}$$

Intuitively, α and β measures a pointwise marginal loss. These expressions are closely related to those of Rochet and Choné(1998). The intuition is that in this nonlinear pricing problem, the revenue from each buyer is additively separable and the cost function is the same, so the objective function decomposes into the sum of revenues from one buyer problems.

Since the marginal loss L is a linear continuous operator, there exists a measure

$$L^m(U;h) = \int h(t_i)d\mu^m(t_i)$$

This observation leads to a following characterization of the solution of nonlinear pricing problem:

Lemma 2. *At the nonlinear pricing mechanism, the solution U^m is C^1, and partitions the type set Ω into three regions of Ω_0^m, Ω_B^m, and Ω_1^m with the following properties:*

- *In the nonparticipation region Ω_0^m, $U^m(t_i) = 0$. and $\mu^m(\Omega_0^m) = 1$.*
- *In the nonbunching region $\Omega_1^m, \alpha^m(t_i) = \beta^m(t_i) = 0$ and U^m is strictly convex.*
- *In the bunching region Ω_B^m, Ω_B^m is divided into a bunch with the same allocation q, and satisfies $\mu_+^{\Omega(q)} = T\mu_-^{\Omega(q)}$.*

Let me review the intuition of Rochet and Choné(1998). An intuition that nonparticipation region has a positive measure is that, as in Armstrong (1994), a marginal loss from exclusion is of a higher order (order of k) than the revenue gain from charging buyers a higher price. Bunching is robust among the conflict between the exclusion, the first order condition, and continuity.

On one hand, the seller wishes to exclude the buyers, and on the other hand, the seller wishes to price discriminate as much as possible by the first order condition. If there is no bunching, there will be a jump at the boundary of the exclusion region and the nonbunching region. To reconcile these three factors, the seller uses bunching.

Then what will be the difference between the nonlinear pricing problems of Rochet and Choné and my multiple buyer problem ? The only difference lies in the fact that the cost function c^m, actually depends on the number of buyers through $\overline{Q}(B)$. Recall the formula $\overline{Q}(B) \equiv \frac{1-P(\Omega/B)^N}{N}$. This bound decreases as the number of buyers, N, increases. It is natural, as the number of buyers increases, there will be more competition, so that each buyer's expected allocation should decrease.

I note that the reserve price (or the exclusion set) is the same regardless of the number of buyers: the intuition is that, at the exclusion region the expected allocation is zero, so the cost is zero. So the cost structure will not matter.

Proposition 3. *Ω_0^m is independent of N* and *m (up to a set of measure 0).*

Proof. By setting $h(t_i) = 1$, I get $L^m(U;1) = \int_\Omega \alpha^m(t_i)dt_i + \int_{\partial\Omega} \beta^m(t_i)d\sigma(t_i) = 1$. Since $\alpha^m(t_i) = \beta^m(t_i) = 0$ at each point of t_i of Ω_1^m and each bunch $\Omega(q)$ in expectation, $\int_{\Omega_0^m} \alpha^m(t_i)dt_i + \int_{\partial\Omega_0^m} \beta^m(t_i)d\sigma(t_i) = 1$. Note that at Ω_0^m, $Q(t_i) = \nabla U(t_i) = 0$. Thus, for any l, $\int_{\Omega_0^m} \alpha^l(t_i)dt_i + \int_{\partial\Omega_0^m} \beta^l(t_i)d\sigma(t_i) = 1$. This implies that $\Omega_0^m = \Omega_0^l$.

5.2 Bunching

In the previous subsection, I considered the nonlinear pricing problem. By applying the convergence result, I can characterize an optimal auction mechanism.

Proposition 4. *At an optimal auction mechanism, the solution U^A partitions the type set Ω into three regions of Ω_0^A, Ω_B^A, and Ω_1^A with the following properties:*

- *In the nonparticipation region Ω_0^A, $U^A(t_i) = 0$, and $\Omega_0^A = \Omega_0^m$.*
- *In the nonbunching region $\Omega_1^A, \alpha^A(t_i) = \beta^A(t_i) = 0$ and U^A is strictly convex.*

- *In the bunching region Ω_B^A, Ω_B^A is divided into a bunch with the same allocation q, and satisfies $\mu_+^{\Omega(q)} = T\mu_-^{\Omega(q)}$.*

The first result is that the seller sets nontrivial reserve prices to exclude some buyers. Moreover, the set of exclusion is convex. Otherwise, the allocation would be non-monotonic, which violates the incentive compatibility. Let $\underline{S}$ be the convex hull of S.

Proposition 5. $\Omega_0^A = \underline{\Omega}_0^A$up to a set of measure zero.

Proof. Suppose not. Then $\exists B \subset \underline{\Omega^A}_0 \backslash \Omega_0^A$ with $\mu(B) > 0$. By Borel regularity, $\exists t_i \in B$ and $\epsilon > 0$ with $Ball_\epsilon(t_i) \in B$. There, $Q^A(t_i) > 0$ $\forall t_i \in Ball_\epsilon(t_i)$. This implies there exists a coordinate whose partial increase in that coordinate causes the strict decrease in Q^A. On the other hand, by convexity of U^A, $\forall t_i, t_i' \in \Omega^A$, $(t_i' - t_i)(Q^{A,k}(t_i') - Q^{A,k}(t_i)) \geq 0$.

The second result is about bunching. As I explained in the previous section, bunching is robust in multidimensional nonlinear pricing problem. But in the optimal auction mechanism, these allocation is not continuous. But on the other hand, due to quantity constraints, the variety of quantities that can be offered is also limited. So bunching still persists. In an example of section 6, all bunch takes place at $q = 1$.

Finally, I consider efficiency at the boundary condition. It is because, for every m, at the upper boundary of the type space, $\alpha^m(t_i) = \beta^m(t_i) = 0$. Thus, by taking the limit, I have $\alpha^A(t_i) = \beta^A(t_i) = 0$.

6 Examples

In the previous section I explained the theoretical structure of the optimal mechanism. In this section I consider some examples for a single buyer case and multiple buyer cases.

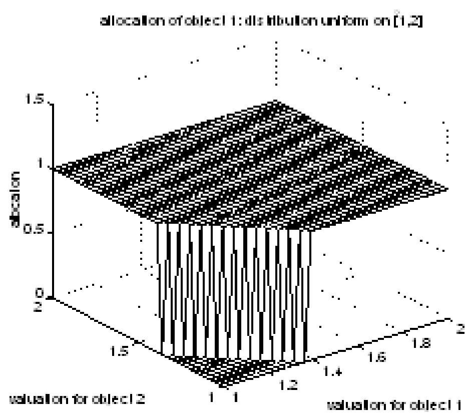
6.1 The Single Buyer Case

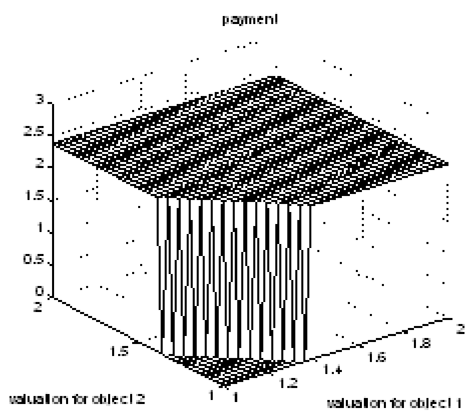
Consider the case of two objects. Let the support of the distribution be $\Omega = [1,2]^2$ with $f(t_i) = 1$. Let $c^m(q) = (q^1)^m + (q^2)^m$. Then $\alpha^m = div[(t_i^1 - m(q^1)^{m-1}, t_i^2 - m(q^2)^{m-1})] + 1 = 3 - m(m-1)(q^1)^{m-2}(q^1)' - m(m-1)(q^2)^{m-2}(q^2)'$and $\beta^m(t_i) = (t_i^1 - m(q^1)^{m-1}, t_i^2 - m(q^2)^{m-1})$. The exclusion region is of the form $\{(t_i^1, t_i^2) : t_i^1 + t_i^2 \leq t^*\}$where t^* is the reserve price. At

$\Omega_0, \alpha^m(t_i) = 3, \beta^m(t_i) = 1$. The exclusion takes place at $t_i^1 + t_i^2 \leq (4+\sqrt{10})/3$ for every m. So the allocation in the optimal auction is

$$\begin{aligned} q^A(t_i^1, t_i^2) &= (0,0) \text{ if } t_i^1 + t_i^2 \leq \frac{4+\sqrt{10}}{3} \\ & \quad (1,1) \text{ otherwise} \end{aligned}$$

Alternatively, the reserve price is obtained by directly maximizing the profit from the buyer with the sale $t^*(1-(t^*-2)^2/2)$. The following computations are done using CPLEX.

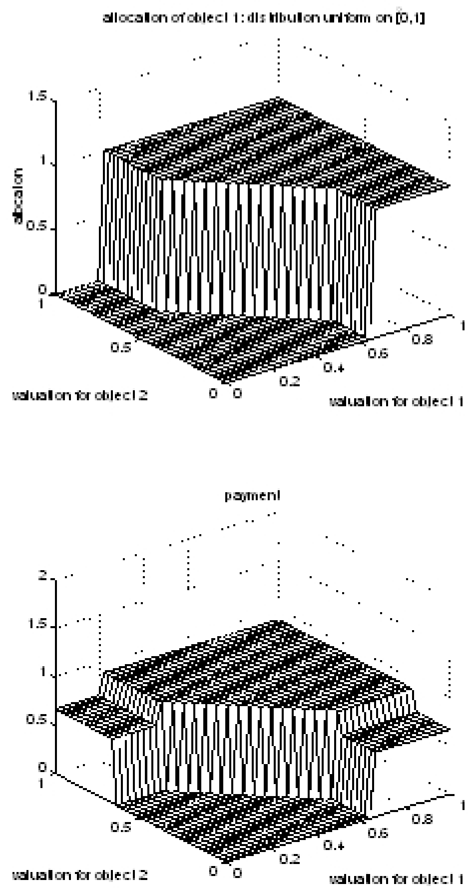




Pure Bundling Auctions

Now, for an example of mixed bundling, consider the case of $K = 2$, and $\Omega = [0,1]^2$. If the seller chooses pure bundling, the reserve price is $\sqrt{6}/3$

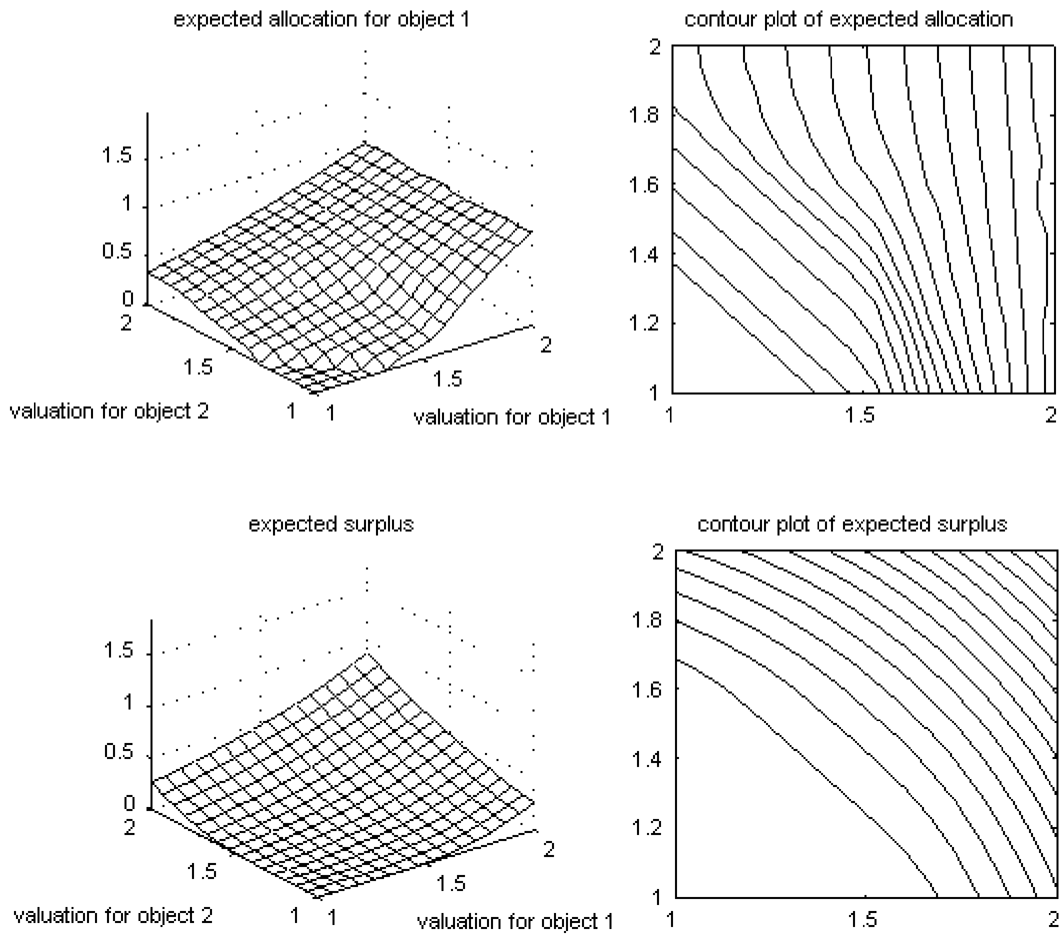
with profit 0.5443. If the seller chooses mixed bundling, the reserve price is 0.6667 for a single unit, and 0.8619 for the package. The seller's profit is 0.5492. Thus the optimal auction is of mixed bundling.



Mixed Bundling Auctions

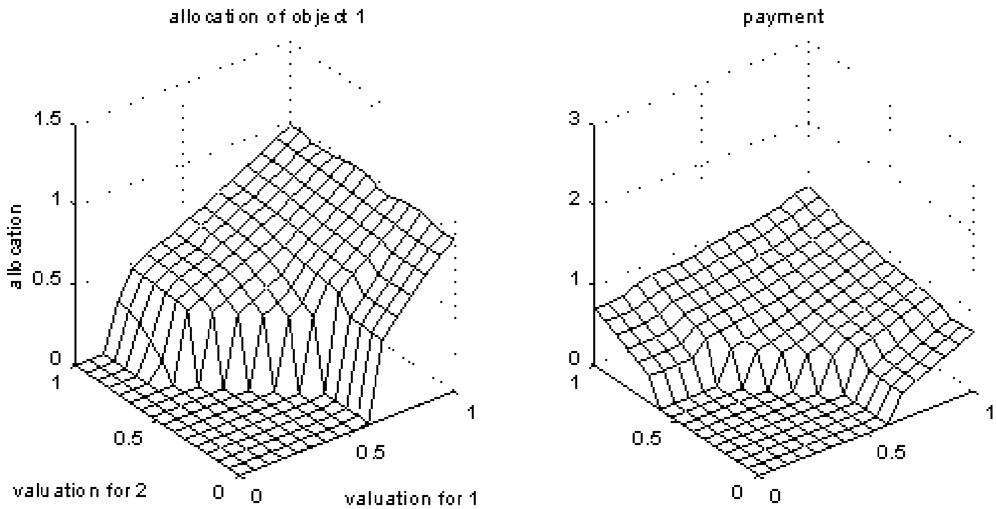
6.2 Multiple Buyers

I present some computational results. First I consider the computational result for two buyers case for the distribution uniform $[1,2]^2$. The reserve prices are identical to the case of a single buyer. The allocation schedule shows a pattern of pure bundling at the boundary of the reserve prices, and then a pattern of independent allocation at the boundary. This suggests the allocation schedule of the optimal auction mechanism can be complex.



Pure Bundling Auctions

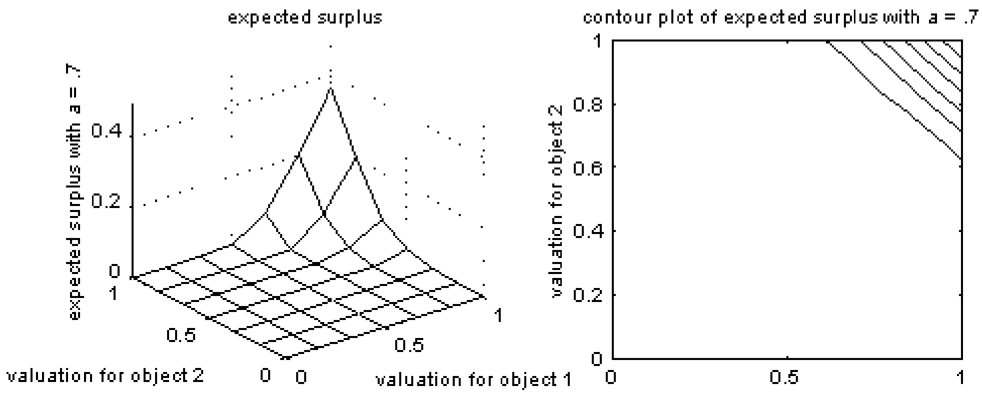
Next I report the result for the distribution uniform $[0,1]^2$. The reserve price is identical to the case of a single buyer, and has the pattern of mixed bundling. The pattern of bundling exhibits downward distortion.



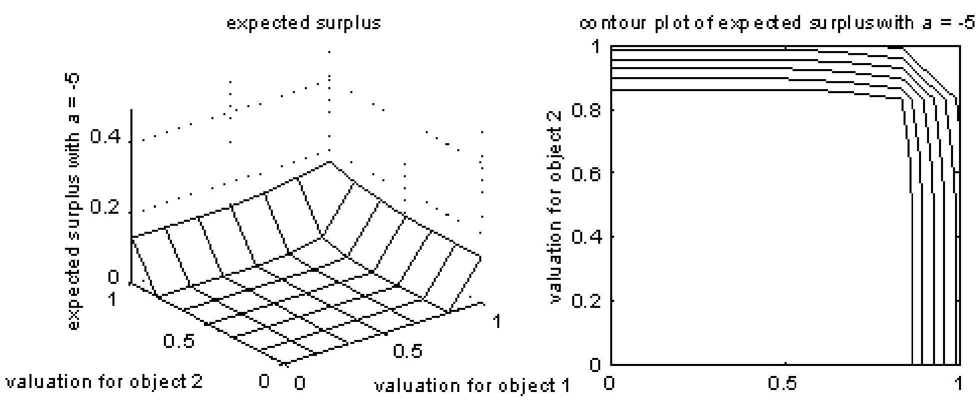
Mixed Bundling Auctions

6.3 Complements and substitutes

Even if a payoff has positive or negative complementarity, if preferences are linear in types, sweeping characterization goes through. Consider a simplest case with two goods and bidder's preference to be $t \cdot q + aq^1q^2$.As $a \to \infty$, the seller allocates both objects to the bidder with the highest marginal revenue for the bundle. That is, the optimal auction is a pure bundling auction. If $a < 0$ and as the number of bidders goes to infinity , then it is optimal to allocate the object to the bidder with the highest marginal value for each object.



Optimal Auction for Substitues



Optimal Auction for Complements

7 Conclusion

In this paper I presented a new convergence result between nonlinear pricing mechanisms and optimal auction mechanisms. For example, this convergence result gives a clean connection between the optimal auction problem of Myerson (1981) and the nonlinear pricing mechanisms in Mussa and Rosen (1978). By applying this result, I obtain first characterization results of the optimal auction mechanisms for heterogeneous objects and multidimensional types. Numerical results show complexities of the optimal auction mechanisms. Future research questions will include the design of tractable mechanisms which will approximate these optimal mechanisms.

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8 Appendix

8.1 Proof of Proposition 1.

Let me first show the existence of the solution for each m. Let me begin by considering a discretized type space. Let the grid size be $\frac{1}{L}$. The discretized type space is given by a collection of grid points $\Omega_L = \{(0,...,0),(0,...,\frac{1}{L}),...,(1,...,1)\}$. Define

$$f_L(t_i) = \int_{\min\{t_i^1 - \frac{1}{2L},0\}}^{\max\{t_i^1+\frac{1}{2L},1\}} ... \int_{\min\{t_i^K - \frac{1}{2L},0\}}^{\max\{t_i^K+\frac{1}{2L},1\}} f(t_i)dt_i$$

From this distribution, I can compute an expected mechanism by $Q^k(t_i) = \sum q^k(t_i,t_{-i})f_L(t_{-i})$,$P(t_i) = \sum p(t_{,i},t_{-i})f_L(t_{-i})$,and $U(t_i) = t_iQ^k(t_i) - P(t_{,i})$.

Then I write the cost function similarly:

$$\begin{aligned} c^{m,k,B,l}(Q) &= (\sum_{t_i\in B} Q^k(t_i)f(t_i)/\overline{Q}(B))^m \\ \text{if}(\sum_{t_i\in B} Q^k(t_i)f(t_i)/\overline{Q}(B))^m &\leq 10NK) \\ &= 10NK\text{else} \end{aligned}$$

with $c^{m,k,l}(Q) = \sum_B c^{m,k,B,l}(U)f(B)$ and $c^{m,l}(Q) = \sum_k c^{m,l,k}(Q)$. Then the seller's problem is

$$\max_{p,q} \sum_l (\sum_{i=1}^N P(t_i) - c^{m,l}(Q))f_L(t).$$

subject to $U(t_i) \geq t_i \cdot Q(t_i') - P(t_i')$, $\forall t_i, t_i'$ and $U(t_i) \geq 0$, $\forall t_i$, where, c^m is the cost function of the original mechanism design problem.

Since the objective function is continuous and the set of constraints is compact in a finite dimensional Euclidean space, there exists a solution to this problem, denoted by $P^{l,m}$ and $Q^{l,m}$.

I now construct a candidate mechanism. I first extend $P^{l,m}$ and $Q^{l,m}$ to a function over $[0,1]^K$. Define $Q'^{l,m}(t_i) = Q^{l,m}(t_i')$ with t_i' such that for a coordinate k, $|t_i - t_i'| \leq \frac{1}{2L}$ with t_i' in Ω_L. Similarly construct $P'^{l,m}$. There exists such t_i' by construction of the discretization of the type space.

I note $P'^{l,m}$ and $Q'^{l,m}$ are bounded (pointwise). By construction, there is an upper bound on the type. This implies that the marginal willingness to pay of buyers for an additional unit is bounded. On the other hand, by construction, the marginal cost will diverge to infinity. Thus the amount of units the seller wishes to sell to the buyer is bounded. Since the buyers have individual rationality constraints, the payment the buyer is willing to make is bounded. By taking expectations, noting that the density is bounded, $P'^{l,m}$ and $Q'^{l,m}$ bounded.

Since these functions are bounded, these functions are of bounded variation. Since the domain of these function is k dimensional, by a multidimensional version of Helly Selection Principle by Leonov (1998), theorem 4, there exists a subsequence limit P^m and Q^m which converges pointwise. For notational simplicity, take m be the convergent subsequence.

I now want to check that P^m and Q^m satisfy the incentive and individual rationality constraints. First, since each of $P^{l,m}$ and $Q^{l,m}$ satisfy these constraints, and $P'^{l,m}$ and $Q'^{l,m}$ are obtained by spreading $P^{l,m}$ and $Q^{l,m}$ in the rectangle of the length $1/2L$, so they still satisfy these constraints in the original nonlinear pricing problem. By taking the limit of these constraints, P^m and Q^m also satisfy these constraints.

Then, since the cost function c^m is continuous, by applying the dominated convergence theorem, the limit P^m and Q^m is a solution of the nonlinear pricing problem with the original type space.

Now I show that, P^m and Q^m are Cauchy. Suppose not. Then, there exists ϵ such that for every M for any $r, s \geq M$, $\int |P^r(t_i, t_{-i}) - P^s(t_i, t_{-i})| f(t) dt > \varepsilon$ and $\int |Q^{k,r}(t_i) - Q^{k,s}(t_i)| f(t_i) dt_i > \varepsilon$. For reasonably large M, $\int |c^r(Q^r(t_i)) - c^s(Q^s(t_i))| f(t_i) dt_i < \varepsilon$ since in the region where the constraints are not binding, the pointwise difference between c^r and c^s can be uniformly bounded above, and this bound can be made arbitrary small number by taking large M. As M becomes larger, it is getting costly to violate constraints, so for sufficiently large m I can restrict attention to the area where the constraints are not violated. It will lead to contradiction about the optimality of either at r or s. Since L^1 is complete, P^m and Q^m converge. Let P^* and Q^* be the limit.

I need to check that P^* and Q^* are a solution of the optimal auction problem. As an intermediate step, let me define an intermediate cost function c^{A1} as follows:

$$
\begin{aligned}
c^{A1,k,B}(Q) &= 0 \text{ if } \int_B Q^k(t_i) f(t_i) dt_i < \overline{Q}(B) \\
&= 1 \text{ if } \int_B Q^k(t_i) f(t_i) dt_i = \overline{Q}(B) \\
&= 10NK \text{ else}
\end{aligned}
$$

and $c^{A1,k}(U(t_i)) = \int_{\mathbf{F}} c^{A1,k,B}(U) d\mu(B)$ and $c^{A1,k}(Q) = \sum_{k=1}^{K} c^{A1,k,B}(Q)$. The only difference between c^A and c^{A1} is at B with $\int_B Q^k(t_i) f(t_i) dt_i = \overline{Q}(B)$. It is immediate to note that c^m converges to c^{A1} for every B.

Now let me study that P^* and Q^* are the solution of the problem with a cost function c^{A1}. From the optimally of P^m and Q^m, for each P and Q which satisfies the incentive constraints and the individually rational constraints,

$$\int (\sum_{i=1}^{N} P^m(t_i) - c^m(Q^m)) f(t) dt_i \geq \int (\sum_{i=1}^{N} P(t_i) - c^m(Q)) f(t) dt.$$

By taking a limit,

$$\int (\sum_{i=1}^{N} P^*(t_i) - c^{A1}(Q^*)) f(t_i) dt_i \geq \int (\sum_{i=1}^{N} P(t_i) - c^{A1}(Q)) f(t_i) dt_i.$$

Finally I move from c^{A1} to c^A. Take an arbitrary P and Q which satisfies the incentive constraints and individually rational constraints. Let me claim

$$\int (\sum_{i=1}^{N} P^*(t_i) - c^{A1}(Q^*)) f(t) dt \geq \int (\sum_{i=1}^{N} P(t_i) - c^A(Q)) f(t) dt.$$

Suppose not. That is, there exists P' and Q' such that

$$\int (\sum_{i=1}^{N} P'(t_i) - c^A(Q')) f(t) dt > \int (\sum_{i=1}^{N} P^*(t_i) - c^{A1}(Q^*)) f(t) dt$$

Then consider a following mechanism P'' and Q'': for all B such that $\int_B \nabla U^k(t_i) f(t_i) dt_i = \overline{Q}(B)$, reduce the allocation by a very small number to make the constraint nonbinding and at the same time $P'(t_i)$ will be reduced to satisfy incentive and individual rationality constraints. With the cost function c^{A1}, $\int (\sum_{i=1}^{N} P''(t_i) - c^{A1}(Q'')) f(t_i) dt_i$ is arbitrary close to $\int (\sum_{i=1}^{N} P'(t_i) - c^A(Q')) f(t_i) dt_i$. It is a contradiction to the assumption that P^* and Q^* are optimal under a cost function c^{A1}.

Thus we have the desired inequality:

$$\begin{aligned} \int (\sum_{i=1}^{N} P^*(t_i) - c^A(Q^*)) f(t) dt &\geq \int (\sum_{i=1}^{N} P^*(t_i) - c^{A1}(Q^*)) f(t) dt \\ \geq \quad \int (\sum_{i=1}^{N} P(t_i) - c^A(Q)) f(t) dt & \end{aligned}$$

where the first inequality comes from the fact that $c^A(Q) \le c^{A1}(Q)$ for all Q.

By the incentive compatibility constraints, $Q(t_i)$ is a subgradient of $U(t_i)$. The buyer's payoff function is differentiable in (P, Q) and satisfies single crossing condition. By the envelope theorem (Milgrom and Segal (2002, theorem 2 and footnote 10, also see Krishna and Maenner (2001)), $U^n(t_i) = U^n(0) + \int Q^m d\alpha$ and $U^A(t_i) = U^A(0) + \int Q_i^A d\alpha$ where α is a smooth path joining 0 and t_i. Since $Q^m(t_i) \to Q^m(t_i)$ in L^1, $U^n(t_i) \to U^A(t_i)$.