Internet Trading Mechanisms and Rational Expectations

Michael Peters and Sergei Severinov
University of Toronto and Duke University
First Version -Feb 03

April 1, 2003

Abstract

This paper studies an internet trading mechanism similar to the one described in (Peters and Severinov 2001) in a market where traders values are interdependent. It is shown that under reasonable conditions this mechanism has a perfect Bayesian equilibrium which supports allocations that converge to rational expectations equilibrium allocations. In particular, this equilibrium supports allocations that are ex post efficient. We show how to construct the rational expectations equilibrium from the market outcome. The mechanism is also compared to a double auction.

This paper models competition in an auction market where buyers' and sellers' valuations are interdependent. Rather than having sellers compete in reserve prices, as has been commonly assumed in the previous literature,[1] we assume that the competition occurs in an *internet auction market* similar to the one studied by (Peters and Severinov 2001). Sellers cannot commit to public reserve prices, but they are free to bid in any auction (including their own) at any time during the bidding process. We show that under reasonable conditions that resemble those found in a very large market, there is an equilibrium for the internet auction market for which trade is post efficient and occurs at a uniform price.

Internet auctions provide a rich source of readily accessible data on bidder behavior in auctions. One way to exploit this is to use the outcomes in the different auctions as separate data points that can be used to test particular predictions of auction theory.[2] One of the main implications of (Peters and Severinov 2001) is that a market based approach to the data may be more effective than an auction theoretic approach. So for example, the price that prevails in some sellers' internet auction will vary with aggregate demand and supply, but will otherwise not be sensitive to the number of actual bidders that participate in the seller's auction.

[1]For example (McAfee 1993, Peters and Severinov 1997).

[2]For example, (Bajari and Hortacsu 2002) measure the winners' curse in ebay auctions by measuring the impact that the number of bidders has on the trading price in eBay auctions.

We show that a similar conclusion can be drawn when values are interdependent. The equilibria that we derive for the internet auction market can be associated with a *rational expectations equilibrium* for the market in which all traders' beliefs conditional on any particular price are just the traders' posterior beliefs when the internet auction markets leads to trade at that price. So individual auction data is again better understood by applying rational expectations reasoning to the whole auction market, than it is by considering the auctions in isolation.[3]

From the opposite perspective, the internet trading mechanism also provides some insight into rational expectations equilibrium. Rational expectations is based on a fixed point argument that is conceptually difficult. Beliefs are based on prices. Equilibrium occurs if the beliefs associated with every possible *market clearing* price coincide with the true distribution over the unknown state conditional on that price. Though the fixed point argument characterization itself is well defined, there is no conceptual procedure, like the tantonnment process for prices, to explain how beliefs might come to have this property. As prices adjust along the equilibrium path in our model, traders' beliefs change as the update their prior beliefs using Bayes rule. We show how the beliefs that traders hold in each terminal information set of the bidding process are related to rational expectations beliefs. In this sense, traders beliefs along the equilibrium path converge to the appropriate rational expectations beliefs in a manner that is similar to a tatonnement.

In the model that we study below, traders bid repeatedly in a collection of different auctions in a manner that is similar to what happens on eBay. At each moment, each auction has a high bidder and a standing bid equal to the second highest bid submitted. In the equilibrium we analyze, traders always bid where standing bids are lowest, and always submit the lowest bid that could conceivably make them high bidder at the auction. Under some conditions, this simple bidding strategy constitutes a weak perfect Bayesian equilibrium where efficient trade occurs at a uniform price. The strategies that our traders use cause prices to adjust in a manner that is similar to some of the ascending price algorithms that have been studied previously (for example (Ausubel and Milgrom 2001) and (Roth and Sotomayor 1990)). However, the market analysis that we are trying to provide requires that we consider an number of significant extensions of the algorithmic arguments. Most fundamentally, we suppose that traders valuations are interdependent. Beyond this, however we also need to deal with the fact that typical algorithmic price adjustment procedures leave incentives for certain traders to manipulate the mechanism. So, for example, in (Roth and Sotomayor 1990) the price adjustment procedure leading to the minimum competitive price is incentive compatible for buyers but not for sellers. Finally, prices are not adjusted by the mechanism designer in our internet auction, rather traders themselves adjust prices as the mechanism plays out. We view this as an important feature of this indirect mechanism because it allows

[3] One implication of both (Peters and Severinov 2001) and this paper is that it would be an error to treat data from different eBay auctions, for example, as independent draws from some common distribution.

traders to make inferences about other traders' types from the actions that they take during the course of the bidding. At the same time, this provides opportunities for individual traders to manipulate the mechanism. For this reason we need to establish the sequential rationality of our bidding behavior. This forces us to verify the best reply property of the price adjustment procedure in information sets that do not occur on the path associated with these strategies.

Our result suggests that internet auctions can function as effective multilateral trading mechanisms. A natural question to ask is whether internet auctions have any advantages over the better known *double auction*.[4] We show that the internet trading mechanism improves on the double auction in a way exactly analogous to the way that an English auction improves upon a sealed bid auction in the single seller case. The internet auction allows traders to glean some of the other traders information when they choose to drop out of the bidding process.

We look more deeply for connections between the two mechanisms. Our results show that the internet auction mechanism shares an important property of double auctions, in particular, all trades occur at a common price. An important distinction is that the common price is a consequence of equilibrium behavior in the internet mechanism, while it is imposed exogenously in a double auction. We focus on the rational expectations interpretation and show by example that the double auction may have equilibria in which the price in the double auction cannot effectively aggregate the other traders information. Information that traders glean in the course of bidding in the internet auction resolves this problem.

(Reny and Perry 2002) study the convergence of the allocation associated with a double auction to a rational expectations equilibrium. The environment we consider is different, as is the indirect mechanism that we study. In a double auction, traders don't formally have expectations based on price, but they do know the equilibrium strategies of the other traders. Conditional on price and knowledge of the equilibrium strategy rule, every trader can calculate the set of states that would generate the observed price, and this determines a set of price conditional beliefs. (Reny and Perry 2002) show that these beliefs converge to appropriate rational expectations beliefs as the market gets large.

Our results have a similar interpretation. Yet there are a couple of practical differences. In the ascending price formulation, traders actually do form price contingent beliefs in the course of bidding in the auction market, and these beliefs are Bayesian updates of prior beliefs based on current prices. This belief formation process seem more straightforward conceptually. Our theorem shows that there is an relatively simple equilibrium bidding rule where traders' actions in each information set are based on the prices they see and a posterior belief that is essentially a truncated version of the prior belief. So the rational expectations belief function can be calculated in practise using prior beliefs and current prices,

[4]The desirable efficiency properties of double auctions in multilateral trading environments with many buyers and sellers have been studied by (Rustichini, Satterthwaite, and Williams 1994) for the private value case, and more recently by (Reny and Perry 2002) for the interdependent value case.

without having to work out the equilibrium bidding strategies.[5] The practical disadvantage is that traders beliefs are based on slightly more information than just the current price. We illustrate with an example later in the paper when this can make a difference.

We do not invoke large numbers directly in our results, though our assumptions have a large number flavor to them. In this sense our results are related to the existence and characterization results for double auctions with large numbers. In the private value case, for example, (Rustichini, Satterthwaite, and Williams 1994) prove a theorem about convergence to efficiency with independently distributed valuations. Cripps and Swinkels (02) prove a corresponding theorem for the case where valuations are correlated. We have already mentioned the connection of the model in this paper to (Peters and Severinov 2001) who prove a convergence result for the private (but correlated) value case that uses a internet auction framework that closely resembles the approach that we use here.

Finally, we mention one earlier model of convergence to rational expectations equilibria, Wolinsky (Wolinsky 1988). In his model traders are pairwise matched and attempt to 'negotiate' a trading price with whoever they are matched with. Traders who meet a series of tough bargaining partners infer that the value of the commodity being traded is higher. His results are negative - as the discount factor goes to one so that is becomes easier for traders to collect information by meeting more partners, the quality of the information conveyed by each meeting deteriorates, so trading prices do not converge to full information prices. Our result contrasts with his, though transactions costs are much different in our model than in his.

1 The Model

1.0.1 Fundamentals

There are n sellers and m buyers trading in a market. Let $\mathcal{I}$ denote the set of traders (both buyers and sellers). Each seller has one unit of a homogeneous good, while each buyer has an inelastic demand for one unit of this good.

Buyers' and sellers' values are determined by the types of all traders. We assume that types and values come from finite sets. The grid of values is $\mathcal{V} \subset \mathbb{R}_+$ which contains η real numbers. The distance between any two points on the grid of values is δ^v. The minimum and maximum grid points are $\underline{u}$ and $\overline{u}$ respectively. The grid of types is also a subset $\mathcal{X}$ of $\mathbb{R}$ with the distance between any two grid points equal to δ^x, with minimum and maximum points $\underline{x}$ and $\overline{x}$. For any vector $y \in \mathbb{R}^n$ and any $i \leq n$, let $y_{(i)}$ be the i^{th} lowest element in the vector.

We assume that a trader i with type $x_i \in \mathcal{X}$ has utility $u(x_i, x_{-i})$ where $x_{-i} \in \mathbb{R}^{n+m-1}$ is the array of types of all the *other* traders. The function u is

[5] Our theorem provides the equilibrium bidding strategies, formulated in such a way that they depend only on current prices and these truncated beliefs.

assumed to generate values that lie in the grid $\mathcal{V}$, to be strictly increasing in the agent's own type,and to be non-decreasing in the types of the other agents.

We assume that traders share a common prior belief about the joint distribution of types. This common prior is assumed to have the property that two traders who share the same type have the same belief about the distribution of the types of the other traders.

An internet auction market consists of n sellers, m buyers, and a joint distribution F_{mn} over the types of the traders.

1.0.2 The bidding Process

Buyers and sellers are indexed in such a way that $i < j$ means that *i bids before j*. At the beginning of the game, each seller is considered to be high bidder at his own auction. The standing bid at each auction is the lowest grid point $\underline{v}$. The traders then take 'turns' bidding. The first buyer in the list of traders has the first turn to bid.

At the beginning of any player i's turn to bid, *each* of the traders who are not high bidders at some auction simultaneously announce whether they wish to remain active or to pass on the chance to bid during the current turn. The game ends if all these traders announce that they want to pass or if all standing bids have risen to the highest grid point $\overline{v}$.

If trader i (whose turn it is to bid) announces that he is active then he is given the opportunity to bid. If i says that he wants to pass, then the opportunity to bid is given to the 'next' active trader j who is not yet a high bidder.[6] Traders who pass on some turn can become active again whenever they like. After some active player j submits a bid, it becomes player $j + 1$'s turn to bid (or player 1's turn to bid if $j = m + n$).

If a trader is given the opportunity to bid, then he must submit a bid with some seller that lies in the grid $\mathcal{V}$, and that strictly exceeds the current *standing bid* at the auction where it is submitted.

When a new bid is received it is compared with the current high bid in the auction. If the new bid is higher than the current high bid at an auction, the bidder is made the new *high bidder.* One exception is that a seller who bids in his own auction becomes high bidder if he matches the existing high bid. Whatever bid is submitted, the standing bid at the auction is adjusted so that it equals the second highest bid that has been submitted to the auction. The identity of the high bidder and the value of the current standing bid are then made public.

When the game ends the high bidder at each auction pays the seller running the auction the standing bid or second highest bid that has been submitted to that auction. Of course, any seller who is high bidder at his own auction does not sell.

[6]The term 'next' means the player with the smallest index $j \geq i$ who is active, or if no such traders exist, it means the player with the smallest index who is active.

1.0.3 Assumptions

We are searching for conditions under which the internet auction mechanism will have an ex post efficient equilibrium. There are a number of conditions. Two of them are substantive. We begin with the following strenghtening of the usual single crossing condition.

Assumption 1 *(Single Crossing Condition) If $x_i > x_j$ where x_j is the j^{th} component of x_{-i}. Then $u(x_i, x_j, \{x_{-ij}\}) \geq u(x', x_i, \{x_{-ij}\}) \, \forall x' < x_i$.*

This strenghtens the usual idea that every trader's own signal is more important than any other trader's signal (which in turn ensures that the traders with the highest types also have the highest values).[7] We are trying with this assumption to incorporate the following stronger idea. Let d be a frequency distribution defined on $\mathcal{X}$ in the sense that $\sum_{x \in \mathcal{X}} d(x) = 1$. It is natural to assume that the utility of any trader depends on his own type, and on the distribution of types of the other traders. If the number of traders is large a natural continuity assumption is that utility does not depend 'much' on the type of any other single trader. Since monotonicity ensures that

$$u(x_i, d) - u(x', d) = \triangle > 0$$

for each $x' < x_i$, a natural continuity assumption is that

$$u(x_i, d) - u\left(x_i, d - \frac{e_{x'}}{m+n-1} + \frac{e_{\underline{x}}}{m+n-1}\right) \leq \triangle$$

for some n and m large enough (where e_x is a vector of zeros with a 1 in the x^{th} position). This is the single crossing condition provided m and n are chosen large enough so that this inequality holds uniformly.

The second substantive assumption is the following:

Assumption 2 *For each i, x_i and vector $\{x_1 \leq x_2 \cdots \leq x_{m-1} \leq x'\}$, consider the event*

$$E = \left\{\tilde{x}_{-i} : \{\tilde{x}_{-i}\}_{(1)} = x_1, \ldots \{\tilde{x}_{-i}\}_{(m-1)} = x_{m-1}, \{\tilde{x}_{-i}\}_{(m)} \geq x'\right\}.$$

Then $\Pr\left(\{\tilde{x}_{-i}\}_{(m)} = x' | E\right) = 1$.

[7] Two trivial examples of utility functions satisfying this restriction are as follows:

$$u(x_i, x_{-i}) = x_i + \sum_{j \neq i} \max[x_i, x_j]$$

If $W : \mathbb{R} \to \mathbb{I}$ is the function that gives for each real x the largest integer which is less than or equal to x, then

$$u(x_i, x_{-i}) = x_i + W\left(\frac{1}{n+m-1} \sum_{j \neq i} x_j\right)$$

satisfies Assumption 1 for all x_i if the maximum grid point in $\mathcal{X}$ divided by $n+m-1$ is less δ.

Sellers are allowed to bid in the rules of our proceedure. The price at which trades occurs is always the highest losing bid. So a seller can sometimes affect the price at which he trades. To use a terminology from another literature, sellers can exert market power whenever they are *pivotal*. When sellers set reserve prices ex ante, raising the reserve price will sometimes raise the sellers' trading price and sometimes lose the seller a trade that he would have made at the higher price anyway. Convergence theorems for private value markets, like the one in (Peters and Severinov 2001) or (Cripps and Swinkels 2002) impose conditions that ensure that the latter effect dominates the former in very large markets. In an ascending price procedure of the kind described in this paper, sellers may bid in information sets in which they know that the latter effect (losing trades that would otherwise have occurred) is not present. There are many ways to proceed with this - use epsilon equilibrium, or approximate efficiency. Since these approaches can be understood from other sources we use this more straightforward approach.[8]

We also require a number of technical assumptions that arise because we want to force bids and values into a finite grid. Lotteries over x_{-i} generate expected *utilities* that need not lie in the grid $\mathcal{V}$ even if the utility function itself generates values that lie in the grid. To circumvent this, we simply assume that for any pair of lotteries F and F' over x_{-i}, the lottery F is strictly preferred to the lottery F' if and only if the maximum grid point in $\mathcal{V}$ that is less than or equal to $\mathbb{E}_F u(x_i, \tilde{x}_{-i})$ is *strictly larger* than the maximum grid point in $\mathcal{V}$ that is less than or equal to $\mathbb{E}_{F'} u(x_i, \tilde{x}_{-i})$. So traders use expected utility to choose between lotteries, but differentiate two lotteries only if the expected utilities they generate lie in different intervals as defined by the grid $\mathcal{V}$. We refer to this as the *finite payoffs* assumption.

The following assumptions restrict the amount by which expected values can change when other traders types vary by a single grid point.

Assumption 3 *(Local Invariance) For any vector of types (x_i, x_{-i}) let x^+_{-i} be the vector of types formed by replacing any $x_j \in x_{-i}$ such that $x_j = x_i$ with $x_i + \delta^x$. Similarly, let x^-_{-i} be the vector of types formed by replacing each $x_j \in x_{-i}$ such that $x_j = x_i$ with $x_i - \delta$. Then*

$$u(x_i, x_{-i}) = u\left(x_i, x^+_{-i}\right) = u\left(x_i, x^-_{-i}\right)$$

Assumption 4 *For any x_i and vector $y \in \mathbb{R}^m$*

$$\max_{p'} \left\{ p' \leq \mathbb{E}\left[u(x_i, \tilde{x}_{-i}) \mid \{\tilde{x}_{-i}\}_{(1)} = y_{(1)}, \ldots \{\tilde{x}_{-i}\}_{(m)} = y_{(m)} \right] \right\} =$$

$$\max_{p'} \left\{ p' \leq \mathbb{E}\left[u(x_i, \tilde{x}_{-i}) \mid \ldots, \{\tilde{x}_{-i}\}_{(m)} = y_{(m)}, \{\tilde{x}_{-i}\}_{(m+1)} \geq y_{(m)} + \delta^x \right] \right\}$$

[8] We provide examples below. However, on special case that is worth mentioning occurs when traders think that they know one anothers types exactly. In that case, the assumption requires that there be at least two traders who have the m^{th} lowest type.

1.0.4 Values

To state the theorem, we begin by defining the 'values' of the traders.

Definition 5 *The implicit value v_i for bidder i with type x_i is the largest grid point in $\mathcal{V}$ that is less than or equal to*

$$\mathbb{E}\left[u\left(x_i, \tilde{x}_{-i}\right) \mid \{\tilde{x}_{-i}\}_{(1)} = \min\left[x_i, \{x_{-i}\}_{(1)}\right], \ldots \{\tilde{x}_{-i}\}_{(m)} = \min\left[x_i, \{x_{-i}\}_{(m)}\right]\right]$$

To simplify the notation slightly, we leave out any explicit conditioning on trader i's own type x_i. This condition should be assumed in each of the formulas that follow. In the calculation of the implicit value, i is assumed to know his own type and either the types of all traders whose valuations are no higher than his, or the types of the traders with the lowest m types.

Let $v = \{v_1, \ldots v_{m+n}\}$ be the entire array of 'values' computed this way. Then $v_{(m)}$ is the m^{th} (lowest) order statistic among the vector of values for all the traders.

2 Efficient Bidding

Let $\iota(i)$ be any information set for trader i. Trader i shares public information on standing bids, the identities of the high bidders and other publicly available information about the bidding history. He has private information about his own type and the unobservable parts of his own bidding history. To decide whether or not to bid, the bidder has to form beliefs about the types of the other bidders and the high bids that they have submitted. We refer to the unknown types and high bids as the *state* of the market. As we have formulated it, the state is a random element in $\mathbb{R}^{2n+m}$. We wish to describe a bidding rule in which traders behavior in each information set depends only on the current array of standing bids, and a summary statistic for their beliefs.[9] Since beliefs are a summary statistic for the bid history, it is convenient to express the bidding rule itself as something that depends on these beliefs. We compress all the relevant information contained in traders' beliefs into something called the trader's *willingness to pay*, which depends on beliefs. We then specify the common bidding strategy as if it depends on this willingness to pay.

Say that a bid is *successful* if the bidder who makes the bid becomes high bidder. A bidder i' is *inactive* in some information set, if he is not high bidder at any auction in that information set, and if he chose to pass on the last k chances to bid, where $k \geq 1$. All other bidders are said to be *active*. For each inactive bidder i', let $q_{i'}$ be the lowest standing bid that prevailed on his k^{th} to last chance to bid. In this case, we say that bidder i' *dropped out at price* $q_{i'}$. Suppose that r bidders are inactive in the information set. Let q_{r+1} be the lowest standing bid in the information set. Let $q = \{q_1, \ldots q_{r+1}\}$ be the vector

[9] In the private value case this is even more striking. Traders don't care about other traders valuations so their beliefs don't matter and bidding strategies don't depend on the bidding history at all. See (Peters and Severinov 2001).

consisting of the low standing bids that prevailed when each of the inactive bidders dropped out, along with the lowest standing bid in the information set.

Among the active bidders, only some of them will be high bidders. A high bidder is *vulnerable* in an information set if he is the seller running an auction where no bids have been submitted, or if unsuccessful bids have been submitted against him at the auction where he is high bidder in the information set. A high bidder is *secure* otherwise.

At each information set where trader i bids, he has to formulate beliefs about the types of the other bidders (because they affect his value) and the high bids at each of the auctions. These beliefs are assumed to have the following properties:

Axiom 6 (B.1) *In any information set, bidder i believes that the high bid at each auction where the high bidder is vulnerable is equal to the standing bid in that auction, while the high bid at each auction where the high bidder is secure is exactly one grid point above the standing bid in the auction.*

We now describe the beliefs that traders hold about the types of other bidders. For inactive bidders, the definition is inductive. In any information set where all traders are active, if trader i chooses to pass, then all traders other than i should infer that i's type is

$$\hat{x}_i = \max\left\{x : \mathbb{E}\left[u\left(x_i, \tilde{x}_{-i}\right) \mid \{\tilde{x}_{-i}\}_{(1)} = \cdots = \{\tilde{x}_{-i}\}_{(m)} = x\right] \leq q_{r+1}\right\}$$

In the special case where $q_{r+1} = \underline{u}$ the inference should be that i's types is less than or equal to $\hat{x}_i$. Then in any information set where there are inactive bidders, suppose that each of the inactive bidders has been assigned a type $\hat{x}_i$ for $i = 1, \ldots r$. Then if trader i drops out in the information set, all traders other than i should infer that i's type is

$$\hat{x}_i = \max\{x :$$

$$\mathbb{E}\left[u\left(x_i, \tilde{x}_{-i}\right) \mid \{\tilde{x}_{-i}\}_{(1)} = \hat{x}_1; \{\tilde{x}_{-i}\}_{(r)} = \hat{x}_r; \{\tilde{x}_{-i}\}_{(r+1)} = \cdots = \{\tilde{x}_{-i}\}_{(m)} = x\right]$$

$$\leq q_{r+1}\}$$

where again, the inference is that i's type is less than or equal to $\hat{x}_i$ in the special case where $q_{r+1} = \underline{u}$. Let $\hat{x}$ be the vector of such types.

Axiom 7 (B.2) *In any information set, the type of the i'^{th} inactive bidder is exactly $\hat{x}_{i'}$ unless i' dropped out at $\underline{u}$ in which case i''s type is less than or equal to $\hat{x}_{i'}$.*

Observe that there must always be n high bidders. There are exactly $m - r$ bidders who are not high bidders but who are still active. For any $q \geq q_{r+1}$, any vector x_{-i} of types such that x_{-i} and $\hat{x}$ have the same first r order statistics, and any $k = r, \dots m$ define

$$\hat{x}^p = \max \{x :$$

$$\mathbb{E}\left[u\left(x_i, \tilde{x}_{-i}\right) | \left\{\tilde{x}_{-i}\right\}_{(1)} = \hat{x}_1; \dots \left\{\tilde{x}_{-i}\right\}_{(r)} = \hat{x}_{(r)}; \left\{\tilde{x}_{-i}\right\}_{(r+1)} = \dots = \left\{\tilde{x}_{-i}\right\}_{(m)} = x\right]$$

$$\leq q_{r+1}\} \tag{1}$$

In any information set, the active bidders are of four possible types - they are either secure high bidders at an auction with the lowest standing bid, vulnerable high bidders at an auction with the lowest standing bid, high bidders at an auction where the standing bid is strictly higher than the lowest standing bid, or not high bidders at all. Let $\hat{x}^a = \hat{x}^p + \delta_x$.

Axiom 8 (B.3) *Traders who are secure high bidders have types at least as high as $\hat{x}^p + \delta_x = \hat{x}^a$.*

Axiom 9 (B.4) *Traders who are vulnerable high bidders have types at least $\hat{x}^p$.*

Axiom 10 (B.5) *Active traders who are not high bidders are assumed to have types at least $\hat{x}^a_{r+1}$ if they have declared themselves to be active at the current low standing bid, and to have types at least $\hat{x}^p_{r+1}$ otherwise.*

In any information set ι, let $S_i(\iota)$ be the set of states that bidder i thinks are possible if his beliefs obey (6) through (10). Observe that bidding ends once there are m inactive bidders.

Definition 11 *Define the* willingness to pay *for bidder i in information set ι to be*

$$\mathbb{E}\left[u\left(x_i, \tilde{x}_{-i} \vee \left\{\tilde{x}_{(r)} = \tilde{x}_{(m)} = \hat{x}^p_{r+1}\right\}\right) | S_i(\iota)\right]$$

if i is inactive and

$$\mathbb{E}\left[u\left(x_i, \tilde{x}_{-i} \vee \left\{\tilde{x}_{(r+1)} = \tilde{x}_{(m)} = \hat{x}^p_{r+1}\right\}\right) | S_i(\iota)\right]$$

if i is active. In this formulation $\tilde{x}_{-i} \vee \left\{\tilde{x}_{(k)} = \tilde{x}_{(m)} = \hat{x}^p_{r+1}\right\}$ is the vector formed by replacing the k^{th} through m^{th} order statistic in $\tilde{x}_{-i}$ with $\hat{x}^p_{r+1}$.

We now describe the strategy rule that all traders will follow in the equilibrium we are interested in.

Definition 12 *The symmetric strategy σ^* is defined as follows: in any information set $\iota(i)$*

(a) *if i has to decide whether to pass or continue in $\iota(i)$, then if i's willingness to pay is less than or equal to the lowest standing bid, he should pass. Otherwise he should remain active;*

(b) *if i has to bid in $\iota(i)$ and there is a unique lowest standing bid, then i should submit a bid with the auction where the standing bid is lowest. The bid should be equal to the lowest value on the grid that exceeds this low standing bid;*

(c) *otherwise if i has to bid in $\iota(i)$ and there are many auctions where the standing bid is lowest, then i should submit the same bid as in (b) randomly choosing among auctions where the high bidder is vulnerable. If all high bidders are secure at all auctions where the standing bid is lowest, the trader should choose randomly among them and bid as in (b).*

Theorem 13 *If the internet auction market satisfies 1,2, 4 and 3, then there is a weak perfect Bayesian equilibrium in which all traders to use the bidding strategy σ^*. For any array of types x, on the path associated with σ^*, each buyer whose type is above $x_{(m)}$ trades for sure with some seller whose type is no larger than $x_{(m)}$. Each seller whose type is below $x_{(m)}$ trades for sure with some buyer whose type is at least $x_{(m)}$. All trades occur at the common price $v_{(m)}$.*

3 Bidding in Internet Auctions

The following lemma is the core of our argument.

Lemma 14 *Suppose that all trader i's beliefs satisfy 6 through 10. Let ι_1 be an information set belonging to trader i and let ι_1 be any information set that i believes occurs with positive probability on the continuation path when all traders follow σ^*. Then every trader's willingness to pay in ι_1 exceeds the lowest standing bid in ι_1 if and only if every traders' implicit value exceeds the lowest standing bid in every state that i believes is possible in ι_1.*

Proof. Trader j's willingness to pay in $\iota_1$is given by Definition 11 as

$$\mathbb{E}\left[u\left(x_j, \tilde{x}_{-j} \vee \left\{\tilde{x}_{(r)} = \tilde{x}_{(m)} = \hat{x}^p_{r+1}\right\}\right)|S_j(\iota)\right] =$$

$$\mathbb{E}_{x^a}\left[u\left(x_j, \tilde{x}_{-j}\right)|\ldots\{\tilde{x}_{-j}\}_{(r-1)} = \{\hat{x}_{-j}\}_{(r-1)}, \{\tilde{x}_{-j}\}_{(r)} = \cdots = \{\tilde{x}_{-j}\}_{(m)} = x^p_{r+1}\right]$$

if j is inactive and

$$\mathbb{E}\left[u\left(x_j, \tilde{x}_{-j} \vee \left\{\tilde{x}_{(r+1)} = \tilde{x}_{(m)} = \hat{x}^p_{r+1}\right\}\right)|S_j(\iota)\right] =$$

$$\mathbb{E}_{x^a}\left[u\left(x_j, \tilde{x}_{-j}\right)|\ldots\{\tilde{x}_{-j}\}_{(r)} = \{\hat{x}_{-j}\}_{(r)}, \{\tilde{x}_{-j}\}_{(r+1)} = \cdots = \{\tilde{x}_{-j}\}_{(m)} = x^p_{r+1}\right]$$

if j is still active. In these expressions conditioning on x^a simply means that i incorporates any information in the current information set about the active bidders whose types are at least $\hat{x}^p_{r+1}$.

Suppose first that $x_j > x^p_{r+1}$ in some state that i believes is possible. Then since i expects all traders to follow σ^* in the continuation, j must be active in ι_1. Since

$$\mathbb{E}\left[u\left(x_j, \tilde{x}_{-j}\right) | \ldots \left\{\tilde{x}_{-j}\right\}_{(r)} = \left\{\hat{x}_{-j}\right\}_{(r)}, \left\{\tilde{x}_{-j}\right\}_{(r+1)} = \cdots = \left\{\tilde{x}_{-j}\right\}_{(m)} = x^p_{r+1}; \left\{x_{-j}\right\}_{(m+1)} \geq x^a_{r+1}\right]$$

$$\geq \mathbb{E}_{x^a}\left[u\left(x_j, \tilde{x}_{-j}\right) | \ldots \left\{\tilde{x}_{-j}\right\}_{(r)} = \left\{\hat{x}_{-j}\right\}_{(r)}, \left\{\tilde{x}_{-j}\right\}_{(r+1)} = \cdots = \left\{\tilde{x}_{-j}\right\}_{(m)} = x^p_{r+1}\right]$$

$$\geq \mathbb{E}\left[u\left(x_j, \tilde{x}_{-j}\right) | \ldots \left\{\tilde{x}_{-j}\right\}_{(r)} = \left\{\hat{x}_{-j}\right\}_{(r)}, \left\{\tilde{x}_{-j}\right\}_{(r+1)} = \cdots = \left\{\tilde{x}_{-j}\right\}_{(m)} = x^p_{r+1}\right]$$

by the monotonicity of u, j's willingness to pay is equal to

$$\mathbb{E}\left[u\left(x_j, \tilde{x}_{-j}\right) | \ldots \left\{\tilde{x}_{-j}\right\}_{(r)} = \left\{\hat{x}_{-j}\right\}_{(r)}, \left\{\tilde{x}_{-j}\right\}_{(r+1)} = \cdots = \left\{\tilde{x}_{-j}\right\}_{(m)} = x^p_{r+1}\right]$$

by Assumption 4. This is at least

$$\mathbb{E}\left[u\left(x^a_{r+1}, \tilde{x}_{-j}\right) | \ldots \left\{\tilde{x}_{-j}\right\}_{(r)} = \left\{\hat{x}_{-j}\right\}_{(r)}, \left\{\tilde{x}_{-j}\right\}_{(r+1)} = \cdots = \left\{\tilde{x}_{-j}\right\}_{(m)} = x^p_{r+1}\right]$$

$$> q_{r+1} \tag{2}$$

by monotonicity and the definition of x^a_{r+1}.

So in any state that i believes is possible j's implicit value is

$$\mathbb{E}\left[u\left(x_j, \tilde{x}_{-j}\right) | \ldots \left\{\tilde{x}_{-j}\right\}_{(r)} = \left\{x_{-j}\right\}_{(r)} \ldots\right.$$

$$\left.\ldots \left\{\tilde{x}_{-j}\right\}_{(r+1)} = \min\left[x_j, \left\{x_{-j}\right\}_{(r+1)}\right] \ldots, \left\{\tilde{x}_{-j}\right\}_{(m)} = \min\left[x_j, \left\{x_{-j}\right\}_{(m)}\right]\right] \geq$$

$$\mathbb{E}\left[u\left(x^a_{r+1}, \tilde{x}_{-j}\right) | \ldots \left\{\tilde{x}_{-j}\right\}_{(r)} = \left\{x_{-j}\right\}_{(r)}, \left\{\tilde{x}_{-j}\right\}_{(r+1)} = \cdots = \left\{\tilde{x}_{-j}\right\}_{(m)} = x^p_{r+1}\right]$$

If trader i is active in ι_1 then in any state that i thinks is possible in ι_1, $x_k = \hat{x}_k$ for each inactive bidder. So this last expression strictly exceeds q_{r+1} by (2). If trader i is inactive in ι_1 then in this last expression $x_k = \hat{x}_k$ for every trader except trader i. If $x_i \geq \hat{x}_i$ then the expression again strictly exceeds q_{r+1}. If $x_i < \hat{x}_i$ then the inequality follows from the single crossing condition and the fact that $\hat{x}_i \leq x^p_{r+1} < x^a_{r+1}$.

If trader j's type is less than or equal to x^p_{r+1} then j's willingness to pay (using Assumption 4) is less than or equal to

$$\mathbb{E}\left[u\left(x^p_{r+1}, \tilde{x}_{-j}\right) | \ldots \left\{\tilde{x}_{-j}\right\}_{(r)} = \left\{\hat{x}_{-j}\right\}_{(r)}, \left\{\tilde{x}_{-j}\right\}_{(r+1)} = \cdots = \left\{\tilde{x}_{-j}\right\}_{(m)} = x^p_{r+1}\right]$$

by monotonicity.[10] So j's willingness to pay is less than or equal to q_{r+1}. Since j's type is less than or equal to x_{r+1}^p trader j's implicit value is less than or equal to

$$\mathbb{E}\left[u\left(x_{r+1}^p, \tilde{x}_{-j}\right) | \ldots \{\tilde{x}_{-j}\}_{(r)} = \{x_{-j}\}_{(r)}, \{\tilde{x}_{-i}\}_{(r+1)} = \cdots = \{\tilde{x}_{-i}\}_{(m)} = x_{r+1}^p\right]$$

Since the state used in this calculation must be one that i believes is possible in ι_1, this is less than or equal to

$$\mathbb{E}\left[u\left(x_{r+1}^p, \tilde{x}_{-j}\right) | \ldots \{\tilde{x}_{-j}\}_{(r)} = \{\hat{x}_{-j}\}_{(r)}, \{\tilde{x}_{-i}\}_{(r+1)} = \cdots = \{\tilde{x}_{-i}\}_{(m)} = x_{r+1}^p\right]$$

if $\hat{x}_i \geq x_i$ by monotonicity. If $x_i < \hat{x}_i$ the same inequality follows from the single crossing condition.

This argument establishes that both the willingness to pay and the implicit value strictly exceed q_{r+1} for any type $x_j > x_{r+1}^p$ and that both these numbers are less than or equal to q_{r+1} otherwise. ■

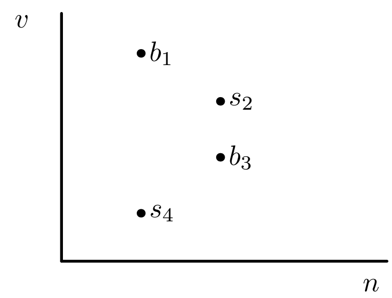
The essential argument is that when any trader deviates from σ^* he can compute the outcome state by state. For each state that he thinks is possible he simply calculates the vector of implicit values associated with that state, then proceeds to trace out the outcome by assuming that traders bid using σ^* but use their implicit values to determine whether or not to bid. The advantage is that these implicit values do not vary with the information set as the willingness to pay does. Furthermore, these implicit values (which determine the other traders behavior in the continuation) do not depend directly on trader i's actions, though the bidding behavior that they support may depend on conditions in the information set that trader i can influence.

We begin by illustrating how to use this theorem (and more generally how to think about bidding in the internet auction) by providing an example, given in Figure 1. The horizontal axis just indexes the number of bidders, while the vertical axis describes the bidders' implicit values.

Theorem 13 says that if the 'state' of the market is such that traders' types generate these implicit values, buyer 1 (whose implicit value is b_1) and seller 2 (whose implicit value is s_2) should each win an auction and the prices at both auctions will be b_3.

Lemma 14 states that we can predict bidding behavior using information about implicit values. We briefly describe the process that leads to this outcome, assuming to make things slightly simpler, that s_4 is the lowest grid point, and that the only other feasible bids are equal to the traders implicit values. Initially, the two sellers are high bidders at their own auctions with standing bids equal to s_4. If buyer 1 bids first, then he might submit a bid b_3 with seller 4. Buyer 2, following σ^* would then bid with seller 2 and become high bidder. Neither bid will change the standing bid at the corresponding auction. Seller 4 no longer has

[10] There are two parts to this inequality. j's type is less than or equal to x_{r+1}^p by hypothesis, and if j is inactive, then in the calculation his dropout x_j is replaced by x_{r+1}^p.



any interest in bidding according to σ^*. Seller 2 on the other hand is no longer high bidder at his auction. He doesn't want to sell at price s_4 so he can bid at either his own or seller 4's auction. Suppose that he chooses his own auction, and bids b_3 there, as σ^* requires. He then becomes high bidder at the auction (since he is the seller responsible for the auction) and drives the standing bid at the auction up to b_3. Buyer 3, who has been displaced as high bidder will bid with seller 4 where the standing bid is still s_4. The bid will not be successful, but it will drive up the standing bid there to b_3 as well. Buyer 3 will lose interest in bidding at this point and bidding will stop as specified by the theorem.

4 Beliefs and Values

Our method of proof is completely constructive. We specify beliefs in each information set, then construct willingness to pay functions based on these beliefs that define a strategy rule. The willingness to pay functions derived from these beliefs bear a close relationship with the traders' implicit values, which depend only on the types of the other traders, as is shown in Lemma 14. Axioms 6 through 10 describe a set of beliefs for traders in each information set. In information sets that lie off the equilibrium path, we are free to set beliefs in any desired way. So we first point out that if all traders are using σ^*, then along the path associated with this strategy, beliefs satisfying 6 through 10 are consistent with Bayes rule.

Consider first, the information set in which trader i gets his first opportunity to bid. If i is the first trader to bid, then i believes that all high bidders are vulnerable, and that no seller has a high bid above the lowest standing bid. So beliefs satisfying 6 through 10 coincide with i's original beliefs. If i is not the first to bid, then one or more traders may have submitted bids, while one or more traders might have passed. If some trader j passes on his first chance to bid, then by Lemma 14, it must be that j has an implicit value that is no higher than the lowest grid point $\underline{u}$, i.e.,

$$\mathbb{E}\left\{u\left(x, \tilde{x}_{-i}\right) : \left\{\tilde{x}_{-i}\right\}_{(1)} = \ldots \left\{\tilde{x}_{-i}\right\}_{(m)} = x\right\} \leq \underline{u}$$

which coincides with i's inference according to 6. Let $\underline{x}$ be the highest grid point that satisfies this condition. If no such grid point exists, then j has deviated from σ^* and any inference is legitimate. Otherwise each trader who has passed must have a type less than or equal to $\underline{x}$.

In all other information sets, suppose that i has already correctly inferred the types of bidders who dropped out in previous information sets. Let i be any bidder who passes for the first time in the current information set (bidders using σ^* will not pass then re-enter). Then from the first property of σ^* and Lemma 14, the highest type that this player could have is given by $\hat{x}^p_{r+1}$. On the other hand, by monotonicity, lowering i's type by one grid point, lowers i's value by at least one grid point in every state. So if i's type is even one grid point lower, i would have passed at the previous price (if the lowest standing bid jumps by many grid points, traders are not using σ^*).

Observe that a trader with type $x^a_{r+1} = x^p_{r+1} + \delta^x$ must have an implicit value that strictly exceeds q_{r+1}. If this isn't the case, then

$$\mathbb{E}\left[u\left(x^p_{r+1}+\delta^x, \tilde{x}_{-i}\right) \mid \{\tilde{x}_{-i}\}_{(1)} = \hat{x}_1; \ldots \{\tilde{x}_{-i}\}_{(r)} = \hat{x}_{(r)}; \{\tilde{x}_{-i}\}_{(r+1)} = \cdots = \{\tilde{x}_{-i}\}_{(m)} = \hat{x}^p_{r+1}\right] =$$

$$\mathbb{E}\left[u\left(x^p_{r+1}+\delta^x, \tilde{x}_{-i}\right) \mid \{\tilde{x}_{-i}\}_{(1)} = \hat{x}_1; \ldots \{\tilde{x}_{-i}\}_{(r)} = \hat{x}_{(r)}; \{\tilde{x}_{-i}\}_{(r+1)} = \cdots = \{\tilde{x}_{-i}\}_{(m)} = \hat{x}^p_{r+1}+\delta^x\right]$$

$$\leq q_{r+1}$$

by the local invariance property of preferences. This contradicts the fact that x^p_{r+1} is the largest type for which this inequality is supposed to hold. Thus any secure high bidder must have a type at least x^a_{r+1} if he is using σ^*,as must any bidder who has declared himself to be active at the current price. Bidders who become high bidders at auctions where the standing bid strictly exceeds the lowest standing bid are not using σ^* so any inference is consistent with weak perfect Bayesian equilibrium.[11]

We can now give a heuristic account of the proof of the theorem. We need to show that it will always pay each trader to follow σ^* if he thinks the other traders are following σ^* in *every conceivable information set* including those that are never attained when traders use σ^*. The key argument is Lemma 14, which has two implications. The first is that if other traders are using σ^* then every traders behavior in each future information set can be predicted from knowledge of his implicit value, which is complete determined in each state that a deviating trader thinks is possible. Second, once a trader has decided that a

[11] The specification of beliefs given by 8 and 9 may not be consistent with 'perfect Bayesian Equilibrium' which, depending on the source, may involve restrictions on off equilibrium beliefs. The complication occurs when these beliefs are applied to high bidders at auctions where the standing bid is strictly above the lowest standing bid. 8 and 9 require that in this case traders make the same inference that they do for active traders who are not high bidders, or for traders who are high bidders at the lowest standing bid. An implication of this is that beliefs about these traders will change as the lowest standing bid goes up - as if the types of the active bidders and these off equilibrium bidders were correlated.

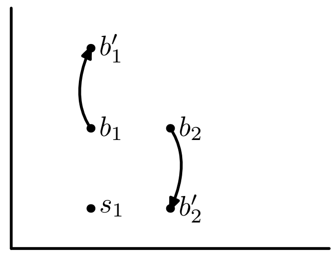


Figure 1: Figure 2

state is possible, each traders' implicit value, and therefore his future behavior, is independent of the path the trader followed to arrive in that information set. So each trader can do a state by state computation of the outcome that will occur if he uses the action specified by σ^* in the information set following this action, for every state that he thinks is possible. Since the set of states he believes are possible is independent of the action that he takes, he can then do a state by state comparison of this outcome with the outcome that prevails if he chooses some alternative action. The main logic of our proof takes this idea and shows that in each state, if all traders follow σ^* then all trades will occur at the lowest 'competitive' price consistent with the implicit values of the traders in that state (when appropriately qualified by the conditions in the initial information set). The argument is similar to (Roth and Sotomayor 1990). The complicated part here is to show that this result can be applied in *any* information set, not just from the outset of the game as is done in (Roth and Sotomayor 1990). The details of this argument are left to the appendix.

The lowest competitive trading is always the implicit value of a trader who fails to win an auction,[12] so by Lemma 14, buyers cannot lower the price at which they trade. If buyers trade at a price above their implicit value, then the price must also be at least as high as their willingness to pay by Lemma 14, so following σ^* is at least as good for buyers as is any deviation in every information set, and in every state the trader thinks is possible in that information set.

The argument is slightly different for sellers since they can affect the lowest competitive trading price when they are outbid in their own auction. Figure 2 illustrates the benefits and costs to a seller who deviates from σ^*.

The Figure illustrates a situation where there is one seller and two buyers. The points b_1, b_1', b_2, and b_2' all represent possible implicit values for buyers 1 and 2 respectively. Suppose that seller 1 finds himself in a situation where the standing bid at his auction is currently s_1, his implicit value. He is supposed to stop bidding according to σ^* if one of the buyers, say buyer 1, is high bidder at his auction. Suppose that he decides to stay active and try to submit a bid.

[12] As shown in the appendix, this assertion is not quite right, since off the equilibrium path the price may be determined by a bid that is above the implicit value of the trader who submits the bid. This complication is discussed there.

Whether buyer 1 is secure or vulnerable, seller 1 will believe that buyer 1's high bid is no higher than b_1 according to property 6 of beliefs. So he should expect to become high bidder at his own auction with a bid of b_1.

Seller 1 initially believes that the buyers have implicit values at least s_1 by Property 10 of beliefs. There are a number of possibilities that seller 1 needs to consider. If both buyers have implicit values equal to s_1 they will both drop out in the current round, and the seller will win his auction anyway. It makes no difference in this case whether or not the seller submits a bid. If both buyers have values above s_1 they will bid the price up to at least b_1 whether or not seller 1 submits a bid. It doesn't pay the seller to submit a bid in this case, because he might eliminate a profitable trade that he otherwise could have had if both buyers have implicit values equal to b_2.[13]

The potential gain comes when one of the buyers drops out when the standing bid is s_1. If the seller also drops out, the game ends and the seller trades at price s_1. If he submits a bid and becomes high bidder, he will gain if the other buyer has a valuation high enough for him to want to displace the seller at price b_1. This is where Assumption 2 comes into play, since the seller has to believe that if the $2nd$ highest implicit value among the buyers is at least b_1 then it must be exactly equal to b_1. If this is true, seller 1 must believe that buyer 1 will never be willing to submit a bid high enough to displace him as high bidder. The bid may then cost the seller, but can never gain the seller any surplus with positive probability.

5 Rational Expectations

An outcome for any individual is a price and a decision about whether or not to trade at that price. An outcome function describes a probability distribution over outcomes for all individuals for each array of signals that the traders' possess. Given an outcome function and a particular price and recommendation about whether or not to trade, each trader could calculate a posterior belief about the state conditional on the announced price and recommended trade. If the recommended trade for each trader is optimal at the announced price given this posterior belief for every price and recommended trade, then we say that the outcome function has the *rational expectations property.*[14]

It is well known that the *full information price function* will have this property if it is fully revealing. That is one possibility in the interdependent value case.[15] Part of the contribution of the present paper is to show that there is

[13]In that case, neither of the buyers will be willing to displace the seller by bidding a price above b_1.

[14]This condition is different from ex post individual rationality, but similar to it in spirit. The primary difference is that the direct mechanism might map many different arrays of types for the other traders into the same outcome for an individual trader, so this trader could not discover the true state simply by observing his own outcome.

[15]Though in the private value version of our model, it is not hard to see that the full information price function is never revealing.

an outcome function that has the rational expectations property when the full information price function is not revealing.

To show that the outcome function associated with the internet trading mechanism has the rational expectations property, fix the assumptions required for Theorem 13 to hold. Let (p, k) be an outcome for trader i with $k = 0, 1$ meaning that i does not win, or does win respectively some auction at the end of the bidding process where the standing bid is p. Let $\mathcal{H}$ be a set of histories for the bidding process, and $\gamma(\cdot)$ a function that assigns an outcome to each trader for each history. The strategy σ^* along with the prior distribution over $\mathcal{X}^{m+n}$ induces a probability distribution on $\mathcal{H}$ for trader i conditional on x_i. Since the set of possible histories is finite, a conditional probability distribution exists for each event in $\mathcal{H}$. Let $O_i(\cdot)$ denote the mapping from histories into outcomes for trader i. Again exploiting finiteness, the mapping O_i is measurable, so $H(p, k) \equiv \{h : O_i(h) = (p, k)\}$ defines an event in $\mathcal{H}$,which in turn defines a conditional probability distribution over types. Let $\mathbb{E}\{\cdot|\,(p, k)\}$ denote the expectation of a random variable with respect to this conditional distribution function.

Theorem 15 *If the assumptions required by Theorem 13 hold, then the outcome function $O(\cdot)$ associated with the internet trading mechanism has the rational expectations property.*

Proof. Let $(x_1, \ldots x_{m+n})$ be a vector of types that has positive probability given $(p, 1)$. Then by Theorem 13 $v_{(m)} = p$. Furthermore, if i trades there cannot be n traders other than i whose types strictly exceed $x_{(m)}$, since in that case each of them must win one of the n auctions by Theorem 13. So $x_i \geq x_{(m)}$ and $v_i \geq v_{(m)}$. This means that we can partition the histories in $H(p, k)$ according to the dropout decisions m traders other than i. Let h_d be a subhistory in which m traders drop out of the bidding, and write the expectation

$$\mathbb{E}\{u(x_i, \tilde{x}_{-i})\,|\,(p, 1)\}$$

as

$$\mathbb{E}_{h_d}\left\{\mathbb{E}_{x_{(1)},\ldots x_{(m)}}\left\{\mathbb{E}\left\{u(x_i, \tilde{x}_{-i})\,|\,\{\tilde{x}_{-i}\}_{(1)} = x_{(1)}, \ldots \{\tilde{x}_{-i}\}_{(m)} = x_{(m)}|h_d\right\}|h_d\right\}|\,(p, 1)\right\} \tag{3}$$

Since beliefs given by 7 through 10 follow Bayes rule along the equilibrium path, the inner expectation is i's willingness to pay in the terminal information set. By Lemma14, this must be no less than p since $v_i \geq p$. So the entire expectation is no less than p and trade is at least as good not trading. The argument is identical for the event $(p, 0)$. ■

There is more to this theorem than a simple demonstration that an outcome function having the rational expectations property exists. The theorem also shows that this outcome function is *implementable* in the weak sense that there is a non-cooperative game that supports it as an equilibrium. So if traders play this equilibrium when they participate in the internet trading mechanism, they act exactly as if they were playing a rational expecations equilibrium where belief functions are given as described before Theorem 15.

There are other ways to do this. The paper by (Reny and Perry 2002), for example, considers *double auctions* and shows that in the limit the equilibrium price function is fully revealing. For this reason, it satisfies the rational expectations property. The environment they consider is substantially different from the one here. In particular, the equilibrium price in a double auction for our environment is unlikely to be fully revealing except under much stronger conditions than we have imposed. However, it is a reasonable conjecture that the equilibrium pricing function for the double auction should have the rational expecations property. We consider the issue in more detail below. Even if it does, the belief function in the rational expectations equilibrium is somewhat mysterious, since it is derived by inverting the equilibrium outcome, in much the same way that a rational expecations equilibrium is calculated when the full information price function is revealing. Equation 3 in the proof of Theorem 15 illustrates a useful conceptual property of the rational expectations belief function associated with the internet trading mechanism since it is found by taking expectations of beliefs that traders actually compute using Bayes rule in various histories of the bidding process.

The rational expecations approach also adds something to the analysis of internet auctions. To see this, consider the set

$$E\left(p,1\right)=\left\{\left(x_i,x_{-i}\right):v_{(m)}=p;\left\{x_{-i}\right\}_{(m)}\leq x_i\right\}$$

and define the beliefs of trader i when the auctioneer announces price p and recommends that i consume to be the posterior beliefs conditional on the event $E\left(p,1\right)$. The value of a trade is now

$$\mathbb{E}\left\{u\left(x_i,x_{-i}\right)|E\left(p,1\right)\right\}=$$

$$\mathbb{E}_{x_{(1)},\ldots x_{(m)}}\left\{\mathbb{E}\left\{u\left(x_i,\tilde{x}_{-i}\right)|\left\{\tilde{x}_{-i}\right\}_{(1)}=x_{(1)},\ldots\left\{\tilde{x}_{-i}\right\}_{(m)}=x_{(m)}\right\}|E\left(p,1\right)\right\}$$

Since the inner expectation is just trader i's implicit value, and since this is at least p in every state in $E\left(p,1\right)$ this expectation is at least p. Since the outcome function associated with the internet trading mechanism has the rational expectations property, the expected value of trade given the true distribution of states given $(p,1)$ also exceeds p. Thus traders who act according to this simple belief function will announce the same demands as traders who have the rational expectations belief function. In this sense, the rational expectations approach helps to explain behavior in the internet auction.

5.1 Example

In this section we provide a brief example to illustrate the relationship between our internet trading mechanism and the more common double auction.

The traders in the internet mechanism adjust their beliefs as they observe the bidding behavior of other traders. The ability to observe this behavior before making trading decisions sometimes allows the internet trading mechanism to achieve efficiency when a double auction cannot. The advantages of the internet mechanism are exactly analogous to the advantages that the ascending price auction has over a sealed bid second price auction in the single seller case. The example also illustrates why beliefs need to depend on more than simply the price that the auctioneer announces.

In the first example there are 6 traders, one of them is a seller, the others are buyers. There are only two possible arrays of types for the traders given by

$$\{9, 8, 8, 8, 8, 8\}$$

and

$$\{9, 10, 10, 3, 3, 3\}$$

Ex ante, each of these arrays of types is equally like and every permutation of the identities of the traders is equally likely, so each trader can have any one of four possible types. Beliefs are symmetric. In this formulation, traders actually learn the true state from knowledge of their own type, except in the case where they have type 9. Payoffs are given by

$$u\left(x_i, \tilde{x}_{-i}\right) = x_i + W\left(\frac{\sum_{j \neq i} x_j}{5}\right)$$

where $W(x)$ is the largest integer that is less than or equal to x.

The important characteristic of this example is the trader with type 9 learns the true state once he knows the types of the traders whose values are below his. In particular, this means that the values of the traders in each state coincide with the implicit values, and are given by

$$\{17, 16, 16, 16, 16\}$$

and

$$\{14, 15, 15, 8, 8, 8\}$$

respectively.

It isn't hard to verify (brute force) that if the grid of types is restricted to $\{3, \ldots, 10\}$, then the single crossing condition holds. The very special probability distribution ensures that assumption 2 holds. The technical assumptions follow from the properties of the function $W(\cdot)$.

The restrictions of Theorem 13 all hold. So there is a weak perfect Bayesian equilibrium for the internet trading mechanism in which trade occurs at price 16 in the first state, and 15 in the second. Observe that the trader who has type 9 cannot deduce the state from knowledge of the price alone. However he consumes one unit of the good in the first state, and consumes nothing in the second. So the outcome of the internet trading mechanism fully reveals the state. The rational expectations property is immediate in this example.

Now let us establish that the ex-post efficient outcome cannot be attained in a double auction. Recall that in a double auction all traders simultaneously submit bids or asks, the market maker orders them in a single vector of values and executes trades that maximize *apparent gains from trade* by allocating the goods to the sellers and buyers whose bids and asks are above the m-the lowest value in this vector. A whole class of double auctions can be defined depending on the way in which the final price is set between the m-th and the $m+1$-st lowest bids.[16] Yet, since all the auctions are outcome-equivalent, we can focus on the *seller's offer* double auction where the price is set equal to the m-th lowest value (from the bottom)

Let us show that the seller's offer double auction has no ex-post efficient equilibrium. If the seller, for example, has type 9 then she will submit the same bid in both states. All of the other traders know their own values at this point. If the bid is above 16 then the seller will surely win in both states, if it is below *14* then she will surely lose in both states. Neither outcome is ex post efficient. So if there is an ex post efficient outcome, the seller must submit the bid either 14, 15 or 16. Whichever of these bids she submits, she will win in the wrong state..

The example illustrates a case where the internet auction achieves a better outcome than the double auction. It also illustrates why - the internet auction allows traders to observe when other traders drop out of the bidding. The advantages of the internet auction over the English auction are exactly analogous to the advantages that the English auction has over a sealed bid second price auction.

A slight modification of the example illustrates the connection between internet auctions, double auctions, and a rational expectations equilibrium. Suppose that the types of the traders in the second state are changed to $\{9, 11, 11, 3, 3, 3\}$. Then the vector of implicit values is given by $\{15, 16, 16, 10, 10, 10\}$. The internet auction then generates the same trading price in both state, though the outcome for the seller is different in each state, since he wins his auction in the first and loses in the second. Notice that the price alone will not reveal the state, though once the trader conditions on his own outcome, the allocation has the rational expectations property.

What of the double auction? In the seller's offer double auction, it is well known that bidding true value is a weakly dominant strategy for buyers. So suppose that all buyers who know their values bid their values. Then a seller with type 9 cannot affect the price that will prevail in the double auction. His bid only determines whether or not he trades. Whatever he chooses to do in equilibrium, the equilibrium price in the double auction will be 16 in both states. In particular, the equilibrium price in the double auction will not reveal the true state. Furthermore, conditioning on the outcome of trade does not help either

[16]These mechanisms have been studied by (Rustichini, Satterthwaite, and Williams 1994) in the private value case.

since the seller either wins of loses in both states. The only way for traders to deduce the true state from the outcome is for the seller to bid 16, then let the auctioneer allocate the unit of output in a way that depends on all the bids. The seller can then infer the state from the outcome that the auctioneer selects for him.

6 Appendix

Proof of Theorem 13:

Start with some terminology and notation. Let ι be an arbitrary information set. Since ι describes the observable history of bidding, it describes the list of standing bids and the identities of all high bidders. A *state of the game* is an information set ι along with a list of the unobservables associated with that information set - the types of all the traders and the list of high bids.

6.0.1 Part 1:The outcome associated with σ^*

Define trader i's *augmented* implicit value to be the maximum of his true implicit value and the highest winning bid that he has in any auction. Let $\hat{v}$ be *augmented vector of implicit values.*

We begin with some simple properties of the augmented vector of implicit values.

Claim 16 *Let v be a vector in $\mathbb{R}^{m+n}$. Then the number of components of v which are at or below $v_{(m)}$ must be at least as large as the number of components that strictly exceed $v_{(m)}$.*

Proof. The number of componentes of v at or below $v_{(m)}$ must be at least m since $v_{(m)}$ is the mth lowest value in v. Let b_1, b_2, b_3 and s_1, s_2, s_3 be the numbers of components that are below, at or above $v_{(m)}$ respectively. Then

$$b_1 + b_2 + s_1 + s_2 \geq m = b_1 + b_2 + b_3$$

Subtracting the common terms from both sides of the inequality gives $s_1 + s_2 \geq b_3$ ■

Claim 17 *Let $\hat{v}$ be the vector of augmented implicit values associated with some state that i believes is possible in information set $\iota(i)$. The number of traders whose augmented implicit values strictly exceed $\hat{v}_{(m)}$ and who are not already high bidders at auctions where the standing bid exceeds $\hat{v}_{(m)}$ can be no larger than the number of auctions where the standing bid is at or below $\hat{v}_{(m)}$.*

Proof. Of the n auctions, let n_1 be the number where the standing bid strictly exceeds $\hat{v}_{(m)}$. By definition, the augmented implicit values of the high bidders at each of these auctions must be strictly higher than $\hat{v}_{(m)}$. So there are n_1 traders whose values exceed $\hat{v}_{(m)}$ who are high bidders at standing bids above $\hat{v}_{(m)}$. In the $n - n_1$ other auctions, the standing bids are less than or equal to $\hat{v}_{(m)}$. By the definition of $\hat{v}_{(m)}$ there cannot be more than n traders with augmented implicit values strictly above $\hat{v}_{(m)}$, and so at most $n - n_1$ of the traders who are not already high bidders where standing bids exceed $\hat{v}_{(m)}$ can have implicit values that strictly exceed $\hat{v}_{(m)}$. ■

Lemma 18 *Consider any information set and state in $S_i(\iota)$. Suppose that all bidders use strategy σ^*in the continuation. Then there are two cases: (i) any trader who does not have a high bid above $\hat{v}_{(m)}$ in ι will pay a price that is no higher than $\hat{v}_{(m)}$ in any auction he or she wins in the continuation; (ii) any trader who has a high bid above $\hat{v}_{(m)}$ in ι will win each of the auctions where she has submitted those bids.*

Proof. If either (i) or (ii) is false, then some trader using σ^* must be willing to submit a bid at price $p > \hat{v}_{(m)}$. This requires three things to be true. First this bidder must have an implicit value strictly larger than $\hat{v}_{(m)}$. Second, this bidder cannot be a high bidder at the time that she submits her bid. Third, no other auction can have a lower standing bid. All n of the auctions must have high bidders. In any state that i thinks is possible in $S_i(\iota)$, each of these high bidders, with the possible exception of i has an implicit value that strictly exceeds $\hat{v}_{(m)}$. In case (i) no bidder will bid at a price above $\hat{v}_{(m)}$ unless his adjusted bid price exceeds $\hat{v}_{(m)}$. Thus there must then be $n + 1$ bidders with valuations strictly above $\hat{v}_{(m)}$. This contradicts the definition of $\hat{v}_{(m)}$. In case (ii), in the initial information set, the number of traders whose implicit values strictly exceed $\hat{v}_{(m)}$ and who are not high bidders at auctions where the standing bid exceeds $\hat{v}_{(m)}$ can be no larger than the number of auctions where the standing bid is at or below $\hat{v}_{(m)}$ by Claim 17. Thus there can be no more than $n - 1$ traders other than i whose implicit values exceed $\hat{v}_{(m)}$. Since $n - 1$ such traders are already high bidders at standing bids above $\hat{v}_{(m)}$ in the information set where the price p is bid, there is no trader to submit this bid. ■

Lemma 19 *Beginning in any information set and state in $S_i(\iota)$. Suppose that all buyers other than i use σ^* in the continuation. Then trader i cannot win an auction at a price strictly below $\hat{v}_{(m)}$.*

Proof. Suppose that i wins at price $p < \hat{v}_{(m)}$. Then there is at least one auction where the standing bid is strictly less than $\hat{v}_{(m)}$ in the terminal information set. Suppose there are r_1 such auctions. The terminal information set is regular for i since all traders other than i use σ^* in the continuation. Of the n traders whose adjusted bid prices are at or above $\hat{v}_{(m)}$, at most $n - r_1$ of them can be high bidders at standing bids at or above $\hat{v}_{(m)}$. So there must be at least r_1 traders whose implicit values are at least $\hat{v}_{(m)}$ who are either high bidders at standing bids below $\hat{v}_{(m)}$ or who aren't high bidders at all. If no bids are submitted at the price p, then because the other b bidders are using σ^*, they must all have implicit values strictly below $\hat{v}_{(m)}$. This contradicts the definition of $\hat{v}_{(m)}$. We conclude that there are strictly more traders whose implicit values exceed p and who either aren't high bidders, or are high bidders at auctions where the standing bid is less than $\hat{v}_{(m)}$,than there are auctions with standing bids below $\hat{v}_{(m)}$. If the price p is not bid up, then at least one of these traders must pass when his implicit value exceeds p, or must submit a bid at an auction where the standing bid exceeds p. Either of these events is inconsistent with σ^*. ■

Corollary 20 *Consider some information set ι and state in $S_i(\iota)$. Suppose that all buyers use σ^* in the continuation. Then trader i will win some auction and pay $\hat{v}_{(m)}$ if his implicit value is strictly larger than $\hat{v}_{(m)}$. Alternatively if i's implicit value is strictly less than $\hat{v}_{(m)}$ then trader i will not win an auction.*

Proof. If i wins an auction, then he will pay $\hat{v}_{(m)}$ by Lemmas 19 and 18. Trader i will bid at $\hat{v}_{(m)}$ if and only if his implicit value exceeds $\hat{v}_{(m)}$. ■

6.1 σ^* is a best reply for buyers

The argument uses Lemma 14 and Corollary 20 to compare the outcome when the buyer uses the action specified by σ^* compared to any other action. The argument simply checks each deviation in each type of information set. Lemma 14 shows that buyer i cannot change the implicit values of the other traders in any state that he thinks is possible. Lemma 20 provides the outcome associated with any array of implicit values when all traders follow σ^*.

To begin, consider an information set where the buyer has to decide whether to pass or continue bidding. There are only two possibilities, either his implicit value is at or below the lowest standing bid or not. In the former case, if he passes as specified by σ^* his payoff in the continuation is zero. If he deviates and decides to bid, he cannot trade at a price below q_{r+1}. Since his implicit value is less than or equal to q_{r+1}, his willingness to pay in the terminal information set cannot exceed q_{r+1} so no trade that he makes with this strategy can be profitable. If his implicit value exceeds the lowest standing bid, his payoff when he follows σ^* is $\max[v_i - v_{(m)}, 0]$ by 20. If he deviates and passes once, then reverts to σ^* in the continuation, his payoff will be the same in each state except the one where all traders who are not high bidders pass on their current turn (so the game ends). In this case the deviation results in a loss of a profitable trade. So for every state the buyer thinks is possible, deviating from σ^* either leaves the payoff unchanged or lowers it.

In an information set where a buyer is supposed to bid, things are more tedious. However, we again do a state by state comparison. Let p^* be the bid specified by σ^* in the information set, and p any alternative bid. Since $p \geq q_{r+1}$ by the rules of the game we can reduce the set of potential outcomes to the following table in the case where $v_i > q_{r+1}$. The top row describes the possible states, the first column the different bids, and the cells within the tables depict the payoffs if i follows σ^* after this information set. The interesting cell is the one on the top right where the buyer bids p and $v_{(m)} < p$. If $p > \hat{v}_{(m)}$ then the buyer will win the auction and pay price at least $\hat{v}_{(m)} \geq v_{(m)}$ (whether or not

$v^i > v_{(m)}$), which is clearly worse than what he gets by bidding p^*. If $p = \hat{v}_{(m)}$ then i may or may not trade. If he does he pays at least $\hat{v}_{(m)}$ by Lemma 19, which is more than $v_{(m)}$. If he doesn't trade he gets either the same payoff as that attainable by bidding p^*, or loses a trade that he might have had for price $v_{(m)}$. So deviations are never profitable for the buyer.

	$p \leq \hat{v}_{(m)}$	$p > \hat{v}_{(m)}$
p	$\max\left[v^i - \hat{v}_{(m)}, 0\right]$	$\leq v^i - \hat{v}_{(m)}$
p^*	$\max\left[v^i - \hat{v}_{(m)}, 0\right]$	$\max\left[v^i - \hat{v}_{(m)}, 0\right]$

It is straightforward that i will submit the lowest possible bid if his value is less than or equal to the lowest standing bid, as we have already established that he cannot gain in any auction he wins in that case.

6.2 Seller's Part

As argued above, there are some states where sellers would like to deviate from σ^*. Focus first on an information set where i has to submit a bid. Since i may submit a bid above his implicit value, he may change the vector of augmented implicit values used to calculate the outcome. Let $\hat{v}'$ denote the vector of augmented implicit values when i submits the bid p, while $\hat{v}$ is the augmented vector of implicit values when i submits the bid p^* that he is supposed to make if he follows σ^*. Using the notation for buyers above, the effects of a deviation are captured in the following table:

	$p < \hat{v}_{(m)}$	$p = \hat{v}'_{(m)}; p^* \leq \hat{v}_{(m)}$	$p = \hat{v}'_{(m)}; \hat{v}_{(m)} > p^*$	$p > \hat{v}'_{(m)}$
p	$\max\left[\hat{v}_{(m)} - v_i, 0\right]$	E^+	E^-	0
p^*	$\max\left[\hat{v}_{(m)} - v_i, 0\right]$	$\max\left[\hat{v}_{(m)} - v_i, 0\right]$	$\max\left[\hat{v}_{(m)} - v_i, 0\right]$	$\max\left[\hat{v}_{(m)} - v_i, 0\right]$

If $\hat{v}_{(m)}$ exceeds i's bid, then other traders will bid up the price past p and if i reverts to σ^* his payoff will be as given in the table, whether he bids p^* or p. At the other extreme if $p > \hat{v}_{(m)}$ then i must win the auction where he submits the bid by Lemma 18. If he uses the action specified by σ^* then he only wins an auction if his implicit value exceeds the common trading price. So the seller cannot gain in this event.

One potential gain for the seller occurs when a sale occurs at price p which would have occurred at price $v_{(m)} < p$ if the seller has observed σ^*. This occurs with positive probability in the event

$$E^+ = \left\{x : \hat{v}'_{(m)} = p; \hat{v}_{(m)} = p^*\right\}$$

as illustrated in the box. For this to occur, both p^* and p must be the m^{th} highest implicit value. If p^* is the m^{th} highest implicit value, then each of the traders with the n highest values has value at least equal to p^*. By assumption 2, at least one of these traders must have value equal to p^* so event E^+ occurs with zero probability.

The second is that a profitable sale that would have occurred at price p for sure if i had observed σ^* will be lost if i bids p. This occurs with positive probability in the event

$$\hat{E} = \left\{x : \hat{v}_{(m)} = p; v_{(m)} \neq p^*\right\}$$

and again, this cannot benefit the seller. Finally, in the event

$$E^- = \left\{x : \hat{v}_{(m)} = v_{(m)} = p^*\right\}$$

the seller will lose any trade that might have occurred at price p^*. These latter trades might not be profitable for the seller, but if they are, they again involve a loss for the seller.

Now this argument can simply be extended to information sets where the seller has to decide whether or not to bid. If he enters and bids when he is supposed to stop, then by the previous argument, he cannot gain in the continuation. If he stops when he is supposed to bid, he gives up the opportunity to make a profitable trade if all other traders choose to pass.

References

Ausubel, L. M., and P. Milgrom (2001): "Ascending Auctions with Package Bidding," University of Maryland Working paper.

Bajari, P., and A. Hortacsu (2002): "The Winner's Curse, Reserve Prices and Endogenous Entry: Empirical Insights from eBay Auctions," Discussion paper, Stanford University, to appear in Rand Journal of Economics.

Cripps, M., and J. Swinkels (2002): "Depth and Efficiency of Large Double Auctions," Washington University in St. Louis.

McAfee, P. (1993): "Mechanism Design by Competing Sellers," *Econometrica*, 61(6), 1281–1312.

Peters, M., and S. Severinov (1997): "Competition Among Sellers who offer Auctions Instead of Prices," *Journal of Economic Theory*, 75(1), 141–179.

——— (2001): "Internet Auctions with Many Traders," Toronto:University of Toronto, Department of Economics.

Reny, P., and M. Perry (2002): "Information Aggregation In A K-Double Auction with Interdependent Values," paper presented at the North American Winter meetings of the Econometric Society, 2002.

Roth, A. E., and M. A. Sotomayor (1990): *Two-Sided Matching - A Study in Game Theoretic Modelling and Analysis*, Econometric Society Monographs. Cambridge University Press, Cambridge MA.

Rusticchini, A., M. A. Satterthwaite, and S. R. Williams (1994): "Convergence to Efficiency in a Simple Market with Incomplete Information," *Econometrica*, 62(5), 1041–1063.

Wolinsky, A. (1988): "Dynamic Markets with Competitive Bidding," 55(181), 71–84.