Auctions with a Buy Price*

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Abstract

Internet auctions on eBay and Yahoo allow sellers to list their auctions with a *Buy-Now* option. In such auctions the seller sets a *buy price* at which a bidder may purchase the item immediately and end the auction. In the eBay version of a buy-now auction, the buy-now option disappears as soon as a bid is placed, while in the Yahoo version of the auction the buy-now option remains in effect throughout the auction. When bidders are risk neutral there is no advantage to introducing a buy price in either an eBay or a Yahoo auction. When bidders are risk averse, introducing a buy price raises seller revenue in both the eBay and the Yahoo auction for a wide range of buy prices. We show that while the Yahoo format raises more revenue than the eBay format (when the reserve and the buy price are the same in both auctions), the auctions are utility equivalent from the bidders' perspective.

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1 Introduction

The expansion of commerce conducted over the Internet has sparked a surge of interest in auctions and new auction forms. Many online auction sites appeared, and quite a few subsequently disappeared. These online auctions used a variety of formats and rules. In many cases online auctions adapted procedures that had been used for running auctions long before the Internet came into being. In a few cases online auctions introduced features which appear to be new and unique to the online environment. Lucking-Reiley (2000) describes the wide variety of online auction formats that were being used as of 1999.

An example of a new twist in online auction formats appears in Yahoo and eBay auctions. In 1999 Yahoo introduced the *Buy-Now* feature into its ascending bid auctions. The Buy-Now feature allows the seller to set a price, termed a buy price, at which any bidder may purchase the item at any time during the auction.[1] Since the buy price remains in effect throughout the auction, this feature allows the seller to post a maximum price for the item. In 2000 eBay introduced its own version of a fixed price feature into its online auctions via the *Buy It Now* option. In contrast to the Yahoo format, eBay permits bidders to select the buy price only at the opening of the auction, before any bids are submitted, or in the case of an auction with a (secret) reserve, before bids reach the reserve price.[2] We use the expression "buy-now auction" as the generic term for an ascending bid auction with a buy price.

The buy-now auction format has become quite popular at both eBay and Yahoo. Table 1 lists the total number of auctions and the number of buy-now auctions on eBay and Yahoo in selected categories on a recent day (the categories are similar, but not identical across the two auction sites). Overall, about 40% of these eBay auctions and 66% of these Yahoo auctions utilize the buy-now feature. Hof (2001) also cites a

[1] For more on this format see, http://auctions.yahoo.com/phtml/auc/us/promo/buynow.html

[2] See, http://pages.ebay.com/services/buyandsell/buyitnow.html

40% figure for the fraction of eBay auctions that use the buy-now feature.

	Yahoo Auctions		eBay Auctions	
Category	**# buy-now**	**total #**	**# buy-now**	**total #**
Automobiles	157	186	662	2,378
Clothing	148	223	2,678	10,078
DVD Players	86	115	981	2,251
VCR's	91	136	384	1,078
Digital Cameras	337	583	7,179	13,284
TV's	23	39	596	2,073

Table 1: Buy-Now and Total Auctions for Yahoo and eBay
(for auctions on March 27, 2002)

Mathews (2003a) reports that the buy price is taken in about 27% of eBay buy-now auctions of electronic games.

At first glance the prevalence of buy-now auctions is puzzling. After all, an ascending bid auction is intended to elicit high bids from potential buyers. Putting a cap on these bids (as in a Yahoo buy-now auction) or offering a fixed price at the auction open (as in an eBay buy-now auction) would seem to limit the seller's expected revenue. But there are at least two reasons why a buy-now auction might yield greater revenue for the seller. First, offering a buy price may reduce the riskiness of the auction for bidders. Consider a bidder whose value exceeds the buy price. If the bidder does not accept the buy price then he may win the item at a price less than the buy price, but he runs the risk of losing the auction (this would happen if another bidder accepts the buy price, or if he is outbid in the ascending auction). A risk-averse bidder would be willing to pay a buy price which includes a risk premium, rather than face uncertainty regarding whether he wins and how much he pays. Second, with the buy-now format the seller might be able to exploit impatient bidders who are willing to pay a premium included in the buy price in order to end the auction early.

In this paper we analyze the eBay and Yahoo buy-now auctions, focusing on the effects of bidder risk aversion rather than on bidder impatience. We utilize a symmetric independent private values framework with a continuous distribution of values for the n bidders. Bidders are either risk neutral or CARA risk averse. In both auction formats, in the first stage the bidders simultaneously choose whether to accept the buy price or wait. If a bidder accepts the buy price, then the auction ends and he pays the buy price. In the eBay auction, if the bidders wait then the buy-now option disappears and, in the ascending bid auction that follows, the bidder with the highest value wins the item (if his value exceeds the reserve); he pays the maximum of the second highest value and the reserve. In the Yahoo auction, if the bidders wait then there is an ascending clock auction with the bid starting at the reserve; as the clock progresses, at any point a bidder may either remain in the auction, drop out of the auction, or accept the buy price. When all bidders except one drop out, then the remaining bidder wins the auction and pays the current bid. If any bidder accepts the buy price then he wins the item and pays the buy price. In the Yahoo auction the buy price is a price cap since the buy-now option remains in effect throughout the auction, while in the eBay auction the final price may exceed the buy price since the buy-now option disappears after bidding begins.

In section 3 we consider the eBay buy-now auction. Equilibrium strategies in the eBay auction are characterized by a "cutoff" value; a bidder accepts the buy price at the auction open if their value exceeds the cutoff and waits otherwise. We characterize the equilibrium cutoff, and show that the set of types that accept the buy price is decreasing in the buy price, increasing in the reserve price, and increasing as bidders become more risk averse. We also show that when bidders are risk averse then introducing a buy price into an eBay auction raises seller revenue for a wide range of buy prices. For any given buy price, seller revenue is increasing as the bidders become more risk averse, so long as the buy price is not so high that it is never accepted by any bidder type.

In section 4 we consider the Yahoo buy-now auction. In the Yahoo auction equilibrium is characterized by threshold strategies which specify the bid price (or, clock time) at which a bidder will accept the buy price, as a function of his value. We show that the Yahoo buy-now auction is utility equivalent to the eBay buy-now auction for bidders, when the buy price and the reserve are the same in both auctions. The set of bidder types that accepts the buy-price immediately is also the same in both auctions. However, the probability the buy price is taken is higher in the Yahoo than the eBay auction since it is accepted with positive probability in the ascending bid phase of the auction. Section 4 provides a characterization of equilibrium threshold strategies.

We show that introducing a buy price into a Yahoo auction raises seller revenue for a wide range of buy prices. In particular, for any buy price which is not accepted immediately at the auction open, the Yahoo buy-now auction raises more revenue than a Yahoo auction without a buy price when bidders are risk averse and the reserve is the same in both auctions. (If bidders are risk neutral, then the two auctions are revenue equivalent.) Moreover, for any given buy price, seller revenue in the Yahoo buy-now auction is increasing as bidders become more risk averse.

While the eBay and Yahoo buy-now auctions are utility equivalent from the bidders' perspective, in Section 5 we show that if bidders are risk averse then seller revenue is higher in the Yahoo auction. If bidders are risk neutral then the two auction formats are revenue equivalent for the seller. We also provide a numerical example illustrating the similarities and differences of the two buy-now auction formats.

Related Literature

There is a small but growing literature on auctions with a buy price. Mathews (2002) models eBay buy-now auctions with risk-neutral bidders. He shows that a risk-averse seller can raise his expected utility by setting a buy price. Mathews (2003b) explores the role of impatience in eBay buy-now auctions, in a model that allows for either impatient bidders or an impatient seller. He shows that a seller can increase his expected revenue by setting a buy price, thereby exploiting impatient bidders who are willing to pay a premium included in the buy price in order to end the auction early.

Kirkegaard and Overgaard (2003) suggest another rationale for setting a buy price. Their model has two risk-neutral bidders, each of whom demands two units. Two sellers sequentially offer a unit for sale in second-price auctions. Kirkegaard and Overgaard show that if sellers are able to offer the good at a buy price prior to conducting the second-price auction (as in the eBay buy-now auction), then the seller in the first auction can raise his expected revenue by setting a buy price.

Our model of eBay buy-now auctions is most closely related to Mathews (2002). The main difference is that Mathews focuses on a risk-averse seller facing risk-neutral bidders, whereas we focus on a risk-neutral seller facing risk-averse bidders. Our set up is more general, allowing for a general distribution of bidder values and allowing for a reserve price.[3]

Budish and Takeyama (2001) analyze a simple version of a Yahoo buy-now auction with two bidders and two possible valuations for each bidder – high or low. They demonstrate that when bidders are risk averse there is a buy price for which bidders with the high-value accept immediately, bidders with the low-value wait, and which yields more expected revenue to the seller than the ascending bid auction without a buy price.[4] Lopomo (1998), in a model with a general distribution for bidder values (which may be either independent or affiliated), studies a class of auctions he refers to as "simple sequential auctions." This class includes the English ascending bid auction as well as other auctions, like the Yahoo buy-now auction, in which the item auctioned is available to bidders at a constant ask price throughout the course of the auction. Lopomo shows that if bidders are risk neutral, then the English auction is optimal within the class of all simple sequential auctions. Thus, when bidders are risk neutral, a Yahoo buy-now auction cannot yield more revenue for the seller than the English ascending bid auction.

Our analysis of the Yahoo buy-now auction is considerably more general than Budish and Takayama's. We utilize a model with n bidders with independent private values, drawn from a general continuous distribution, and which allows for a reserve price. Our model of the Yahoo auction yields insights into the relationship between a bidder's value and the threshold at which he accepts the buy price in the ascending bid phase of the auction. In contrast to Budish and Takeyama, we find that the buy-now auction raises seller revenue even if the buy price is not accepted immediately by any bidder type. In contrast to Lopomo, we deal with bidder risk aversion in the Yahoo auction.[5] We show that with bidder risk aversion, introducing a buy price in the Yahoo auction raises seller revenue for a wide range of buy prices (and the

[3]Mathews (2002) assumes that bidders' values are uniformly distributed and does not deal with reserve prices.

[4]In Budish and Takeyama the highest buy price for which it is an equilibrium for high-value bidders to accept immediately and low-value bidders wait is denoted by B^*. They claim that this is the seller's optimal buy price, without considering the revenue consequences of a buy price above B^*.

[5]Lopomo utilizes a framework in which the bid price is raised by discrete increments over a sequence of trading periods, as opposed to our model with a continuously rising bid price.

Yahoo buy-now auction raises more revenue than the English auction). In addition we provide a characterization of the equilibrium strategies in Yahoo auctions, whether bidders are risk averse or risk neutral. Our analysis differs also differs from prior work in that we provide an integrated analysis of the eBay and Yahoo buy-now auctions. This allows us to compare the two auctions in terms of bidder payoffs and seller expected revenue.

Several recent papers have studied other novel features of online auctions. Roth and Ockenfels (2002) is an empirical analysis of the effect the auction closing rule – "hard" on eBay and "soft" on Amazon – has on bidding behavior and auction outcomes. Ariely, Ockenfels, and Roth (2002) provide a theoretical and experimental comparison of the effects of a hard and soft close. Peters and Severinov (2002) analyze the equilibrium strategies of sellers and bidders when many auctions are conducted simultaneously, as is often the case for online auction websites. Bajari and Hortascu (2003) document empirical regularities in a sample of eBay coin auctions and estimate a structural model of bidding on eBay.

2 The Model

There are $n \geq 2$ bidders for a single item whose values are independently and identically distributed according to cumulative distribution function F with support $[\underline{v}, \bar{v}]$, where F' is continuous and positive on $(\underline{v}, \bar{v})$. Let $G(v) = F(v)^{n-1}$ be the *c.d.f.* of the highest of $n-1$ values. Denote by v_i the value of bidder i. Let B denote the buy price set by the seller. We assume that $\underline{v} < B < \bar{v}$, since a seller would never wish to set $B \leq \underline{v}$, whereas if $B \geq \bar{v}$ then no bidder will ever take it. Denote by $r \in [\underline{v}, B)$ the minimum bid, or reserve, set by the seller.[6] If a bidder whose value is v wins the item and pays price p then his payoff is $u(v-p)$; he obtains a payoff of zero otherwise. Bidders have constant absolute risk aversion, with $u(x) = (1 - e^{-\alpha x})/\alpha$ for $\alpha \geq 0$. Note $\lim_{\alpha \to 0} u(x) = x$, and hence $\alpha = 0$ corresponds to risk neutrality.

[6] If $r = \underline{v}$ then there is effectively no reserve, while if $r = B$ then the eBay and Yahoo buy-now auctions are both equivalent to a posted price of B.

eBay

At the open of the eBay buy-now auction the bidders simultaneously decide whether to "buy" or "wait." If some bidder chooses to buy, then he wins and he pays the seller B. (If more than one bidder chooses buy, then the winner is randomly assigned among these bidders.) The bidding process that follows if all the bidders wait is not explicitly modeled. Instead, we suppose that if all bidders wait, then the buy-now option disappears, and (i) if at least one bidder has a value of r or greater, then the bidder with the highest value wins the item and he pays the maximum of r and the second highest value, and (ii) if all bidders have a value less than r then the minimum bid is not met and item does not sell.[7] This is consistent with there being, for example, either an ascending clock auction (with the bid price starting at the minimum bid r) or a second-price sealed-bid auction, when all the bidders wait. The eBay auction without a buy price is equivalent to an English ascending bid auction, with the bidder with the highest value winning at a price equal to the maximum of the reserve and the second highest value.

In the eBay buy-now auction a bidder's strategy tells him, for each possible value, whether to buy or wait. We focus on equilibria in "cutoff strategies." A **cutoff strategy** for a bidder is characterized by a value $c \in [B, \bar{v}]$ such that he chooses buy if his value exceeds c and chooses wait if his value is below c. Suppose that a bidder's value is $v > r$ and all his rivals employ the same cutoff c. The bidder's expected payoff if he chooses buy is

$$U^b(v,c) = u(v-B)\sum_{k=0}^{n-1}\binom{n-1}{k}\frac{1}{k+1}(1-F(c))^k F(c)^{n-1-k},$$

where in this expression k is the number of other bidders who also choose to buy. If the bidder waits, then he wins the auction only if all his rivals also wait and he has the highest value. His expected payoff is

$$U^w(v,c) = \int_r^{\min\{v,c\}} u(v-y)dG(y) + u(v-r)G(r).$$

[7]In eBay auctions with a minimum bid (or reserve), the buy-now option disappears as soon as a bid is placed. Although we don't deal with them here, eBay also allows sellers to set a "secret" reserve. In auctions with a secret reserve, the buy-now option remains active until a bid is placed that exceeds the secret reserve.

A cutoff c^* is a **symmetric equilibrium** if $U^w(v, c^*) > U^b(v, c^*)$ for all $v \in [\underline{v}, c^*)$ and $U^w(v, c^*) < U^b(v, c^*)$ for all $v \in (c^*, \bar{v}]$. That is, given that his rivals use the cutoff c^* then it is optimal for a bidder to wait if $v < c^*$ and it is optimal for a bidder to buy if $v > c^*$.

Yahoo buy-now Auctions

We model the Yahoo buy-now auction as an ascending clock auction, in which the bid rises continuously from r to B. As the clock progresses, at any point a bidder may either drop out of the auction or may accept the buy price. The auction ends when either all but one bidder has dropped out, or when a bidder accepts the buy price. In the former case, the remaining bidder wins and pays the current bid price. In the later case, the bidder who accepts the buy price wins and he pays B. (If all bidders drop at the bid of r then the auction ends without a winner.) As the clock progresses, bidders observe only the current bid, and not the number of remaining bidders. (A Yahoo auction without a buy price is equivalent to an English ascending auction.)

Clearly a bidder whose value is less than B never accepts the buy price since by doing so he obtains a negative payoff, whereas he would obtain a payoff of zero by dropping out. We assume that such bidders simply drop out when the bid reaches their value, with bidders whose values are below r dropping out immediately. Similarly, a bidder whose value is above B never drops out since whatever the current bid is, he obtains a positive payoff accepting the buy price but would obtain zero by dropping out. Thus we focus on how bidders whose values are above B choose the bid at which to accept the buy price. A strategy for a bidder is a function which gives for each value v in $[B, \bar{v}]$ a threshold bid price $t(v)$ (in $[r, B]$) at which the bidder accepts the buy price. A function $t : [B, \bar{v}] \to [r, B]$ is a **threshold strategy** if either (i) t is continuous and strictly decreasing on $[B, \bar{v}]$, or (ii) there is a $z \in (B, \bar{v})$ such that t is continuous and strictly decreasing on $[B, z]$, and t jumps down to $t(v) = r$ for $v \in (z, \bar{v}]$. If $t(v) = r$ a bidder with value v accepts the buy price at the auction open.

To see how the bidders' payoffs are determined given a profile of threshold strategies, it is useful to consider Figure 1 which shows the maximum of the other bidders' values (denoted by y) on the horizontal axis. Consider bidder 1 and suppose all the other bidders follow the threshold strategy $t(v)$. Let $[\underline{t}, \bar{t}]$ denote the range of threshold values for which $t(v)$ is strictly decreasing. (For $t(v)$ as in the figure, $\bar{t} = B$ and $\underline{t} = t^{-1}(\bar{v})$.) Suppose bidder 1 chooses the threshold $\tilde{t}$ (shown on the vertical axis). If y is less than r, then all the other bidders drop out at r, bidder 1 wins and he pays r. If y is between r and $\tilde{t}$, then bidder 1 is the last remaining bidder when the bid reaches y, bidder 1 wins and he pays y, the price at which the last of his rivals dropped out. If y is above $\tilde{t}$ but below $t^{-1}(\tilde{t})$, then bidder 1 accepts the buy price when the bid reaches $\tilde{t}$, he wins the auction, and he pays B. Finally, if y is above $t^{-1}(\tilde{t})$, then the bidder with value y accepts the buy price when the bid reaches $t(y)$, he wins the auction, and he pays B.

Hence, if a bidder's value is $v > r$, he chooses the threshold $\tilde{t}$, and the other bidders follow the threshold strategy t (one without a jump down) then the bidder's expected utility is

$$U(\tilde{t}, v; t) = \begin{cases} G(r)u(v-r) + \int_r^{\tilde{t}} u(v-y)dG(y) + [G(t^{-1}(\tilde{t})) - G(\tilde{t})]u(v-B) & \text{if } \tilde{t} \in [\underline{t}, \bar{t}] \\ \\ G(r)u(v-r) + \int_r^{\tilde{t}} u(v-y)dG(y) + [1 - G(\tilde{t})]u(v-B) & \text{if } \tilde{t} < \underline{t}. \end{cases}$$

Note that if $\tilde{t} < \underline{t}$ then the bidder wins for sure, paying r if the maximum value of a rival is less than r, paying the maximum of his rivals' values when this maximum is less than $\tilde{t}$, and paying B otherwise.

We say that t is a **(symmetric) equilibrium in threshold strategies** if for each $v \in [B, \bar{v}]$ we have

$$U(t(v), v; t) \geq U(\tilde{t}, v; t) \quad \forall \tilde{t} \in [\underline{v}, B].$$

In other words, for each value v a bidder's optimal threshold is $t(v)$ when the other bidder follows the threshold strategy t.

While these models capture salient features of buy-now auctions as they are implemented on eBay and Yahoo, there are some differences. eBay auctions end at a predetermined time specified by the seller (i.e., they have a "hard" close). Ariely, Roth, and Ockenfels (2002) show that a hard close, combined with uncertain processing of bids placed in the last minutes of an auction, can lead to a final price less than the second highest value. eBay and Yahoo auctions are not conducted as ascending clock auctions, but on eBay bidders submit "proxy" bids and on Yahoo bidders may bid either a fixed amount or may make proxy bids.[8] Last, here we have supposed that there is a fixed commonly known number of bidders, a condition that is unlikely to prevail in actual Internet auctions.

3 eBay buy-now auctions

In this section we compare eBay auctions with and without buy prices. In characterizing equilibrium of the eBay buy-now auction with reserve r it is useful to first consider an eBay auction with the same reserve, but without a buy price. Consider a bidder whose value is v and who is either risk neutral or CARA risk averse with index of risk aversion $\alpha \geq 0$. If the bidder wins in the auction with no buy-price, he makes a (random) payment of $\max\{r, y\}$, where y denotes the maximum of his rivals' values. The certainty equivalent payment, denote by $\delta_\alpha(v)$, is defined by

$$u(v - \delta_\alpha(v)) = E\left[u(v - \max\{r, y\})|\underline{v} \leq y \leq v\right], \tag{1}$$

where y is distributed according to G. In other words, a bidder with value v is indifferent between winning the auction (and making a random payment of $\max\{r, y\}$) and winning and paying the certain amount $\delta_\alpha(v)$. When bidders are risk neutral (i.e., $\alpha = 0$) then equation (1) reduces to

$$\delta_0(v) = E\left[\max\{r, y\}|\underline{v} \leq y \leq v\right].$$

To simplify notation we suppress the dependence of $\delta_\alpha(v)$ on the reserve price r and the distribution G.

The certainty equivalent payment $\delta_\alpha(v)$ has several important properties: $\delta_\alpha(r) = r$, $\delta_\alpha(v)$ is increasing in v, and $\delta_\alpha(v) < v$ for $v > r$. Furthermore, $\delta_\alpha(v)$ is increasing in α for $v > r$, i.e., as a bidder becomes more risk averse he is willing to pay more to avoid the uncertain payment of the auction.

Proposition 1 characterizes equilibrium in the eBay buy-now auction for risk-neutral and risk averse bidders.

[8] See http://pages.ebay.com/help/buy/proxy-bidding.html for a description of proxy bidding.

Proposition 1: *Suppose bidders are risk neutral* ($\alpha = 0$) *or CARA risk averse with index of risk aversion* $\alpha > 0$. *Consider an eBay auction with reserve price* r *and buy price* B.
(i) If $B \geq \delta_\alpha(\bar{v})$ *then the buy price is never accepted by a bidder in equilibrium, i.e., the unique symmetric equilibrium cutoff value is* $c^* = \bar{v}$.
(ii) If $B < \delta_\alpha(\bar{v})$ *then there is a unique symmetric equilibrium cutoff* $c^* \in (B, \bar{v})$ *that is implicitly defined by*

$$u(c^* - B)Q(F(c^*)) = u(c^* - \delta_\alpha(c^*))G(c^*),$$

where

$$Q(F(c^*)) = \left[\frac{1 - F(c^*)^n}{n(1 - F(c^*))}\right].$$

This cutoff value is increasing in B, *decreasing in* r, *and decreasing in* α. *The equilibrium is inefficient since the high-value bidder is awarded the item with probability less than one.*
Proof: Appendix.

Proposition 1(i) is intuitive. The certainty equivalent of the payment made by a bidder with value $\bar{v}$ is $\delta_\alpha(\bar{v})$, if he and all his rivals reject the buy price. Hence, if the buy price B exceeds $\delta_\alpha(\bar{v})$, then such a bidder prefers to reject the buy price when all his rivals also reject it. When a bidder with the highest value optimally rejects the buy price, then bidders with lower values optimally reject as well. Hence it is an equilibrium for all bidders to reject the buy price when $B \geq \delta_\alpha(\bar{v})$.

If $B < \delta_\alpha(\bar{v})$ then it is no longer an equilibrium for all bidders to reject the buy price. In particular, a bidder with value $\bar{v}$ would optimally accept the buy price if his rivals always rejected it. In equilibrium, a bidder with value c^* (defined implicitly in Proposition 1(ii)) is just indifferent between accepting the buy price and rejecting it, when his rivals follow the strategy of accepting the buy price if their value is above c^* and rejecting it otherwise. The buy-now auction is inefficient when the buy price is set low enough so that some bidder types accept it. The inefficiency is similar to the inefficiency that results when a single item is offered for sale at a fixed price to multiple buyers. If there is no mechanism to put the high-value buyer at the head of the queue of buyers, then there is a positive probability that the high-value buyer will not receive the item.

Proposition 1(ii) also establishes some intuitive comparative statics when $B < \delta_\alpha(\bar{v})$. Bidders are less likely to accept the buy price as the buy price increases (since c^* is increasing in B). Bidders are more likely to accept the buy price as the reserve price increases or as bidders become more risk averse. The increased willingness of more risk averse bidders to accept the buy price follows from two effects: First, acceptance of the buy price reduces the chance that a bidder will "lose", i.e., not be awarded the object, and have a zero surplus. (Acceptance of the buy price does not completely eliminate uncertainty for a bidder, because there is a chance another bidder will also accept; the object is randomly awarded in this case.)[9] A risk averse bidder is willing to trade off this reduced chance of losing (and receiving the lowest possible surplus, zero) for a lower expected surplus. Second, conditional on winning the auction, a more risk averse bidder is willing to make a higher certain payment in order to avoid the random payment of he would make if he won in the ascending bid phase of the auction (i.e., $\delta_\alpha(v)$ is increasing in α).

Seller Revenue

Myerson (1981) shows that when bidders are risk neutral, a first or second-price sealed-bid auction or an English ascending bid auction are each revenue-maximizing mechanisms, provided that the reserve price is set optimally.[10] Hence, when bidders are risk neutral ($\alpha = 0$), there is no advantage to a (risk-neutral) seller to setting a buy price. Indeed, taking the reserve price as given, it's easy to see from Myerson's characterization of the optimal mechanism that, in order to maximize seller revenue, the object must be awarded to the bidder with the highest value (provided that this value exceeds the reserve). In an eBay buy-now auction with $B < \delta_0(\bar{v})$, with positive probability the object is not awarded to the bidder with the highest value. Thus, in this case, the eBay buy-now auction raises strictly less revenue than the eBay auction without a buy price and the same reserve.

[9] Bidders with values $v > c^*$ accept the buy price and win with probability $Q(F(c^*))$. If such a bidder were to instead wait, then he wins in the ascending bid auction that follows with probability $F(c^*)^{n-1}$, i.e., he wins so long as none of his rivals accepts the buy price. In the proof of Proposition 1 it is established that $Q(F(c^*)) > F(c^*)^{n-1}$.

[10] This optimality result requires the regularity assumption that $J(v) = v - (1 - F(v))/F'(v)$ is increasing in v. The optimal reserve price satisfies $J(r) = 0$. See Burguet (2000) for a nice discussion of this result.

Bidder risk aversion has no effect on seller revenue in an eBay auction without a buy price. However, if bidders are risk averse then setting a buy price can be advantageous for the seller. Consider any buy price B which is accepted with positive probability by bidders with index of risk aversion $\alpha > 0$, but which would be rejected if bidders were risk-neutral (that is, consider any B satisfying $\delta_0(\bar{v}) < B < \delta_\alpha(\bar{v})$). Let c^*_α denote the equilibrium cutoff. (Since $B < \delta_\alpha(\bar{v})$ then $c^*_\alpha < \bar{v}$.) An auction with such a buy price and with reserve r raises more revenue than an eBay auction with the same reserve and no buy price. To see this, let bidder 1's value v_1 be fixed and suppose, without loss of generality, that y (the maximum of his rival's values) is less than v_1. If $v_1 < c^*_\alpha$ then the buy price is not accepted by any bidder and seller revenue is $\max\{r, y\}$, the same as in the auction without a buy price. If $v_1 > c^*_\alpha$ then seller revenue is B in the buy-now auction and is $\max\{r, y\}$ in the auction without a buy price. Now, B may be either more or less than $\max\{r, y\}$. However, B is greater than the expectation of $\max\{r, y\}$ since

$$E[\max\{r, y\}|\underline{v} \leq y \leq v_1] = \delta_0(v_1) \leq \delta_0(\bar{v}) < B,$$

where the equality holds by the definition of $\delta_0(v_1)$, the weak inequality holds since $\delta_0(v)$ is increasing in v, and the strict inequality holds by assumption. We have shown that the seller's expected revenue conditional on bidder 1 winning is (i) the same whether or not the seller sets a buy price if $v_1 < c^*_\alpha$, and is (ii) higher in the auction with the buy price if $v_1 > c^*_\alpha$. Since $c^*_\alpha < \bar{v}$, then $v_1 > c^*$ with positive probability, and hence the seller's *ex-ante* expected revenue is higher in the auction with the buy price.

The following corollary summarizes these results.

Corollary 1: *Suppose bidders are CARA risk averse with index of risk aversion* $\alpha > 0$. *Consider an eBay buy-now auction with reserve price* r *and buy price* B. *If* $\delta_0(\bar{v}) < B < \delta_\alpha(\bar{v})$ *then expected seller revenue in the buy-now auction exceeds expected revenue in the eBay auction with the same reserve and no buy price.*

Corollary 1 suggests why eBay introduced the buy-now auction format, and why it has proven to be so popular with sellers – this format has the potential for raising seller revenue and eBay's own auction revenue (which is a percentage of seller revenue) relative to standard eBay auctions.

Corollary 1, while it doesn't identify the optimal buy price, does show the seller how to set a buy price that raises revenue. Given the uncertainty a seller is likely to face regarding the degree of bidder risk aversion and the distribution of bidders' values, providing a range of revenue-improving buy prices may be of more practical use than providing conditions characterizing an optimal buy price.

One can see that for any *given* buy price, seller revenue increases as bidders become more risk averse. Suppose bidders have index of risk aversion α', but the index of risk aversion increases to α''. Consider a buy-now auction with reserve r and buy price B, where $B < \delta_{\alpha''}(\bar{v})$. Let $c^*_{\alpha'}$ and $c^*_{\alpha''}$ denote the equilibrium cutoff for $\alpha = \alpha'$ and $\alpha = \alpha''$, respectively. By Proposition 1(ii) we have $c^*_{\alpha''} < c^*_{\alpha'}$. Let bidder 1's value v_1 be fixed and suppose $v_1 > y$. If $v_1 < c^*_{\alpha''}$ then seller revenue is $\max\{r, y\}$, whether bidders have risk aversion index α' or α''; if $v_1 > c^*_{\alpha'}$ then seller revenue is B whether $\alpha = \alpha'$ or $\alpha = \alpha''$. If $c^*_{\alpha''} < v_1 < c^*_{\alpha'}$ then revenue is B if $\alpha = \alpha''$, whereas if $\alpha = \alpha'$ then expected revenue is

$$E[\max\{r, y\} | \underline{v} \leq y \leq v_1] = \delta_0(v_1) < \delta_{\alpha'}(c^*_{\alpha'}) \leq B,$$

where the first inequality holds since $\delta_\alpha(v)$ is increasing in α and in v. If $c^*_{\alpha'} = \bar{v}$ then by Proposition 1(ii) we have $B \geq \delta_{\alpha'}(\bar{v})$ and the second equality holds immediately. If $c^*_{\alpha'} < \bar{v}$ then $B < \delta_{\alpha'}(\bar{v})$ and

$$u(c^*_{\alpha'} - B)Q(F(c^*_{\alpha'})) = u(c^*_{\alpha'} - \delta_{\alpha'}(c^*_{\alpha'}))G(c^*_{\alpha'}).$$

Since $Q(F(c^*_{\alpha'})) > G(c^*_{\alpha'})$ then $\delta_{\alpha'}(c^*_{\alpha'}) < B$ and, again, the second inequality holds. Hence, seller revenue increases as bidders become more risk averse.

Corollary 2: *Consider an eBay buy-now auction with reserve price r and buy price B. If the index of bidder risk aversion increases from α' to α'' then seller revenue strictly increases unless $B \geq \delta_{\alpha''}(\bar{v})$, i.e., unless the buy price is always rejected even when bidders have the higher index of risk aversion α''.*

The key difference between the eBay and Yahoo buy-now auctions is the temporary nature of the buy price in the eBay auction. To understand incentives in the Yahoo auction it is useful to consider the incentives of a bidder in the eBay auction if he had the (hypothetical) option to accept the buy price once the bid begins to ascend. A bidder who waits and observes the bid price begin to rise above the reserve learns (i) $y < c^*$, i.e., the "good" news no rival have a value above c^*, and (ii) $y > r$, i.e., the "bad" news that at least one rival has a value above r. Hence, a bidder with value $v < c^*$ who waits has, once the ascending bid phase of the eBay auction begins, an expected utility of

$$E[u(v-y)|r < y \leq v]\frac{G(v) - G(r)}{G(c^*) - G(r)},$$

whereas if he could accept the buy price his utility would be $u(v - B)$. We say that there is **no regret** in the buy-now auction if

$$E[u(c^* - y)|r < y \leq c^*] > u(c^* - B). \qquad \text{(No Regret)}$$

This condition states that an eBay bidder with value c^* has a higher expected utility once the bid begins to ascend than he would obtain were he able to accept the buy price. Hence, he would not accept the buy price even if it were available. The no regret condition implies that a bidder with a value $v < c^*$ would also not accept the buy price (if it were available) once the ascending bid phase of the auction begins.

Intuitively, no regret will be satisfied if (i) either the buy price is high, or (ii) not too much bad news is revealed when a bidder doesn't win at the reserve price r; in other words, the probability that $y \in [\underline{v}, r]$ is small. Remark 1 formalizes this idea. No regret will tend to fail as the buy-now auction comes to resemble a posted price mechanism, with the reserve and the buy price close.

Remark 1: *The no regret condition holds if either (i) the buy price is sufficiently high, or (ii) the reserve is sufficiently small or there is no reserve, i.e., if* $r = \underline{v}$.
Proof: Appendix.

As we shall see, the satisfaction of the no regret condition plays an important role in the existence of an equilibrium in threshold strategies in the Yahoo buy-now auction.

4 Yahoo buy-now auctions

In this section we compare the Yahoo buy-now auction to the Yahoo auction without a buy price. We begin by establishing that the eBay and Yahoo buy-now auctions are utility equivalent for bidders. This result will be exploited in order to characterize equilibrium in the Yahoo buy-now auction.

Proposition 2: *Assume bidders are risk neutral* ($\alpha = 0$) *or CARA risk averse with index of risk aversion* $\alpha > 0$. *Consider an eBay auction and a Yahoo auction where the reserve price is* r *and the buy price is* B *for both auctions. Let* t *be an equilibrium threshold function of the Yahoo auction (which is differentiable except possibly at one point* z *where it jumps down), and let* c^* *be the equilibrium cutoff in the eBay auction (see Prop. 1).*
(i) The Yahoo and eBay auctions are utility equivalent for the bidders, i.e., a bidder whose value is v *obtains the same expected utility in the Yahoo auction as in the eBay auction.*
(ii) If $B \geq \delta_\alpha(\bar{v})$ *then the equilibrium threshold strategy* t *has no jump down. If* $B < \delta_\alpha(\bar{v})$ *then* t *jumps down at* c^**, with* $t(v) > r$ *if* $v \leq c^*$ *and* $t(v) = r$ *if* $v > c^*$.
Proof: Appendix.

Proposition 2(i) shows that bidders are indifferent between the eBay and Yahoo buy-now auctions when the reserve and buy prices are the same in both auctions. For sufficiently high buy prices (i.e. $B \geq \delta_\alpha(\bar{v})$), bidders are indifferent between the eBay buy-now auction and the English ascending bid auction.[11] This implies that bidder utility is constant in the Yahoo buy-now auction even as the buy price decreases, so long as it remains above $\delta_\alpha(\bar{v})$. Matthews (1987) shows that CARA bidders are indifferent between the English ascending bid auction and the first-price sealed-bid auction. These results imply that bidders are indifferent between all four auction formats – eBay buy-now, Yahoo buy-now, English ascending bid, and first-price sealed-bid – when in each case the reserve is the same and the buy price is high.

Propositions 1 and 2(ii) shows that the set of values for which a bidder accepts the buy price immediately is the same for the Yahoo and eBay auctions. If $B < \delta_\alpha(\bar{v})$ then bidders in both types of auctions will accept the buy price immediately if their value is above c^*, but wait otherwise. If $B \geq \delta_\alpha(\bar{v})$ then the buy price is not *immediately*

[11] If $B > \delta_\alpha(\bar{v})$ then by Prop. 1(i) the buy price is not accepted by any bidder in the eBay buy-now auction, and hence the auction is equivalent to the English ascending bid auction.

accepted in either auction. In the Yahoo auction, however, the buy price is accepted with higher total probability since it is accepted with positive probability in the ascending bid phase of the auction.

Even though the two auction formats are utility equivalent, the *ex-post* outcomes are generally different. In the eBay auction if all the bidders wait, then the final price is the second highest value (which may be more or less than B). In the Yahoo auction the final price never exceeds B.

Proposition 3: *Assume bidders are CARA risk averse with index of risk aversion $\alpha \geq 0$. Consider a Yahoo auction with reserve price r and buy price B such that "no regret" holds. There is a unique symmetric equilibrium $t(v)$ in threshold strategies that are differentiable (except possibly at one point where the threshold strategy jumps down).*
(i) If $B \geq \delta_\alpha(\bar{v})$ then $t(v)$ is defined implicitly by

$$E[u(v-y)|t(v) \leq y \leq v] = u(v-B), \tag{2}$$

for $v \in [B, \bar{v}]$, or equivalently

$$\int_{t(v)}^{v} (e^{\alpha y} - e^{\alpha B}) dG(y) = 0.$$

(ii) If $B < \delta_\alpha(\bar{v})$ then $t(v)$ is as given above for $v \in [B, c^*]$ and $t(v) = r$ for $v \in (c^*, \bar{v}]$.
Proof: Appendix.

Proposition 3 is established by exploiting the utility equivalence for bidders of the eBay and Yahoo auction. According to equation (2), the equilibrium threshold $t(v)$ has a natural economic interpretation: $t(v)$ makes the buy price B equal to the certainty equivalent of the random payment a bidder would make in an English ascending auction in which the maximum of his rivals' values is known to be between $t(v)$ and v. Equation (2) can be used to easily numerically calculate the equilibrium threshold function.

Seller Revenue

When bidders are risk neutral the Yahoo buy-now auction is revenue equivalent to the Yahoo auction without a buy price, so long as the buy price is high enough that it is not accepted immediately. To see this, assume $B \geq \delta_0(\bar{v})$. Let bidder 1's value $v_1 > r$ be fixed and suppose that bidder 1 has the highest value. If $v_1 < B$, then seller revenue is $\max\{r, y\}$ in a Yahoo auction, with or without the buy price. Similarly, if $v_1 \geq B$ and $y < t(v_1)$ seller revenue is still $\max\{r, y\}$ in both auctions. If $v_1 \geq B$ and $y \geq t(v_1)$ then seller revenue is B in the Yahoo buy-now auction. In the Yahoo auction without a buy price expected revenue is $E[\max\{r, y\}|t(v_1) \leq y \leq v_1]$, which equals $E[y|t(v_1) \leq y \leq v_1]$ since $t(v_1) \geq r$. When bidders are risk neutral then equation (2) reduces to

$$E[y|t(v_1) \leq y \leq v_1] = B,$$

and hence the two auctions are revenue equivalent. Revenue equivalence holds even though the buy price is accepted with positive probability in the ascending bid phase of the buy-now auction. If $B < \delta_0(\bar{v})$, then by Proposition 3(ii) the buy price is accepted immediately with positive probability. For the reasons discussed for the eBay auction with risk-neutral bidders, in this case the introduction of a buy price lowers seller revenue.

Hence, we have the following Corollary.

Corollary 3: *Suppose bidders are risk neutral. Consider a Yahoo buy-now auction with reserve price r and buy price B. If $B \geq \delta_0(\bar{v})$ then expected seller revenue in the Yahoo buy now auction is the same as in the Yahoo auction with the same reserve and no buy price. If $B < \delta_0(\bar{v})$ then expected seller revenue is lower in the Yahoo buy-now auction.*

Intuition would suggest that bidders are quicker to accept the buy price as they are more risk averse. Corollary 4 shows this is indeed the case; bidders in a Yahoo buy-now auction choose lower thresholds when they are more risk averse.

Corollary 4: *The equilibrium threshold function shifts down as bidders become more risk averse. In particular, let $\alpha'' > \alpha' \geq 0$ and let t_α be the equilibrium threshold function of a Yahoo auction with buy price B and reserve r when bidders are CARA risk averse with index of risk aversion α. Then for $v > B$ we have $t_{\alpha''}(v) < t_{\alpha'}(v)$ if $t_{\alpha'}(v) > r$ and $t_{\alpha''}(v) = r$ if $t_{\alpha'}(v) = r$.*

Proof: Appendix.

An immediate consequence of Corollary 4 is that for any given buy price, seller revenue increases as bidders become more risk averse. Figure 2 shows a shift down in the equilibrium threshold function (from $t_{\alpha'}$ to $t_{\alpha''}$) when the index of risk aversion increases from α' to α''. Let bidder 1's value v_1 be fixed and suppose that v_1 exceeds the maximum value y of his rival. For combinations (v_1, y) below $t_{\alpha''}(v_1)$, seller revenue is y whether bidders have index of risk aversion α' or α''. For (v_1, y) above $t_{\alpha'}(v_1)$, seller revenue is B in each case. For (v_1, y) that lie between the two threshold functions, seller revenue is B when bidders are more risk averse (i.e., $\alpha = \alpha''$), and seller revenue is y, where $y < B$, when bidders are less risk averse (i.e., $\alpha = \alpha'$).

Figure 2 goes here.

Hence we have the following Corollary.

Corollary 5: *Consider a Yahoo buy-now auction with reserve r and buy price B. If the index of risk aversion increases from α' to α'' then seller revenue strictly increases.*

By Corollary 3, when bidders are risk neutral and $B \geq \delta_0(\bar{v})$ then the Yahoo buy-now auction yields the same revenue as the auction without a buy price. By Corollary 5 revenue in the buy-now auction increases as bidder become more risk averse. Since revenue in the Yahoo auction without a buy price doesn't depend on risk attitudes, we have the following result.

Corollary 6: *Suppose bidders are CARA risk averse with index of risk aversion $\alpha > 0$. Consider a Yahoo buy-now auction with reserve price r and buy price B, where $B \geq \delta_0(\bar{v})$. Then expected seller revenue in the buy-now auction exceeds expected revenue in the Yahoo auction with the same reserve and no buy price.*

Corollary 6 shows that when bidders are risk averse, introducing a buy price into a Yahoo auction raises revenue for a wide range of buy prices. Corollary's 2 and 5 point out another difference between the eBay and Yahoo buy-now auction. In the eBay auction seller revenue is constant as bidders become more risk averse as long as the buy price is so high that it is not accepted; in the Yahoo buy-now auction seller revenue increases as bidders become more risk averse, for any B less than $\bar{v}$.

5 Comparing eBay and Yahoo

The eBay and Yahoo auctions can be revenue ranked, when the buy price and reserve are the same in both auctions. Suppose bidders are CARA risk averse with index of risk aversion $\alpha > 0$. Suppose also that bidder 1 has the highest value v_1. If $v_1 > c^*$ then seller revenue is B in both the eBay and Yahoo buy-now auctions. If $v_1 \leq B$, then seller revenue is $\max\{r, y\}$ in both auctions, where y is the second-highest value. If $v_1 \in (B, c^*)$ and $y < t(v_1)$, then seller revenue is also $\max\{r, y\}$ in both auctions. Revenue differs between the auctions when $v_1 \in (B, c^*)$ and $t(v_1) \leq y \leq v_1$. In this case, revenue is B in the Yahoo auction and is y in the eBay auction. Expected revenue in the eBay auction is $E[y|t(v_1) \leq y \leq v_1]$. If bidders are risk neutral, then equation (2) reduces to

$$E[y|t(v_1) \leq y \leq v_1] = B,$$

i.e., the eBay and Yahoo auction are revenue equivalent. If bidders are risk averse then since u is concave we have

$$E[u(v_1 - y)|t(v_1) \leq y \leq v_1] < u(E[v_1 - y|t(v_1) \leq y \leq v_1]).$$

This inequality and equation (2) imply that

$$E[v_1 - y|t(v_1) \leq y \leq v_1] > v_1 - B,$$

or $E[y|t(v_1) \leq y \leq v_1] < B$, i.e., expected revenue in the eBay auction is less than expected revenue in the Yahoo auction. Thus, we have established the following Corollary.

Corollary 7: *Consider an eBay auction and a Yahoo auction where the reserve price is r and the buy price is B for both auctions. If bidders are risk neutral then the seller's expected revenue is the same for both the eBay and Yahoo auction. If bidders are CARA risk averse, then the Yahoo auction raises more revenue than the eBay auction.*

As noted earlier, the probability that the buy price is accepted immediately is the same in the eBay and Yahoo buy now auctions. Since the buy price is accepted with positive probability in the ascending bid phase of the Yahoo auction, another testable implication of the model is that the buy price is accepted with higher overall probability in the Yahoo auction. Finally, the set of buy prices we have identified as revenue enhancing (when bidders have index of risk aversion $\alpha > 0$) is larger in the Yahoo auction: in the Yahoo auction any buy price $B \geq \delta_0(\bar{v})$ raises revenue, while in the eBay auction we have shown that any buy price B satisfying $\delta_0(\bar{v}) < B < \delta_\alpha(\bar{v})$ raises revenue.

An Example

We conclude this section with an example which illustrates some of our results. Consider a market in which there are two bidders, with each bidder's value independently distributed $U[.45, 1]$. Given this distribution of values, the seller's optimal reserve in a second-price sealed-bid auction is .5. This is also the optimal reserve in an eBay or a Yahoo auction *without* a buy price, whether bidders are risk neutral or risk averse, since it is a dominant strategy for a bidder in either auction to remain active until the bid reaches his value. Set the reserve price to be $r = .5$. For these parameters, when bidders are risk neutral then $\delta_0(\bar{v}) = .727$; when bidders are risk averse with index of risk aversion $\alpha = 10$ then $\delta_{10}(\bar{v}) = .829$.[12]

Table 2 characterizes equilibrium in the eBay auction when $\alpha = 0$ and $\alpha = 10$, respectively, for several different buy prices. The columns of these tables give (i) the equilibrium cutoff c^*, (ii) the seller's expected revenue, and (iii) the probability that the item is sold at the buy price conditional on a sale.

	$\alpha = 0$			$\alpha = 10$		
Buy Price	Equil. Cutoff c^*	Expected Revenue	Prob. the price is B given a sale	Equil. Cutoff c^*	Expected Revenue	Prob. the price is B given a sale
$B = .80$	1.0	0.634	0	0.938	0.652	0.21
$B = .75$	1.0	0.634	0	0.850	0.661	0.47
$B = .70$	0.9	0.632	0.33	0.771	0.649	0.66

Table 2: Equilibrium in the eBay buy-now auction, $r = .5$.

[12] An index of absolute risk aversion equal to 10 implies an index of relative risk aversion equal to $10x$, where x is the monetary payoff for a bidder. For example, if the buy price is $B = 0.75$ then a bidder with a value equal to the cut-off $c^* = .85$ has a monetary payoff of 0.10 from accepting the buy price. This bidder has a index of relative risk aversion of one. This figure is slightly higher than, but similar to, estimates of relative risk aversion cited in Holt and Laury (2002).

With an optimally chosen reserve, the seller in an eBay auction without a buy price has an expected revenue of .634. Table 2 shows that when $\alpha = 10$, the seller's expected revenue is higher in the eBay buy-now auction for a wide range of buy prices (Corollary 1).

Table 3 characterizes equilibrium in the Yahoo buy-now auction when $\alpha = 0$ and $\alpha = 10$, respectively. (See Reynolds and Wooders (2003) for a closed-formed expression for the equilibrium threshold function for the two bidder case.)

	$\alpha = 0$			$\alpha = 10$		
Buy Price	Equil. Cutoff c^*	Expected Revenue	Prob. the price is B given a sale	Equil. Cutoff c^*	Expected Revenue	Prob. the price is B given a sale
$B = .80$	1.0	0.634	0.27	0.938	0.657	0.37
$B = .75$	1.0	0.634	0.42	0.850	0.662	0.55
$B = .70$	0.9	0.632	0.60	0.771	0.650	0.70

Table 3: Equilibrium in the Yahoo buy-now auction, $r = .5$.

Table 3 illustrates that so long as $B \geq \delta_0(\bar{v})$ then revenue in the Yahoo buy-now auction with risk-neutral bidders is the same as in the Yahoo auction without a buy price (Corollary 3), even though the buy price is accepted with positive probability. Comparing expected revenue when $\alpha = 0$ and $\alpha = 10$ illustrates that as bidders become more risk averse, holding the reserve and buy price fixed, seller revenue in the Yahoo buy-now auction rises (Corollary 5 and 6).

Comparing Tables 2 and 3 illustrates that when bidders are risk neutral the eBay and Yahoo buy-now auctions give the seller the same expected revenue when the reserve and buy price is the same in both auctions. The probability of a sale at the buy price is, however, higher in the Yahoo auction. When bidders are risk averse, then the Yahoo buy-now auction raises more revenue for the seller than the eBay buy-now auction (Corollary 7), although in this example the difference in revenues is small.

6 Conclusion

We have formulated models that capture key features of auctions with buy prices, as implemented on the eBay and Yahoo auction sites. The eBay and Yahoo versions of the buy-now auction differ in the timing of the buy price option; in eBay the buy price is available to bidders only at the beginning of the auction, whereas in Yahoo the buy price is available throughout the auction. We characterized equilibrium strategies for risk neutral and risk averse bidders in buy-now auctions, using an independent private values framework. When bidders are risk neutral, the eBay and Yahoo buy-now auctions yield the same expected seller revenue, given that the buy price and the reserve are the same in the two auctions. These two buy-now auctions also yield the same expected seller revenue as the ascending bid auction, given the same reserve, if the buy price is high enough that it is not accepted at the beginning of the auction (for the eBay buy-now auction, this means that the buy price is never accepted). If the buy price is accepted with positive probability at the beginning of the auction, then the buy-now auctions yield lower expected seller revenue than the ascending bid auction with the same reserve.

An auction with a buy price is a simple mechanism that permits a seller to earn more expected revenue than an ascending bid auction with a reserve, when bidders are risk averse. We focus on the case of bidders with constant absolute risk aversion. Given CARA bidders and a particular reserve, we show that there are wide ranges of reserve prices for eBay and Yahoo buy-now auctions that yield higher expected revenue for the seller than the ascending bid auction. CARA bidders are indifferent between an eBay and Yahoo buy-now auction, if the auctions have the same buy price and the same reserve. However, the seller is not indifferent. Given CARA bidders, a seller earns higher expected revenue in the Yahoo buy-now auction than in the eBay buy-now auction.

There are a number of ways in which this analysis might be extended. Non-CARA risk aversion for bidders could be introduced into the model. One could introduce sequential (possibly random) entry of bidders into the auction or bidder impatience. (Mathews (2003a,b) addresses these issues in eBay buy-now auctions with risk-neutral bidders.) Within the independent private values framework one could consider an uncertain numbers of bidders. It would also be interesting to consider a common or affiliated values setting, since some goods auctioned on the Internet surely have a common value component (see Bajari and Hortascu (2003)).

7 Appendix

Lemma 1: *The certainty equivalent* $\delta_\alpha(v)$ *in* (1) *satisfies*

$$\delta_0(v) = E[\max\{r, y\}|y \leq v],$$

if bidders are risk neutral (i.e., $\alpha = 0$*), and it satisfies*

$$e^{\alpha\delta_\alpha(v)} = \frac{1}{G(v)}\left[G(r)e^{\alpha r} + \int_r^v e^{\alpha y} dG(y)\right], \tag{3}$$

if bidders are CARA risk averse with index of risk aversion $\alpha > 0$*.*

Proof: We prove the lemma only for the case where bidders are risk averse. By the definition of $\delta_\alpha(v)$ we have

$$\frac{1 - e^{-\alpha(v-\delta_\alpha(v))}}{\alpha} = \frac{1}{G(v)}\left[\frac{1 - e^{-\alpha(v-r)}}{\alpha}G(r) + \int_r^v \frac{1 - e^{-\alpha(v-y)}}{\alpha} dG(y)\right],$$

or

$$e^{-\alpha(v-\delta_\alpha(v))} = \frac{1}{G(v)}\left[e^{-\alpha(v-r)}G(r) + \int_r^v e^{-\alpha(v-y)} dG(y)\right].$$

Multiplying both sides by $e^{\alpha v}$ yields the result. □

Lemma 2: *Whether bidders are risk neutral or risk averse, for* $v > r$ *we have*

$$\delta'_\alpha(v) = \frac{G'(v)}{G(v)}\frac{u(v - \delta_\alpha(v))}{u'(v - \delta_\alpha(v))} > 0. \tag{4}$$

Proof: When $\alpha > 0$, then differentiating (3) with respect to v yields

$$\alpha\delta'_\alpha(v)e^{\alpha\delta_\alpha(v)} = \frac{1}{G(v)}e^{\alpha v}G'(v) - \frac{G'(v)}{G(v)^2}\left[G(r)e^{\alpha r} + \int_r^v e^{\alpha y} dG(y)\right].$$

Using (3) again and simplifying yields

$$\delta'_\alpha(v) = \frac{G'(v)}{G(v)}\frac{1 - e^{-\alpha(v-\delta_\alpha(v))}}{\alpha e^{-\alpha(v-\delta_\alpha(v))}},$$

which is (4). Since $v > \delta_\alpha(v)$ for $v > r$, then $\delta'_\alpha(v) > 0$. A symmetric argument establishes the result when $\alpha = 0$. □

Proof of Proposition 1: Consider a bidder with value $v \geq B$, and suppose all his rivals use the cutoff strategy c. (If $c = \bar{v}$ then all rival bidders reject the buy price for all of their values.) If the bidder accepts the buy price, then his expected utility is

$$U^b(v,c) = u(v-B)\sum_{k=0}^{n-1}\binom{n-1}{k}\frac{1}{k+1}(1-F(c))^k F(c)^{n-1-k} \tag{5}$$

$$= u(v-B)Q(F(c)),$$

where for $x \in [0,1)$ we define

$$Q(x) = \frac{1-x^n}{n(1-x)},$$

and where we define $Q(1) = \lim_{x\to 1} Q(x) = 1$. Note that Q is continuously differentiable on $[0,1]$ with $Q'(x) > 0$, and $Q(0) = \frac{1}{n}$. Furthermore, $Q(x) > x^{n-1}$ for $x \in [0,1)$ implies $Q(F(c)) > G(c)$ for $c < \bar{v}$. If the bidder waits, then his expected utility is

$$U^w(v,c) = G(r)u(v-r) + \int_r^{\min\{v,c\}} u(v-y)dG(y). \tag{6}$$

Using the certainty equivalent payment $\delta_\alpha(v)$, we can rewrite $U^w(v,c)$ as

$$U^w(v,c) = u(v - \delta_\alpha(\min\{v,c\}))G(\min\{v,c\}).$$

We prove Prop. 1(ii) first. Assume $B \in (r, \delta_\alpha(\bar{v}))$. A necessary condition for c to be an equilibrium cutoff is

$$U^b(c,c) = U^w(c,c).$$

Rewriting, we obtain

$$u(c-B)Q(F(c)) = u(c-\delta_\alpha(c))G(c), \tag{7}$$

which is the condition given in Prop. 1(ii). We now show there is a unique c satisfying (7). Define $\hat{U}^b(c) = u(c-B)Q(F(c))$ and $\hat{U}^w(c) = u(c-\delta_\alpha(c))F(c)^{n-1}$. We have $\hat{U}^b(B) = 0$, and $B > r$ implies $\hat{U}^w(B) > 0$. Also, $B < \delta_\alpha(\bar{v})$ implies $\hat{U}^b(\bar{v}) = u(\bar{v}-B) > u(\bar{v}-\delta_\alpha(\bar{v})) = \hat{U}^w(\bar{v})$. Therefore, since $\hat{U}^b(c)$ and $\hat{U}^w(c)$ are both continuous, there is some $c \in (B, \bar{v})$ such that $\hat{U}^b(c) = \hat{U}^w(c)$.

We now show that there is a unique such c. We have

$$\frac{d\hat{U}^b(c)}{dc} = Q'(F(c))F'(c)u(c-B) + Q(F(c))u'(c-B)$$

and

$$\begin{aligned}\frac{d\hat{U}^w(c)}{dc} &= G(c)u'(c-\delta_\alpha(c))(1-\delta'_\alpha(c)) + G'(c)u(c-\delta_\alpha(c)) \\ &= G(c)u'(c-\delta_\alpha(c)),\end{aligned}$$

where the second equality follows from Lemma 2. If $\hat{U}^b(c) = \hat{U}^w(c)$ for some $c \in (B, \bar{v})$, i.e., c satisfies (7), then $Q(F(c)) > G(c)$ implies $u(c-B) < u(c-\delta_\alpha(c))$, and hence $u'(c-B) \geq u'(c-\delta_\alpha(c))$ since u is concave. Thus, $Q(F(c))u'(c-B) > G(c)u'(c-\delta_\alpha(c))$ which, together with $Q'(F(c))F'(c)u(c-B) > 0$, implies $\frac{d\hat{U}^b(c)}{dc} > \frac{d\hat{U}^w(c)}{dc}$. We have shown that if $\hat{U}^b(c)$ and $\hat{U}^w(c)$ cross at c, then $\hat{U}^b$ is steeper than $\hat{U}^w$ at c. This establishes there is a unique c at which $\hat{U}^b$ and $\hat{U}^w$ cross.

Next we show that the solution to $\hat{U}^b(c) = \hat{U}^w(c)$ is (a) increasing in B, (b) decreasing in r, and (c) decreasing in α. Since $\hat{U}^b(c)$ is steeper than $\hat{U}^w(c)$ where they cross and since $\hat{U}^b(c)$ shifts down as B increases while $\hat{U}^w(c)$ remains fixed, the solution is increasing in B. As r increases $\hat{U}^b(c)$ remains fixed, while $\hat{U}^w(c)$ shifts down since $\delta_\alpha(c)$ is increasing in r. Hence the solution to $\hat{U}^b(c) = \hat{U}^w(c)$ is decreasing in r.

To establish (c) it is useful to explicitly express the dependence of $\hat{U}^b(c)$ and $\hat{U}^w(c)$ on α, writing $\hat{U}^b_\alpha(c)$ and $\hat{U}^w_\alpha(c)$. Suppose α increases from α' to α''. Denote by c' the solution to $\hat{U}^b_{\alpha'}(c) = \hat{U}^w_{\alpha'}(c)$ and denote by c'' the solution to $\hat{U}^b_{\alpha''}(c) = \hat{U}^w_{\alpha''}(c)$. We have $\hat{U}^b_{\alpha''}(B) = 0 < \hat{U}^w_{\alpha''}(B)$. To establish $c' > c''$ we need to show that $\hat{U}^b_{\alpha''}(c') > \hat{U}^w_{\alpha''}(c')$, which then implies $c'' \in (B, c')$, which is (c). We have $\hat{U}^b_{\alpha'}(c') = \hat{U}^w_{\alpha'}(c')$, i.e.,

$$\frac{1-e^{-\alpha'(c'-B)}}{\alpha'}Q(F(c')) = \frac{1-e^{-\alpha'(c'-\delta_{\alpha'}(c'))}}{\alpha'}G(c'). \tag{8}$$

Since $Q(F(c')) > G(c')$ then $c'-B < c'-\delta_{\alpha'}(c')$. One can show, for x and y fixed and $x < y$, that

$$\frac{1-e^{-\alpha x}}{1-e^{-\alpha y}}$$

is increasing in α. Hence, choosing $x = c'-B$ and $y = c'-\delta_{\alpha'}(c')$ this implies

$$\frac{1-e^{-\alpha''(c'-B)}}{1-e^{-\alpha''(c'-\delta_{\alpha'}(c'))}} > \frac{1-e^{-\alpha'(c'-B)}}{1-e^{-\alpha'(c'-\delta_{\alpha'}(c'))}} = \frac{G(c')}{Q(F(c'))},$$

where the equality holds by (8). Hence

$$\frac{1-e^{-\alpha''(c'-B)}}{\alpha''}Q(F(c')) > \frac{1-e^{-\alpha''(c'-\delta_{\alpha'}(c'))}}{\alpha''}Q(F(c')).$$

Since $\delta_\alpha(c)$ is increasing in α, then

$$\hat{U}^b_{\alpha''}(c') = \frac{1-e^{-\alpha''(c'-B)}}{\alpha''}Q(F(c')) > \frac{1-e^{-\alpha''(c'-\delta_{\alpha''}(c'))}}{\alpha''}Q(F(c')) = \hat{U}^w_{\alpha''}(c'),$$

which establishes (c).

We now show that the cutoff c satisfying (7) is an equilibrium cutoff. We establish that $U^b(v,c) < U^w(v,c)$ for $v \in [B,c)$ and $U^b(v,c) > U^w(v,c)$ for $v \in (c,\bar{v}]$ by showing that $U^b(v,c)$ has a greater slope than $U^w(v,c)$. We have

$$\frac{\partial U^b(v,c)}{\partial v} \equiv U^b_v(v,c) = Q(F(c))u'(v-B). \tag{9}$$

For $v < c$ we have

$$\begin{aligned}\frac{\partial U^w(v,c)}{\partial v} &\equiv U^w_v(v,c) = G'(v)u(v-\delta_\alpha(v)) + G(v)u'(v-\delta_\alpha(v))(1-\delta'_\alpha(v)) \\ &= G(v)u'(v-\delta_\alpha(v)),\end{aligned}$$

where the second equality follows from Lemma 2. For $v > c$ we have

$$U^w_v(v,c) = G(c)u'(v-\delta_\alpha(c)).$$

Case (i): $v < c$. Since $Q(F(c)) > G(c)$ then $B > \delta_\alpha(c)$ by (7). Further, since $\delta_\alpha(v)$ is increasing in v, we have $B > \delta_\alpha(v)$ for all $v < c$. Therefore $u'(v-B) \geq u'(v-\delta_\alpha(v))$ for all $v \in [B,c]$. Thus,

$$U^b_v(v,c) = u'(v-B)Q(F(c)) > u'(v-\delta_\alpha(v))G(v) = U^w_v(v,c) \tag{10}$$

since $Q(F(c)) > G(c) \geq G(v)$. Equation (10) and $U^b(c,c) = U^w(c,c)$ imply that $U^b(v,c) < U^w(v,c)$ for $v < c$.

Case (ii): $v > c$. Since $Q(F(c)) > G(c)$ then $B > \delta_\alpha(c)$. The concavity of u implies $u'(v-B) \geq u'(v-\delta_\alpha(c))$ for all $v > c$. Thus

$$U^b_v(v,c) = Q(F(c))u'(v-B) > G(c)u'(v-\delta_\alpha(c)) = U^w_v(v,c).$$

Therefore, $U^b(v,c) > U^w(v,c)$ for $v > c$. This establishes that c satisfying (7) is an equilibrium cutoff.

We now prove Prop. 1(i). Assume that $B \geq \delta_\alpha(\bar{v})$. We first show that there is no equilibrium cutoff with $c < \bar{v}$. An equilibrium cutoff of $c = B$ implies that $U^w(v,B) < U^b(v,B)$ for all $v \in (B,\bar{v}]$. As v approaches B, however, $U^b(v,B) = (v-B)Q(F(B))$ approaches zero, while $U^w(v,B) = u(v-\delta_\alpha(B))G(B)$ is strictly positive, which contradicts that $c = B$ is an equilibrium cutoff. Suppose there is an equilibrium cutoff c, with $B < c < \bar{v}$; let c be the largest such cutoff. Earlier it was shown that $\hat{U}^b(v)$ is steeper than $\hat{U}^w(v)$ at $v = c$, where the two functions cross. Hence $\hat{U}^b(v) > \hat{U}^w(v)$ $\forall v \in (c,\bar{v})$ and for $v = \bar{v}$ either (i) $\hat{U}^b(\bar{v}) > \hat{U}^w(\bar{v})$, or (ii) $\hat{U}^b(\bar{v}) = \hat{U}^w(\bar{v})$. Since $B \geq \delta_\alpha(\bar{v})$ then

$$\hat{U}^w(\bar{v}) = u(\bar{v} - \delta_\alpha(\bar{v})) \geq u(\bar{v} - B) = \hat{U}^b(\bar{v}),$$

which contradicts (i). If $\hat{U}^b(\bar{v}) = \hat{U}^w(\bar{v})$ then $\hat{U}^b(v)$ is steeper than $\hat{U}^w(v)$ at $v = \bar{v}$, which contradicts $\hat{U}^b(v) > \hat{U}^w(v)$ $\forall v \in (c,\bar{v})$.

We now show that $c = \bar{v}$ is an equilibrium cutoff. Since $B \geq \delta_\alpha(\bar{v})$ then

$$U^w(\bar{v},\bar{v}) = u(\bar{v} - \delta_\alpha(\bar{v})) \geq u(\bar{v} - B) = U^b(\bar{v},\bar{v}).$$

Further, $F(\bar{v}) = 1$ implies $U^w_v(v,\bar{v}) = G(v)u'(v - \delta_\alpha(v))$ and $U^b_v(v,\bar{v}) = u'(v-B)$. For $v < \bar{v}$ we have

$$u'(v-B) \geq u'(v - \delta_\alpha(v)) > G(v)u'(v - \delta_\alpha(v)),$$

where the weak inequality follows from $B \geq \delta_\alpha(\bar{v}) > \delta_\alpha(v)$ and the strict inequality follows from $G(v) < 1$. This establishes that $U^b_v(v,\bar{v})$ is steeper than $U^w_v(v,\bar{v})$ for $v < \bar{v}$. Since $U^w(\bar{v},\bar{v}) \geq U^b(\bar{v},\bar{v})$ then $U^w(v,\bar{v}) \geq U^b(v,\bar{v})$ $\forall v < \bar{v}$, which establishes the result. $\square$

Proof of Remark 1: We first show that no regret holds if $r = \underline{v}$. There are two cases to consider, $B > \delta_\alpha(\bar{v})$ and $B < \delta_\alpha(\bar{v})$. Suppose $B > \delta_\alpha(\bar{v})$. Then $c^* = \bar{v}$ by Proposition 1. The no regret condition holds since

$$E[u(\bar{v} - y)|r < y \leq \bar{v}] = E\left[u(\bar{v} - \max\{r,y\})|\underline{v} < y \leq \bar{v}\right] = u(\bar{v} - \delta_\alpha(\bar{v})) > u(\bar{v} - B).$$

If $B < \delta_\alpha(\bar{v})$, then by Proposition 1(ii) c^* satisfies

$$u(c^* - B)Q(F(c^*)) = u(c^* - \delta_\alpha(c^*))G(c^*).$$

Since $Q(F(c^*)) > G(c^*)$ then $u(c^* - B) < u(c^* - \delta_\alpha(c^*))$. Hence no regret holds as

$$E[u(c^* - y)|r < y \leq c^*] = E\left[u(c^* - \max\{r, y\})|\underline{v} < y \leq c^*\right] = u(c^* - \delta_\alpha(c^*)) > u(c^* - B).$$

In either case, if $r > \underline{v}$ then

$$E\left[u(c^* - \max\{r, y\})|r < y \leq c^*\right] < E\left[u(c^* - \max\{r, y\})|\underline{v} \leq y \leq c^*\right].$$

However, no regret will continue to hold if r is sufficiently close to $\underline{v}$.

No regret will also hold if B is sufficiently large, for given r. Suppose $B > \delta_\alpha(\bar{v})$. Then $c^* = \bar{v}$ and no regret is

$$E[u(\bar{v} - y)|r < y \leq \bar{v}] > u(\bar{v} - B).$$

Since $E[u(\bar{v} - y)|r < y \leq \bar{v}] > 0$, this inequality clearly holds for B sufficiently close to $\bar{v}$. $\square$

Proof of Proposition 2: Let $t(v)$ be an equilibrium in threshold strategies, where t is differentiable except, possibly, at one point $z \in (B, \bar{v})$ where t jumps down. Denote by c^* the equilibrium cutoff of the eBay auction. Utility equivalence of the eBay and Yahoo buy-now auction clearly holds for a bidders whose value is less than the buy price. To establish utility equivalence we need to show that (a) $U(t(v), v; t) = U^w(v, z)$ $\forall\ v \in [B, z]$, (b) $U(r, v; t) = U^b(v, z)$ $\forall\ v \in [z, \bar{v}]$, and (c) if $t(v)$ jumps down at z, then $z = c^*$. Proving (a)-(c) establishes Prop. 2(i).

Proof of (a): We show that for a bidder with value $v \in [B, z]$, his expected payoff in an eBay and Yahoo buy-now auction (with a reserve r and a buy price B in both) is the same as in a second-price sealed-bid auction with the same reserve. In the second-price sealed-bid auction it is a dominant strategy for a bidder with value $v \in [B, z)$ to bid his value, i.e.,

$$v \in \arg\max_{x \geq r}\{G(x)E[u(v - \max\{r, y\})|\underline{v} \leq y \leq x]\}. \tag{11}$$

By (1) we have

$$u(x - \delta_\alpha(x)) = E[u(x - \max\{r, y\}) | \underline{v} \le y \le x].$$

Since u is CARA this equality implies

$$u(v - \delta_\alpha(x)) = E[u(v - \max\{r, y\}) | \underline{v} \le y \le x]. \tag{12}$$

Combining (11) and (12) yields

$$v \in \arg\max_{x \ge r} G(x) u(v - \delta_\alpha(x)). \tag{13}$$

The bidder's payoff in the second price auction is $G(v)u(v - \delta_\alpha(v))$. For $v \le z$ this is equal to $U^w(v, z)$ by (6).

In the Yahoo auction, a bidder with value $v \in [B, z]$ who chooses his threshold as though his true value were $x \in [B, z]$ obtains an expected utility of

$$U(t(x), v; t) = \int_{\underline{v}}^{t(x)} u(v - \max\{r, y\}) dG(y) + \int_{t(x)}^{x} u(v - B) dG(y).$$

Denote the bidder's payment as a function of the maximum of his rivals' values by $p(y)$, where $p(y) = r$ if $y \in [\underline{v}, r)$, $p(y) = y$ if $y \in [r, t(x))$, $p(y) = B$ if $y \in [t(x), x)$, and $p(y) = 0$ if $y > x$. We can rewrite the expression above as

$$U(t(x), v; t) = G(x) E[u(v - p(y)) | \underline{v} \le y \le x].$$

Let $\gamma_\alpha(x)$ be the certainty equivalent of a buyer with value x for the price he would pay conditional on winning in the Yahoo auction, i.e.,

$$u(x - \gamma_\alpha(x)) = E[u(x - p(y)) | \underline{v} \le y \le x].$$

By CARA

$$u(v - \gamma_\alpha(x)) = E[u(v - p(y)) | \underline{v} \le y \le x]. \tag{14}$$

Since t is an equilibrium threshold strategy then

$$v \in \arg\max_{x \in [B, z]} G(x) E[u(v - p(y)) | \underline{v} \le y \le x]. \tag{15}$$

Substituting (14) into (15) yields

$$v \in \arg\max_{x \in [B, z]} G(x) u(v - \gamma_\alpha(x)). \tag{16}$$

Hence in the Yahoo auction $U(t(v), v; t) = G(v)u(v - \gamma_\alpha(v))$.

Equations (13) and (16) provide conditions on the certainty equivalents of payments made in equilibrium in a second-price auction and in a Yahoo auction. The two equations are identical except for the certainty equivalent functions $\delta_\alpha(x)$ and $\gamma_\alpha(x)$. We now show that $\delta_\alpha(v) = \gamma_\alpha(v)$ for all $v \in [B, z]$. A bidder with value $v = B$ has the same equilibrium expected utility in both the second-price auction and the Yahoo auction, i.e., $G(B)u(B - \delta_\alpha(B)) = G(B)u(B - \gamma_\alpha(B))$. Hence $\delta_\alpha(B) = \gamma_\alpha(B)$. Differentiating (13) with respect to x yields the first-order condition

$$G'(v)u(v - \delta_\alpha(v)) - (G(v) - G(r))u'(v - \delta_\alpha(v))\delta'_\alpha(v) = 0,$$

for the second-price auction, or

$$\delta'_\alpha(v) = \frac{G'(v)u(v - \delta_\alpha(v))}{(G(v) - G(r))u'(v - \delta_\alpha(v))}.$$

This is an ordinary differential equation in δ_α. Differentiating (16) shows the γ_α function satisfies exactly the same ordinary differential equation. Both equations have the same initial condition at $v = B$ since $\delta_\alpha(B) = \gamma_\alpha(B)$. Hence for $v \in [B, z]$, we have $\delta_\alpha(v) = \gamma_\alpha(v)$ and so

$$U^w(v, z) = G(v)u(v - \delta_\alpha(v)) = G(v)u(v - \gamma_\alpha(v)) = U(t(v), v; t).$$

This proves (a).

Proof of (b): For $v \in [z, \bar{v}]$ it is trivial to see that $U(r, v; t) = U^b(v, z)$.

Proof of (c). Suppose that t jumps down at $z \in (B, \bar{v})$. A necessary condition for t to jump down at z is that

$$U(t(z), z; t) = U(r, z; t),$$

i.e., a bidder with value z is indifferent between the threshold $t(z)$ and r. By parts (a) and (b) this is equivalent to

$$U^w(z, z) = U^b(z, z),$$

which is (7) with c replaced by z. In other words, the value at which t jumps down is defined by the same condition as the eBay equilibrium cutoff. This proves Prop. 2(i).

If $B \geq \delta_\alpha(\bar{v})$ then the equilibrium cutoff is $c^* = \bar{v}$ by Prop. 1, and hence the threshold function t has no jump down. If $B \in (r, \delta_\alpha(\bar{v}))$ then there is a unique equilibrium cutoff $c^* \in (B, \bar{v})$, and hence t jumps down at $z = c^*$. The proves Prop. 2(ii). $\square$

Proof of Proposition 3: We have shown that if $t(v)$ is an equilibrium threshold function, then bidder equilibrium expected utilities are the same in the eBay and the Yahoo buy-now auctions. We now show that an equilibrium threshold function exists and is unique.

Consider a bidder with value $v \in [B, c^*]$. In the Yahoo buy-now auction the bidder's equilibrium expected utility is

$$G(r)u(v-r) + \int_r^{t(v)} u(v-y)dG(y) + [G(v) - G(t(v))]u(v-B).$$

In the eBay buy-now auction his equilibrium expected utility is

$$G(r)u(v-r) + \int_r^{v} u(v-y)dG(y).$$

By Prop. 2(i) the difference of these two utilities is zero, i.e.,

$$\int_{t(v)}^{v} [u(v-y) - u(v-B)]dG(y) = 0.$$

If $v = B$ this equation implies $t(v) = B$. If $v > B$ there is the trivial solution $t(v) = v$, which we dismiss since a threshold cannot exceed B. For $t(v) < v$ this equality can be re-written as

$$E[u(v-y)|t(v) \leq y \leq v] = u(v-B). \tag{17}$$

We show that (17) defines $t(v)$ uniquely, and $t(v)$ is decreasing in v. Clearly,

$$E[u(v-y)|B \leq y \leq v] < u(v-B). \tag{18}$$

The "no regret" assumption means

$$E[u(c^*-y)|r \leq y \leq c^*] > u(c^*-B).$$

Since $v \leq c^*$ this implies

$$E[u(c^*-y)|r \leq y \leq v] > u(c^*-B).$$

Since bidders have CARA preferences, this is equivalent to

$$E[u(v-y)|r \leq y \leq v] > u(v-B). \tag{19}$$

Since $E[u(v-y)|t \leq y \leq v]$ is continuous and strictly increasing in t, equations (18) and (19) imply there is a unique $t(v) \in (r, B)$ satisfying (17). To see that $t(v)$ is decreasing, note that (17) can be re-written as

$$\int_{t(v)}^{v} (e^{\alpha y} - e^{\alpha B}) dG(y) = 0.$$

For $t(v)$ fixed, the LHS is increasing in v since $v > B$. For v fixed, the LHS is also increasing in $t(v)$ since $t(v) < B$. Hence $t(v)$ must be decreasing in v for the equality to hold as v increases. Let $t(v) = r$ for $v > c^*$.

We now show that $t(v)$, as constructed above, is an equilibrium. Consider a bidder with value $v \in [B, c^*]$. In the proof of Prop. 2 it is established for $v \in [B, c^*]$ that

$$G(x)u(v - \delta_\alpha(x)) = G(x)u(v - \gamma_\alpha(x)) \ \forall x \in [B, c^*],$$

where $\delta_\alpha(x)$ is the certainty equivalent of the payment made by the winning bidder with value x in an eBay auction without a buy price (and also a second-price auction), and $\gamma_\alpha(x)$ is the certainty equivalent in the Yahoo buy-now auction. Value-bidding is optimal in the second-price auction, i.e.,

$$v \in \arg\max_{x \in [B, c^*]} G(x)u(v - \delta_\alpha(x)).$$

Hence $v \in \arg\max_{x \in [B, c^*]} G(x)u(v - \gamma_\alpha(x))$, i.e., a bidder with value $v \leq c^*$ in the Yahoo buy-now auction obtains a higher payoff with a threshold $t(v)$ than with any other threshold $t(x) \in [t(c^*), B]$. Clearly a threshold $t \in (r, t(c^*))$ is not optimal, since a threshold of $t(c^*)$ yields a higher expected payoff.

A bidder with value $v \in [B, c^*]$ obtains a higher payoff with a threshold of $t(v)$ than with a threshold of r, as we now show. If the bidder chooses a threshold of r his payoff is

$$U(r, v; t) = u(v - B)Q(F(c^*)) = U^b(v, c^*).$$

Since c^* is an equilibrium cutoff and $v \leq c^*$, then $U^b(v, c^*) \leq U^w(v, c^*)$. For $v \leq c^*$, by payoff equivalence of the eBay and Yahoo buy-now auctions we have

$$U^w(v, c^*) = U(t(v), v; t).$$

These equalities and inequalities yield the result $U(r, v; t) \leq U(t(v), v; t)$. We have established that $t(v)$ is an optimal threshold for a bidder with value $v \in [B, c^*]$.

We complete the proof by showing that $t(v) = r$ is an optimal threshold for a bidder with value $v > c^*$. Clearly, any threshold $\tilde{t} \in (r, t(c^*))$ is dominated by the threshold $t(c^*)$. Consider a threshold $\tilde{t} \in [t(c^*), B]$. We will show that $U(r, v; t) > U(\tilde{t}, v; t)$. For $v > c^*$ by payoff equivalence of the Yahoo and eBay buy-now auction we have $U(r, v; t) = U^b(v, c^*)$. Since c^* is an equilibrium cutoff and since $v > c^*$ then $U^b(v, c^*) > U^w(v, c^*)$. Let $\tilde{v}$ be such that $t(\tilde{v}) = \tilde{t}$; note that $\tilde{v} \in [B, c^*]$. We also have

$$\begin{aligned} U^w(v, c^*) &= G(r)u(v - r) + \int_r^{c^*} u(v - y)dG(y) \\ &\geq G(r)u(v - r) + \int_r^{\tilde{v}} u(v - y)dG(y), \end{aligned}$$

where the equality is by the definition of $U^w(v, c^*)$ and where the inequality follows from $v > c^* \geq \tilde{v}$. Since $\tilde{v} \leq c^*$, by payoff equivalence we have $U^w(\tilde{v}, c^*) = U(t(\tilde{v}), \tilde{v}; t)$. Since bidders are CARA risk averse, this equality can be rewritten as

$$G(r)u(v - r) + \int_r^{\tilde{v}} u(v - y)dG(y) = U(t(\tilde{v}), v; t).$$

These inequalities yield $U(r, v; t) > U(t(\tilde{v}), v; t)$, which completes the proof. □

Proof of Corollary 4: Suppose that $\alpha'' > \alpha' \geq 0$. Let t_α be the equilibrium threshold function of the Yahoo auction when bidders are CARA risk averse with index of risk aversion α. By Proposition 3, if $t_\alpha(v) > r$ then

$$E[u(v - y)|t_\alpha(v) \leq y \leq v] = u(v - B).$$

If $t_{\alpha'}(v) > r$ and $\alpha' > 0$ then this equality can be re-written as

$$E[e^{\alpha' y}|t_{\alpha'}(v) \leq y \leq v] = e^{\alpha' B}.$$

Raising both sides to the power α''/α' yields

$$\left(E[e^{\alpha' y}|t_{\alpha'}(v) \leq y \leq v]\right)^{\frac{\alpha''}{\alpha'}} = e^{\alpha'' B}.$$

Since $x^{\alpha''/\alpha'}$ is convex, we have

$$\left(E[e^{\alpha' y}|t_{\alpha'}(v) \leq y \leq v]\right)^{\frac{\alpha''}{\alpha'}} < E[\left(e^{\alpha' y}\right)^{\frac{\alpha''}{\alpha'}} |t_{\alpha'}(v) \leq y \leq v] = E[e^{\alpha'' y}|t_{\alpha'}(v) \leq y \leq v].$$

Hence,

$$e^{\alpha'' B} = E[e^{\alpha'' y}|t_{\alpha''}(v) \leq y \leq v] < E[e^{\alpha'' y}|t_{\alpha'}(v) \leq y \leq v],$$

which implies $t_{\alpha''}(v) < t_{\alpha'}(v)$, i.e., more risk averse bidders choose lower thresholds.

If $t_{\alpha'}(v) > r$ and $\alpha' = 0$ then

$$B = E[y|t_{\alpha'}(v) \leq y \leq v].$$

Also,

$$e^{\alpha'' B} = E[e^{\alpha'' y}|t_{\alpha''}(v) \leq y \leq B] > e^{\alpha'' E[y|t_{\alpha''}(v) \leq y \leq B]},$$

where the equality holds by (2) and the inequality holds since $e^{\alpha'' y}$ is convex. Hence $B > E[y|t_{\alpha''}(v) \leq y \leq B]$. Thus

$$E[y|t_{\alpha''}(v) \leq y \leq B] < E[y|t_{\alpha'}(v) \leq y \leq B],$$

and so $t_{\alpha''}(v) < t_{\alpha'}(v)$.

If $t_{\alpha'}(v) = r$ then $t_{\alpha''}(v) = r$ since the c^*, the value at which the threshold jumps down, is decreasing in α by Proposition 1. □

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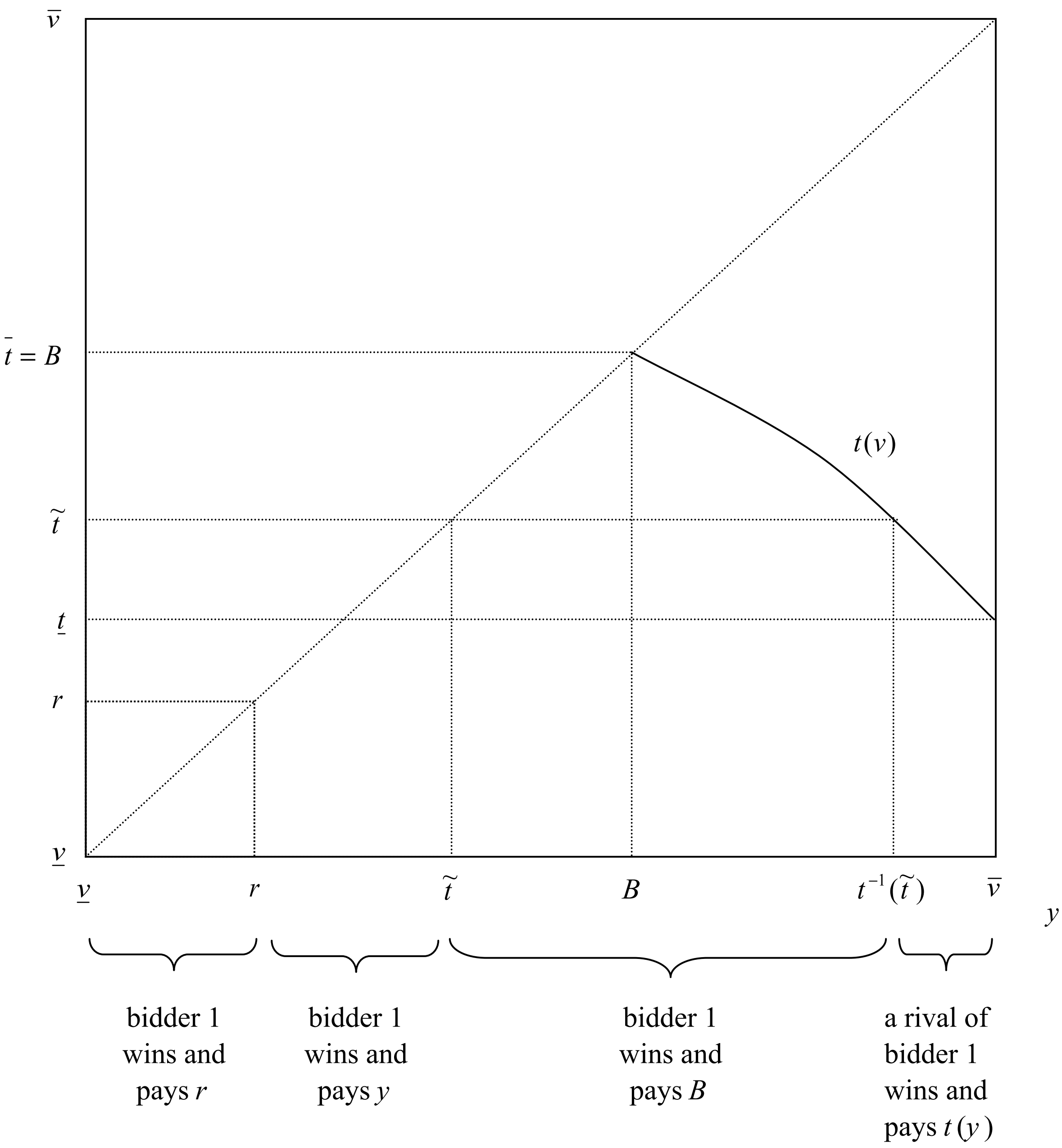
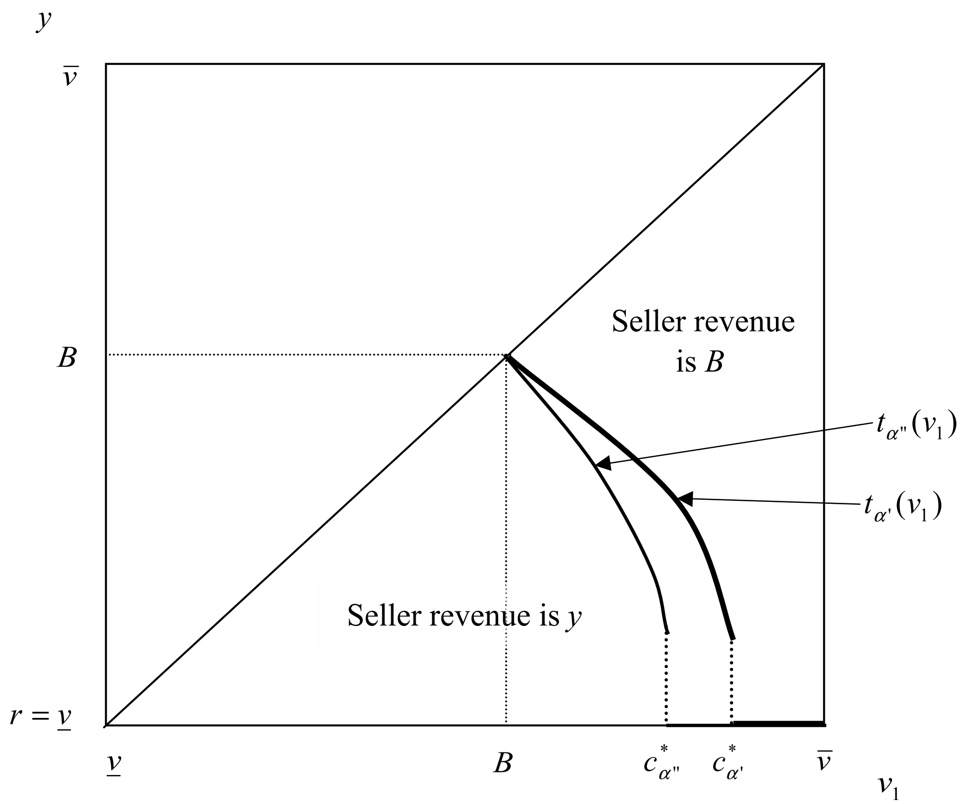


$\bar{v}$
$\bar{t} = B$
$\tilde{t}$
$\underline{t}$
r
$\underline{v}$
$t(v)$
$\underline{v}$
r
$\tilde{t}$
B
$t^{-1}(\tilde{t})$
$\bar{v}$
y
bidder 1 wins and pays r
bidder 1 wins and pays y
bidder 1 wins and pays B
a rival of bidder 1 wins and pays $t(y)$

Figure 1



y
$\bar{v}$
B
$r = \underline{v}$
Seller revenue is B
$t_{\alpha''}(v_1)$
$t_{\alpha'}(v_1)$
Seller revenue is y
$\underline{v}$
B
$c^*_{\alpha''}$
$c^*_{\alpha'}$
$\bar{v}$
v_1

Figure 2