Price screening with network effects and entry deterrence

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Abstract: This paper presents a model of price screening for goods with network effects, by a monopoly seller, and by an entry-deterring monopolist. These products are used in variable quantities by heterogeneous customers, the magnitude of network effects is influenced by gross consumption, rather than simply by user base, and the value derived from the network effects may be related to individual consumption and to customer type. Conditions under which fulfilled-expectations contracts exist and are unique are established. While network effects generally raise total prices in the optimal contracts, accompanying changes in consumption vary widely. Under the standard assumption of homogeneous network effects, there are no changes in consumption; in contrast, usage-dependent network effects increase the consumption of all types. Additionally, type-dependent network effects can cause downward distortions in consumption levels across a subset (and in special cases, all) types. The direction and extent of these distortions depend on the relative rates of variation in marginal intrinsic value and marginal network value with type.

When network effects are homogeneous, the optimal entry-deterring contract is a flat fee that results in the efficient outcome. In contrast, when the network effects are usage-dependent, consumption increases only for a subset of lower types; however, when these network effects are strong enough, there are no changes in total surplus induced by the entry threat, but merely a transfer of surplus from the seller to its customers. The presence of network effects and of an entry threat also increase distributional efficiency by reducing the disparity in relative value captured by different customer types. Managerial, regulatory and policy implications of these results are discussed.

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1. Introduction

This paper presents a model in which products displays positive network effects, individual consumption varies across heterogeneous customers, and the magnitude of network effects depend on gross consumption. The principal goals of the paper are to characterize the optimal price screening contracts for different kinds of network effects, and under the potential threat of competitive entry, and to study their consumption and welfare properties.

Standard theories of network effects typically assume that each customer purchases a maximum of one unit of the product, that the value of the network effect is proportionate to the total size of the product's eventual user base, and that all customer benefit equally from the network effects (Katz and Shapiro, 1985, Farrell and Saloner, 1985). However, there are a number of products that display network effects (henceforth termed *network goods*) which are consumed in variable quantities by different customers, and for which the magnitude of the network effects may depend on the total quantity consumed across customers, rather than simply the total number of adopters. In addition, the value each customer derives from the network good may depend on their individual consumption, which in turn depends on the intrinsic value they place on the product. Extending the standard theory to incorporate these observations has important implications for designing pricing contracts for network goods, as well as for the regulatory analysis of industries with network effects.

The relevance of these observations can be illustrated through a few common examples of products that display network effects. Consider, for instance, the purchase of PC operating systems software by corporate customers. The (simplest) pricing problem faced by a seller in this market is one of choosing a pricing schedule, where quantity is measured by number of user licenses, and each corporate customer purchases a variable quantity of licenses. The network effects are caused largely by the higher availability and quality of complementary goods (applications software, compatible accessories) as the total number of OS installations increases. Consequently, the magnitude of the network effects are proportionate to the total number of licenses sold (the gross consumption), rather than simply the number of corporations who adopt the OS. Moreover, a corporation which has a higher number of licenses, and/or who uses these licenses more intensively, benefits more from the increased quality and availability of the complementary goods – in other words, the value realized from the network effects also depends on individual consumption and on customer type. A similar argument can be made for back-end or enterprise software used in variable quantities by different companies (Oracle's database software and Siebel's CRM software being two examples), or for networking equipment like routers and switches. In these cases, network effects are driven by the ease with which one can find qualified support or administration engineers, trained employees, compatible software, or compatible equipment[2].

[2]While open networking standards do form the basis for most networking equipment, many vendors like Cisco Systems use proprietary operating systems. Moreover, the ease of interoperability between equipment from competing vendors varies widely.

Network goods sold directly to individuals consumers may also display the same properties. For example, electronic marketplaces like eBay are widely recognized as displaying positive network effects, which stem from increased liquidity, as well as a wider availability of robust systems supporting marketplace services (reputation, escrow, payment, settlement, dispute resolution). The magnitude of the network effects increases not just with the number of participants in the market, but with the extent to which each participant actually buys and sells; moreover, an individual who participates more realizes higher benefits from them. Even for products used as canonical examples of network goods, such as telephone service, usage varies across consumers, network effects dependent on total consumption as well as installed base, users with higher consumption levels benefit more from the network effects, and pricing is often nonlinear.

The ubiquity of variable consumption and heterogenous value from network goods underlines the importance of developing a model that incorporates these properties. This paper provides such an model, characterizing the optimal pricing schedule offered by a monopolist selling a network good which explicitly displays the properties highlighted in the examples above. Three cases are analyzed successively. First, *homogeneous* network effects whose magnitude depends on just gross realized consumption are studied. Subsequently, *usage-based* network effects whose magnitude is heterogeneous across customers (by virtue of depending on both gross consumption and individual consumption) are modeled. Finally, *type-dependent* network effects that result in value increases that depend not just on individual and gross consumption, but also on customer type are analyzed.

The changes in consumption induced by the network effects are shown to vary significantly across the cases. When network effects are homogeneous, there are no changes in consumption induced by their presence, and the monopolist captures all the additional surplus generated by a simple increase in price across all participating types. On the other hand, when network effects are usage-based, their presence induces a strict increase in consumption across all customer types. The monopolist captures a portion of this increase in surplus, which is shown to be higher than the direct increase in surplus due to the presence of the network effects. Additionally, there is also an increase in informational rents across types, though this is bounded above than the surplus increase due to the increases in consumption induced by the network effects .

Interestingly, when network effects are type-dependent, their presence may induce a further downward distortion in consumption for a lower fraction of customers. In an extreme case – when the presence of network effects does not alter the efficient levels of consumption – this downward distortion will be for all except the highest type. A simple way to characterize the direction of this distortion, by comparing the rates of variation in marginal intrinsic value and marginal network value as customer type θ increases, is presented. In contrast with Segal (2003), these distortions arise despite customers being heterogeneous, when there is asymmetric information, and when total welfare depends on the distribution of consumption across customers, rather than on total consumption (since value is concave in usage for each agent). There are also interesting variations in the manner in which the value generated by the network effects is distributed across the different customers, which are illustrated through two examples in Appendix B. .

In addition to pure monopoly pricing, this paper also analyzes pricing by an entry-deterring monopolist. Many markets for technology goods feature dominant sellers with market power, and there has been substantial recent interest in whether (and how) the potential threat of entry affects their pricing choices. For instance, in the recent U.S. versus Microsoft case, both parties agreed that Microsoft's pricing was not consistent with monopoly profit maximization, and Schmalensee (1999) argued that Microsoft underprices in order to reduce the desirability of entry by competing firms into the market for operating systems. Fudenberg and Tirole (2000) develop a formal model of limit pricing that supports this argument, in which installed base plays an entry-deterring role analogous to that of excess capacity (Spence, 1977, Dixit, 1980).

This paper proposes and analyzes an alternate representation, in which to successfully deter a threat of entry, the monopolist must provide each customer with surplus equal to at least the maximum intrinsic value they could get from a competing product. This limits the price each customer pays under the monopolist's nonlinear pricing schedule to being no more than the network value they get from the monopolist's product; network value plays the role of being the primary source of profits for a monopolist who prices to successfully deter entry. On the face of it, this has promising welfare implications, since one would expect a threat of entry to induce a substantial increase in consumption. Indeed, flat-fee pricing that results in the efficient outcome is optimal when network effects are homogeneous. While there are generally (though not always) consumption changes induced by the entry threat for usage-dependent network effects, the surplus increases are exclusively from a lower subset of types (which may be empty). Additionally, responding to the threat of entry induces the monopolist to even out the relative distribution of surplus across different customer types.

This paper draws from and adds to two lines of research. The first is the literature on monopoly pricing of products with positive network externalities. Related papers with monopoly models include Rohlfs (1974), Oren, Smith and Wilson (1982), Economides (1996a), Cabral, Salant and Woroch (1999) and Segal (1999, 2003). Modeling network goods for which the network effects depend on gross consumption (rather than the number of adopters) is new, as is the analysis of heterogeneity in the value of the network effects across customers. The notion of fulfilled-expectations equilibrium is extended to the case of customers purchasing variable quantities in a monopoly market. A related area of research is the literature on monopoly with *negative* consumption externalities, specifically in the context of congestion in queuing and service systems (Mendelson, 1985, Dewan and Mendelson, 1990, Mendelson and Whang, 1990, Westland, 1992).

The second line of research this paper adds to is the literature on single-dimensional price screening. It contributes new results to the theory by characterizing how positive network effects of different kinds affect optimal nonlinear pricing, by establishing conditions under which optimal nonlinear pricing schedules that satisfy fulfilled-expectations exist and are unique, and by examining the effects of an entry threat on the manner in which the monopolist price-discriminates. The scope of this model is narrower than Segal (1999), who analyzes the effects of a very general specification of inter-agent externalities on optimal contracting. Unlike Segal (2003), the inter-agent coordination game is not explicitly modeled, and the nature of price discrimination modeled is more conventional. However, the focus on positive network effects allows a richer specification of the relationship between the network effects and customer value – this can depend on individual consumption, total consumption across customers as well as on customer type.

The rest of this paper is organized as follows. Section 2 specifies the basic model, defines the optimal contracts, and characterizes the model's description of entry deterrence. Section 3 presents the analysis of the monopoly with homogeneous network effects, Section 4 analyzes the case of usage-dependent network effects, and Section 5 examines type-dependent network effects. Both sections 3 and 4 examine pricing and consumption changes induced by network effects, examine some welfare properties, and establish how the contracts are affected by the threat of entry, while Section 5 analyzes consumption distortions induced by type-dependent network effects. Section 6 discusses the results further, discusses the model's assumptions, and concludes with an outline of open research questions raised. Appendix A presents proofs of the main results, and Appendix B presents a couple of examples that illustrate the model's results.

2. Model

2.1. Firm and customers

A monopolist sells a homogeneous product which may be used by consumers in varying quantities. The variable costs of production are assumed to be zero (though Section 6.2 describes how the model's results are robust to relaxing this assumption). Customers are heterogeneous, indexed by their type $\theta \in [\underline{\theta}, \overline{\theta}]$. The monopolist does not observe the type of any customer, but knows $F(\theta)$, the probability distribution of types in the customer population. $F(\theta)$ is assumed to be strictly increasing and absolutely continuous, and therefore the corresponding density function $f(\theta)$ exists and is strictly positive for all $\theta \in [\underline{\theta}, \overline{\theta}]$. In addition, $\frac{1-F(\theta)}{f(\theta)}$, the reciprocal of the hazard rate, is assumed to be non-increasing for all θ. Each customer knows their own type θ. The total number of customers in the market is normalized to *1*.

The preferences of a customer of type θ are represented by the linearly separable utility function

$$V(q, \theta, Q, p) = W(q, \theta, Q) - p, \tag{2.1}$$

where q is the quantity of the product used by the customer (often referred to as *individual consumption*), Q is the total quantity of the product used by all customers in the market (often referred to as the *gross consumption*) and p is the total price paid by the customer. $W(q, \theta, Q)$ is often referred to as the *value function*.

The value function when $Q = 0$ is denoted $U(q, \theta)$, and is referred to as the *intrinsic value* from the network good for customer type θ. That is:

$$U(q, \theta) = W(q, \theta, 0) \tag{2.2}$$

for all q, θ. At any positive Q, the expression $[W(q, \theta, Q) - U(q, \theta)]$ is referred to as the *network value* from the network good for customer type θ.

The value function $W(q, \theta, Q)$ is assumed to have the following properties:

1. $W_{11}(q, \theta, Q) < 0$, $W_2(q, \theta, Q) > 0$, $W_{12}(q, \theta, Q) > 0$.
2. $W_3(q, \theta, Q) \geq 0$, $W_{13}(q, \theta, Q) \geq 0$, $W_{23}(q, \theta, Q) \geq 0$
3. $\frac{d}{d\theta}(\frac{-W_{11}(q, \theta, Q)}{W_1(q, \theta, Q)}) < 0$, $W_{122}(q, \theta, Q) \leq 0$.
4. $\beta(\theta, Q) = \arg\max_q W(q, \theta, Q)$ is finite and unique for all θ. $W_1(q, \theta, Q) > 0$ for $q < \beta(\theta, Q)$, and $W_1(q, \theta, Q) < 0$ for $q > \beta(\theta, Q)$.

Numbered subscripts to functions denote partial derivatives with respect to the corresponding variable. The first and third sets of properties are standard. The second set of properties characterizes the nature of the network effects – the gross value from the network effects is non-decreasing in gross consumption, and the marginal value from an increase in gross consumption is (weakly) higher at a higher level of individual consumption, and is (weakly) higher for higher types. The source of these network effects are not modeled explicitly. The final set of properties simply state that there is a consumption level beyond which the value from additional consumption decreases. It reflects the reality that customers consume a finite quantity of any network good, even if the marginal price of additional consumption is zero (under a site license, for instance). This is because value from usage is typically bounded by a constraint on some related resource – attention or computing power being two common examples – and the implicit presence of a substitute use for this resource[3]. Analogously, sometimes the increased consumption of the product may necessitate the purchase of additional necessary complementary assets (more powerful computer hardware for increased software usage, for instance). The quantity that maximizes *intrinsic value* is denoted $\alpha(\theta)$ – that is, $\alpha(\theta) = \beta(\theta, 0)$.

[3]All the results of this paper are available for the case where W is strictly increasing and has a finite upper bound. Since variable costs are assumed to be zero, the issue that arises in a model of that kind is that the efficient consumption level is infinite. The contracts are still well-defined and specified as presented in this paper (so long as W is bounded above) – however, the presence of network effects cannot increase the efficient consumption levels, which misses a potentially important effect. A more formal discussion of this is available on request.

Each customer of type θ is assumed to have reservation utility $\hat{U}(\theta) \geq 0$. The functions $F(\theta)$, $W(q,\theta,Q)$, $U(q,\theta)$, and $\hat{U}(\theta)$ are common knowledge. The monopolist chooses a *contract* (also called a *pricing schedule*), represented by the standard menu of incentive-compatible quantity-price pairs $(q(t),\ \tau(t))$ for each $t \in [\underline{\theta}, \overline{\theta}]$.

2.2. Sequence of events and expectations formation

The interaction between the monopolist and their customers is according to the following sequence:

1. The monopolist announces their menu of quantity-price pairs $q(t), \tau(t)$.

2. Customers observe $q(t), \tau(t)$, and form an expectation about what the gross consumption under this pricing schedule will be. All customers have access to the same relevant information[4], and are assumed to form the same expectation Q^E, which is also known to the monopolist.

3. Based on their type θ and the expectation of gross consumption Q^E, each customer determines their optimal individual consumption $q(t(\theta))$, where $t(\theta) = \arg\max_t [W(q(t), \theta, Q^E) - \tau(t)]$. If the customer gets at least their reservation utility, that is, if:

$$W(q(t(\theta)), \theta, Q^E) - \tau(t(\theta)) \geq \hat{U}(\theta), \tag{2.3}$$

   then the customer chooses to consume $q(t(\theta))$ and pay $\tau(t(\theta))$. If not, the customer does not participate, and purchases zero quantity.

4. The monopolist gets a payoff of

$$\int_{\theta \in \mathbf{\Theta}} \tau(t(\theta)) f(\theta) d\theta, \tag{2.4}$$

   where $\mathbf{\Theta}$ is the set of participating types. Each participating customer gets a payoff of

$$W(q(t(\theta)), \theta, Q^A) - \tau(t(\theta)), \tag{2.5}$$

   where Q^A is the actual realized gross consumption. Each customer that does not participate gets a payoff of $\hat{U}(\theta)$.

2.3. Contracts

This subsection defines the different contracts that are used in subsequent analysis. To simplify notation, the definition of the following contracts is based on the assumption of *full participation*

[4]The customer's unique knowledge of their own type does not affect their expectation of gross consumption, which is completely determined by $f(\theta)$, the pricing schedule, and the functions $U(q,\theta)$, $W(q,\theta,Q)$, and $\hat{U}(\theta)$ (all of which are common knowledge at this stage).

– that is, that all customers find it optimal to purchase under the contract, if it specifies a non-negative allocation for their type. Inducing full participation is always optimal for the monopolist in the models analyzed in sections 3 through 5.

Q-feasible contracts: Given any expectation of gross consumption Q, a *Q-feasible contract* is a menu of of quantity-price pairs $(q^F(t,Q), \tau^F(t,Q))$ which is incentive-compatible and ensures participation.

$$[IC] \quad : \quad \theta = \arg\max_t W(q^F(t,Q),\theta,Q) - \tau^F(t,Q)\ \forall\theta \tag{2.6}$$

$$[IR] \quad : \quad W(q^F(\theta,Q),\theta,Q) - \tau^F(\theta,Q) \geq \hat{U}(\theta)\ \forall\theta \tag{2.7}$$

Q-optimal contracts: Given any expectation of gross consumption Q, an *Q-optimal contract* $(q(\theta,Q), \tau(\theta,Q))$ is a Q-feasible contract that solves the monopolist's profit maximization problem:

$$\max_{q^F(t,Q),\tau^F(t,Q)} \int_{\underline{\theta}}^{\overline{\theta}} \tau^F(t,Q) f(t) dt, \tag{2.8}$$

over all $(q^F(t,Q), \tau^F(t,Q))$ that satisfy [IC] and [IR].

Optimal fulfilled-expectations contracts: A *optimal fulfilled-expectations contract* is a menu of price-quantity pairs $q^*(\theta), \tau^*(\theta)$ such that the contract $q(\theta,Q), \tau(\theta,Q)$ defined by

$$\begin{aligned} Q &= \int_{\underline{\theta}}^{\overline{\theta}} q^*(\theta) f(\theta) d\theta \\ q(\theta,Q) &= q^*(\theta) \\ \tau(\theta,Q) &= \tau^*(\theta) \end{aligned} \tag{2.9}$$

is a Q-optimal contract.

Based on the definitions above, note that if any Q-optimal contract $q(\theta,Q), \tau(\theta,Q)$ satisfies *fulfilled-expectations [FE]* for some Q:

$$[FE] : Q = \int_{\underline{\theta}}^{\overline{\theta}} q(\theta,Q) f(\theta) d\theta, \tag{2.10}$$

then the contract $q^*(\theta) = q(\theta,Q), \tau^*(\theta) = \tau(\theta,Q)$ is an optimal fulfilled-expectations contract.

The solution that the monopolist seeks is a optimal fulfilled-expectations contract. The conditions for the existence and possible uniqueness of these contracts are described independently in each subsection.

2.4. Participation constraints and entry deterrence

The monopolist in the model may face a threat of entry from an entrant[5], whose product is intrinsically a perfect substitute for the monopolist's product. By virtue of being the incumbent, the monopolist's product generates *positive network value* for all customers. The entrant's product, on the other hand, is assumed to provide only its *intrinsic value* to the customers. The fixed cost of entry is assumed to be zero.

The purpose of this subsection is to establish that the problem of pricing to deter entry under the threat of costless entry is equivalent to a problem of pricing in the absence of the entry threat, but instead with specific type-dependent individual rationality constraints.

At a gross consumption level Q, the utility of a customer of type θ who purchases a quantity q of the monopolist's product for a payment p is $(W(q,\theta,Q)-p)$, and the utility of a customer of type θ who purchases a quantity q of the entrant's product for a payment p is $(U(q,\theta)-p)$. Given a set of prices, and an expectation Q of gross consumption of the monopolist's product, customers choose the product and quantity that maximizes their utility. Customers indifferent between the monopolist's and the entrant's products are assumed to choose the monopolist's product.

A complete characterization of the entry game is not provided. Rather, the analysis focuses on the characteristics of pricing schedules for the monopolist that successfully *deter entry*. Since the fixed cost of entry is assumed to be zero, these are pricing schedules for the monopolist under which any pricing schedule offered by the entrant results in zero profits for the entrant.

Recall that

$$\alpha(\theta) = \arg\max_q U(q,\theta), \tag{2.11}$$

and that

$$\beta(\theta,Q) = \arg\max_q W(q,\theta,Q). \tag{2.12}$$

Suppose the entrant offered the constant pricing scheme $p(q)=\varepsilon$, where ε is small. Under this pricing scheme, each customer would choose their intrinsic-value maximizing level of consumption $\alpha(\theta)$, and would realize surplus of $(U(\alpha(\theta),\theta)-\varepsilon)$. If customers of type θ expected surplus of less than $(U(\alpha(\theta),\theta)-\varepsilon)$ from the monopolist's product, they would buy the entrant's product, and the entrant would receive non-zero profits. Therefore, in order to deter entry, the monopolist's pricing scheme must provide customers of type θ with a surplus of at least $(U(\alpha(\theta),\theta)-\varepsilon)$, for all $\varepsilon>0$. Clearly, this cannot be achieved unless the monopolist's pricing scheme provides customers of type θ with surplus of at least $U(\alpha(\theta),\theta)$. Since $U(\alpha(\theta),\theta)$ is the maximum surplus that a customer of type θ can get from the entrant's product under *any* pricing scheme, ensuring that customers get this level of surplus is both necessary and sufficient for the monopolist to deter entry.

As a consequence, when the fixed cost of entry is zero, deterring entry simply imposes a lower

[5] The analysis would not change if there were multiple identical entrants.

bound on the surplus each customer type must receive. Analytically, this is identical to the problem of choosing a pricing scheme with type-dependent individual rationality constraints (Jullien, 2000). In other words, setting $\hat{U}(\theta) = U(\alpha(\theta), \theta)$ in equation (2.7) ensures that any Q-feasible contract deters entry, and the definitions of all the other contracts in section 2.3 remain the same.

When faced with a threat of entry, the monopolist's problem is therefore to choose the optimal fulfilled-expectations contract, with $\hat{U}(\theta) = U(\alpha(\theta), \theta)$. In the following sections, the monopolist's problem is solved both in the absence of an entry threat, as well as in its presence.

2.5. Preliminary results

The purpose of this subsection is to present two preliminary results used in the subsequent analysis. The first result characterizes the optimal contract offered by the monopolist in the absence of network effects – that is, when $W(q, \theta, Q) = U(q, \theta)$ for all Q. This is termed the *base case*, and is used as a benchmark in sections 3 and 4. The second result describes the structure of the unique Q-optimal contract.

In the base case, since there are no network effects, fulfilled-expectations do not play a role, and one gets the standard optimal contract for one-dimensional types (Maskin and Riley, 1984).

Lemma 1. *When $W(q, \theta, Q) = U(q, \theta)$, the monopolist offers the contract $q^0(\theta), \tau^0(\theta)$ which satisfies the following conditions for all θ:*

$$\frac{U_1(q^0(\theta), \theta)}{U_{12}(q^0(\theta), \theta)} = \frac{1 - F(\theta)}{f(\theta)}; \tag{2.13}$$

$$\tau^0(\theta) = U(q^0(\theta), \theta) - \int_{\underline{\theta}}^{\theta} U_2(q^0(x), x) dx \tag{2.14}$$

This contract defined by (2.13) and (2.14) is unique. Moreover, for all θ such that $q^0(\theta) > 0$, it satisfies $q^0_1(\theta) > 0$, $\tau^0_1(\theta) > 0$.

Similarly, for an exogenously specified level of consumption Q, the optimal contract takes the following familiar form:

Lemma 2. *If $\hat{U}(\theta) = 0$, for every exogenously specified expectation of consumption Q, the Q-optimal contract $q(\theta, Q), \tau(\theta, Q)$ is unique, and is defined by the following conditions:*

$$\frac{W_1(q(\theta, Q), \theta, Q)}{W_{12}(q(\theta, Q), \theta, Q)} = \frac{1 - F(\theta)}{f(\theta)}, \tag{2.15}$$

and

$$\tau(\theta, Q) = W(q(\theta, Q), \theta, Q) - \int_{\underline{\theta}}^{\theta} W_2(q(\theta, Q), x, Q) dx. \tag{2.16}$$

3. Homogeneous network effects

This section analyzes network effects that depend on just gross consumption, and discusses some properties of consumption, pricing and welfare under the optimal fulfilled-expectations contract. The value function $W(q, \theta, Q)$ is assumed to be linearly separable in intrinsic value and network value, and to take the following form

$$W(q, \theta, Q) = U(q, \theta) + w(Q). \tag{3.1}$$

This specification of network effects is similar to that in many standard models (for instance, Katz and Shapiro, 1985), wherein the heterogeneity of customers is only with respect to the intrinsic value they get from the product, though with the additional feature of variable (rather than unit) consumption by each customer.

3.1. Pure monopoly pricing

In the absence of an entry threat (which is referred to as *pure monopoly*, to distinguish it from the subsequent *entry-deterring monopoly*), the following proposition establishes that the unique solution to the monopolist's problem is very similar to that of the base case. Proofs of all the main results are available in Appendix A.

Proposition 1. *If $W(q, \theta, Q) = U(q, \theta) + w(Q)$, then the optimal fulfilled-expectations contract takes the form:*

$$q^*(\theta) = q^0(\theta); \tag{3.2}$$

$$\tau^*(\theta) = \tau^0(\theta) + w(Q^0), \tag{3.3}$$

where $q^0(\theta)$ and $\tau^0(\theta)$ are specified in (2.13) and (2.14), and $Q^0 = \int_{\underline{\theta}}^{\overline{\theta}} q^0(\theta) f(\theta) d\theta$. A contract of this form exists and is unique for any function $w(Q)$.

Proposition 1 shows that when the network value function depends on just gross consumption, the monopolist finds it optimal to induce levels of consumption from each customer type that are identical to those in the absence of network effects, and to simply increase the total price charged to every type by an amount equal to the network value. The intuition behind this result is straightforward. For any common expectation Q of gross consumption, the value functions of all customer types are shifted up by the same constant amount $w(Q)$. Since there is no change in the marginal properties of the utility functions, the monopolist's optimal allocation $q^*(\theta)$ remains the same for all types. This is illustrated in Figure 3.1.

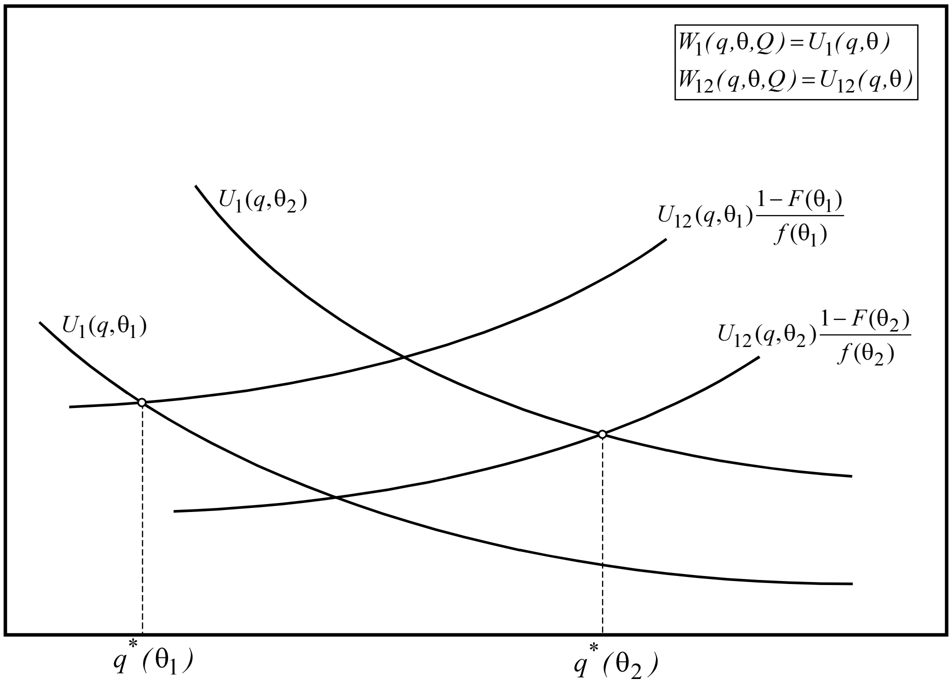

Figure 3.1: Illustrates the optimal consumption of two types θ_1 and θ_2 ($\theta_1 < \theta_2$) with *homogeneous* network effects under *pure monopoly*. First-order necessary conditions are met for each type at the intersection of the $U_1(q, \theta)$ and the $U_{12}(q, \theta)\frac{1-F(\theta)}{f(\theta)}$ curves. As a consequence, $q^*(\theta) = q^0(\theta)$.

It is evident from (3.3) that the monopolist captures all of the increase in surplus from the network effects. In addition, customer surplus does not change for any customer type relative to the base case. This outcome changes substantially when there is an entry threat, as established in the following subsection.

3.2. Entry-deterring monopoly pricing

This subsection specifies the optimal fulfilled-expectations contracts in the presence of an entry threat that is successfully deterred. The main result establishes that the unique solution to the monopolist's problem in this case is to specify a quantity-independent (flat fee) pricing schedule:

Proposition 2. *If $W(q, \theta, Q) = U(q, \theta) + w(Q)$, then the optimal fulfilled-expectations contract that deters entry takes the form:*

$$q^*(\theta) = \alpha(\theta); \tag{3.4}$$

$$\tau^*(\theta) = w(Q^*), \tag{3.5}$$

where $Q^* = \int_{\underline{\theta}}^{\overline{\theta}} \alpha(\theta) f(\theta) d\theta$. *A contract of this form exists and is unique for any network value function* $w(Q)$.

Proposition 2 establishes that when network effects depend on just gross consumption, the optimal entry-deterring pricing scheme results in all customers choosing the level of consumption that maximizes total surplus[6]. Intuitively, a contract that separates any subset of types (in order to price-discriminate) would need to induce consumption levels that are strictly lower than $\alpha(\theta)$ for all but the highest type in this subset. This would result in a strict decrease in profits for the monopolist, on account of having to share some portion of the network value $w(Q^*)$ with the customers in this subset in order to satisfy $[IR]$ and ensure that customer surplus is at least $U(\alpha(\theta), \theta)$. The accompanying reduction in Q^* accentuates the reduction in monopoly profits further. As a consequence, it is strictly profit-reducing to price-discriminate, and the monopolist offers the flat fee that maximizes profits.

4. Usage-dependent network effects

This section models network effects that depend on both gross consumption and individual consumption. The value function $W(q, \theta, Q)$ is assumed to be linearly separable in intrinsic value and network value, and to take the following form

$$W(q, \theta, Q) = U(q, \theta) + qw(Q).$$

4.1. Pure monopoly pricing

In the absence of an entry threat, the following proposition establishes the main characteristics of the optimal fulfilled-expectations contracts:

Proposition 3. *(a) If $W(q, \theta, Q) = U(q, \theta) + qw(Q)$, then any optimal fulfilled-expectations contract satisfies the following conditions:*

$$\frac{U_1(q^*(\theta), \theta) + w(Q^*)}{U_{12}(q^*(\theta), \theta)} = \frac{1 - F(\theta)}{f(\theta)}, \tag{4.1}$$

and

$$\tau^*(\theta) = U(q^*(\theta), \theta) + q^*(\theta)w(Q^*) - \int_{\underline{\theta}}^{\theta} U_2(q^*(x), x)dx, \tag{4.2}$$

where $Q^* = \int_{\underline{\theta}}^{\overline{\theta}} q^*(\theta)f(\theta)d\theta$.

(b) If $w(Q)$ has a finite upper bound $\overline{w}$, then an optimal fulfilled-expectations contract always exists. In addition, if $w_1(Q) < -U_{11}(q, \overline{\theta})$ for all Q and q, then (4.1) and (4.2) specify the unique optimal fulfilled-expectations contract.

[6] Note that since $W_1(q, \theta, Q) = 0$ in this case, $\beta(\theta, Q) = \alpha(\theta)$.

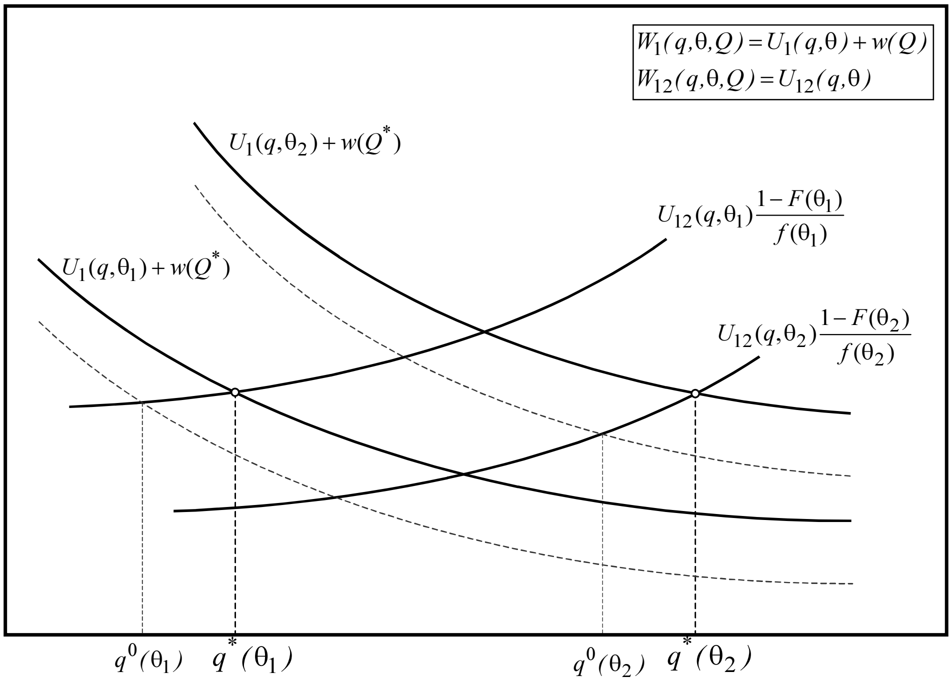

Figure 4.1: Illustrates the optimal consumption of two types θ_1 and θ_2 ($\theta_1 < \theta_2$) with *usage-dependent* network effects under *pure monopoly*. The marginal value curves $W_1(q, \theta, Q)$ are higher than the corresponding $U_1(q, \theta)$ curves, by a constant amount $w(Q)$. This results in a strict increase in consumption for all types, relative to the base case.

(c)For all θ, $q^*(\theta) > q^0(\theta)$, and $\tau^*(\theta) > \tau^0(\theta)$.

Sufficient conditions for the existence of an optimal fulfilled-expectations equilibrium are fairly mild – all that is required is that the marginal benefit from the network effects $w(Q)$ be bounded. The condition for uniqueness requires that in general, marginal network value not grow too fast relative to marginal intrinsic value. However, even if the solution is not unique, this is not unduly troubling, since multiple possible equilibrium outcomes are not uncommon in models of network goods. The monopolist simply needs to pick the optimal fulfilled-expectations contract that provides the highest profits[7]. It is important to note that the results in part (c) of the proposition (and those in Proposition 5) *do not* rely on uniqueness.

The network effects shift the customer value functions up by $qw(Q^*)$ for all types. Since this shift is proportionate to individual consumption, it results in optimal quantities that are different from those of the base case. Part (c) of the proposition establishes that this is a strict increase for all types, and is illustrated in Figure 4.1, for two candidate types. Correspondingly, prices also go up for all customers. Appendix B discuss the changes in the division of total surplus further through a simple example.

[7]Recall that customer expectations are formed after the contract is specified.

4.2. Entry deterring monopoly pricing

The analysis of Proposition 3 is now extended to the case where a threat of entry is successfully deterred. Some new notation is introduced (though mostly in the proof of Proposition 4, which is in the appendix).

Let $q^m(\theta, Q)$ denote the Q-optimal contract under *pure monopoly*. Applying Lemma 2, this allocation is defined for each θ by the necessary conditions

$$\frac{U_1(q^m(\theta,Q),\theta) + w(Q)}{U_{12}(q^m(\theta,Q),\theta)} = \frac{1-F(\theta)}{f(\theta)}, \tag{4.3}$$

and is unique for a fixed value of Q. Also, from Proposition 3, we know that there is an optimal fulfilled-expectations equilibrium – that is, there is a value of gross consumption such that

$$Q^m = \int_{\underline{\theta}}^{\overline{\theta}} q^m(\theta, Q^m) f(\theta) d\theta. \tag{4.4}$$

The following proposition establishes that the monopolist's pricing scheme results in individual consumption that is either of the form $q^m(\theta, Q)$, or that maximizes *intrinsic value* for the customer:

Proposition 4. *Suppose $W(q,\theta,Q) = U(q,\theta) + qw(Q)$. Assume that the uniqueness condition $w_1(Q) < -U_{11}(q,\overline{\theta})$ from Proposition 3 is met. Define:*

$$\begin{aligned} Q^\alpha &= Q : q^m(\underline{\theta}, Q) = \alpha(\underline{\theta}), \text{ and} && (4.5)\\ \hat{\theta}(Q) &= \theta : q^m(\theta, Q) = \alpha(\theta). && (4.6) \end{aligned}$$

(a) If $Q^\alpha \le Q^m$, then the unique optimal fulfilled-expectations contract is:

$$\begin{aligned} q^*(\theta) &= q^m(\theta, Q^m) && (4.7)\\ \tau^*(\theta) &= U(q^m(\theta),\theta) + q^m(\theta)w(Q^m) - U(\alpha(\theta),\theta) - [\int_{\underline{\theta}}^{\theta} (U_2(q^*(x),x) - U_2(\alpha(x),x))dx] && (4.8) \end{aligned}$$

(b) If $Q^\alpha > Q^m$, then the unique optimal fulfilled-expectations contract is:

$$\begin{aligned} q^*(\theta) &= \alpha(\theta) && (4.9)\\ \tau^*(\theta) &= \alpha(\theta)w(Q^*) && (4.10) \end{aligned}$$

for $\theta \leq \hat{\theta}(Q^*)$, *and*

$$q^*(\theta) = q^m(\theta, Q^*) \tag{4.11}$$

$$\tau^*(\theta) = U(q^*(\theta),\theta) + q^*(\theta)w(Q^*) - U(\alpha(\theta),\theta) - [\int_{\hat{\theta}(Q)}^{\theta} (U_2(q^*(x),x) - U_2(\alpha(x),x))dx] \tag{4.12}$$

for $\theta \geq \hat{\theta}(Q^*)$, *where* Q^* *is the unique solution to:*

$$Q = \int_{\underline{\theta}}^{\hat{\theta}(Q)} \alpha(\theta)f(\theta)d\theta + \int_{\hat{\theta}(Q)}^{\overline{\theta}} q^m(\theta,Q)f(\theta)d\theta. \tag{4.13}$$

Proposition 4 establishes that the same conditions that ensure uniqueness of the optimal fulfilled-expectations contract in the absence of an entry threat are sufficient to ensure uniqueness under the threat of entry. It also establishes that the optimal fulfilled-expectations contract that deters entry can be elegantly characterized using a combination of Q-optimal contracts under pure monopoly, and the contract that implements allocations of $\alpha(\theta)$ for each type θ.

If $q^m(\underline{\theta}, Q^m) > \alpha(\underline{\theta})$ for the lowest type $\underline{\theta}$, an immediate corollary of the proposition is that the presence of the entry threat does not change the individual consumption of any of the types (since $q^m(\underline{\theta}, Q^m) > \alpha(\underline{\theta})$ implies that $Q^\alpha < Q^m$). This is likely to happen when the marginal network value $w(Q)$ is high relative to marginal intrinsic value, or equivalently, if network effects are substantial for all types. This is illustrated further in Appendix B.

Under the conditions of part (b) of the proposition, there are substantial changes in individual consumption (relative to pure monopoly). However, $\hat{\theta}(Q^*)$ is always an interior point of $[\underline{\theta}, \overline{\theta}]$. This implies that the larger increases in individual consumption (to the level $\alpha(\theta)$ which maximizes intrinsic value) will always be for a subset of 'lower' types, and that there will always be a subset of higher types whose individual consumption is still of the form $q^m(\theta, Q^*)$. It is easily shown that under part (b) of the proposition, $Q^* > Q^m$, which implies that consumption increases for all customer types (but more substantially for the lower subset).

4.3. Welfare analysis

This subsection characterizes how the monopolist and its customers share the surplus generated by the network effects under pure monopoly, and also discusses surplus division under entry-deterring monopoly.

Suppose $q^*(\theta), \tau^*(\theta)$ is an optimal fulfilled-expectations contract for some value function $W(q,\theta,Q)$, with realized gross consumption $Q^* = \int_{\underline{\theta}}^{\overline{\theta}} q^*(\theta)f(\theta)d\theta$. Relative to the base case, the net change in *total surplus* as a consequence of the network effects is therefore:

$$\int_{\underline{\theta}}^{\overline{\theta}} [W(q^*(\theta), \theta, Q^*)] f(\theta) d\theta - \int_{\underline{\theta}}^{\overline{\theta}} U(q^0(\theta), \theta) f(\theta) d\theta. \tag{4.14}$$

The *direct change in surplus* from a customer of type θ as a consequence of the network effects is defined as:

$$s^n(\theta) = W(q^0(\theta), \theta, Q^0) - U(q^0(\theta), \theta), \tag{4.15}$$

where $Q^0 = \int_{\underline{\theta}}^{\overline{\theta}} q^0(\theta) f(\theta) d\theta$. Similarly, define the *indirect change in surplus* from a customer of type θ as a consequence of the network effects as

$$s^q(\theta) = W(q^*(\theta), \theta, Q^*) - W(q^0(\theta), \theta, Q^0) \tag{4.16}$$

$s^n(\theta)$ measures the change in surplus as a consequence of having the *increase in value* from the network effects, without accounting for any of the changes in consumption. $s^q(\theta)$ measures the changes in surplus that arise indirectly as a consequence of the changes in *consumption* (both individual and gross) that the network effects induce. The total change in surplus across all types, as specified in (4.14), can now be equivalently expressed as $\int_{\underline{\theta}}^{\overline{\theta}} [s^n(\theta) + s^q(\theta)] f(\theta) d\theta$.

Proposition 5. *Under pure monopoly, the monopolist always captures all of the direct increase in surplus, and shares some of the indirect increase in surplus with the customers. That is:*

$$\int_{\underline{\theta}}^{\overline{\theta}} \tau^*(\theta) f(\theta) d\theta - \int_{\underline{\theta}}^{\overline{\theta}} \tau^0(\theta) f(\theta) d\theta \geq \int_{\underline{\theta}}^{\overline{\theta}} s^n(\theta) f(\theta) d\theta, \tag{4.17}$$

and

$$\int_{\underline{\theta}}^{\overline{\theta}} \tau^*(\theta) f(\theta) d\theta - \int_{\underline{\theta}}^{\overline{\theta}} \tau^0(\theta) f(\theta) d\theta < \int_{\underline{\theta}}^{\overline{\theta}} [s^n(\theta) + s^q(\theta)] f(\theta) d\theta, \tag{4.18}$$

where $s^n(\theta)$ and $s^q(\theta)$ are as defined in (4.15) and (4.16).

While proved for usage-dependent network effects, this result holds trivially for homogeneous network effects, since under Proposition 1, there is no indirect increase in surplus, and the monopolist captures all the direct surplus increase. Proposition 5 establishes that for usage-dependent network effects, the monopolist continues to get all the direct increase in surplus from the network effects, and that any increase in customer surplus are driven by increases in consumption.

Under entry-deterring monopoly, the division of direct and indirect increases in surplus is less relevant – all customers of type θ get surplus at least equal to $U(\alpha(\theta), \theta)$, which implies that they capture all of the intrinsic value that they create. Moreover, the customer types whose optimal consumption is of the form $q^m(\theta, Q^*)$ (that is, all customers under part (a), and the higher subset under part (b) of Proposition 4) capture a fraction of the network value that they create. Since $U(\alpha(\theta),\theta) > U(q^*(\theta),\theta)$ for $q^*(\theta) > \alpha(\theta)$, the monopolist needs to give up network value to the customer if they raise consumption beyond $\alpha(\theta)$. The negative terms in square brackets at the end of equations (4.8) and (4.12) represent the surplus type θ gets beyond $U(\alpha(\theta),\theta)$, which implies that these customers are capturing a fraction over and above this reservation level.

5. Type-dependent network effects

This section models network effects that depend on gross consumption, individual consumption and consumer type. The value function $W(q,\theta,Q)$ is assumed to be linearly separable in intrinsic value and network value, and to take the following general form:

$$W(q,\theta,Q) = U(q,\theta) + w(q,\theta,Q).$$

5.1. Pure monopoly pricing

In the absence of an entry threat, the following proposition establishes the structure of the optimal fulfilled-expectations contracts:

Proposition 6. *If $W(q,\theta,Q) = U(q,\theta) + w(q,\theta,Q)$, then any optimal fulfilled-expectations contract satisfies the following conditions:*

$$\frac{U_1(q^*(\theta),\theta) + w_1(q^*(\theta),\theta,Q^*)}{U_{12}(q^*(\theta),\theta) + w_{12}(q^*(\theta),\theta,Q^*)} = \frac{1-F(\theta)}{f(\theta)}, \tag{5.1}$$

and

$$\tau^*(\theta) = U(q^*(\theta),\theta) + w(q^*(\theta),\theta,Q^*) - \int_0^{\theta} [U_2(q^*(x),x) + w_2(q^*(x),x,Q^*)]dx, \tag{5.2}$$

where $Q^* = \int_{\underline{\theta}}^{\overline{\theta}} q^*(\theta) f(\theta) d\theta.$

5.2. Distortions in consumption induced by network effects

The comparison of individual consumption levels that are induced by the optimal contract $q^*(\theta), \tau^*(\theta)$ with those that occur in the absence of network effects is characterized by the following result:

Proposition 7. *For each θ, and for the contract $q^*(\theta), \tau^*(\theta)$ specified in Proposition 6:*

(a) *If* $\dfrac{U_{12}(q^*(\theta),\theta)}{U_1(q^*(\theta),\theta)} < \dfrac{w_{12}(q^*(\theta),\theta,Q^*)}{w_1(q^*(\theta),\theta,Q^*)}$, *then* $q^*(\theta) < q^0(\theta)$.

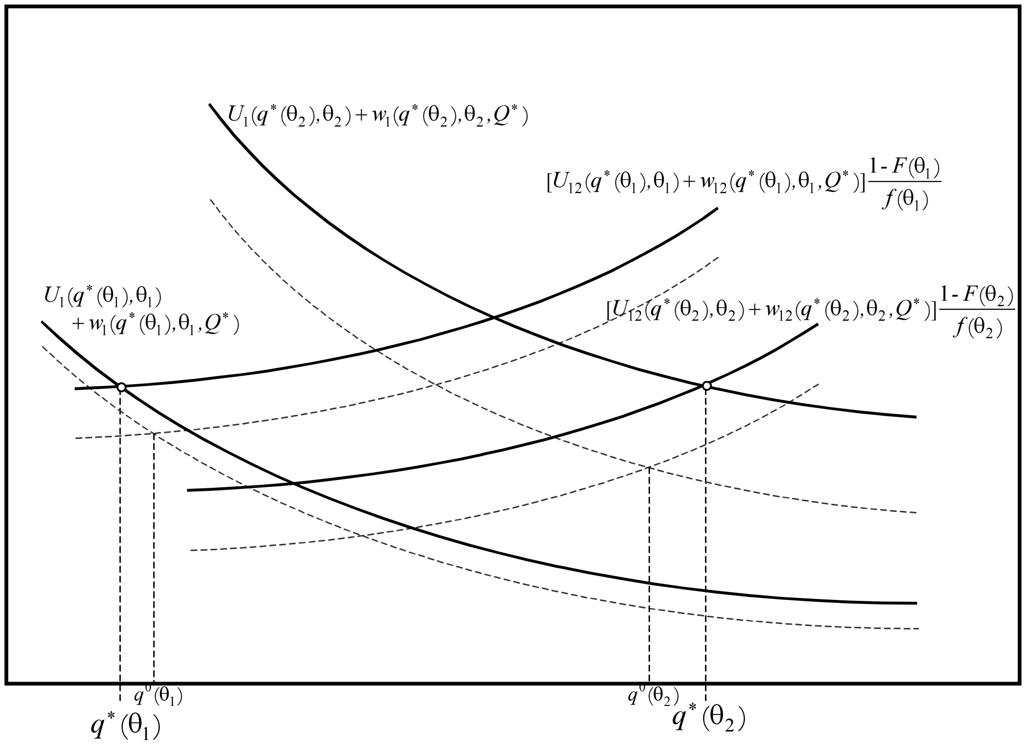

Figure 5.1: Illustrates the distortions in consumption for *type-dependent* network effects, when marginal network value changes more slowly than marginal intrinsic value. The resulting increased heterogeneity between types results in a reduction in consumption for θ_1 when the monopolist optimally price-discriminates.

(b) If $\dfrac{U_{12}(q^*(\theta),\theta)}{U_1(q^*(\theta),\theta)} > \dfrac{w_{12}(q^*(\theta),\theta,Q^*)}{w_1(q^*(\theta),\theta,Q^*)}$ *then* $q^*(\theta) > q^0(\theta)$.

Proposition 7 shows that the general case of type-dependent network effects admits a wider variety of consumption distortions. The result makes sense intuitively if one recognizes $\frac{U_{12}(q,\theta)}{U_1(q,\theta)}$ as the percentage change in marginal intrinsic value that results from a marginal increase in type, and $\frac{w_{12}(q,\theta,Q)}{w_1(q,\theta,Q)}$ as the percentage change in marginal network value for a marginal increase in type. When the network effects *increase* the marginal impact of an increase in type on total marginal value (or loosely, makes the customer types more different from each other), the monopolist changes pricing in a manner that *increases* the differences in consumption across types, which reduces consumption for a subset of types[8]. This distortion occurs despite the fact that the positive network effects increase value from consumption for all types, at all consumption levels. Any increase in consumption for a subset of types would increase the total surplus generated by all types, but at the cost of reducing the monopolist's ability to price-discriminate.

[8]In contrast, the usage-dependent network effects analyzed in Section 4 did not depend on type, and therefore, $w_{12}(q,\theta,Q)$ was zero everywhere. As a consequence, condition (b) of the proposition holds everywhere, which confirms the strict increase in usage across types derived in Proposition 3.

There will be a reduction in individual consumption across *all* types if the efficient individual consumption levels is unaltered by the network effects (if it is infinite, for instance). More customarily, if the efficient level for the highest type strictly increased on account of the network effects, this would ensure an increase in consumption for a positive measure of types. This subset would be an interval of higher types; the reduction in individual consumption would be induced for the remaining interval of (lower) types, so long as the increase in the efficient level of consumption is highest for the highest type. Network effects of this kind therefore increase consumption and surplus disparities across types, and while benefiting higher customer types, may have a net negative effect on the other (lower-type) customers.

6. Discussion

A number of new results on price screening with network effects have been derived in Sections 3 through 5. This section discusses some of these results, examines some of the model's assumptions, and concludes with an outline of open questions raised by the analysis.

6.1. Discussion of results

Contracting is especially complicated in industries with network effects. The challenges include setting complex pricing schedules for variable quantity purchases, designing optimal quantity discounts, taking into account heterogeneity in network value across different customers, and also incorporating the reality that entry threats and 'comparables' from potential competitors play an important role in limiting the amount customers can be charged. Network effects pose an additional unique challenge[9], since there is the trade-off between designing prices that increase value from higher gross consumption, and prices that enables the seller to capture as much of this value as possible.

When network effects do not vary across customers, Proposition 1 establishes that an increase in network effects induces no change in consumption, and that all surplus from the network effects is appropriated by the monopolist. A threat of entry changes pricing substantially – a flat fee is offered to all customer types, and the outcome is socially optimal. While the specification of network effects in section 3 is simple, it would apply to industries in which the primary network value stems from a common fixed-cost reduction – for instance, the cost of finding the appropriate hosting infrastructure, or qualified technical support. These results also indicate that if competing products are anything but perfectly compatible – for instance, through policy that mandates a shared standard for the network good, or policy that forces an operating system vendor to release its source code – any oligopoly outcome will be socially inferior to the entry-deterring monopoly

[9] A further contracting challenge, not addressed by this paper but studied by Segal (2003), is ensuring coordination between the agents adopting the network good.

outcome. In other words, from a regulatory perspective, policy that ensures a credible threat of entry may be more socially efficient than actually inducing entry, if entry results in any future imperfection in product compatibility.

When the value realized from network effects varies with individual consumption, Proposition 3 establishes a strict increase in individual consumption across all customer types. In any price screening model, there is always a trade-off between value creation and price discrimination, and the consumption of any customer type is limited by the monopolist' desire to extract rent from higher types. The issue of value creation is accentuated further when there are network effects, since increases in consumption from *any* subset of customer types increases the value created by *all* customer types. The trade-off still exists, though, and while pricing is redesigned to induce usage increases from both lower and higher customer types, customers still consume at an inefficient level. However, the relative distribution of surplus improves for lower customer types, implying that the network effects benefit lower-usage customers disproportionately, even though the higher-usage customers contribute relatively more to their actual magnitude.

Furthermore, when network effects are usage-based, the effects of an entry threat are less pronounced that those established by Proposition 2. In fact, as shown in Proposition 4(a), the threat of entry may have no effect on consumption or surplus, and may merely result in a price change that redistributes surplus between the monopolist and its customers. Note that this occurs even when entry is not blockaded. This outcome is most likely when, relative to marginal intrinsic value, marginal network value is fairly high across all customers, as illustrated further by the example in Appendix B.

When the network effects are type-dependent, their presence may distort individual consumption in a manner that reduces consumption for a fraction of customer types. The intuition behind this result is simple – the distortion occurs when utility functions are such that the presence of network effects effectively increases the heterogeneity between customers. Proposition 7 presents a necessary and sufficient condition to characterize whether this increase in heterogeneity actually occurs, by comparing the rates of variation in marginal intrinsic value and marginal network value as customer type θ increases.

Most empirical papers on network externalities (for instance, Gandal, 1995, Brynjolfsson and Kemerer, 1996, Forman, 2001) have studied technology markets – databases, spreadsheets, networking equipment – in which sellers with monopoly power routinely offer nonlinear pricing schedules, sell variable quantities to business customers, and price to deter entry. The results of this paper could form a stronger theory base for future empirical work which aims to estimate the extent and implications of network effects in such markets.

The examples studied in Appendix B highlight the effect of network effects and entry deterrence on the relative distribution of surplus across participating customers. Regulatory agencies often consider implementing policy that affects not just total surplus, but the equity of surplus distribution across customers. For instance, the attention received by the issue of the 'digital divide' illustrates this potential objective clearly. Towards this end, the examples establishes that even if creating a credible threat of entry does not increase total surplus, it will reduce the inequity in surplus division across the different customers who generate the surplus through their consumption. In addition, there will always be accompanying transfer of surplus to all customers. While the outcome never maximizes total surplus, it is still likely that it is more efficient than an oligopoly with incompatible products.

6.2. Discussion of assumptions

The sequence of events specified in section 2.3 assumes that all customers have identical expectations of gross consumption. Under the assumption of rational participants, this is not restrictive – everyone has access to all the information needed to compute the expected consumption, and once the monopolist has specified prices, there is no residual uncertainty about demand. Clearly, in equilibrium, all customers must have the same expectation (the correct one).

However, compared to standard models of nonlinear pricing, this paper places a higher computational burden on customers. Each customer has to know $F(\theta)$, compute the optimal consumption (not just for themselves, but for all customer types), and them calculate the gross consumption. It may be likely that customers of network goods cannot actually compute the true gross consumption immediately, due to a lack of information, or due to bounds on information processing capability. There may be a multi-period adjustment process, in which customers iteratively make a series of guesses which converge to the fulfilled-expectations equilibrium outcome. Alternately, customers may learn the distribution of types from the pricing schedule. Formalizing these notions remains (early-stage) work in progress.

The assumption that $W(q, \theta, Q)$ has a finite maximum q for all θ and Q is non-standard. However, given that marginal costs are zero in the model, it is necessary in order to get a bounded solution. It is also a reflection of reality – that customers do stop using zero marginal price products at a finite level, typically due to the presence of resource constraints, and substitute uses for shared resources, as discussed in Section 2.1. Additionally, all the results of this paper are available for the case where W is strictly increasing and has a finite upper bound. Since variable costs are assumed to be zero, the issue that arises in a model of that kind is that the efficient consumption level is infinite. The contracts are still well-defined and specified as presented (so long as W is bounded above) – however, the presence of network effects cannot increase the efficient consumption levels, which misses a potentially important effect. A more formal discussion of this is available on request.

In addition, slightly modified versions of *all* of the results in this paper continue to hold under the assumption of unbounded value functions and positive convex costs. Consider, for instance, a (standard) specification in which customer utility is $\tilde{W}(q, \theta, Q)$, $\tilde{W}_1(q, \theta, Q) > 0$ for all q (and $\tilde{W}(q, \theta, Q)$ has the other curvature properties attributed to the customer value function in this paper). In addition, suppose the provision of quantity q to each customer has a positive cost $c(q)$, where $c_1(q) > 0$, $c_{11}(q) > 0$. If one defined the total surplus function as:

$$W(q, \theta, Q) = \tilde{W}(q, \theta, Q) - c(q),$$

then $W(q, \theta, Q)$ would have the same properties as it does in this model. More importantly, all the expressions for $q^*(\theta)$ derived in the model would continue to be valid, and so would all the expressions for $\tau^*(\theta)$, if it is treated as the optimal markup rather than the optimal price. In other words, the optimal contracts would be $q^*(\theta), (\tau^*(\theta) + c(q^*(\theta)))$, with the same expressions for $q^*(\theta)$ and $\tau^*(\theta)$ as derived in sections 3 and 4. Therefore, this paper's results are also applicable for technology products that display positive network effects, but which have non-zero marginal costs (networking equipment or handheld computers, for instance)

Some of the paper's results have specified conditions that are necessary to guarantee uniqueness. However, none of the properties of the contracts derived in Propositions 1 through 3 depend on uniqueness, and neither do the results of Proposition 5 or Proposition 7. If there are multiple optimal fulfilled-expectations equilibria, all the monopolist needs to do is choose the one with the highest profits. Proposition 4 relies on uniqueness, though a slightly modified version holds if one assumes that the monopolist always chooses the highest-profit contract.

6.3. Concluding remarks

A more general characterization might be to model the network good as a multiproduct bundle, and characterize customers using a two-dimensional type vector, drawing on Armstrong (1996) and Rochet and Chone (1998). Admitting this extension is current work-in-progress. Industries in which products display network effects are often natural monopolies, especially when competing products are incompatible and marginal costs are near-zero. Moreover, entry-deterrence appears to play a significant role in practice (as illustrated by the Microsoft antitrust case). The analysis of entry-deterring monopoly is therefore likely to be very important for these industries. In light of the results obtained in this paper, a natural (and open) question that arises is how *non-zero* entry costs affects outcomes. Clearly, monopoly profits will increase, and entry deterrence will still be an optimal strategy – however, it is likely that profits will increase by less than the entry cost.

The analysis of entry deterrence also suggests the feasibility of solving a general model of nonlinear pricing for competing network goods. If customers expect the competing products to have different levels of gross consumption, they would view them as vertically differentiated products, as in Stole (1995), which would admit pricing other than the zero-markup contracts in Mandy (1992). Similar issues have been analyzed in a model of coalition formation by Economides and Flyer (1998). Price reductions that increase network effects would become 'quality' investments, and the issue of how competitive intensity is affected by these investments becomes relevant, especially since Section 4.2 suggests that in a general model, the equilibrium profits of the smaller network are likely to be zero. I hope to address some of these questions in the near future.

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A. Appendix: Proofs

The following definitions and intermediate results are used in the proofs of the paper's results:

Definition 1. *A feasible fulfilled-expectations contract is a menu of price-quantity pairs* $q^{FE}(\theta), \tau^{FE}(\theta)$ *such that the contract* $q^F(t,Q), \tau^F(t,Q)$ *defined by*

$$Q = \int_{\underline{\theta}}^{\overline{\theta}} q^{FE}(\theta) f(\theta) d\theta$$

$$\begin{aligned} q^F(t,Q) &= q^{FE}(t) \\ \tau^F(t,Q) &= \tau^{FE}(t) \end{aligned}$$

is a Q-feasible contract.

Note that if any Q-feasible contract $q^F(t,Q), \tau^F(t,Q)$ satisfies fulfilled-expectations at some Q:

$$Q = \int_{\underline{\theta}}^{\overline{\theta}} q^F(t,Q) f(t) dt, \tag{A.1}$$

then the contract $q^{FE}(\theta) = q^F(\theta,Q), \tau^{FE}(\theta) = \tau^F(\theta,Q)$ is a feasible fulfilled-expectations contract.

Definition 2. *The gross consumption function is defined as:*

$$\Gamma(Q) = \int_{\underline{\theta}}^{\overline{\theta}} q(\theta,Q) f(\theta) d\theta, \tag{A.2}$$

where $q(\theta,Q)$ *is part of the unique Q-optimal contract associated with an expected gross consumption Q.*

Two immediate consequences of this definition are:

Lemma 3. *(a) For any Q-optimal contract* $q(\theta,Q), \tau(\theta,Q)$*, if* $\Gamma(Q) = Q$*, then the contract defined by*

$$\begin{aligned} q^*(\theta) &= q(\theta,Q); && \text{(A.3)} \\ \tau^*(\theta) &= \tau(\theta,Q) && \text{(A.4)} \end{aligned}$$

is an optimal fulfilled-expectations contract.

(b) $\Gamma(0) > 0$.

Proof. Part (a) follows directly from the definition of $\Gamma(Q)$ and of an optimal fulfilled-expectations contract, and part (b) from the fact that $U(q,\theta) > 0$ for $q > 0$. ■

The next lemma, which establishes the strict monotonicity of $\frac{U_1(q,\theta)}{U_{12}(q,\theta)}$, follows from decreasing absolute risk aversion:

Lemma 4. *If* $\frac{d}{d\theta}(\frac{-U_{11}(q,\theta)}{U_1(q,\theta)}) < 0$*, then* $\frac{d}{dq}(\frac{U_1(q,\theta)}{U_{12}(q,\theta)}) < 0$.

Proof.

$$\frac{d}{d\theta}(\frac{-U_{11}(q,\theta)}{U_1(q,\theta)}) = \frac{-U_{112}(q,\theta)U_1(q,\theta) + U_{11}(q,\theta)U_{12}(q,\theta)}{(U_1(q,\theta))^2}, \tag{A.5}$$

and

$$\frac{d}{dq}(\frac{U_1(q,\theta)}{U_{12}(q,\theta)}) = \frac{U_{11}(q,\theta)U_{12}(q,\theta) - U_{112}(q,\theta)U_1(q,\theta)}{(U_{12}(q,\theta))^2}. \tag{A.6}$$

The denominators of the RHS (A.5) and (A.6) are both strictly positive, and the numerators are identical. The result follows. ■

Proof of Proposition 1

Suppose $q^*(\theta), \tau^*(\theta)$ is an optimal fulfilled-expectations contract. From the definition of an optimal fulfilled-expectations contract, the contract defined by:

$$Q^* = \int_{\underline{\theta}}^{\overline{\theta}} q^*(\theta)f(\theta)d\theta \tag{A.7a}$$

$$q(\theta, Q^*) = q^*(\theta) \tag{A.7b}$$

$$\tau(\theta, Q^*) = \tau^*(\theta) \tag{A.7c}$$

must be a Q-optimal contract. Since $w(Q)$ does not depend directly on θ or q, from Lemma 2, $q^*(\theta, Q^*), \tau^*(\theta, Q^*)$ must satisfy:

$$\frac{U_1(q^*(\theta), \theta, Q^*)}{U_{12}(q^*(\theta), \theta)} = \frac{1 - F(\theta)}{f(\theta)}, \tag{A.8}$$

and

$$\tau^*(\theta, Q) = U(q^*(\theta), \theta) + w(Q^*) - \int_{x=\underline{\theta}}^{\theta} U_2(q^*(x), x)dx \tag{A.9}$$

It follows from (A.7b), (A.7c), (A.8) and (A.9) that $q^*(\theta), \tau^*(\theta)$ satisfies:

$$\frac{U_1(q^*(\theta), \theta)}{U_{12}(q^*(\theta), \theta)} = \frac{1 - F(\theta)}{f(\theta)}, \tag{A.10}$$

and

$$\tau^*(\theta) = U(q^*(\theta), \theta) + w(Q^*) - \int_{x=\underline{\theta}}^{\theta} [U_2(q^*(x), x)]dx. \tag{A.11}$$

Comparing (A.10) with (2.13) yields

$$q^*(\theta) = q^0(\theta), \tag{A.12}$$

since Lemma 4 has established that $\frac{U_1(q,\theta)}{U_{12}(q,\theta)}$ is strictly monotonic in q. Therefore, $Q^* = Q^0$. Consequently, comparing (A.11) with (2.14) yields:

$$\tau^*(\theta) = \tau^0(\theta) + w(Q^0). \tag{A.13}$$

Now, from Lemma 2, it is clear that

$$q(\theta, Q) = q^0(\theta) \tag{A.14}$$

for all Q and θ. Therefore, $\Gamma(Q) = Q^0$ for all Q. Clearly, $\Gamma(Q)$ always has a unique fixed point Q^0, which completes the proof.

Proof of Proposition 2

Consider any expectation of gross consumption Q, and any Q-feasible contract $q^F(\theta, Q), \tau^F(\theta, Q)$. [IR] implies that:

$$U(q^F(\theta, Q), \theta) + w(Q) - \tau^F(\theta, Q) \geq U(\alpha(\theta), \theta). \tag{A.15}$$

Since $U(q^F(\theta, Q), \theta) \leq U(\alpha(\theta), \theta)$, this implies that

$$\tau^F(\theta, Q) \leq w(Q) \tag{A.16}$$

for all θ. Consequently, a pricing scheme that provides the monopolist with a total price of $w(Q)$ from each type has equal or higher profits than any Q-feasible contract. Now, if payments from each type are constant across types, incentive-compatibility requires that a customer of type θ be allocated $\alpha(\theta)$. Since marginal costs are zero, this means that

$$q(\theta, Q) = \alpha(\theta); \tag{A.17}$$

$$\tau(\theta, Q) = w(Q) \tag{A.18}$$

is Q-optimal. Defining

$$Q^* = \int_{\underline{\theta}}^{\overline{\theta}} \alpha(\theta) f(\theta) d\theta, \tag{A.19}$$

this implies that $\Gamma(Q) = Q^*$ for all Q, which in turn implies that there is a unique optimal fulfilled-expectations contract, and the result follows.

Proof of Proposition 3

(a) Suppose $q^*(\theta), \tau^*(\theta)$ is an optimal fulfilled-expectations contract. By definition, the contract defined by:

$$Q^* = \int_{\underline{\theta}}^{\overline{\theta}} q^*(\theta) f(\theta) d\theta \quad \text{(A.20a)}$$

$$q(\theta, Q^*) = q^*(\theta) \quad \text{(A.20b)}$$

$$\tau(\theta, Q^*) = \tau^*(\theta) \quad \text{(A.20c)}$$

must be a Q-optimal contract. Applying Lemma 2 for $W(q, \theta, Q) = U(q, \theta) + qw(Q)$ yields:

$$\frac{U_1(q^*(\theta), \theta, Q^*) + w(Q^*)}{U_{12}(q^*(\theta), \theta)} = \frac{1 - F(\theta)}{f(\theta)}, \quad \text{(A.21)}$$

and

$$\tau^*(\theta, Q) = U(q^*(\theta), \theta) + q^*(\theta) w(Q^*) - \int_{x=\underline{\theta}}^{\theta} U_2(q^*(x), x) dx \quad \text{(A.22)}$$

Consequently, $q^*(\theta), \tau(\theta)$ satisfies:

$$\frac{U_1(q^*(\theta), \theta) + w(Q^*)}{U_{12}(q^*(\theta), \theta)} = \frac{1 - F(\theta)}{f(\theta)}, \quad \text{(A.23)}$$

and

$$\tau^*(\theta) = U(q^*(\theta), \theta) + w(Q^*) - \int_{x=\underline{\theta}}^{\theta} [U_2(q^*(x), x)] dx. \quad \text{(A.24)}$$

(b) From Lemma 2, for any Q, individual consumption in the unique Q-optimal contract satisfies:

$$\frac{U_1(q(\theta, Q), \theta) + w(Q)}{U_{12}(q(\theta, Q), \theta)} = \frac{1 - F(\theta)}{f(\theta)} \quad \text{(A.25)}$$

Existence: If $w(Q)$ is bounded, this implies that $W_1(q(\theta, Q), \theta, Q)$ is bounded for all θ, which in turn implies that $q(\theta, Q)$ is bounded for all θ, since $U_{11}(q, \theta) < 0$. Therefore $\Gamma(Q)$ is bounded. Since $\Gamma(0) > 0$ (from Lemma 3), this implies that a fixed point for $\Gamma(Q)$ exists.

Uniqueness: Differentiating both sides of (A.25) with respect to Q and rearranging yields:

$$q_2(\theta, Q) = \frac{w_1(Q)}{U_{112}(q(\theta, Q), \theta) \frac{1 - F(\theta)}{f(\theta)} - U_{11}(q(\theta, Q), \theta)}. \quad \text{(A.26)}$$

Since $U_{112}(q(\theta, Q), \theta) \geq 0$, this implies that

$$q_2(\theta, Q) \leq \frac{w_1(Q)}{-U_{11}(q(\theta, Q), \theta)}. \quad \text{(A.27)}$$

From the conditions of the proposition for uniqueness, we know that $w_1(Q) < -U_{11}(q,\theta)$, which when combined with (A.27) implies that

$$q_2(\theta, Q) < 1 \tag{A.28}$$

Now, differentiating both sides of (A.2) with respect to Q yields

$$\Gamma_1(Q) = \int_{\underline{\theta}}^{\overline{\theta}} q_2(\theta, Q) f(\theta) d\theta, \tag{A.29}$$

which when combined with (A.28), implies that $\Gamma_1(Q) < 1$ for all Q. This in turn implies that $\Gamma(Q)$ is a contraction, and since $\Gamma(0) > 0$ from Lemma 3, it has a unique and strictly positive fixed point.

(c) When $w(Q) > 0$, (4.1) implies that

$$\frac{U_1(q^*(\theta), \theta)}{U_{12}(q^*(\theta), \theta)} < \frac{1 - F(\theta)}{f(\theta)} \tag{A.30}$$

From Lemma 1, we know that

$$\frac{U_1(q^0(\theta), \theta)}{U_{12}(q^0(\theta), \theta)} = \frac{1 - F(\theta)}{f(\theta)} \tag{A.31}$$

and therefore

$$\frac{U_1(q^0(\theta), \theta)}{U_{12}(q^0(\theta), \theta)} > \frac{U_1(q^*(\theta), \theta)}{U_{12}(q^*(\theta), \theta)} \tag{A.32}$$

From Lemma 4, we know that $\frac{U_1(q,\theta)}{U_{12}(q,\theta)}$ is strictly decreasing in q for all θ, which when combined with (A.32) proves that $q^0(\theta) < q^*(\theta)$ for all θ.

Proof of Proposition 4

This proof uses three intermediate results (Lemmas 5, 6 and 7) which are stated and proved in the body of the proof. Some new notation is also introduced, which follows Jullien (2000), since Proposition 3 of that paper is used to establish Lemma 7.

For any expectation of gross consumption Q, define:

$$l(\gamma, \theta, Q) = \arg\max_q U(q, \theta) + qw(Q) - U_2(q, \theta) \frac{\gamma - F(\theta)}{f(\theta)}. \tag{A.33}$$

It follows immediately that

$$l(1, \theta, Q) = q^m(\theta, Q), \tag{A.34}$$

and that $l(0, \theta, Q) \geq \beta(\theta, Q)$ for all θ, which implies that $l(0, \theta, Q) > \alpha(\theta)$ for all θ.

Next, the unique incentive-compatible contract that implements $\hat{U}(\theta)$ – that is, the unique contract under which the surplus of type θ is $\hat{U}(\theta)$ – can be shown to be:

$$\begin{aligned} \hat{q}(\theta) &= \alpha(\theta); && \text{(A.35)} \\ \hat{\tau}(\theta) &= \alpha(\theta)w(Q). && \text{(A.36)} \end{aligned}$$

Therefore, the set $\Theta = \{\theta : l(1, \theta, Q) \leq \hat{q}(\theta) \leq l(0, \theta, Q)\}$ reduces to:

$$\Theta = \{\theta : q^m(\theta, Q) \leq \alpha(\theta)\}. \tag{A.37}$$

Also, define

$$\hat{\gamma}(\theta, Q) = \gamma : \hat{q}(\theta) = \arg\max_q U(q, \theta) + qw(Q) - U_2(q, \theta)\frac{\gamma - F(\theta)}{f(\theta)} \tag{A.38}$$

Since $\hat{q}(\theta) = \alpha(\theta)$, and $U_1(\alpha(\theta), \theta) = 0$, first-order conditions for (A.38) yield:

$$\hat{\gamma}(\theta, Q) = F(\theta) + \frac{f(\theta)w(Q)}{U_{12}(\alpha(\theta), \theta)} \tag{A.39}$$

Finally, define:

$$H(\gamma, \theta) = \frac{\gamma - F(\theta)}{f(\theta)}$$

The following intermediate results can now be stated and proved.

Lemma 5. *If $\frac{1-F(\theta)}{f(\theta)}$ is non-increasing for all θ, then $H_2(\gamma, \theta) \leq 0$ for all θ, γ such that $H(\gamma, \theta) \geq 0$.*

Proof. Assume the converse – that for some γ, $\frac{\gamma - F(\theta)}{f(\theta)}$ is increasing in some interval $[\theta_1, \theta_2]$. This implies that

$$\frac{\gamma - F(\theta_1)}{f(\theta_1)} < \frac{\gamma - F(\theta_2)}{f(\theta_2)}. \tag{A.40}$$

Since $F(\theta_1) < F(\theta_2)$, this implies that $f(\theta_1) > f(\theta_2)$, which in turn implies that

$$\frac{1-\gamma}{f(\theta_1)} < \frac{1-\gamma}{f(\theta_2)} \tag{A.41}$$

Adding (A.40) and (A.41) yields $\frac{1-F(\theta_1)}{f(\theta_1)} < \frac{1-F(\theta_2)}{f(\theta_2)}$, which contradicts the fact that $\frac{1-F(\theta)}{f(\theta)}$ is non-increasing, and the result follows. ■

Lemma 6. *If $\frac{\partial}{\partial\theta}(\frac{U_{11}(\alpha(\theta),\theta)}{U_{12}(\alpha(\theta),\theta)}) \leq 0$, then $l_2(\hat{\gamma}(\theta, Q), \theta, Q) \leq \hat{q}_1(\theta)$*

Proof. By the definition of $\hat{\gamma}(\theta, Q)$ in (A.38), and of $l(\gamma, \theta, Q)$ in (A.33), we know that

$$l(\hat{\gamma}(\theta, Q), \theta, Q) = \hat{q}(\theta) = \alpha(\theta). \tag{A.42}$$

From the first order conditions for (A.33), we know that

$$U_1(l(\gamma, \theta, Q), \theta) + w(Q) = U_{12}(l(\gamma, \theta, Q), \theta)\frac{\gamma - F(\theta)}{f(\theta)}. \tag{A.43}$$

Differentiating both sides of (A.43) with respect to θ and rearranging yields:

$$l_2(\gamma,\theta,Q) = \frac{U_{12}(l(\gamma,\theta,Q),\theta)(1-H_2(\gamma,\theta)) - U_{122}(l(\gamma,\theta,Q),\theta)H(\gamma,\theta)}{U_{112}(l(\gamma,\theta,Q),\theta)H(\gamma,\theta) - U_{11}(l(\gamma,\theta,Q),\theta)}. \tag{A.44}$$

Substituting in $\hat{\gamma}(\theta,Q)$ and using (A.42) yields:

$$l_2(\hat{\gamma}(\theta,Q),\theta,Q) = \frac{U_{12}(\alpha(\theta),\theta)(1-H_2(\hat{\gamma}(\theta,Q),\theta)) - U_{122}(\alpha(\theta),Q),\theta)H(\hat{\gamma}(\theta,Q),\theta)}{U_{112}(\alpha(\theta),\theta)H(\hat{\gamma}(\theta,Q),\theta) - U_{11}(\alpha(\theta),\theta)}. \tag{A.45}$$

Since $\hat{q}(\theta) = \alpha(\theta)$, and $U_1(\alpha(\theta),\theta) = 0$ by definition, it follows that:

$$\hat{q}_1(\theta) = \frac{U_{12}(\alpha(\theta),\theta)}{-U_{11}(\alpha(\theta),\theta)}. \tag{A.46}$$

Comparing equations (A.45) and (A.46), and using the fact that $U_{11}(q,\theta)U_{12}(q,\theta)H_2(\gamma,\theta) \geq 0$, it follows that $l_2(\hat{\gamma}(\theta,Q),\theta,Q) \leq \hat{q}_1(\theta)$ if:

$$U_{12}((\alpha(\theta),\theta))U_{112}(\alpha(\theta),\theta) - U_{11}(\alpha(\theta),\theta)U_{122}(\alpha(\theta),\theta) \leq 0, \tag{A.47}$$

which is precisely the condition implied by $\frac{\partial}{\partial\theta}(\frac{U_{11}(q,\theta)}{U_{12}(q,\theta)}) \leq 0$. The result follows. ■

It was shown in Section 2.5 that the problem of finding an optimal contract which deters entry was equivalent to finding an optimal contract with type-dependent participation constraints. The following result therefore holds, based on Jullien (2000):

Lemma 7. *The Q-optimal contract which deters entry satisfies:*

(a) If $\theta \in \Theta$,

$$\begin{aligned} q(\theta,Q) &= \alpha(\theta); \\ \tau(\theta) &= \alpha(\theta)w(Q). \end{aligned} \tag{A.48}$$

(b) If $\theta \notin \Theta$, then:

$$\begin{aligned} q(\theta,Q) &= q^m(\theta,Q); \\ \tau(\theta) &= U(q^m(\theta,Q),\theta) + q^m(\theta,Q)w(Q) - U(\alpha(\theta),\theta) - [\int_{\hat{\theta}(Q)}^{\theta}(U_2(q^m(x,Q),x) - U_2(\alpha(x),x))dx], \end{aligned} \tag{A.49}$$

where $\Theta = \{\theta : q^m(\theta,Q) \leq \alpha(\theta)\}$, and $\hat{\theta}(Q) = \theta : q^m(\theta,Q) = \alpha(\theta)$.

In addition, if Θ is empty, then for all θ:

$$\begin{aligned} q(\theta,Q) &= q^m(\theta,Q), \\ \tau(\theta) &= U(q^m(\theta,Q),\theta) + q^m(\theta,Q)w(Q) - U(\alpha(\theta),\theta) - [\int_{\underline{\theta}}^{\theta}(U_2(q^m(x,Q),x) - U_2(\alpha(x),x))dx]. \end{aligned} \tag{A.50}$$

Proof. Lemma 6 ensures that the problem of finding a Q-optimal contract satisfies all the conditions for Proposition 3 of Jullien (2000) to apply. The expressions for $q(\theta, Q)$ follow immediately. The expressions for $\tau(\theta, Q)$ follow by imposing incentive-compatibility and profit maximization by the monopolist. ■

Now, recall the definitions of Q^m, Q^α from the statement of Proposition 4, and $\Gamma(Q)$ from Definition 2:

$$
\begin{aligned}
Q^m &= Q : Q = \int_{\underline{\theta}}^{\overline{\theta}} q^m(\theta, Q) f(\theta) d\theta && \text{(A.51)} \\
Q^\alpha &= Q : q^m(\underline{\theta}, Q) = \alpha(\underline{\theta}), && \text{(A.52)} \\
\Gamma(Q) &= \int_{\underline{\theta}}^{\overline{\theta}} q^m(\theta, Q) f(\theta) d\theta && \text{(A.53)}
\end{aligned}
$$

Clearly, Q^m is a fixed point of $\Gamma(Q)$. Also define $\hat{\Gamma}(Q)$ as

$$
\begin{aligned}
\hat{\Gamma}(Q) &= \int_{\underline{\theta}}^{\hat{\theta}(Q)} \alpha(\theta) f(\theta) d\theta + \int_{\hat{\theta}(Q)}^{\overline{\theta}} q^m(\theta, Q) f(\theta) d\theta, \text{ if } \hat{\theta}(Q) \text{ exists;} && \text{(A.54)} \\
\hat{\Gamma}(Q) &= \Gamma(Q) \text{ otherwise.} && \text{(A.55)}
\end{aligned}
$$

Now, from (A.26). we know that

$$
q_2^m(\theta, Q) = \frac{w_1(Q)}{U_{112}(q^m(\theta, Q), \theta) \frac{1-F(\theta)}{f(\theta)} - U_{11}(q^m(\theta, Q), \theta)}. \tag{A.56}
$$

The RHS of (A.56) is strictly positive since $w_1(Q) > 0$, $U_{112}(q, \theta) \geq 0$, and $U_{11}(q, \theta) < 0$. Therefore, $q_2^m(\theta, Q) > 0$ for all Q. As a consequence, if $Q^\alpha < Q^m$, then $q^m(\underline{\theta}, Q^\alpha) < q^m(\underline{\theta}, Q^m)$, which in turn implies that $q^m(\underline{\theta}, Q^m) > \alpha(\underline{\theta})$. The set Θ is therefore empty, and this establishes part (a), based on Lemma 7.

Now, differentiating both sides of (A.54) and (A.55) with respect to Q yields:

$$
\begin{aligned}
\hat{\Gamma}_1(Q) &= \int_{\hat{\theta}(Q)}^{\overline{\theta}} q_1^m(\theta, Q) f(\theta) d\theta, \text{ if } \hat{\theta}(Q) \text{ exists;} && \text{(A.57)} \\
\hat{\Gamma}_1(Q) &= \Gamma_1(Q) \text{ otherwise.} && \text{(A.58)}
\end{aligned}
$$

Since $q_2^m(\theta, Q) > 0$, this implies that $\hat{\Gamma}_1(Q) \leq \Gamma_1(Q)$, and the inequality is strict if $\hat{\theta}(Q) > \underline{\theta}$. Consequently, under the conditions for uniqueness in Proposition 3, both $\Gamma(Q)$ and $\hat{\Gamma}(Q)$ have unique strictly positive fixed points.

If $Q \geq Q^\alpha$, $\hat{\Gamma}(Q) = \Gamma(Q)$. As a consequence, if $Q^\alpha > Q^m$, this means that the fixed point of $\hat{\Gamma}(Q)$ has to lie in (Q^m, Q^α), because we know that $\hat{\Gamma}(Q) > \Gamma(Q)$ for $Q < Q^\alpha$, which means it cannot have a fixed point at Q^m, which in turn implies that if it has a fixed point greater than Q^α, this violates the uniqueness of the fixed point of $\Gamma(Q)$ that was established in Proposition 3.

Using the fact that the unique optimal fulfilled-expectation equilibrium has gross consumption that is the fixed point of $\hat{\Gamma}(Q)$, part (b) of the result follows. This completes the proof.

Proof of Proposition 5

Consider the contract:

$$q^{FE}(\theta) = q^0(\theta) \tag{A.59a}$$

$$\tau^{FE}(\theta) = \tau^0(\theta) + q^0(\theta)w(Q^0) \tag{A.59b}$$

It is straightforward to establish that this is a *feasible* fulfilled-expectations contract. Under this contract, the monopolist's profits would be

$$\Pi = \int_{\underline{\theta}}^{\overline{\theta}} \tau^0(\theta)f(\theta)d\theta + \int_{\underline{\theta}}^{\overline{\theta}} q^0(\theta)w(Q^0)f(\theta)d\theta \tag{A.60}$$

Using the definition of $s^n(\theta)$ from (4.15), this implies that

$$\Pi = \int_{\underline{\theta}}^{\overline{\theta}} \tau^0(\theta)f(\theta)d\theta + \int_{\underline{\theta}}^{\overline{\theta}} s^n(\theta)f(\theta)d\theta \tag{A.61}$$

Profits under the optimal fulfilled expectations contract must be at least as high as Π. Based on (A.61), this yields:

$$\int_{\underline{\theta}}^{\overline{\theta}} \tau^*(\theta)f(\theta)d\theta \geq \int_{\underline{\theta}}^{\overline{\theta}} \tau^0(\theta)f(\theta)d\theta + \int_{\underline{\theta}}^{\overline{\theta}} s^n(\theta)f(\theta)d\theta,$$

which proves the first part. Now denote the surplus of type θ under the optimal fulfilled-expectations contract as $s^*(\theta)$, and the surplus of type θ under the base case contract as $s^0(\theta)$. We know that

$$s^*(\theta) = \int_{x=\underline{\theta}}^{\theta} [U_2(q^*(x), x)]dx,$$

and

$$s^0(\theta) = \int_{x=\underline{\theta}}^{\theta} [U_2(q^0(x), x)]dx.$$

Since $q^*(\theta) > q^0(\theta)$, and $U_{12}(q, \theta) > 0$, this implies that $s^*(\theta) > s^0(\theta)$ for all θ. Therefore, the monopolist does not appropriate all the surplus generated by the network effects, and the second part of the result follows.

Proof of Proposition 6

(a) Suppose $q^*(\theta), \tau^*(\theta)$ is an optimal fulfilled-expectations contract. By definition, the contract defined by:

$$Q^* = \int_{\underline{\theta}}^{\overline{\theta}} q^*(\theta) f(\theta) d\theta \tag{A.62a}$$

$$q(\theta, Q^*) = q^*(\theta) \tag{A.62b}$$

$$\tau(\theta, Q^*) = \tau^*(\theta) \tag{A.62c}$$

must be a Q-optimal contract. Applying Lemma 2 for $W(q, \theta, Q) = U(q, \theta) + w(q, \theta, Q)$ as in Proposition 3 yields:

$$\frac{U_{12}(q^*(\theta), \theta) + w_{12}(q^*(\theta), \theta, Q^*)}{U_1(q^*(\theta), \theta) + w_1(q^*(\theta), \theta, Q^*)} = \frac{1 - F(\theta)}{f(\theta)}, \tag{A.63}$$

and

$$\tau^*(\theta) = U(q^*(\theta), \theta) + w(q^*(\theta), \theta, Q^*) - \int_0^{\theta} [U_2(q^*(t), t) + w_2(q^*(t), t, Q^*)] dt. \tag{A.64}$$

Proof of Proposition 7

Lemma 4, establishes that $\dfrac{U_1(q, \theta)}{U_{12}(q, \theta)}$ is decreasing in q,and therefore:

$$\frac{d}{dq} \frac{U_{12}(q, \theta)}{U_1(q, \theta)} > 0. \tag{A.65}$$

Now, suppose $\dfrac{U_{12}(q^*(\theta), \theta)}{U_1(q^*(\theta), \theta)} < \dfrac{w_{12}(q^*(\theta), \theta, Q^*)}{w_1(q^*(\theta), \theta, Q^*)}$. Straightforward algebra[10] yields

$$\frac{U_{12}(q^*(\theta), \theta) + w_{12}(q^*(\theta), \theta, Q^*)}{U_1(q^*(\theta), \theta) + w_1(q^*(\theta), \theta, Q^*)} > \frac{U_{12}(q^*(\theta), \theta)}{U_1(q^*(\theta), \theta)}. \tag{A.66}$$

(A.63) and (A.66) yield:

$$\frac{U_{12}(q^*(\theta), \theta)}{U_1(q^*(\theta), \theta)} < \frac{f(\theta)}{1 - F(\theta)}. \tag{A.67}$$

(2.13) and (A.67) yield:

$$\frac{U_{12}(q^*(\theta), \theta)}{U_1(q^*(\theta), \theta)} < \frac{U_{12}(q^o(\theta), \theta)}{U_1(q^o(\theta), \theta)}, \tag{A.68}$$

and part (a) of the result follows from (A.65). A parallel set of steps yields part (b).

[10] For $a, b, c, d > 0$, $\frac{a}{b} < \frac{c}{d} \Rightarrow bc - ad > 0$. Therefore, $\frac{a+c}{b+d} - \frac{a}{b} = \frac{bc - ad}{b(b+d)} > 0$, or $\frac{a+c}{b+d} > \frac{a}{b}$.

Base case

Optimal contract:	$q^0(\theta) = 2\theta$; $\tau^0(\theta) = 2\theta - \theta^2$

Pure monopoly

Q-optimal contract:	$q(\theta, Q) = 2\theta$; $\tau(\theta, Q) = 2\theta - \theta^2 + wQ$
Optimal fulfilled-expectations contract:	$q^*(\theta) = 2\theta$; $\tau^*(\theta) = 2\theta - \theta^2 + w$
Surplus functions:	$s^*(\theta) = \theta^2$; $s_F(\theta) = 3\theta^2$

Entry-deterring monopoly

Q-optimal contract:	$q(\theta, Q) = \theta + 1$; $\tau(\theta, Q) = wQ$
Optimal fulfilled-expectations contract:	$q^*(\theta) = \theta + 1$; $\tau^*(\theta) = \dfrac{3w}{2}$
Surplus functions:	$s^*(\theta) = \dfrac{(\theta+1)^2}{2}$; $s_F(\theta) = \dfrac{3(\theta+1)^2}{7}$

Table B.1: Optimal contracts and surplus expressions from the example with homogeneous network effects

B. Illustrative examples

Two examples are presented to illustrate the results of the paper further, and to examine how network effects and the threat of entry changes the surplus distribution across customers. In order to perform the latter analysis, define the *customer surplus function* as:

$$s^*(\theta) = W(q^*(\theta), \theta, Q^*) - \tau^*(\theta). \tag{B.1}$$

$s^*(\theta)$ is the surplus that customers of type θ get under the optimal fulfilled-expectations contract. Also, define the *surplus distribution function* $s_F(\theta)$ as:

$$s_F(\theta) = \frac{s^*(\theta)}{\int s^*(\theta) f(\theta) d\theta}. \tag{B.2}$$

$s_F(\theta)$ measures how is the total customer surplus is distributed across the different customer types. It enables one to examine how changes in network effects affect the *relative* levels of surplus that different customer types get.

B.1. Homogeneous network effects

The example uses a simple quadratic value function, and uniformly distributed customer types. The value function is assumed to take the following form:

$$W(q, \theta, Q) = (\theta + 1)q - \frac{1}{2}q^2 + wQ, \tag{B.3}$$

Base case

Optimal contract:	$q^0(\theta) = 2\theta$; $\tau^0(\theta) = 2\theta - \theta^2$

Pure monopoly

Q-optimal contract:	$q(\theta, Q) = 2\theta + wQ$, $\tau(\theta, Q) = 2\theta - \theta^2 + \dfrac{wQ(2 + wQ)}{2}$
Optimal fulfilled-expectations contract:	$q^*(\theta) = 2\theta + \dfrac{w}{1-w}$, $\tau^*(\theta) = 2\theta - \theta^2 + \dfrac{w(2-w)}{2(1-w)^2}$
Surplus functions:	$s^*(\theta) = \theta(\theta + \dfrac{w}{1-w})$, $s_F(\theta) = \dfrac{6\theta(\theta(1-w)+w)}{2+w}$

Table B.2: Optimal contracts and surplus in the example with usage-dependent network effects, under pure monopoly

and customer types are assumed to be uniformly distributed between 0 and 1, which implies that $f(\theta) = 1$ and $F(\theta) = \theta$.

The contracts and surplus values that result from applying Propositions 1 and 2, and equations (B.1) and (B.2) are summarized in Table B.1. Under pure monopoly, consistent with Proposition 1, consumption is unaffected by the network effects, and prices increase by an amount equal to the network value. Under entry deterring monopoly, individual consumption increases for all customers, and a flat fee equal to the network value is charged to each customer.

Figures B.1 (a) and (b) illustrate how $q^*(\theta)$ and $\tau^*(\theta)$ vary with type. By substituting $q^*(\theta)$ into $\tau^*(\theta)$, one can derive the explicit pricing function $p(q) = q - \frac{q^2}{4}$, which implies a progressively increasing quantity discount.

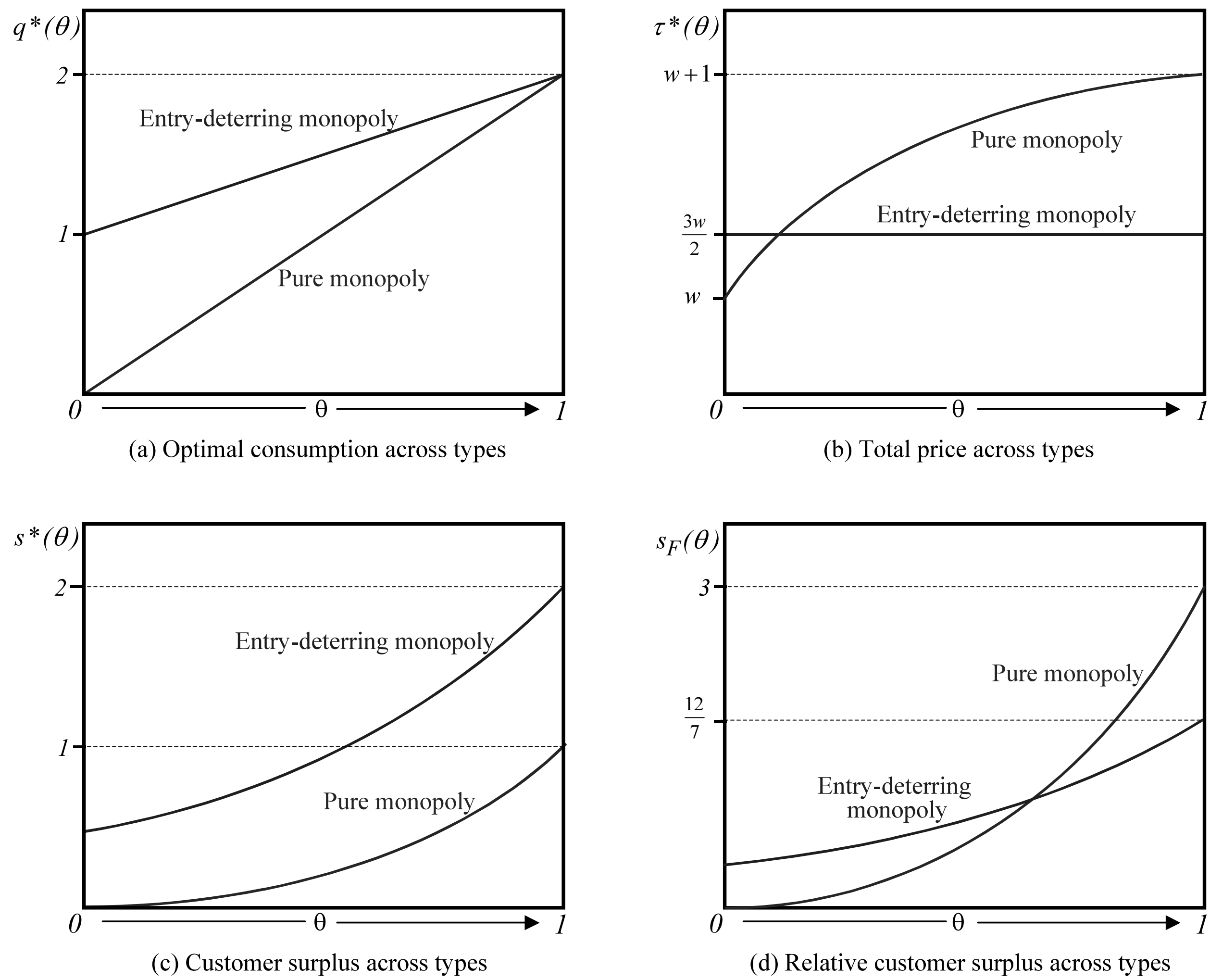
Under entry-deterring monopoly, prices increase for a subset of lower types. However, so does customer surplus, as indicated in Figure B.1 (c). Furthermore, Figure B.1 (d) shows that when there is a threat of entry, the relative distribution of surplus across different customer types is less skewed in favor of higher-usage customers. This is despite the increase in total price for the lower-usage customers, relative to the higher-usage customers.

B.2. Usage-dependent network effects

Next, consider an example in which the value function takes the following form:

$$W(q, \theta, Q) = (\theta + 1)q - \frac{1}{2}q^2 + wqQ, \tag{B.4}$$

and as before, customer types are assumed to be uniformly distributed between 0 and 1. Since $U_{11}(q, \theta) = -1$, the uniqueness condition in Propositions 3 and 4 reduces to $w < 1$. Tables B.2 and B.3 summarizes the solutions for the optimal contracts and surplus functions under this condition. As expected from Proposition 3, both quantities and prices increase under pure monopoly, relative to the base case. Figures B.2 (a) and (b) illustrate the optimal contract for two different values of marginal network value w. In addition, by substituting $q^*(\theta)$ into $\tau^*(\theta)$, one can obtain the explicit



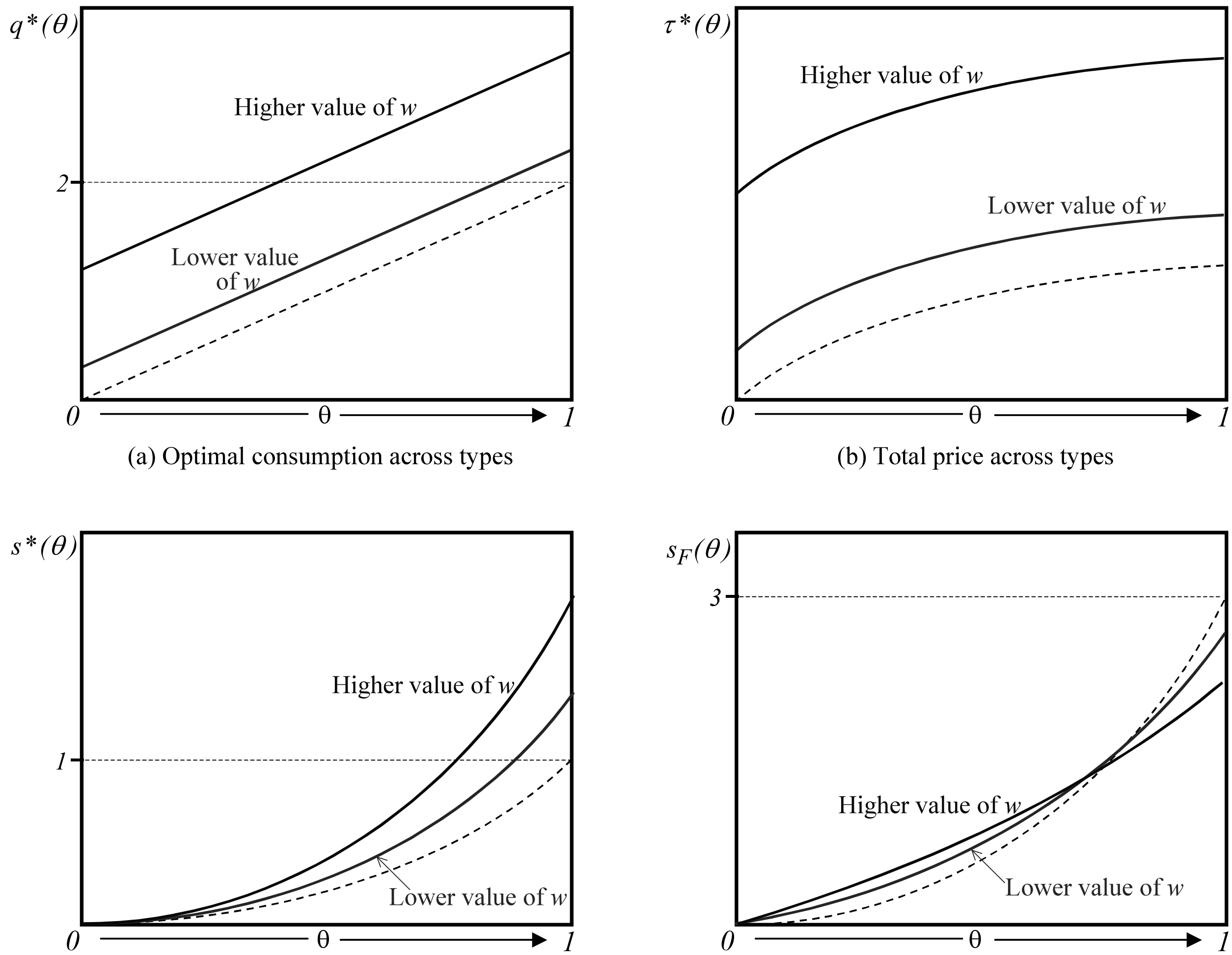
(a) Optimal consumption across types

(b) Total price across types

(c) Customer surplus across types

(d) Relative customer surplus across types

Figure B.1: Illustrate the optimal fulfilled-expectation contracts and corresponding customer surplus and relative surplus, in the example when network effects are homogeneous across types.



(a) Optimal consumption across types

(b) Total price across types

(c) Customer surplus across types

(d) Relative customer surplus across types

Figure B.2: Illustrates the optimal fulfilled-expectation contracts and corresponding customer surplus and relative surplus for *pure monopoly*, in the example when network effects are usage-dependent. In each figure, the dotted curve represents the base case (when network value is zero).

pricing function:

$$p(q) = \frac{w^2}{4(1-w)^2} + \frac{2-w}{2(1-w)}q - \frac{q^2}{4}. \tag{B.5}$$

The optimal pricing function is therefore a nonlinear two-part tariff, with a fixed component that increases with the marginal network value w, and a quantity discount that is progressively increasing. Moreover, differentiating (B.5) with respect to q indicates that $p_1(q) = \frac{2-w}{2(1-w)} - \frac{q}{2}$, which is strictly increasing in w for $w < 1$. Therefore, absolute prices at any level of consumption always increase with w.

As shown in Figure B.2 (c), an increase in w increases customer surplus for all customer types. What is particularly interesting is that as w increases, the relative distribution of surplus across customer types is less convex. This is illustrated in Figure 4.2 (d), and indicates that at higher levels of network effects, surplus is distributed more evenly across customers of different types. This

Entry-deterring monopoly

Intermediate variables:	$Q^{\alpha} = \frac{1}{w}$; $Q^{m} = \frac{1}{1-w}$; $\hat{\theta}(Q) = 1 - wQ$

When $w \geq \frac{1}{2}$

Optimal fulfilled-expectations contract:	$q^*(\theta) = 2\theta + \frac{w}{1-w}$, $\tau^*(\theta) = \frac{1}{2(1-w)^2} - (1-\theta)^2$
Surplus functions:	$s^*(\theta) = \theta(\theta + \frac{w}{1-w}) + \frac{1}{2}$, $s_F(\theta) = \frac{3((2\theta^2+1)(1-w)+2\theta w)}{5-2w}$

When $w \leq \frac{1}{2}$

Optimal fulfilled-expectations contract:	$\theta \leq \hat{\theta}(Q^*)$: $q^*(\theta) = \theta + 1$, $\tau^*(\theta) = wQ^*(1+\theta)$ $\theta \geq \hat{\theta}(Q^*)$: $q^*(\theta) = 2\theta + wQ^*$, $\tau^*(\theta) = 2wQ^* - (1-\theta)^2$
Surplus functions:	$\theta \leq \hat{\theta}(Q^*)$: $s^*(\theta) = \frac{(1+\theta)^2}{2}$, $s_F(\theta) = \frac{3(1+\theta)^2}{7+(wQ^*)^3}$ $\theta \geq \hat{\theta}(Q^*)$: $s^*(\theta) = \frac{(1+\theta)^2+(\theta-(1-wQ^*))^2}{2}$, $s_F(\theta) = \frac{3s^*(\theta)}{7+(wQ^*)^3}$

Note: When $w \leq \frac{1}{2}$, $Q^* = \frac{1-\sqrt{1-3w^2}}{w^2}$ *and* $\hat{\theta}(Q^*) = \frac{w-1+\sqrt{1-3w^2}}{w}$

Table B.3: Optimal contracts and surplus expressions in thet example with heterogeneous network effects, under entry-deterring monopoly

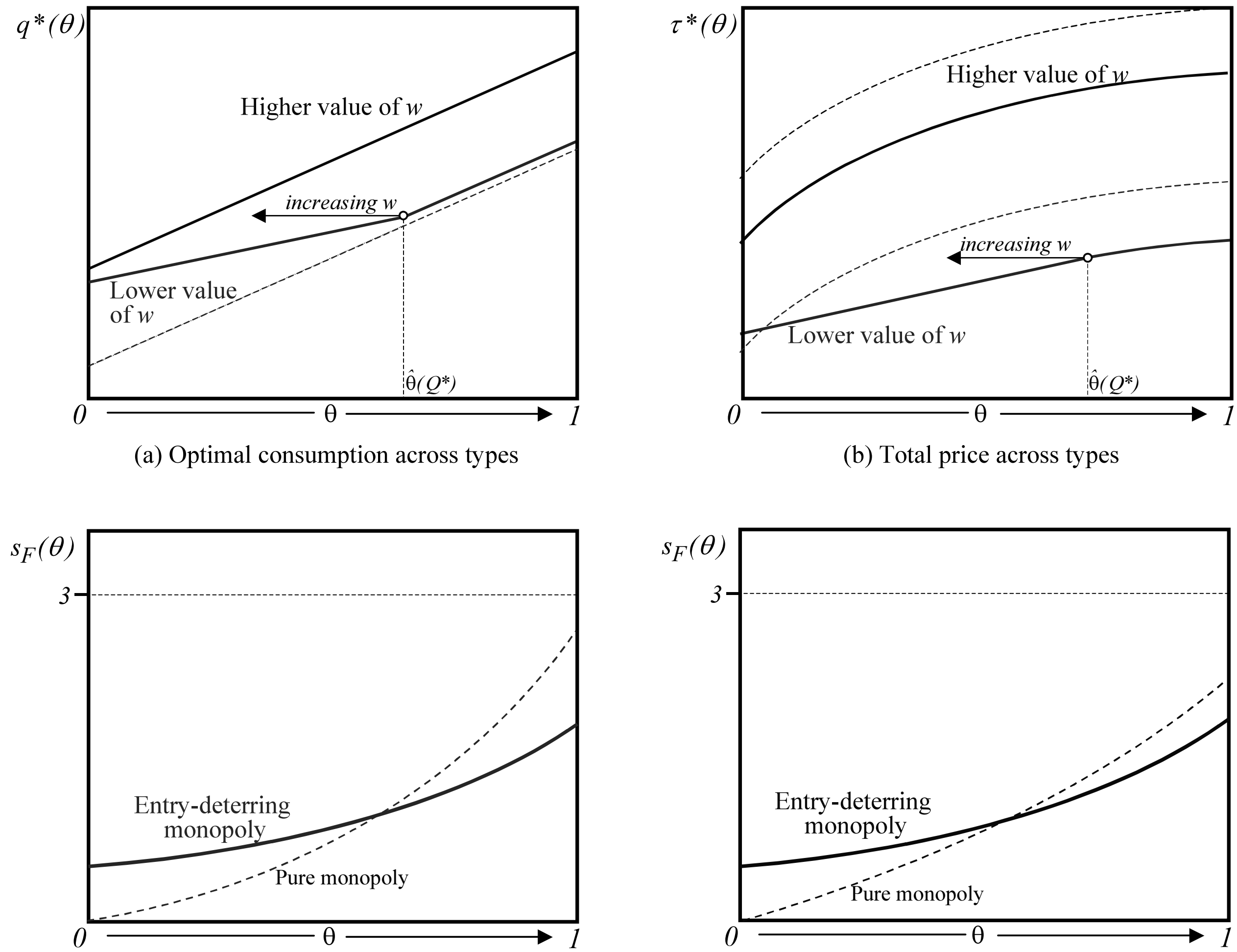
is a socially favorable result, because it suggests higher distributional equity of the value created, across customers who differ in their usage levels.

Under entry-deterring monopoly, equating the expressions for Q^{α} and Q^{m} indicate that part (a) of Proposition is applicable for $w \geq \frac{1}{2}$, and part (b) applies for $w \leq \frac{1}{2}$. This confirms that the entry threat induces changes in total surplus (via an induced change in optimal consumption) for *lower* levels of network effects, but not at higher levels.

As illustrated in Figure B.3 (a), as w increases, optimal consumption is raised (relative to the corresponding levels under pure monopoly) for an increasingly smaller fraction of customer types, and when $w \geq \frac{1}{2}$, consumption is unaltered for all types (though total prices reduce by a fixed amount across all types).

At fairly low values of w, total price may increase for a subset of lower types. This is because the changes in consumption are substantial for these lower customer types, relative to the case of pure monopoly. Average prices (per unit of consumption) always decrease with a threat of entry, across all types. Clearly, customer surplus also increases, across all types.

Figures B.3 (c) and (d) further highlight the socially desirable effect of a threat of entry that was noted in section 3.3 – the flattening of the relative distribution of surplus across types. This accentuates the increased distributional equity from increasing network effects that was illustrated in Figure 4.2(d). The former effect is more pronounced when network effects are lower. This is



(a) Optimal consumption across types

(b) Total price across types

(c) Relative customer surplus across types (lower *w*)

(d) Relative customer surplus across types (higher *w*)

Figure B.3: Illustrates the optimal fulfilled-expectation contracts and corresponding relative customer surplus for *entry-deterring monopoly*, in the example when network effects are usage-dependent. In each figure, the dotted curves represent the corresponding values in the case of *pure monopoly.*

not surprising, since the latter effect is more pronounced when network effects are higher (and as a consequence, there is already less inequity across customers to begin with).

C. Proof of Lemma 2

The argument for Lemma 2 mirrors the standard derivation of single-dimensional nonlinear pricing. It is presented for completeness, just so that it is clear that the dependence on an exogenously specified Q does not alter optimal consumption.

Given a expectation of gross consumption Q, any Q-feasible contract $q^F(t,Q), \tau^F(t,Q)$ satisfies [IC] if:

$$\theta = \arg\max_{t\in[\underline{\theta},\overline{\theta}]} W(q^F(t,Q),\theta,Q) - \tau^F(t,Q), \tag{C.1}$$

for all θ. The necessary and sufficient conditions for (C.1) are:

$$[W_1(q^F(\theta,Q),\theta,Q)]q_1^F(\theta,Q) - \tau_1^F(\theta,Q) = 0 \; \forall\theta; \tag{C.2}$$

$$[W_{11}(q^F(\theta,Q),\theta,Q)](q_1^F(\theta,Q))^2 + [W_1(q^F(\theta,Q),\theta,Q)]q_{11}^F(\theta,Q) - \tau_{11}^F(\theta,Q) \leq 0 \; \forall\theta. \tag{C.3}$$

Differentiating (C.2) with respect to θ and substituting (C.3) yields modified sufficient conditions:

$$[W_{12}(q^F(\theta,Q),\theta,Q)]q_1^F(\theta,Q) \geq 0 \; \forall\theta. \tag{C.4}$$

By assumption, $W_{12}(q,\theta,Q)$ is strictly positive, which means that (C.2) and (C.4) reduce to:

$$\tau_1^F(\theta,Q) = [W_1(q^F(\theta,Q),\theta,Q)]q_1^F(\theta,Q), \tag{C.5}$$

$$q_1^F(\theta,Q) \geq 0, \tag{C.6}$$

for all θ.

Now, under the contract $q^F(t,Q), \tau^F(t,Q)$, the surplus of type θ is

$$s(\theta) = W(q^F(\theta,Q),\theta,Q) - \tau^F(\theta,Q). \tag{C.7}$$

Differentiating (C.7) with respect to θ, and substituting (C.5) yields:

$$s_1(\theta) = W_2(q^F(\theta,Q),\theta,Q). \tag{C.8}$$

Since reservation utility $\hat{U}(\theta) = 0$ for all types, if IR is satisfied for the lowest type $\underline{\theta}$, it is satisfied for all others. Therefore, $s(\underline{\theta}) = 0$, and

$$s(\theta) = \int_{x=\underline{\theta}}^{\theta} W_2(q^F(x,Q),x,Q)dx. \tag{C.9}$$

Combining (C.7), and (C.9), the objective function whose maximizer is the optimal contract $q(\theta,Q), \tau(\theta,Q)$ can be written as:

$$\int_{\theta=\underline{\theta}}^{\overline{\theta}} [W(q^F(\theta,Q),\theta,Q) - \{\int_{x=\underline{\theta}}^{\theta} W_2(q^F(x,Q),x,Q)dx\}]f(\theta)d\theta. \tag{C.10}$$

Integrating the second part of (C.10) by parts and rearranging yields:

$$q(\theta,Q) = \arg\max_{q^F(\theta,Q)} \int_{\theta=\underline{\theta}}^{\overline{\theta}} [W(q^F(\theta,Q),\theta,Q) - W_2(q^F(\theta,Q),\theta,Q)\frac{1-F(\theta)}{f(\theta)}]f(\theta)d\theta, \tag{C.11}$$

subject to $q(\theta, Q) \geq 0$, and that

$$\tau(\theta, Q) = W(q(\theta, Q), \theta, Q) - \int_{x=\underline{\theta}}^{\theta} W_2(q(x, Q), x, Q)dx. \tag{C.12}$$

If the unconstrained problem has a unique solution for which $q(\theta, Q) \geq 0$, then this is the solution to the constrained problem as well.

Define

$$H(\theta) = \frac{1 - F(\theta)}{f(\theta)} \tag{C.13}$$

First-order conditions for the unconstrained problem are therefore:

$$W_1(q(\theta, Q), \theta, Q) = [W_{12}(q(\theta, Q), \theta, Q)]H(\theta) \; \forall \theta, \tag{C.14}$$

and are sufficient if the point-wise profit function:

$$\pi(q, \theta, Q) = W(q, \theta, Q) - [W_2(q, \theta, Q)]H(\theta) \tag{C.15}$$

is strictly concave in q. Differentiating (C.15) with respect to q twice yields:

$$\pi_{11}(q, \theta, Q) = W_{11}(q, \theta, Q) - [W_{112}(q, \theta, Q)]H(\theta), \tag{C.16}$$

which verifies that $\pi(q, \theta, Q)$ is strictly concave, since $W_{11} < 0$, and $W_{112} \geq 0$. This ensures that for the unconstrained problem, first-order conditions (C.14) yield the unique solution. These conditions can be rearranged as:

$$\frac{W_1(q(\theta, Q), \theta, Q)}{W_{12}(q(\theta, Q), \theta, Q)} = \frac{1 - F(\theta)}{f(\theta)}. \tag{C.17}$$

Now, differentiating both sides of (C.14) with respect to θ yields:

$$[W_{11}(q(\theta, Q), \theta, Q)]q_1(\theta, Q) + [W_{12}(q(\theta, Q), \theta, Q)] = \\ [W_{112}(q(\theta, Q), \theta, Q)]H(\theta)q_1(\theta, Q) + [W_{122}(q(\theta, Q), \theta, Q)]H(\theta) + [W_{12}(q(\theta, Q), \theta, Q)]H_1(\theta)$$

which implies that:

$$q_1(\theta, Q) = \frac{[W_{12}(q(\theta, Q), \theta, Q)][1 - H_1(\theta)] - [W_{122}(q(\theta, Q), \theta, Q)]H(\theta)}{[W_{112}(q(\theta, Q), \theta, Q)]H(\theta) - W_{11}(q(\theta, Q), \theta, Q)}. \tag{C.18}$$

Since $\pi(q, \theta, Q)$ has been shown to be strictly concave in q, the denominator of (C.18) is strictly positive. Also, the reciprocal of the hazard rate is non-increasing in θ, which implies that $H_1(\theta) \leq 0$. Therefore, so long as $W_{122}(q, \theta)$ is non-positive, the numerator of (C.18) is strictly positive, and $q_1(\theta, Q) > 0$.