Consumption, Portfolio Policies, and Dynamic Equilibrium in the Presence of Preference for Ownership

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Abstract

In this article we construct a model in which agents exhibit preference for ownership with respect to a durable (house). Ownership is modeled as a continuous function of debt service scaled by the house price. We study the utility optimization problem of an investor not endowed with the durable. In other words, the agent borrows against his endowment stream in order to be able to purchase the house immediately. However, due to liquidity constraints, which prevent the agent from selling the fraction of his endowment that corresponds to the house price, i.e. shorting financial assets, the purchase of the house involves payments over a long horizon. This work presents effects of preference for ownership on demands, interest rate and asset risk premium. The agent accumulates quickly wealth in the durable and postpones consumption. It is shown that the poor and liquidity constrained agent does not hold financial assets until the durable is completely acquired. We also show that the equilibrium interest rate depends, in particular, on the debt payments. Conditions under which preference for ownership decreases the equilibrium interest rate are provided. Nonseparabilities can cause a higher equity risk premium.

1 Introduction

The basic idea we pursue in this work is that agents are not only concerned about the return characteristics and the flow of services of a house but also about owning the durable. So there is a "pride of ownership" effect. We analyze this effect of preference for ownership on demands and equilibrium in a pure exchange economy with two agents. Ownership is modeled as the normalized capital share of a house that belongs to the investor under consideration.

The average US household portfolio is composed as follows: residential property accounts for 27% of total assets and other durables are about 7.5%, whereas stocks are only 28% of total assets [Source: Flow of Funds Accounts, 1998.] Notice also that 68% of the US households are homeowner [Dietz and Haurin (2002)]. On the other hand we know that compared to investment in stocks or bonds a house purchase has three major drawbacks. These are forgone diversification, indivisibility, and illiquidity [see Berkovec and Fullerton (1992) and Shiller (1993) for housing risk]. Building on these observations we assume that ownership matters for economic individuals. This means that utility derived from ownership[1] compensates for some of the disadvantages of a house purchase[2].

In an influential work Mankiw and Zeldes (1991) report not only the very low stock market participation of investors but also the surprisingly low position in stocks of relative wealthy individuals. A decade later, thanks to enduring promotion in the mass media, 401(k) plans, and most importantly the sustained growth in stock prices, the relative importance of financial assets and, in particular, of stocks increased substantially. On the other hand not much has changed given that around 50% of all stock wealth is owned by the top 1% of households ranked by non-human wealth and only 50% of all households hold stocks. Even worse, this does not say anything

[1] We do not claim that preference for ownership is a phenomenon that can be observed only with respect to houses. It is evident that owners of small firms may have similar intentions and behave in comparable ways as homeowner in our model. But for the sake of convenience we assume here that the dispersion of the stock in our model is so high that ownership does not matter.

[2] Tax benefits are another form of compensation but frictions on the rental market might be involved as well. Additionally, homeownership was and will be a good hedge against unanticipated inflation. Further, a mortgage is one of the easiest ways to borrow (large stakes) against future labor income and a decent hedge, though extremely expensive, against fluctuations in lodging services.

about the importance of stocks in the portfolio of the average investor. On top of that, most homeowner are by nature over invested in their residential property, and this fact will play a prominent role in our model, hence, there cannot be any doubt that it is premature to conclude that nonparticipation in the stock market is no longer a puzzle.

In this paper we consider a model with a perishable and a durable good. The durable good is interpreted as a house. Since we want to focus on the ownership decision it is assumed that there is only one house size, the durable is indivisible, and lodging services are constant over time. We analyze the consumption and portfolio behavior of a "poor" agent not endowed with a house. The poor investor has to finance his consumption as well as his ownership needs out of an endowment stream. To prevent the poor agent from selling the fraction of his endowment that corresponds to the house price, i.e. shorting the stock or the bond, which certainly is not very common, the following endogenous constraint arises. The poor agent's liquid wealth process needs to be positive until the debt burden associated with the house purchase is completely eliminated. The rich agent in this model is endowed with the stock and two houses[3]. We assume that agents behave competitively in the markets. So poor buys a house from rich. All this implies that the house is purchased by the poor investor only with a fraction of its (relative) price. We characterize consumption demand, optimal ownership, and investment behavior of the poor agent. Our analysis reveals that the "pride of ownership" effect induces a path dependent debt policy. As a consequence even if the utility function is additively separable consumption will interplay with the level of ownership. Consumption is decreased in early age and postponed either until a substantial level of ownership is achieved or probably even after the mortgage is paid down. More importantly, it is shown that preference for ownership pushes the liquid wealth into negative values. Clearly, the nonnegativity constraint on liquid wealth pushes it back. Therefore, liquid wealth has to be zero until the durable is completely acquired. This insight has dramatic implications for the investment policy of the poor agent. There is only one way such that a portfolio strategy leads to zero liquid wealth at all times and this requires not to invest into stocks at all.

In essence, the nonparticipation in the stock market depends on the as-

[3] The way how we label the agents is somewhat arbitrary. In general, we do not need to assume that the labor income of the poor agent is smaller than the two house prices plus the value of the stock. However, it is not only convenient but also reasonable to think about the agent not endowed with the house being poor.

sumption that the endowment stream does not introduce a new source of risk. Hence, it is perfectly correlated with the stock market in this economy. The argument, however, will go through as long as incremental changes in the endowment stream are positively correlated with stock returns.

Based on the assumption that the exogenous supply of houses is fixed over time we characterize also dynamic equilibrium. Another consequence of the introduction of preference for ownership is that the equilibrium interest rate directly depends on the debt service and the divergence of risk aversion across agents, poor and rich, where the risk aversion is to be understood with respect to ownership normalized by perishable consumption risk, i.e. composite risk. Conditions under which preference for ownership decreases the equilibrium interest rate are provided. The model also has implications for the equity premium puzzle. The expression for the premium suggests that even though the model cannot account for an additional term in the formula, it still can affect the premium through composite risk. The pure single beta consumption CAPM, where only perishable consumption matters, is retrieved if and only if the utility functions are additively separable. An additional contribution of the paper, which emerges from equilibrium analysis, is a formula for the value of a mortgage.

Even though we are not aware of other work on preference for ownership several different strands of the asset pricing literature are, of course, related to our work. Grossman and Laroque (1990) derive optimal portfolio choice in the presence of an illiquid durable consumption good. The durable consumption good comes in stocks of various sizes and is indivisible. Furthermore agents do not have additional utility from holding multiple units of the good. To change consumption the agent must sell the durable and buy a new one. The durable holding process is discontinuous because consumers do face some adjustment costs. Because of the adjustment costs the consumption CAPM fails even though equilibrium asset prices satisfy the CAPM. This durable good can be interpreted as a house or a car. In a complementary work Cuoco and Lui (2000) examine the intertemporal optimal consumption and investment problem in continuous time. The durable is divisible. As in Grossman and Laroque (1990) the optimal consumption policy does not satisfy the usual first order conditions due to transaction costs. Again the traditional CAPM holds while the consumption CAPM fails. In this natural extension, the durable is regarded as furniture or clothing. Hindy and Huang (1993) study the optimal consumption and investment process where investors receive utility from an irreversible purchase of a durable. Services are produced from past purchases. In their model agents invest more in the risky asset than in a comparable economy with no dependence on past consumption paths. One characteristic of this kind of economy is that the models abstract completely from nondurable consumption. Detemple and Giannikos (1996) examine an economy with perishable and durable commodities. The durable provides not only services but also status. They examine the effects of the durables attributes on demands and on equilibrium. Our work is also related to the impressive calibration exercise of Caroll and Dunn (1997). They developed a model where debt financed durables purchases depends on labor income uncertainty. When labor income uncertainty increases agents postpone purchases until their financing condition improves. Due to the complexity of the model it is not possible to solve it in a general equilibrium setting. Our work can be understood as a simplified version of their model where uncertainty is only one dimensional and thus labor income is spanned by traded assets. Cocco (2000) analyses the demand for perishable goods and durables (housing). He shows that housing can crowd out stockholding. Flavin and Yamashita (2002) analyze another generalization of Grossman and Laroque (1990) by focusing on intervals within the household chooses not to sell the current house. Agents maximize expected lifetime utility of a perishable consumption good and the house. They show that young households tend to hold bonds or pay down their mortgage as opposed to hold stocks. Flavin (2002) confirms the finding of Grossman and Laroque (1990) that with adjustment costs and nonseparabilities the CAPM holds, while the consumption CAPM does not. Hu (2002) analyzes the choice between owning and renting. In the model equity holdings are reduced by 50% under the assumption that owning provides more utility than renting. Notice that all models from Caroll and Dunn (1997) to Hu (2002) take price processes as given. Piazzesi, Schneider and Tuzel (2002) show, as in Detemple and Giannikos and in our model, that composite risk, the expenditure share of perishable consumption and housing services, can at the same time increase the equity premium and lower the riskfree rate. However, they do neither address portfolio choice nor nonparticipation in the stock market. Finally, Davidoff (2003) addresses the influence of income and house price covariation on housing choice.

The outline of the paper is as follows. In Section 2, we present the economy. Section 3 focuses on the optimization problem of the poor agent. From the first order conditions we deduce in Section 4 equilibrium allocations and valuation formulas for financial assets. In Section 5, interest rate and risk premium are examined. Section 6 concludes the paper. Proofs are in the Appendix.

2 The economy

We consider an exchange economy in continuous time. The uncertainty is represented by a complete probability space $\left(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,\infty]}, P\right)$ supporting a standard one-dimensional Brownian Motion, W, over the finite time horizon $[0,T]$, where Ω is the state space, $\mathcal{F}$ is the σ-algebra representing measurable events, and P is the probability measure. The information structure $\mathcal{F}_{(.)}$ is generated by the natural filtration and we assume that $\mathcal{F}_T = \mathcal{F}$. All the stochastic processes to appear in the paper are progressively measurable[4] with respect to $\mathcal{F}$ and all (in)equalities involving random variables are understood to hold $P - a.s.$

2.1 Consumption space

We assume that the consumption good is perishable and that the consumption set is given by a nonnegative process satisfying $\int_0^T c_t dt < \infty$.

2.2 Financial markets

Two types of investment opportunities are available: a locally riskless asset and a risky asset[5]. The riskless asset pays an interest rate r. The risky asset pays dividends D such that,

$$dD_t = D_t\left[\gamma_t dt + \lambda_t dW_t\right] \tag{1}$$

D_0 given. The price of this risky asset which we will refer to be the stock satisfies the following equation,

$$dS_t + D_t dt = S_t\left[\mu_t dt + \sigma_t dW_t\right] \tag{2}$$

[4] A process $X = \{X_t, t \in [0,T]\}$ is p.m. with respect to $\mathcal{F}_{(.)}$ if for each $t \in [0,T]$ the *map* $(s,w) \longrightarrow X(s,w)$, defined on $[0,T] \times \Omega$, is measurable with respect to the product $B_t \times \mathcal{F}_t$ where B_t denotes the Borel sigma field on $[0,T]$.

[5] None of the results will be altered if we introduce a set of risky assets.

where S_0 is given and we assume that only 1 share of the stock is available. The process μ represents the expected return and σ is the volatility coefficient. Note that the dividend process is exogenously given whereas the interest rate r and the stock price parameters μ and σ are equilibrium values.

In addition there is also a durable good available in this economy which we will refer to be the house. In this model it is assumed that the asset house is available with one size only, is indivisible, and generates services automatically. The services can be interpreted as lodging and are generated without any differences in terms of quality over the whole lifetime of the economy. At date zero the house is available for a (relative) price of H_0.

Before we proceed, let us introduce two important processes which will be employed quite frequently. Consider the variable

$$\theta_t = \sigma_t^{-1} (\mu_t - r_t) \tag{3}$$

which represents the market price of risk[6]. The corresponding state price density process is

$$\xi_t = \eta_t b_t \tag{4}$$

where b_t can be interpreted as a money market account,

$$b_t = \exp\left[-\int_0^t r_s ds\right] \tag{5}$$

the process η_t

$$\eta_t = \exp\left[-\int_0^t \theta_s dW_s - \frac{1}{2}\int_0^t (\theta_s)^2 ds\right] \tag{6}$$

is an exponential martingale associated with the market price of risk, and Arrow-Debreu prices are $\xi_t dP$. For the optimization problem, Section 3, we will treat the market price of risk and the state price density process as given, while in Section 4 the processes represent equilibrium values.

2.3 The notion of ownership

Ownership of one house at date t is defined as the ratio of the date zero discounted value of all payments up to and including the t-th payment divided

[6] The market price of risk θ is a progressively measurable process and satisfies $Eexp\left[\frac{1}{2}\int_0^T \parallel \theta_t \parallel^2 dt\right] < \infty$ P-a.s..

by the initial price of the house. In the same vein ownership is the normalized capital share of the house that belongs to the investor under consideration. More formally, the level of ownership is

$$o_t = \min\left(\frac{\int_0^t \xi_s l_s ds}{H_0}, 1\right) \tag{7}$$

where l_s is the debt service at time s and H_0 is the (relative) price of the house at date 0. The minimum function guarantees both that ownership for one house never becomes larger than one and that the seller is fully compensated on all paths the economy eventually follows.

2.4 The agents and their endowment

The agents in this economy can be characterized by VNM utility function (U) :

$$U(c,o) = E\left[\int_0^T (\rho_t)^{-1} u(c_t, o_t)\, dt\right] \tag{8}$$

where $(\rho_t) = \exp\left[\int_0^t \beta_v dv\right]$, $0 < \beta < 1$, representing the subjective discount rate, $u(.,.)$ is the instantaneous utility function, c_t denotes the consumption at time t and o_t the achieved level of ownership[7]. The utility function $u(.,.)$: $[0,\infty) \times [0,\infty) \longrightarrow (-\infty,\infty)$ satisfies the following conditions:

- $u(.,.)$ is a twice continuously differentiable function, non-decreasing, and strictly concave in each argument (and concave in (c,o)),
- $u(.,.)$ is separable, time additive, and state independent,
- $u(.,.)$ satisfies the Inada conditions with respect to the first and second argument: $u_1(0,.) = \infty$, $\forall o \in \Re^+$; $u_2(.,0) = \infty$, $\forall c \in \Re^+$; $u_1(\infty,.) = 0$, $\forall o \in \Re^+$; $u_2(.,\infty) = 0$, $\forall c \in \Re^+$[8].

[7] Since services are constant over time we suppress them in the analysis.

[8] $u_1(.,.)$, and $u_2(.,.)$ denote the derivatives of $u(.,.)$ with respect to the first and second argument respectively.

Let us define $I^1(y^1, o)$ and $I^2(y^1, y^2)$ as the inverse function of $u_1(c, o)$ with respect to c for o given and the inverse of $u_2(I^1(y^1, o), o)$ with respect to o for y^1 fixed, i.e.

$$u_1\left(I^1\left(y^1, o\right), o\right) = y^1 \tag{9}$$

$$u_2\left(I^1\left(y^1, I^2\left(y^1, y^2\right)\right), I^2\left(y^1, y^2\right)\right) = y^2. \tag{10}$$

We assume that the functions I^1 and I^2 exist, are continuously differentiable and have well defined limiting values[9].

This economy is populated with a poor agent and a rich agent. The poor agent receives only an endowment stream. The endowment stream e is a progressively measurable process, such that,

$$de_t = e_t\left[\mu_t^e dt + \sigma_t^e dW_t\right] \tag{11}$$

where e_0 is given and the processes μ^e and σ^e represent the drift and the volatility coefficients. Progressive measurability implies that the income process is spanned by market assets and therefore is not a source of new uncertainty[10]. The reason for the introduction of the endowment process is simple. We need to give a meaning to the strategy of purchasing the house with debt. Additionally, to prevent the poor agent from selling the fraction of his endowment that corresponds to the house price, i.e. shorting the stock or the bond, at date 0 and purchasing the house in one shot, it is required that the poor agent's liquid wealth process, X, is positive until the purchase of the house is completed. The rich agent's endowment is composed out of the stock S_0 and two houses H_0. Agents behave competitively in the markets. In particular, the rich agent sells one of his houses to the poor. This assumption requires, of course, that the poor agent is not too poor but has sufficient future wealth on each path the economy eventually takes. In the following we will focus on the optimization problem of the poor agent. However, in the equilibrium analysis both agents are considered.

A triplet (π, c, l) of investment in the stock, consumption of the perishable good, and purchase of "ownership" is admissible for the poor agent if and

[9] Limiting values are, $lim_{y^1\uparrow\infty} I^1\left(y^1, o\right) = 0$ and $lim_{y^1\downarrow 0} I^1\left(y^1, o\right) = \infty$, $lim_{y^2\uparrow\infty} I^2\left(y^1, y^2\right) = 0$ and $lim_{y^2\downarrow 0} I^2\left(y^1, y^2\right) = \infty$.

[10] Clearly, this is not very realistic. As in Detemple and Serrat (1998 and 2003), El Karoui and Jeanblanc-Picqué (1998), and He and Pagés (1993) this assumption is needed to keep the model tractable. Note that if the income stream is driven by a second Brownian Motion and the stream is not marketed, then, agents face incomplete markets. See also Carroll and Dunn (1997) for a more realistic but almost intractable model.

only if the liquid wealth process X satisfies the nonnegativity constraint, that is,

$$X_t \geq 0 \quad \text{P-a.s.} \tag{12}$$

where X solves the stochastic differential equation

$$dX_t = (r_t X_t + e_t - c_t - l_t)\, dt + \pi_t \left[(\mu_t - r_t)\, dt + \sigma_t dW_t \right] \tag{13}$$

subject to the initial condition $X_0 = 0$. The portfolio π_t is a progressively measurable process which satisfies the integrability condition $\int_0^T \pi_t (\mu_t - r_t)\, dt + \int_0^T (\pi_t \sigma_t)^2\, dt < \infty$ P-a.s. and represents the amount, denominated in the consumption good, invested in the risky asset. A consequence of the liquidity constraint, $X_t \geq 0$, on the poor agent is that the bankruptcy condition, $X_t + E \int_t^T \xi_{t,s} (e_s - c_s)\, ds \geq 0$, usually needed for this kind of problem to hold will never bind[11].

Notice that although formally the poor agent is prevented from borrowing against his future income, in fact this condition does not hold when a durable is purchased. We interpret this as collateralization.

Recall, that the objectives of the paper are threefold: First, we want to show that a solution to the maximization problem, maximum of (8) s.t. (12) and (13), of the poor agent exist. Second, optimal consumption, optimal ownership, optimal debt service, and portfolio policy are characterized. Third, we analyze the implications of preference of ownership on dynamic equilibrium. The paper focuses, in principle, on the solution as long as the poor agent is still coping with the payment stream associated with the house purchase. Whenever the poor investor emerges from the debt burden, then, we are back to a version of a Lucas (1978) economy.

3 Optimization

In this section we analyze the problem of the poor agent with standard martingale methods. The static optimization problem associated with the dynamic optimization problem of the investor is (Cox and Huang (1989,

[11] Of course, the condition holds also under the present setting. Via an application of Ito's lemma and the martingale representation theorem one can show that the liquid wealth process is a local martingale and bounded from below. Hence, by Fatou's lemma a supermartingle which proves the condition.

1991), Karatzas, Lehocky and Shreve (1987))

$$\sup_{c,l} U(c,o) \quad \text{s.t.} \tag{14}$$

$$E\int_0^T \xi_t c_t dt \leq E\int_0^T \xi_t e_t dt - H_0 \tag{15}$$

$$H_0 = E\int_0^T \xi_t l_t dt \tag{16}$$

$$X_t = E_t \int_t^T \xi_s (c_s + l_s - e_s) ds \geq 0 \quad s \in [0,T]\, 1\!\!I_{(o_s<1)} \tag{17}$$

$$c, o \geq 0. \tag{18}$$

The intuition underlying the static optimization approach is the following: The present value of all planned expenditures cannot exceed the value of endowment. Since the market is complete there exists a (unique) trading strategy supporting the desired consumption path. Thus it is sufficient to optimize c and l in a static framework. Note that l_t does not appear directly in the static budget constraint (15) of the agent. It is also required, equation (16), that the agents discounted debt service equals the initial house price[12]. This constraint ensures that the seller is compensated for the loss of utility associated with the exchange of the house[13]. Equation (17) is the liquidity constraint. The function $1\!\!I_{(o_t<1)}$ is an indicator function. To proceed with the optimization problem we define y^1 and y^2 to be the Lagrange multipliers associated with the static problem and k the multiplier for the liquidity constraint. The following theorem provides the first order conditions for the optimization problem .

Theorem 1 *The policy (c,l) is optimal if and only if (c,l,y^1,y^2,k) solves,*

$$u_1(c_t, o_t) = \rho_t \xi_t \left(y^1 - k_t\right) \tag{19}$$

[12]For simplicity we assume that $H_0 < E\int_0^T \xi_t e_t dt < 2H_0$ holds.

[13]Here there is no minimum down-payment to be made initially. We make this simplification because we did not want to burden the argument of the paper with the technical difficulties a constraint like this induces, since it is not important for the essential point we want to make. Moreover, there is no default in our model and thus requiring a down-payment seems not necessary.

$$E_t \int_t^T (\rho_s)^{-1} u_2(c_s, o_s)\, ds = \left[y^2 - k_t\right] H_0 \tag{20}$$

$$c_t \geq 0, o_t \geq 0, t \in [0, T], y^1 > 0, y^2 > 0, k_0 = 0 \tag{21}$$

$$E \int_0^T \xi_t c_t dt \leq E \int_0^T \xi_t e_t dt - H_0 \tag{22}$$

$$H_0 = \int_0^\tau \xi_t l_t dt. \tag{23}$$

$$E_t \int_t^\tau \xi_s (c_s + l_s - e_s)\, ds \geq 0 \tag{24}$$

$$E \int_0^\tau X_t dk_t = 0 \tag{25}$$

where c, l and o are progressively measurable and square-integrable processes[14].

Equation (19) characterizes the standard optimality condition in a complete market: marginal utility of consumption equals marginal cost of consumption in every state of nature. However, the cost structure here involves, in contrast to the standard model, an adjustment, k, that accounts for the liquidity constraint. This implies that even when the utility function is additively separable the choice of perishable consumption will always depend on the choice of ownership in that particular state through the multiplier k. Due to the structure of ownership, equation (20) is slightly more involved. The left hand side is composed of expected marginal utility of ownership from t to T associated with an incremental increase of ownership at date t. The right hand side represents the cost of this policy. Obviously, the cost structure of the ownership policy is also affected by the liquidity constraint. Notice that both first order conditions represent optimal policies under the assumption that the debt service is active. Whenever $t > \tau$, i.e. $o_t = 1$, then, the agent restarts the optimization problem anew, however, without the liquidity constraint and a flat ownership at unity. Equation (22) is the static budget constraint. Whereas equation (23) is due to the arbitrary constraint that this agent can finance only one house during his lifetime. The fifth condition

[14] Note that, initially, it is not required that l is nonnegative. Thus adjusting the optimal level of ownership, in principle, is not constrained. Clearly, under the present setting neither l nor o ever becomes negative.

is the liquidity constraint and, lastly, equation (25) is the complementary slackness condition.

It is worth to point out that the aforementioned multiplier process will induce a particular pattern on the consumption behavior of the poor agent. Consumption in early age will be reduced (The standard Lagrangian, y^1, associated with consumption of the perishable good will be higher than in an economy without a Lagrangian process k.) and, consequently, consumption will be postponed. Over time the Lagrangian process will increase, which decreases the shadow cost of consumption. When the mortgage is paid down, then, an additional drop in the price of consumption occurs.

Another consequence of preference for ownership is the forward-backward structure of the first order condition with respect to ownership. Before we focus on this equation in detail and turn to the analysis of the shadow costs of ownership, k, we illustrate the striking nature of liquid wealth for the poor investor.

Theorem 2 *The optimal liquid wealth of the poor and, therefore, liquidity constrained investor is given by*

$$X_t = 0 \quad t \in [0, \tau] . \tag{26}$$

This result has three strong implications. First, there is no investment in the stock

$$\pi_t = 0 \quad t \in [0, \tau] . \tag{27}$$

Second, the poor agent performs no intertemporal transformation and, therefore, he cannot influence the terminal date for the debt payments by accumulating wealth in the stock market [Substitution from consumption towards ownership, however, is possible.]. Notice here that this is his optimal strategy. Third, in contrast to models with liquidity constraints[15] but without agents deriving utility from ownership the optimization involves the stochastic multiplier process dk that is not only predictable and nondecreasing but also strictly positive until $o_t = 1$[16].

[15] El Karoui and Jeanblanc-Picque (1997) solve the control problem of a liquidity constraint investor as a stopping time problem. Detemple and Serrat (1998 and 2003) characterize the path-dependency of consumption and portfolio policies as well equilibrium prices and allocations. Most importantly they document a singular component in the risk free rate, which occurs whenever the liquidity constraint is binding.

[16] Unfortunately, this does not imply that the liquidity constraint is differentiable with respect to time, i.e. we cannot write $k_t = \int_0^t z_t dt$ for some positive z.

In an influential work Mankiw and Zeldes (1991) report not only the very low stock market participation of investors but also the surprisingly low position in stocks of relative wealthy individuals. As a response Basak and Cuoco (1998) show that restricted market participation has the potential to simultaneously increase the market price of risk and decrease the risk free rate. Heaton and Lucas (1997) show, however, that in general low market participation is not sufficient to explain the magnitude of the equity premium[17]. An unsatisfactory feature of models based on restricted participation is the exogenous nature of the constraint. In our framework, participation or in fact nonparticipation is an endogenous outcome. To our knowledge their is no other model based on expected utility that generates this important feature[18].

3.1 The shadow price of ownership

The control problem of the poor and liquidity constrained agent depends on the identification of the multiplier process k_t. To solve the optimization problem presented above we need to characterize the multiplier such that

$$\xi_t X_t = E_t \int_t^{\mathcal{T}} \xi_s \left(I^1 \left(\left(y^{1*} - k_s \right) \rho_s \xi_s, o_s \left(y^{2*} \right) \right) - e_s \right) ds$$

$$+ E_t \int_t^{\mathcal{T}} H_0 dI^2 \left(o_s \left(y^{2*} \right), c_s \left(y^{1*} \right) \right) \geq 0 \tag{28}$$

$$X_t dk_t = 0 \tag{29}$$

$$\int_0^{\mathcal{T}} \xi_s \left(I^1 \left(\left(y^{1*} - k_s \right) \rho_s \xi_s, o_s \left(y^{2*} \right) \right) - e_s \right) ds = -H_0 \tag{30}$$

$$1 = \int_0^{\mathcal{T}} dI^2 \left(o_s \left(y^{2*} \right), c_s \left(y^{1*} \right) \right). \tag{31}$$

In light of Theorem (2) the conditions (28-29) are always satisfied and due to the modified budget constraint (30) the last condition is redundant. To simplify notation we assume here that $u(c, o) = u(c) + u(o)$. To compute the

[17] See also Brav, Constantinides, and Geczy (2002).

[18] See Ang, Bekaert and Liu (2002) for a model with optimal nonparticipation in the stock market with Disappointment Aversion preferences however. Transaction costs can also induce non-participation. Vissing-Jorgensen (2002) shows that a relative small per period (fixed) cost is enough to explain the nonparticipation of half the nonparticipants.

consumption policies and the Lagrange multiplier we need to find a solution to the following problem

$$I^1\left(\left(y^{1*}-k_t\right)\rho_t\xi_t\right)+l_t-e_t=0 \quad \forall t\in[0,\tau] \tag{32}$$

$$E_t\int_t^T\left(\rho_s\right)^{-1}u_2\left(o_s\right)ds=\left[y^2-k_t\right]H_0 \tag{33}$$

$$\int_0^\tau \xi_s\left(e_s-I^1\left(\left(y^{1*}-k_s\right)\rho_s\xi_s\right)\right)dt=H_0. \tag{34}$$

In short, first note that (32-34) still involve the controls l and/or o that we want to solve for. Now, observe that in contrast to l the level of ownership is purely backward looking[19]. So that, at the terminal date we can characterize ownership in terms of exogenous variables, i.e. probabilities and state-prices, as well as in terms of Lagrange multipliers and the multiplier process k. By successive replacement of levels of ownership while moving backwards along the space of ownership, for instance in a binomial economy, all levels of ownership can be expressed in terms of exogenous processes and the set of Lagrangian. We report these expressions for our numerical example with power utility in the Appendix and generalizations to other utility functions and economies are straightforward. Then, notice that the debt service, l, is given by

$$l_t=\frac{H_0}{\xi_t}do_t \tag{35}$$

after inserting into (32-34) we are left with three equations in three unknowns, per state of nature. The debt service, however, will be forward-backward looking. The recovery of the two Lagrangian and the Lagrange multiplier process have to be carried out numerically.

Before the optimal policies can be calculated by backward induction the stopping time τ must be determined. Again this is a numerical task.

[19] Detemple and Giannikos (1996) provide in their appendix a cookbook solution method for this kind of problem. See also Detemple and Zapatero (1991) and (1992), Duffie and Epstein (1992), Duffie and Lions (1993), Duffie and Skiadas (1992), Duffie, Geoffard, and Skiadas (1992) and for existence and uniqueness Antonelli (1993) and Perdoux and Peng (1990).

4 Equilibrium

In this section we demonstrate how a competitive equilibrium can be achieved. Notice that the equilibrium is a characterization of an economy in which the poor investor did not pay down the mortgage. The equilibrium interest rate as well as the equity premium are examined in terms of the poor and the rich agent.

It is important to remark that the nonparticipation of the poor investor endogenously generates market incompleteness. Nevertheless, a unique state price density exists. Barbachan (2001), Basak and Cuoco (1998), Cuoco and He (1994) as well as Detemple and Serrat (2003) characterize equilibrium with endogenous market incompleteness in a similar way as below. These models define the representative agent as a stochastically weighted average of individual utility functions. In this model the stochastic Lagrange multiplier serves as the weighting process. Notice that changes in the Lagrange multiplier process represent changes in the wealth of the two investors.

Let $C = D + e$ denote aggregate consumption[20]. Market clearing and individual rationality imply,

$$C_t = I_1^1\left(\left(y_1^1 - k_t\right)\rho_{1t}\xi_t, o_{1t}\left(y_1^2\right)\right) + I_2^1\left(y_2^1\rho_{2t}\xi_t, o_{2t}\right) \tag{36}$$

$$O_t = o_{1t} + o_{2t} = 2 \tag{37}$$

$$E\int_0^T \xi_t I_1^1\left(\left(y_1^1 - k_{1t}\right)\rho_{1t}\xi_t, o_{1t}\left(y_1^2\right)\right)dt + H_0 = E\int_0^T \xi_t e_t dt \tag{38}$$

$$E\int_0^T \xi_t I_2^1\left(\left(y_2^1 - k_{2t}\right)\rho_{2t}\xi_t, o_{2t}\right)dt = S_0 + \int_0^\tau \xi_t l_t dt \tag{39}$$

$$H_0 = \int_0^\tau \xi_t l_t dt \tag{40}$$

$$y_1^1 > 0, y_2^1 > 0, y_1^2 > 0, dk > 0 \text{ for } o_t < 1, k_o = 0$$

the subscript 1 stands for the poor agent and 2 for the rich agent[21] and $k_{2t} = 0$. Equation (36) is the resource constraint for each date and each state.

[20] Since services are constant we suppress them in the equilibrium analysis. The drift and volatility of aggregate consumption are defined to be $C\mu^c = D\gamma + e\mu^e$ and $C\sigma^c = D\lambda + e\sigma^e$.

[21] Recall that, I_1^1 and I_2^1 are the inverse of u_1 with respect to c for given o. I_1^2 and I_2^2 will represent the inverse of u_2 with respect to o for y^1 fixed. The maps I^1, I^2 (for both agents) exist, are unique, and are strictly decreasing in their first argument. Limiting values are, $lim_{y^1\uparrow\infty} I^1\left(y^1, o\right) = 0$ and $lim_{y^1\downarrow 0} I^1\left(y^1, o\right) = \infty$, $lim_{y^2\uparrow\infty} I^2\left(y^1, y^2\right) = 0$ and $lim_{y^2\downarrow 0} I^2\left(y^1, y^2\right) = \infty$.

Note that in an exchange economy there is no intertemporal transformation of resources. Equation (37) ensures that aggregated ownership is constant over time. Furthermore, equations (38-39) represent the budget constraint of the poor and the rich agent. Finally, equation (40) is the constraint on debt service associated with the house purchase. For the rich agent we do not need a second Lagrange multiplier, since constraining the poor agent is sufficient to ensure optimality with respect to ownership for the rich agent. Another important observation is that, by Walras law, one of the budget constraints is redundant. Before we present our next result it is convenient to define the aggregated relative risk aversion coefficient $\left(R_t^{1A}\right)$ with respect to consumption, and the aggregated relative prudence coefficient $\left(P_t^{1A}\right)$ with respect to consumption. These are respectively given by,

$$R_t^{1A} = C_t \left[\sum_i \frac{c_{it}}{R_{it}^1} \right]^{-1} \tag{41}$$

$$P_t^{1A} = \frac{\left(R_t^{1A}\right)^2}{C_t} \left[\sum_i \frac{P_{it}^{1A} c_{it}}{\left(R_{it}^1\right)^2} \right]^{-1} \tag{42}$$

where $i = 1, 2$ and $R_{it}^1 = -\frac{u_{11}(c_{it}, o_{it})}{u_1(c_{it}, o_{it})} c_{it}$. Now, equilibrium for this model can be computed in terms of relative risk aversion and relative prudence and is stated in the following theorem.

Theorem 3 *Consider the economy described above. Suppose that,*

$$\theta_t = R_t^{1A} \sigma_t^C \tag{43}$$

satisfies the condition,

$$Eexp \left[\frac{1}{2} \int_0^T \parallel \theta_t \parallel^2 dt \right] < \infty. \tag{44}$$

then equilibrium Arrow-Debreu prices are,

$$\xi_t = f\left(C_t; y_1^1 - k_t; y_1^2; y_2^1\right) \tag{45}$$

where f is the inverse aggregate demand function with respect to the state price density[22]*, and the (exogenous) initial price of the house is set such that*

$$H_0 < \int_0^T \xi_t e_t dt \text{ and } E \int_0^T \xi_t e_t dt < 2H_0 \tag{46}$$

holds. Finally, the equilibrium interest rate and the risk premium can be expressed as follows,

$$dR_t = r_t dt + dK_t$$

$$r_t = \beta_t^A + R_t^{1A}\mu_t^c + R_t^{2A}\mu_t^o - \frac{1}{2}R_t^{1A}P_t^{1A}\left(\sigma_t^c\right)^2 \tag{47}$$

$$dK_t = -\frac{R_t^{1A}}{R_{1t}^1}\frac{c_{1t}}{C_t}\frac{dk_t}{y_1^1 - k_t}$$

$$\mu_t - r_t = R_t^{1A}\sigma_t\sigma_t^c. \tag{48}$$

The drift of ownership μ_t^o across agents is $\frac{\xi_t l_t}{H_0}$.

Recall that the process θ_t is the market price of risk. Equation (44) is standard and ensures that the equivalent martingale measure evaluated at equilibrium allocations is well defined. The theorem states the following results: State prices, equation (45), are equal to the aggregated marginal utility of the perishable consumption good. This is because the consumption good is the numeraire in this economy. The state prices, in this model, depend not only on current aggregates (C_t, O_t) but also on past allocations. In particular, past distributions of ownership across agent will be reflected in state prices via the stochastic Lagrange multiplier process k. This will be the case even if both agents utility function is additively separable. Condition (46) can be interpreted as a restriction on the exogenous processes of the economy. The initial relative value of the house coincides also with the value of the debt service at date 0. In equilibrium the locally riskless interest rate, equation (47), has four components. The aggregated relative risk aversion coefficient $\left(R_t^{2A}\right)$ with respect to ownership will be discussed in detail in the next section. We also know that the equilibrium interest rate is the negative of the expected rate of change in the state price density (Theorem 1 in Cox, Ingersoll, Ross (1985a)). Further dK_t is a singular component that affects the cumulative

[22] The function f is the unique solution of $C_t = I_1^1\left(\left(y_1^1 - k_t\right)\rho_{1t} f\left(C_t; y_1^1 - k_t; y_1^2; y_2^1\right), o_{1t}\left(y_1^2\right)\right) + I_2^1\left(y_2^1\rho_{2t} f\left(C_t; y_1^1 - k_t; y_1^2; y_2^1\right), o_{2t}\right)$.

rate. Clearly, the appearance of such a singular process is due to the liquidity constraint which, in our model, always binds prior to τ. The risk premium, equation (48), is standard. Again we have the general statement that the asset risk premium is given by the covariation between the expected rate of change in the state price density and asset rates of returns. The next section discusses how the dynamics of the interest rate and the risk premium can be interpreted. Valuation formulas for implicitly traded financial assets are of the form[23],

Theorem 4 *Equilibrium prices of financial assets in this economy are,*

$$S_t = E_t \int_t^T \xi_{s,t} D_s ds \tag{49}$$

$$M_t = E_t \int_t^\tau \xi_{s,t} l_s ds \tag{50}$$

where M_t is the present value of the debt associated with a leveraged purchase of the house, i.e. a mortgage.

The closed form presentations for the stock (equation 49) and the mortgage (equation 50) have common style. However dividends and housing services are exogenous to this economy whereas the debt services associated with the leveraged purchase of the house in equation (50) are endogenous equilibrium values. By way of contrast real valuation of the house takes place only at date zero, the equilibrium price of the house coincides with the value of the mortgage at date 0. Preference for ownership, therefore, makes the asset house to an illiquid investment[24].

5 Asset risk premium and interest rate

In this section we take a closer look at equilibrium asset risk premium and interest rate. In this model the asset risk premium takes a form, which has

[23] Appendix B contains explicit solutions for the volatilities of all implicit traded financial assets. Indirect effects of ownership distribution across agents are briefly discussed.

[24] Note that even in a world with different types of houses/services the demand for the asset houses will not be a simple function of wealth (y^1 and y^2) because higher (lower) levels of services have to be compensated with lower (higher) level of ownership. If the space of services is discontinuous the illiquid character of the house will remain.

been already observed in the standard Lucas economy. The following proposition summarizes the behavior of risk premium in the present economy.

Proposition 5 *The asset risk premium is positively related to the covariation between consumption growth and asset rates of returns.*

The proposition is standard. However, whenever consumption and ownership interplay with each other, for instance due to nonseparabilities, the premium can, in principle, be much higher than in the standard model.

One of the key implications stemming from preference for ownership is the appearance of debt services in the expression for interest rates $(R_t^{2A}\frac{1}{H_0}\xi_t l_t)$. The behavior of the interest rate is therefore dependent on these services. The ultimative impact of ownership on the equilibrium interest rate depends on the aggregate relative risk aversion coefficient $\left(R_t^{2A}\right)$ with respect to ownership. Since the definition of the aggregate relative risk aversion is not exactly the same as proposed in the literature[25] we show now how a reasonable risk aversion coefficient can be constructed. Recall that equation 36 relies on the aggregation of the poor and rich agent and note that debt service (purchase of ownership) is a two sided business. This means that a positive debt service of the poor agent decreases at the same time ownership for the rich agent. Now the aggregate relative risk aversion with respect to ownership takes the form

$$R_t^{2A} = \left[\frac{R_{1t}^2}{R_{1t}^1} c_{1t} o_{1t} - \frac{R_{2t}^2}{R_{2t}^1} c_{2t} o_{2t}\right] \left[\sum_i \frac{c_{it}}{R_{it}^1}\right]^{-1} \tag{51}$$

where $R_{it}^2 = -\frac{u_{12}(c_{it}, o_{it})}{u_1(c_{it}, o_{it})} o_{it}$. This means that we have shown how different levels of ownership interacting with possibly not identical utility functions can drive the interest rate. It should also be perceived that initial wealth distributions, which are not necessarily the same as different levels of ownership, affect the interest rate not only through the endogenous process l_t but also through ownership dynamics. Apparently, the drift of ownership (μ_t^o) across agents weighted by the relative aggregate risk aversion $\left(R_t^{2A}\right)$ may contribute to explain the variate shapes of the term structure. In the next proposition we state the overall behavior of the interest rate, r.

[25] See Kihlstrom and Mirman (1981) for a definition of risk aversion with respect to multivariate (consumption) risk.

Proposition 6 *First recall that (aggregate) $u_1 > 0$, $u_{11} < 0$. Now the interest rate is positively related to the expected consumption growth rate. Furthermore positively (negatively) related to the expected growth in debt services if the aggregate cross partial derivative u_{12} is negative (positive). The volatility of consumption growth increases (decreases) the interest rate if (aggregate) u_{111} is negative (positive).*

Since it is of interest, if the equilibrium interest rate can be altered in such a way, that the resulting interest rate lies always below the interest rate in standard Lucas economies (Weil (1989)), Proposition 7 shows how this can be achieved.

Proposition 7 *The cumulative interest rate in the present economy where agents utility function exhibits preference for ownership is lower than in a comparable economy where preference for ownership does not matter if and only if*

$$dK_t + \frac{R_{1t}^2}{R_{1t}^1} c_{1t} o_{1t} > \frac{R_{2t}^2}{R_{2t}^1} c_{2t} o_{2t}. \tag{52}$$

Proposition (7) relies on the assumption that both investors cross partial derivative u_{12} is positive. Overall a positive cross partial derivative is natural. It is hard to think of an example in which the consumption of an additional unit of a good decreases the utility of an additional unit of a second good.

6 Conclusion

In this paper we presented a general equilibrium model in which agents have preference for ownership. We motivate our assumption that agents exhibit preference for ownership with puzzling consumer behavior. Especially for most homeowners, the house is the single most important consumption and investment good. Thus it is a natural to assume that some of the characteristics of a house enter the utility function. Since services can be obtained without direct investment into a house we assume that ownership matters. We define ownership as a function of past debt service. It is shown how the demand for ownership affects consumption demand and directly translates into debt service. A surprising and quite strong result is that preference for ownership in combination with liquidity constraints lead to a zero investment into stocks if investors are not endowed with a house.

Our equilibrium analysis demonstrates that the interest rate contains components originating from preference for ownership. Thus the incorporation of preference for ownership into the standard exchange economy may then contribute to the resolution of the riskfree rate puzzle. A further decrease in the cumulative rate is demonstrated through a singular component with negative impact on the risk free rate. Additionally, composite risk can induce a higher risk premium without raising risk aversion.

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7 Appendices

7.1 Appendix A: proofs

In this section we provide the proofs of all theorems appearing in the paper. First, let us recapitulate how to move from the dynamic optimization problem to the static. According to Karatzas, Lehocky and Shreve (1987) admissibility of the triplet (π, c, l) can be defined as follows

$$(\pi, c, l) \in A \text{ iff } X_t + E \int_t^T \xi_{t,s} e_s ds \geq 0 \tag{53}$$

where $t \in [0, T]$. The dynamic consumption-ownership-investment problem is

$$(\pi, c, l) \text{ s.t. } \max_{\pi,c,l} U(c, o) : (\pi, c, l) \in A. \tag{54}$$

This means that if this optimal controls exist, i.e. (π, c, l) is optimal, there exist no

$$\left(\hat{\pi}, \hat{c}, \hat{l}\right) \in A \text{ s.t. } U(\hat{c}, \hat{o}) > U(c, o). \tag{55}$$

Now, recall the (static) budget constraint

$$B = \left\{ c, l : E \int_0^T \xi_t (c_t + l_t) dt \leq E \int_0^T \xi_t e_t dt \right\} \tag{56}$$

where we have replaced the house price with the stream of debt payments. Next we show that

$$(\pi, c, l) \in A \Longrightarrow (c, l) \in B. \tag{57}$$

The reverse can be proved along the same lines with help of the martingale representation theorem. From Section 2 we know the two processes

$$d\xi_t = -\xi_t [r_t dt + \theta_t dW_t] \tag{58}$$

$$dX_t = (r_t X_t + e_t - c_t - l_t) dt + \pi_t [(\mu_t - r_t) dt + \sigma_t dW_t]. \tag{59}$$

The product of the processes is

$$d(\xi_t X_t) = \xi_t (e_t - l_t - c_t) dt + \xi_t [\pi_t \sigma_t - X_t \theta_t] dW_t. \tag{60}$$

Integrating from 0 to T on both sides gives

$$\xi_T X_T = \xi_0 X_0 + \int_0^T \xi_t \left(e_t - l_t - c_t\right) dt + \int_0^T \xi_t \left[\pi_t \sigma_t - X_t \theta_t\right] dW_t. \tag{61}$$

Recall that $X_0 = 0$ and let us assume that $l_t > 0$ at all times. Note also that $\int_0^T \xi_t \left[\pi_t \sigma_t - X_t \theta_t\right] dW_t$ is a local martingale. Since $(\pi, c, l) \in A \Longrightarrow X_T \geq 0$, and $(\pi, c, l) \in A \Longrightarrow E \int_0^T \xi_t \left(e_t - l_t - c_t\right) dt \geq X_0 = 0$ a non negative local martingale is a super martingale. Taking expectations leads to

$$E\left[\xi_T X_T + \int_0^T \xi_t \left(c_t + l_t\right) dt\right] \leq E \int_0^T \xi_t e_t dt \tag{62}$$

$$E \int_0^T \xi_t \left(c_t + l_t\right) dt \leq E \int_0^T \xi_t e_t dt \tag{63}$$

$$E \int_0^T \xi_t c_t dt \leq E \int_0^T \xi_t e_t dt - H_0 \tag{64}$$

$$(c, l) \in B. \tag{65}$$

Proof. of Theorem 1:

(i) necessity: the utility gradient of (8) is given by (19-20). (19-23) are standard saddle-point conditions.

(ii) sufficiency: consider an alternative feasible policy (c', l'). By concavity of the utility function we have

$$u\left(c_t, o_t\right) \geq u\left(c_t', o_t'\right) + u_1\left(c_t, o_t\right)\left(c_t - c_t'\right) + u_2\left(c_t, o_t\right)\left(o_t - o_t'\right). \tag{66}$$

Multiplying by $\left(\rho_t\right)^{-1}$ and integrating over the product measure $dP \times dt$ yields,

$$E \int_0^T \left(\rho_t\right)^{-1} u\left(c_t, o_t\right) dt \geq E \int_0^T \left(\rho_t\right)^{-1} u\left(c_t', o_t'\right) dt + E \int_0^T \left(\rho_t\right)^{-1} u_1\left(c_t, o_t\right)\left(c_t - c_t'\right) dt + E \int_0^T \left(\rho_t\right)^{-1} u_2\left(c_t, o_t\right)\left(o_t - o_t'\right) dt. \tag{67}$$

To complete the proof we have to show that the three last terms are non-negative. After substituting $y^1\rho_t\xi_t$ for $u_1(c_t, o_t)$ and $\frac{\int_0^t \xi_s(l_s - l'_s)ds}{H_0}$ for $(o_t - o'_t)$ into (67) and rearranging we have

$$U(c, o) \geq U(c', o') + y^1 \left[E \int_0^T \xi_t (e_t - c'_t) dt - H_0 \right]$$

$$+ y^2 \left[H_0 - E \int_0^T \xi_t l'_t dt \right]. \quad (68)$$

By the budget constraint, the constraint that present value of all debt payments is equal to the initial house price, and the constraints $y^1 > 0, y^2 > 0$ optimality of (c, l) follows.

■

Proof. of Theorem 2:

The FOC without the liquidity constraint is

$$E_t \int_t^T (\rho_s)^{-1} u_2(c_s, o_s) ds = y^2 H_0. \quad (69)$$

Since the left hand side is a constant the right hand side can hold only once, e.g. $E_0 \int_0^T (\rho_t)^{-1} u_2(c_t, o_t) ds = y^2 H_0$. By assumption $e_0 < H_0$ negativity of X follows. If $X \geq 0$ is enforced, then, by the FOC in (69) $X = 0$ holds. At $t = 1$ (69) implies negativity of X. If $X \geq 0$ is enforced, then, $X = 0$ holds. Continuing until $t = \tau$ yields the theorem. ■

Proof. of Theorem 3:

Applying Ito's lemma on both sides of equation (36) yields

$$C_t [\mu_t^c dt + \sigma_t^c dW_t] = \sum_i I_i^{1\prime} \times \left(\left(y_i^1 - k_{it} \right) \rho_{it}\xi_t \right) \left[\frac{d\rho_{it}}{\rho_{it}} + \frac{d\xi_t}{\xi_t} - \frac{dk_{it}}{y_i^1 - k_{it}} \right]$$

$$+ \frac{1}{2} \sum_i I_i^{1\prime\prime} \times \left(y_i^1 \rho_{it} \xi_t \right)^2 \left[\frac{d \langle \xi_t \rangle}{(\xi_t)^2} \right]$$

$$+ \sum_i I_i^{1\,\hat{}} \times (o_t) \left[\frac{do_{it}}{o_{it}} \right] \quad (70)$$

where $k_{2t} = 0$, $\langle\rangle$ denotes the quadratic variation process, (') and (^) denote the derivatives of $I_i^1(.,.)$ with respect to the first and second argument respectively. All derivatives are evaluated at equilibrium allocations. Substituting the processes and equating terms in dW yields the asset risk premium,

$$\theta_t = -C_t\sigma_t^C\left[\sum_i \frac{c_{it}}{R_{it}^1}\right]^{-1} \tag{71}$$

whereas equating terms in dt yields the interest rate,

$$\begin{aligned} r_t = & \left[\sum_i \frac{c_{it}}{R_{it}^1}\beta_{it}\right]\left[\sum_i \frac{c_{it}}{R_{it}^1}\right]^{-1} - C_t\mu_t^C\left[\sum_i \frac{c_{it}}{R_{it}^1}\right]^{-1} \\ & + \left[\frac{R_{2t}^2}{R_{2t}^1}c_{2t}o_{2t} - \frac{R_{1t}^2}{R_{1t}^1}c_{1t}o_{1t}\right]\left[\sum_i \frac{c_{it}}{R_{it}^1}\right]^{-1}\mu_t^o \\ & + \frac{1}{2}C_t\left[\sum_i \frac{c_{it}}{R_{it}^1}\right]^{-1}\frac{\left(R_t^{1A}\right)^2}{C_t}\left[\sum_i \frac{P_{it}^{1A}c_{it}}{\left(R_{it}^1\right)^2}\right]^{-1}\left(\sigma_t^C\right)^2 \\ & - \frac{R_t^{1A}}{R_{1t}^1}\frac{c_{1t}}{C_t}\frac{dk_{1t}}{y_1^1 - k_{1t}}. \end{aligned} \tag{72}$$

Note that the following relationship for the individual inverse function with respect to the first argument (consumption) $(I_i^{1\,\hat{}})$, i.e. the first derivative with respect to the second argument (ownership), $I^{1\,\hat{}} = \frac{-u_{12}}{u_{11}}$ holds, where we have suppressed the subscripts for the different agents.

■

Proof. of Theorem 4:

The comovements of state prices and the stock takes the form

$$d(\xi S) = \xi dS + S d\xi + d\xi dS. \tag{73}$$

Substituting the processes leads to

$$\begin{aligned} d(\xi_t S_t) = & \ \xi_t\left(S_t\left[\mu_t dt + \sigma_t dW_t\right] - D_t dt\right) + S_t\left(-\xi\left[r_t dt + \theta_t dW_t\right]\right) \\ & + S_t\sigma_t\xi\theta_t dt. \end{aligned} \tag{74}$$

After minor manipulation we take expectations and integrate from t to T. This leads to

$$E\xi_T S_T - \xi_t S_t = -E\int_t^T \xi_s D_s ds. \tag{75}$$

Since $S_T = 0$ the first part of the proof is complete. The value of the mortgage can be done along the same lines. We only need to assume that l_t follows

$$dl_t = l_t \left[\mu_t^l dt + \sigma_t^l dW_t\right] \tag{76}$$

where the drift and the volatility has to be determined at equilibrium. Now the mortgage process with endogenous drift and volatility terms satisfies

$$dM_t + l_t dt = D_t \left[\mu_t^M dt + \sigma_t^M dW_t\right]. \tag{77}$$

■

7.2 Appendix B: Equilibrium volatilities

In this appendix we characterize the volatility of the stock and the mortgage. For simplicity we assume that we already solved for a representative agent and thus utility terms (u) are to be understood as aggregate utility.

Theorem 8 *Equilibrium volatilities for the stock and the mortgage are,*

$$S_t \sigma_t = (u_1)^{-1} \left[\rho_t^{-1} E_t \mathbb{D}_t B - u_{11} \left(D_t \lambda_t + e_t \sigma_t^e\right) S_t\right] \tag{78}$$

$$M_t \sigma_t^M = (u_1)^{-1} \left[\rho_t^{-1} E_t \mathbb{D}_t F - \left(u_{11} \left(D_t \lambda_t + e_t \sigma_t^e\right)\right) M_t\right] \tag{79}$$

where,

$$\mathbb{D}_t B = \int_t^T \rho_v^{-1} [\left(-\int_t^v \mathbb{D}_t \beta_s ds\right) u_1 D_v + u_{11} \left(\mathbb{D}_t \left(D_v + e_v\right)\right) D_v$$
$$+ u_1 \left(\mathbb{D}_t D_v\right) + u_{12} \left(\mathbb{D}_t o_v\right) D_v] dv, \tag{80}$$

$$\mathbb{D}_t F = \int_t^T \rho_v^{-1} [\left(-\int_t^v \mathbb{D}_t \beta_s ds\right) u_1 l_v + u_{11} \left(\mathbb{D}_t \left(D_v + e_v\right)\right) l_v$$
$$+ u_1 \left(\mathbb{D}_t l_v\right) + u_{12} \left(\mathbb{D}_t o_v\right) l_v] dv, \tag{81}$$

and,

$$\mathbb{D}_t D_v = D_v \left[\lambda_t + \int_t^v \mathbb{D}_t \left(\gamma_s - \frac{1}{2} \lambda_s^2\right) ds + \int_t^v \mathbb{D}_t \lambda_s dW_s\right] \tag{82}$$

$$\mathbb{D}_t e_v = D_v \left[\sigma_t^e + \int_t^v \mathbb{D}_t \left(\mu_s^e - \frac{1}{2} \left(\sigma_s^e\right)^2\right) ds + \int_t^v \mathbb{D}_t \sigma_s^e dW_s\right] \tag{83}$$

$$\mathbb{D}_t l_v = D_v \left[\sigma_t^l + \int_t^v \mathbb{D}_t \left(\mu_s^l - \frac{1}{2} \left(\sigma_s^l \right)^2 \right) ds + \int_t^v \mathbb{D}_t \sigma_s^l dW_s \right] \tag{84}$$

$$\mathbb{D}_t o_v = \frac{1}{H_0} \int_t^v \left[\left(\mathbb{D}_t \xi_s \right) l_s + \xi_s \left(\mathbb{D}_t l_s \right) \right] ds \tag{85}$$

$$\mathbb{D}_t \xi_s = -\xi_s \left[\theta_t + \int_t^s \mathbb{D}_t r_u du + \int_t^s \mathbb{D}_t \theta_u dW_u + \int_t^s \left(\mathbb{D}_t \theta_u \right) \theta_u du \right] \tag{86}$$

$$\mathbb{D}_t l_s = D_s \left[\sigma_t^l + \int_t^s \mathbb{D}_t \left(\mu_u^l - \frac{1}{2} \left(\sigma_u^l \right)^2 \right) du + \int_t^s \mathbb{D}_t \sigma_u^l dW_u \right]. \tag{87}$$

$\mathbb{D}_t$ is the Malliavin derivative. The volatilities depend on the current volatilities of the endowment processes, the volatility of the endogenous debt service, and on perturbations in the Brownian motion process W_t.