Bidder Collusion

Robert C. Marshall — Penn State University
Leslie M. Marx* — Duke University

First draft: February 2002
This draft: December 30, 2002

Duke/UNC Micro-Theory Working Paper #13

Abstract

Within the heterogeneous independent private values model, we analyze bidder collusion at first and second price single-object auctions, allowing for within-cartel transfers. Our primary focus is on (i) coalitions that contain a strict subset of all bidders and (ii) collusive mechanisms that do not rely on information from the auctioneer, such as the identity of the winner or the amount paid. To analyze collusion, a richer environment is required than that required to analyze non-cooperative behavior. We must account for the possibility of shill bidders as well as mechanism payment rules that may depend on the reports of cartel members or their bids at the auction. We show there are cases in which a coalition at a first price auction can produce no gain for the coalition members beyond what is attainable from non-cooperative play. In contrast, a coalition at a second price auction captures the entire collusive gain. For collusion to be effective at a first price auction we show that the coalition must submit two bids that are different but close to one another, a finding that has important empirical implications.

Keywords: auctions, collusion, bidding rings, shill

*We are grateful to Andreas Blume, Paul Heidhues, Vijay Krishna, and Andy Skrzypacz for helpful discussions. We thank the participants at the 2002 North American Summer Meetings of the Econometric Society and seminar participants at the Federal Trade Commission, New York University, Penn State University, University of Maryland, University of Pittsburgh, and University of Rochester for useful comments. The authors can be reached at Department of Economics, 613 Kern Graduate Bldg., University Park, PA 16802, rcm10@psu.edu and Fuqua School of Business, Duke University, Durham, NC 27708, marx@duke.edu.

1 Introduction

Auctions are a prevalent mechanism of exchange.[1] It is natural for bidders to attempt to suppress rivalry and thus capture some of the rents that would be transferred to the seller if their bidding were non-cooperative. Case law is replete with examples of Section 1 violations of the Sherman Act for bid rigging—and these cases are just the bidders who were apprehended. For many bidders, a potential Section 1 violation is just the cost of doing business. As a casual observation, whenever new auction mechanisms are proposed or designed, there seems to be remarkably little attention paid to the issue of bidder collusion. Yet, in terms of foregone revenue, bidder collusion is probably the most serious practical threat to revenue.

Within the heterogeneous independent private values model, we analyze bidder collusion at first and second price single-object auctions, allowing for within-cartel transfers. Our primary interest is in coalitions that contain a strict subset of the bidders. We focus attention on "pre-auction mechanisms"—those in which the collusive mechanism does not rely on any information from the auction itself, such as the identity of the winner or the amount paid. To analyze collusion, we require a richer environment, as compared to non-cooperative behavior, to account for shill bidders,[2] as well as mechanism payment rules that may depend on the reports of cartel members or their bids at the auction. We demonstrate that the auction format (first price versus second price) leads to dramatically different results in terms of the viability and profitability of collusion.

One contribution of this paper is to specify elements of an auction/mechanism environment that are critical to the study of collusion. Some of these elements are entirely inconsequential for the study of non-cooperative behavior. For example, we

[1] So are procurements. Our results apply to procurements, but we refer to auctions throughout the paper.

[2] In this paper, a shill bidder is an incentiveless perfect agent of another bidder. A shill provides a way for a bidder to disguise the identity of a "second" bid. Of course, the bidding behavior of a shill may lead to inferences that the bid was not that of a "true bidder".

show that at a first price auction, if within-ring[3] transfers only depend on the reports ring members make to the collusive mechanism prior to the auction (and not on bids the ring members or their shills submit), then a ring mechanism that asks all ring members with less than the highest value to bid zero cannot produce a surplus in excess of that obtainable via non-cooperative play. A natural response to this finding is to think that the bids of ring members will be observable to those who run the ring mechanism, and thus it should be possible to condition payments on that information. Introducing the observability of the ring members' bids does not affect results for a non-cooperative setting, but does have an important effect on the profitability of collusion.

In addition to specifying whether a ring can observe the bids of ring members, in order to study collusion we must also consider the use of shill bidders. If we think of non-cooperative play within the independent private values model, it is difficult to imagine any role for a shill bidder, especially if the auctioneer is non-strategic. However, in a collusive environment, ring members are asked to submit specific bids at the auction, and they may have better alternative bids. Further, it may be the case that the bids *they* (as opposed to their shills) submit are observable by those who run the ring, who can penalize them for inappropriate bids. The ability of a ring member to use a shill bidder relaxes the constraint imposed on him by the ring regarding his behavior at the auction. When ring members can use shill bidders at the auction, our results for second price auctions are unaffected, but collusive payoffs are weakly reduced for first price auctions.

We show that there are cases in which a coalition at a first price auction can produce no gain for coalition members beyond what is attainable from non-cooperative play. However, there are also cases in which collusion at a first price auction may be profitable. In these cases, unless the mechanism can directly control ring bids, the mechanism must be such that at least two ring bidders are sent to the auction by

[3] Bidder coalitions are often referred to as "rings" (see Cassady 1967, Chapter 13).

the ring, where one is instructed to submit a bid that is just below the other. The lower bid is required to prevent the high bidder from deviating to a lower bid—if such a deviation were allowed to occur, it would result in violation of an incentive compatibility constraint. This result has potentially important empirical implications. Specifically, if we frequently observe two sequential bids within one bid increment of each other, such as that the highest losing bid being within one bid increment of the winning bid (as opposed to being tied), and if there is no obvious market-based reason for such bids, then these bids are suggestive of collusive bidding.

With regard to the comparison between first price and second price auctions with respect to bidder collusion, there has been some intuition offered for years that goes as follows. At a second price auction, a bidder cartel must suppress the bids of all members except the bidder with highest value. The cartel bidder with highest value goes to the auction and bids as he would were he acting non-cooperatively. Any cartel member who thinks of breaking ranks and competing at the auction faces the highest cartel bidder and the highest non-cartel bidder, each submitting bids that are the same as if all were acting non-cooperatively. Thus, there is no gain to deviant behavior. The first price auction is quite different. In order to secure a collusive gain the ring member with the highest value must lower his bid below what he would have bid acting non-cooperatively, and other ring members must suppress their bids. But when the highest-valuing ring member lowers his bid, the non-coalition bidders optimally lower theirs in response, and the opportunity is created for a non-highest-valuing coalition member to enter an aggressive bid at the auction, either on his own or through a shill, and secure an item that he may not have been able to win acting non-cooperatively. This possibility jeopardizes the feasibility of a coalition at a first price auction.

In addition, the optimal reduction in bids by non-cartel bidders implies that some of the collusive gain "leaks out" to them. This inability of the cartel to keep all of the collusive gain, which the cartel can do at a second price auction, further jeopardizes the feasibility of the ring at a first price auction.

In this paper we provide scenarios in which the above intuitions are borne out.

The paper proceeds as follows. The literature review is in Section 2, the model is in Section 3, and the results are in Section 4. A discussion of the results is in Section 5. The case of the all-inclusive coalition is in an appendix, as are the proofs for the results in the main text.

2 Literature review

Perhaps the starting point for auction theory is the work of Vickrey (1961, 1962). Three papers in the early 1980's—Riley and Samuelson (1981), Myerson (1981), and Milgrom and Weber (1982)—resolve major conundrums and provide benchmark results from which much progress has been made in the ensuing two decades. One modeling framework that has received much attention in the auction literature is called the "independent private values" model (IPV) where it is assumed that bidders independently draw values from the same distribution F and that each bidder knows his value but not the value of any other bidder (values are private information).[4] The central result within the IPV framework is the revenue equivalence theorem—a broad class of auction mechanisms, including those most commonly used in practice, produce identical revenue for the seller when bidders are risk neutral and act non-cooperatively. Much ensuing research in auction theory addresses the relaxation of the underlying modeling assumptions to determine the impact on expected revenue for different auction schemes. For example, when bidders are risk averse, the first price auction outperforms the second price auction (Matthews 1983, 1987). One vein

[4] We refer to this particular variant of the IPV model as "symmetric IPV" or simply "IPV". In Myerson, the IPV framework allows for the possibility that bidders independently draw private values from different distributions, F_i. We refer to this generalization of the IPV model as "heterogeneous IPV" or "asymmetric IPV". We focus on a variant of the heterogeneous IPV model that requires all value distributions to have the same lower and upper support.

of work in this regard focuses on the relaxation of the non-cooperative assumption.[5]

Within an IPV, single-object framework, Graham and Marshall (GM, 1987) provide a profitable mechanism for a coalition of any size at a second price or English auction.[6] In their framework, k of n bidders are in the ring where $k \leq n$. Prior to the auction the k ring members each receive a fixed ex ante non-contingent payment from a "center".[7] Each ring member makes a report r_i to the center. The center recommends that the $k-1$ ring members with lowest reports bid below the reserve price at the auction,[8] while the ring member with highest report bids up to his report at the auction. If the ring member wins the auction, he pays the center nothing if the auction price is greater than the second-highest report from the ring. If the second-highest ring report exceeds the price paid at the auction then the winning ring bidder pays the center the difference between the second-highest report and the price at the auction. The ex ante expectation of this payment to the center, divided by k, is the fixed ex ante non-contingent payment made by the center to each ring member (thus the mechanism is ex ante balanced budget). GM show that this mechanism is incentive compatible (ring members report truthfully to the center and follow her recommendations). Also, each bidder wants to join the coalition and each ring member wants a potential new member to join. The mechanism is efficient in that the

[5] For repeated auctions, collusion by an all-inclusive ring can be sustained in some environments. Fudenberg, Levine, and Maskin (1994) prove a folk theorem for the case in which bidders can communicate prior to each auction and can observe each others' bids but cannot make transfers. They show that as the discount factor increases to one, the optimal collusive scheme is efficient. Even without communication or the ability to observe bids, Blume and Heidhues (2001) and Skrzypacz and Hopenhayn (2001) show that for discount factors sufficiently large, the ring can do better than noncooperative play or a bid rotation scheme by using implicit transfers of equilibrium continuation payoffs, although efficiency cannot be achieved.

[6] For heterogeneous IPV bidders at a second price auction, Mailath and Zemsky (1991) find an optimal mechanism. Graham, Marshall, and Richard (1990) show that a side payment scheme that is commonly employed by practicing rings when its members are heterogeneous allocates each ring member his Shapley value.

[7] The notion of a center, an incentiveless agent who facilitates implementation of the mechanism, was introduced by Myerson (1983).

[8] In a later section of the paper, Graham and Marshall describe optimal "disguised" bids by the $k-1$ lowest-valuing ring members. These meaningless "competitive" bids are submitted by the ring so that the auctioneer cannot infer whether bids are coming from a ring or non-ring bidder.

winner is always the bidder with the highest value. Finally, there is no alternative mechanism that all ring members would prefer. A critical implicit assumption of GM is that the designated ring bidder cannot circumvent payment to the center when, both, he wins and the second-highest report is greater than the price paid at the auction.

McAfee and McMillan (1992) provide an analysis of collusion within an IPV framework for a first price auction, where emphasis is on the surplus division game for an all-inclusive cartel. When the cartel members cannot make internal transfers (weak cartel), McAfee and McMillan show that the outcome of the auction is potentially inefficient in that a cartel member is selected at random (from those willing to pay in excess of the reserve price) to be the sole bidder at the auction. When side payments are possible (strong cartel), then the members conduct an ex ante first price auction, where the winning bid is equally distributed to all losers and the winner is the sole bidder at the main auction.[9] Strong cartels produce efficient allocations, provided the highest value exceeds the reserve price.[10] With regard to individual rationality constraints, McAfee and McMillan offer some characterizations, but because of the analytic intractability that emerges from the heterogeneity implied by collusion within an IPV model, results are only provided for a special discrete case.

The existence of equilibrium in a heterogeneous IPV setting has been demonstrated by a number of authors, including Athey (2001), Maskin and Riley (2000b), and Lebrun (1996).[11] Bidding behavior and expected revenue within an asymmetric IPV framework has been analyzed by Maskin and Riley (2000a).[12] A remarkable

[9]Lyk-Jensen (1997a) shows there exist several efficient, ex post budget balanced, pre-auction mechanisms for an all-inclusive ring.

[10]Relaxing the IPV assumption, Lyk-Jensen (1996) shows that an all-inclusive ring can sustain collusion using the second price pre-auction knock-out of Graham and Marshall (1987) or the first price pre-auction knock-out and McAfee and McMillan (1992) in the general symmetric model with affiliated values (see Milgrom and Weber (1982)). In this case, efficiency is not achieved with a first or second price pre-auction knock-out, but can be achieved with a pre-auction knock-out that allows information sharing and is ex ante budget balanced (Lyk-Jensen (1996)) or ex post budget balanced (Lyk-Jensen (1997b)).

[11]The comprehensive contribution of Athey (2001) covers heterogeneous IPV as a special case.

[12]The working paper circulated for nearly a decade prior to publication and thus influenced work

non-result emerges from this work—it is extremely difficult to provide any meaningful general analytic characterization as to the conditions under which one auction scheme will outperform another in terms of expected revenue.

Marshall, Meurer, Richard, and Stromquist (MMRS, 1994) provide numerical methods for obtaining solutions to the differential equations that implicitly define bids when k of n IPV bidders ($k \leq n$) collude and the remaining bidders act non-cooperatively (a specific kind of asymmetric IPV). Because of numerical instabilities at the origin the solutions involve "backward shooting" methods. The appendix of their paper provides an exact analytic solution for the terminal point of the bid functions for a special case. Unfortunately, for most situations the terminal condition must also be numerically obtained.

Maskin and Riley (1996a), Bajari (1997, 2001), and Lebrun (1999) analyze a heterogeneous IPV model in which each bidder's distribution has common lower and upper support and shows that there is a unique equilibrium.[13] This implies that the bid functions in MMRS are unique. Further, Bajari (2001) implies that whatever mechanism is used by a cartel at a first price auction, if the designated cartel bidders and non-cartel bidders arrive at the auction with values consistent with a heterogeneous IPV model, then the equilibrium is unique. This result will be used in this paper.

3 Model

We first provide the ingredients of the heterogeneous IPV model and restate known results. We then discuss the additional structure needed to analyze collusion within this framework. Compared to non-cooperative behavior, a ring member has a richer set of questions to confront. Can he increase his expected payoff by misrepresenting his report to the center? Can he increase his expected payoff by deviating from the

published much earlier.

[13]For results regarding non-common supports see Lebrun (2002).

ring's recommended bid, perhaps to take advantage of suppressed ring bids to win an item that he would not win if he followed the center's recommendation? Can he profitably make use of a shill bidder?

3.1 Heterogeneous IPV model[14]

We consider a single object auction within a heterogeneous IPV framework with a non-strategic seller.[15] In the case of a tie, we assume the object is randomly allocated to one of the bidders with the high bid. There are n risk neutral bidders where bidder i independently draws a value v_i from a distribution F_i.

Assumption 1 *For all i, $F_i(v_i)$ has support $[\underline{v}, \bar{v}]$, where $\underline{v} \geq 0$. The probability density function $f_i(v_i)$ is continuously differentiable and, for all i, $f_i(v_i)$ is bounded away from zero on $[\underline{v}, \bar{v}]$.*[16]

Lemma 1 *Under Assumption 1, an equilibrium at a first price auction exists in pure strategies, the bid function is strictly increasing and differentiable, and the equilibrium is unique.*[17]

Furthermore, the unique equilibrium bid functions have the feature that a bidder with value $\underline{v}$ chooses a bid equal to $\underline{v}$, and so (regardless of the reserve price) no bidder chooses a bid less than $\underline{v}$ (see, e.g., Lebrun (1999)).

[14]The initial model characterization in this section borrows from Bajari (2001), with straightforward conversions from costs to values.

[15]Although we assume the seller is non-strategic, i.e., does not set a reserve price greater than the lower support of the bidders' value distributions in order to extract additional surplus from bidders, the lower support is common knowledge, so we treat it as a binding reserve price.

[16]An alternative assumption generating Lemma 1 can be found in Lebrun (2002). For all i, F_i has support $[\underline{v}, \bar{v}]$, where $\underline{v} \geq 0$ and is differentiable over $(\underline{v}, \bar{v}]$ with a derivative f_i locally bounded away from zero over this interval, and there exists $\delta > 0$ such that F_i is strictly log-concave over $(\underline{v}, \underline{v} + \delta)$. The function f_i is locally bounded away from zero if for all v in $(\underline{v}, \bar{v}]$, there exists $\epsilon > 0$ such that $f_i(w) > 0$, for all w in $(v - \varepsilon, v + \varepsilon)$.

[17]See citations in Section 2.

3.2 General features of a collusive mechanism

We are interested in the existence of collusive mechanisms that generate an expected surplus for each ring member that strictly exceeds the expected surplus each ring member could attain acting non-cooperatively. For the remainder of the paper this is what we mean by "profitable collusion". In particular, we are interested in the case in which there are $n \geq 3$ bidders, and k of those bidders are eligible to participate in a ring, where $2 \leq k \leq n-1$ (see Appendix A for results for an all-inclusive ring, i.e., $k = n$). We use indices $1, ..., k$ to denote ring members and $k+1, ..., n$ to denote outside bidders.

The game is as follows: First, a ring mechanism is announced (there is credible commitment to the mechanism). Both potential ring members and outside bidders observe the announcement. Second, ring members decide whether to join. All bidders observe whether all potential ring members join or not. Third, if not all potential ring members join, then the ring mechanism does not operate and all bidders submit their bids. If all potential ring members join, then ring members participate in the mechanism and all bidders submit their bids.

When a potential ring member decides whether to join the ring, we assume he joins if and only if his ex ante (before learning his own value) expected payoff from participation in the mechanism is greater than or equal to his ex ante (before learning his own value) expected payoff from non-cooperative play.[18] One can view this assumption as assuming that potential ring members must decide whether to join the

[18] There are two features of ring membership. One is individual rationality—does a given bidder want to participate in the ring? The other is whether a set of $k-1$ ring members want to include the k^{th} bidder as a ring member. This latter issue raises the question whether it might be necessary to provide different ex ante non-contingent payments to the different ring members, reflecting their different marginal values to the ring. This consideration does not affect our results. It is not relevant for our negative results for first price auctions because the failure of the ring is not due to membership considerations. It is not relevant for our positive results for first price auctions because they assume symmetry, and so equal ex ante non-contingent payments are appropriate. For second price auctions, membership issues are potentially a concern, but Graham, Marshall, and Richard (1990, Theorem 7) show how to construct the ex ante non-contingent payments to deal with this concern.

ring before they learn their values, or one can assume that potential ring members decide after they learn their values but that the auction is the stage game for an infinitely repeated game in which values are independently drawn at the beginning of each period.[19] In Section 4.3 (and Appendix A), we consider the more restrictive assumption of interim individual rationality, under which a ring member decides whether to participate in the ring after learning his value.

If one or more potential ring members chooses not to join the ring, the ring does not operate, and all bidders participate in the auction non-cooperatively.[20] If all the potential ring members decide to join the ring, then a "center", the standard Myerson (1983) incentiveless mechanism agent,[21] makes payments to all ring members (could be zero to all).[22] Then each ring member makes a report to the center. Based on these reports the center recommends a bid to be made by each ring member and requires payments from the ring members. In a *bid coordination mechanism,* the center bases required payments only on the reports of the ring members.[23] In a *bid submission mechanism*, the center bases required payments on both the reports of the ring members and their bids at the auction. Thus, in a *bid submission mechanism*, the center can require a very large payment from a bidder who does not bid according to the center's recommendation, guaranteeing that it is a best reply for ring members to follow the center's recommendation.[24] Ring members then decide on what action to take at the auction, which may include the use of shill bidders to bid on their

[19]In this case, for discount factors sufficiently close to one, the participation strategy is supported by Nash reversion to non-cooperative play. To see this, note that bidder i's one-shot gain is bounded above by $\bar{v}$ (more precisely i's expected one-shot gain is bounded above by the non-cooperative surplus of a bidder with value $\bar{v}$), so if the expected stage-game payoff is positive for each player, then for discount factors sufficiently close to one, it is alway individually rational for potential ring members to join the ring.

[20]This is a common, but not innocent, simplifying assumption in the auction literature.

[21]The center in this paper is also a banker when ex ante budget balance is required.

[22]Given that bidders are ex ante heterogeneous, the ex ante non-contingent payments could be different.

[23]At auctions of precious gems, it is sometimes the case that neither the identity of the winner nor the bids submitted by specific bidders are publicly revealed, although the winning price is announced. In this case, the payments to the center can only be a function of the initial ring reports (and perhaps the announced winning price).

[24]One interpretation of this is that the center actually submits a bid on behalf of each ring member.

behalf.[25] Incentive compatibility involves (i) making honest reports to the center, (ii) following the center's recommendation regarding the bid to submit at the auction, and (iii) not using a shill to submit a bid. We require that the center's budget be balanced in expectation.

Thus, an incentive compatible mechanism is $\mu = (x_1, ..., x_k, \beta_1, ..., \beta_k, p_1, ..., p_k)$, where for all $i \in \{1, ..., k\}$, x_i is ring member i's non-contingent payment from the center, $\beta_i(r_1, ..., r_k)$ is his recommended bid as a function of the ring members' reports, and $p_i(r_1, ..., r_k)$ is his required payment to the center in a *bid coordination mechanism* and $p_i(r_1, ..., r_k; b_1, ..., b_k)$ is his required payment in a *bid submission mechanism*, where b_i is ring member i's bid. The mechanism must satisfy incentive compatibility for reports, incentive compatibility for bidding, incentive compatibility for not using a shill,[26] budget balance, and ex ante individual rationality. To be precise, we now write the definition of an incentive compatible *bid coordination mechanism* at a first price auction in detail; the definitions for second price auctions and for *bid submission mechanisms* are similar. To state the definition, we use the convention that v^k denotes the vector of values for the ring members, i.e., $v^k = (v_1, ..., v_k)$, v^k_{-i} denotes the values of ring members other than i, v_{-i} denotes the values of all bidders other than i, and β_i^{nc} denotes player i's noncooperative bid function for a first price auction.

Definition 1 *For a first price auction, an incentive compatible* ***bid coordination mechanism*** *is* $\mu = (x_1, ..., x_k, \beta_1, ..., \beta_k, p_1, ..., p_k)$, *where for all* $i \in \{1, ..., k\}$, $x_i \in \mathbb{R}$ *and* $\beta_i : \mathbb{R}^k \to (-\infty, \bar{v}]$ *and* $p_i : \mathbb{R}^k \to \mathbb{R}$ *are measurable, satisfying:*

[25] It seems unrealistic that a defense contractor for a major project could use a shill bidder. However, it seems quite possible at an antique auction.

[26] For a *bid coordination mechanism*, incentive compatibility for not using a shill (#3 in the definition) is implied by incentive compatibility for bidding (#2 in the definition), but for a *bid submission mechanism*, a ring member's required payment depends on his bid, and so incentive compatibility for bidding no longer implies incentive compatibility for not using a shill. Thus, the constraint that a ring member not use a shill can bind in a *bid submission mechanism*.

1. (incentive compatibility for reports) *for all* $i \in \{1, ..., k\}$,

$$v_i \in \arg\max_{r_i} \max_b E_{v_{-i}} \left(\begin{array}{c} (v_i - b)\, 1_{b \geq \max\left\{\max_{j\in\{1,\ldots,k\}\setminus\{i\}} \beta_j(r_i, v^k_{-i}), \max_{j\in\{k+1,\ldots,n\}} b_j(v_j)\right\}} \\ -p_i(r_i, v^k_{-i}) \mid I_i \end{array} \right),$$

where $I_i = \left(v_i, \beta_i(r_i, v^k_{-i}), p_i(r_i, v^k_{-i})\right)$;

2. (incentive compatibility for bidding) *for all* $i \in \{1, ..., k\}$,

$$\beta_i(v^k) \in \arg\max_b E_{v_{-i}} \left(\begin{array}{c} (v_i - b)\, 1_{b \geq \max\left\{\max_{j\in\{1,\ldots,k\}\setminus\{i\}} \beta_j(v^k), \max_{j\in\{k+1,\ldots,n\}} b_j(v_j)\right\}} \\ -p_i(v^k) \mid I_i \end{array} \right),$$

where $I_i = \left(v_i, \beta_i(v^k), p_i(v^k)\right)$;

3. (incentive compatibility for not using a shill) *for all* $i \in \{1, ..., k\}$,

$$\beta_i(v^k) \in \arg\max_b E_{v_{-i}} \left(\begin{array}{c} \left(v_i - \max\left\{\beta_i(v^k), b\right\}\right) \cdot \\ 1_{\max\left\{\beta_i(v^k), b\right\} \geq \max\left\{\max_{j\in\{1,\ldots,k\}\setminus\{i\}} \beta_j(v^k), \max_{j\in\{k+1,\ldots,n\}} b_j(v_j)\right\}} \mid I_i \end{array} \right),$$

where $I_i = \left(v_i, \beta_i(v^k), p_i(v^k)\right)$;

4. (optimal behavior by outside bidders) *for all* $i \in \{k+1, ..., n\}$,

$$b_i(v_i) \in \arg\max_b E_{v_{-i}} \left((v_i - b) 1_{b \geq \max\left\{\max_{j\in\{1,\ldots,k\}} \beta_j(v^k), \max_{j\in\{k+1,\ldots,n\}\setminus\{i\}} b_j(v_j)\right\}}\right);$$

5. (budget balance) $E_{v^k}\left(\sum_{i=1}^k p_i(v^k)\right) \geq \sum_{i=1}^k x_i$;

6. (ex ante individual rationality) *for all* $i \in \{1, ..., k\}$,

$$\begin{aligned} & E_{\mathbf{v}}\left(\left(v_i - \beta_i(v^k)\right) 1_{\beta_i(v^k) \geq \max\left\{\max_{j\in\{1,\ldots,k\}\setminus\{i\}} \beta_j(v^k), \max_{j\in\{k+1,\ldots,n\}} b_j(v_j)\right\}} - p_i(v^k)\right) + x_i \\ \geq \; & E_{\mathbf{v}}\left(\left(v_i - \beta_i^{nc}(v_i)\right) 1_{\beta_i^{nc}(v_i) \geq \max_{j\neq i} \beta_j^{nc}(v_j)}\right). \end{aligned}$$

To define an incentive compatible *bid submission mechanism*, modify the above definition to allow the payment required from ring member i, p_i, to depend on the vector of reports and on the ring members' bids at the auction, and modify the information I_i available to ring member i so that ring member i is told what his payment would be as a function of his bid assuming all other ring members bid according to their recommended bids.[27] One can show that it is sufficient to allow p_i to depend on the vector of reports and on only ring member i's bid at the auction. This holds because in equilibrium ring member i's information and expected payoff depends only on the other ring members' recommended bids and not on their actual bids. Thus, in a *bid submission mechanism*, one can envision a bidding machine that enters bids for the ring members as a function of their reports, preventing the possibility of deviations from the center's recommended bids. Alternatively, the center might prevent ring members whose recommended bids are not highest from attending the auction. Or, each ring member might post a performance bond that is forfeited if any bid appears at the auction under his name that is different from the center's recommendation.

We say a collusive mechanism is ex post efficient if the highest-valuing bidder, whether a ring member or outside bidder, always wins the object. We now define what it means for the ring to capture the entire collusive gain. To state the result formally, when the vector of bidders' values is $\mathbf{v}$ and play is non-cooperative, let $R^{nc}(\mathbf{v})$ be the joint payoff of the ring members, $O_j^{nc}(\mathbf{v})$ be the payoff of outside bidder j ($j \in \{k+1, ..., n\}$), and $S^{nc}(\mathbf{v})$ be the auctioneer's (seller's) payoff. Similarly, when the vector of bidders' values is $\mathbf{v}$ and bidders $1, ..., k$ participate in collusive mechanism μ, let $R^{\mu}(\mathbf{v})$ be the joint payoff of the ring members, $O_j^{\mu}(\mathbf{v})$ be the payoff of outside bidder j, and $S^{\mu}(\mathbf{v})$ be the auctioneer's payoff.

Definition 2 *We say that the ring* ***captures the entire collusive gain*** *under mechanism μ if for all $\mathbf{v}$, $\forall j \in \{k+1, ..., n\}$, $R^{\mu}(\mathbf{v}) \geq R^{nc}(\mathbf{v})$ and*

$$R^{\mu}(\mathbf{v}) > (=) R^{nc}(\mathbf{v}) \Rightarrow S^{\mu}(\mathbf{v}) \leq (=) S^{nc}(\mathbf{v}) \text{ and } O_j^{\mu}(\mathbf{v}) \leq (=) O_j^{nc}(\mathbf{v}).$$

[27] Formally, we assume that instead of learning his required payment, ring member i learns the mapping $p_i(r_i, v_{-i}^k; \cdot, \beta_{-i}^k(r_i, v_{-i}^k))$ from his bid onto his required payment. Alternatively, our results continue to hold if one assumes that a ring member learns the mapping from the actual bids of all the ring members onto his required payment, $p_i(r_i, v_{-i}^k; \cdot, ..., \cdot))$. In this case the ring member must take the expectation over the bids of the other ring members, which in equilibrium are their recommended bids.

If a ring does not capture the entire collusive gain, then the presence of collusion provides some benefit, relative to non-cooperative play, to parties outside the ring, either the outside bidders or the auctioneer.

4 Results

4.1 Bid coordination mechanisms

We begin by considering mechanisms that result in the highest-valuing ring member's bidding at the auction, but that suppress the bids of the other ring members, for example by having non-highest-valuing ring members bid $\underline{v}$ or not bid at all. We refer to mechanisms of this kind as mechanisms that *suppress all ring competition.* As we show, in this case there is a stark difference between profitability of collusion at a second price versus a first price auction. Proposition 1 shows that the ring can capture the entire collusive gain if the auction is second price, but Proposition 2 shows that there is no profitable collusive mechanism if the auction is first price. Thus, Propositions 1 and 2 formalize the intuition that the effectiveness of collusion can be reduced by using a first price rather than a second price auction.

Proposition 1 *There exists a profitable, ex post efficient* ***bid coordination mechanism*** *for a second price auction that suppresses all ring competition and allows the ring to capture the entire collusive gain.*

Proof. See the Appendix.

Proposition 2 *There does not exist a profitable* ***bid coordination mechanism*** *for a first price auction that suppresses all ring competition.*

Proof. See the Appendix.

Proposition 1 establishes the existence of a profitable collusive mechanism under weaker conditions that in the previous literature. For example, the mechanism of Graham and Marshall (1987), which applies to second price auctions, relies on the identity of the winner and the amount paid at the auction.

The proof of Proposition 1 is by construction. The collusive mechanism proposed specifies that the highest-reporting ring member pay the center an amount equal to the expected surplus for a bidder with value equal to the second-highest report from bidding at the auction against the outside bidders. For example, if bidders are symmetric,[28] the ring member with the highest report pays the center $\tilde{p}(r_2)$, where r_2 is the second-highest report and

$$\begin{aligned}\tilde{p}(r_2) &\equiv E_{v_{k+1},...,v_n}\left(r_2 - \max_{j\in\{k+1,...,n\}} v_j \mid r_2 \geq \max_{j\in\{k+1,...,n\}} v_j\right) \Pr\left(r_2 \geq \max_{j\in\{k+1,...,n\}} v_j\right) \\ &= \int_{\underline{v}}^{r_2} F^{n-k}(x)dx.\end{aligned}$$

Ring members with lower reports pay nothing. The center recommends that the bidder with the highest report bid his report at the auction and that all other ring members bid $\underline{v}$. In equilibrium, ring members truthfully report their values and follow the recommendations of the center. Integrating over the possible second-highest values in the ring, the ex ante non-contingent payment to each ring member is $\frac{1}{k}\int_{\underline{v}}^{\bar{v}} \tilde{p}(x)k(k-1)F^{k-2}(x)(1-F(x))f(x)dx$, which is positive and satisfies budget balance for the center.

To see that interim individual rationality is satisfied (and therefore also ex ante individual rationality), note that if a bidder $i \in \{1, ..., k\}$ with value v joins the ring, he has expected payoff equal to the ex ante non-contingent payment plus

$$\tilde{p}(v) - \frac{\int_{\underline{v}}^{v} \tilde{p}(x)(k-1)F^{k-2}(x)f(x)dx}{F^{k-1}(v)} F^{k-1}(v), \tag{1}$$

where the first term is the ring member's expected payoff from competing at the auction and the second term is the ring member's expected payment to the center. Re-

[28] Proposition 1 also applies to asymmetric bidders, but for the purposes of the example, it is useful to focus on the case of symmetric bidders.

arranging (1) and integrating by parts, it can be shown to be equal to $\int_{\underline{v}}^{v} F^{n-1}(x)dx$, which is the expected payoff to a bidder with value v under non-cooperative play. Thus, a potential ring member bidder strictly prefers to join the ring since he enjoys a payoff that exceeds his non-cooperative payoff by the amount of the ex ante non-contingent payment.[29]

Proposition 1 establishes the profitability of a *bid coordination mechanism* when the auction is second price, and this implies that there is a similarly profitable *bid submission mechanism.*

Corollary 1 *There exists a profitable, ex post efficient* ***bid submission mechanism*** *for a second price auction that suppresses all ring competition and allows the ring to capture the entire collusive gain.*

As has been noted in the literature, the ability of ring members to use shills to place bids on their behalf can affect the profitability of a collusive mechanism.[30] But, because the incentive compatibility constraints for a *bid coordination mechanism* imply that no ring member has an incentive to use a shill bidder to submit a bid at the auction, Proposition 1 implies that a ring at a second price auction is unaffected by the feasibility of shill bidding.

Corollary 2 *The feasibility of shill bidding does not affect the ring's expected payoff in the second price mechanism of Proposition* 1*.*

[29] For a numerical example, consider the case with $n = 3$ and $k = 2$, so there are two bidders in the ring and one outside bidder, and assume values are uniformly distributed on $[0, 1]$. In this case, the ex ante non-contingent payment is $\frac{1}{24}$. Let bidders 1 and 2 be the ring members, and let $v_1 \geq v_2$. In equilibrium, bidder 1 (or a randomly selected ring member if $v_1 = v_2$) pays $\frac{v_2^2}{2}$ to the center and competes against bidder 3 at the auction. Bidder 1 expects payoff $\frac{v_1^3}{3}$ from non-cooperative play if he does not join the ring. If he does join the ring, he gets ex ante payment $\frac{1}{24}$, expects to pay $\frac{v_1^3}{6}$ to the center, and expects surplus $\frac{v_1^3}{2}$ from the auction, for an expected payoff of $\frac{1}{24} + \frac{v_1^3}{3}$, which is greater than his non-cooperative expected payoff by the amount of the ex ante payment.

[30] Some literature uses "shill bidding" to mean bids submitted by the auctioneer (or seller) under the guise of being a regular bidder (see Chakraborty and Kosmopoulou (2001) and Hidvégi, Wang, and Whinston (2001)). We assume a non-strategic auctioneer and use "shill bidding" to mean bids submitted by ring members under a different name, which cannot be traced to them.

In contrast, when the auction is first price, Proposition 2 shows that, when the ring is restricted to use a payment rule that depends only on the reports to the center, there is no profitable collusive mechanism that suppresses the bids of all but the highest-valuing ring member. Because the center cannot penalize deviations from the recommended bids, in any profitable collusive mechanism that suppresses the bids of non-highest-valuing ring member, the highest-valuing ring member bids optimally against the outside bidders implying that he bids strictly less than his value (for all values above $\underline{v}$) and that his bid does not depend on the values of the other ring members. But then, with positive probability, there exists a ring member who is supposed to suppress his bid but who can profitably deviate by competing at the auction against the highest-valuing ring member and the outside bidders, a contradiction.

Because Proposition 2 focuses on mechanisms that suppress the bids of all but the highest-valuing ring member, the result does not rule out the possibility that there exists some other kind of profitable *bid coordination mechanism* at a first price auction; however, it does suggest that *bid coordination mechanisms* have limited benefit. In particular, the proposition implies that no *bid coordination mechanism* can suppress all competition among the ring members. Since optimal non-cooperative bids at a first price auction depend on the number of bidders, one might think that a ring at a first price auction could secure a collusive gain merely by suppressing the bids of some of the ring members. However, as we now show, for symmetric bidders, even if the ring could suppress competition among all ring members other than the first and second-highest-valuing ring members, that would not be sufficient to secure a collusive gain. The proof relies on Assumption 1 and the uniqueness result of Lemma 1.[31]

Proposition 3 *Assume bidders are symmetric. If a* ***bid coordination mechanism***

[31] A similar result holds for second price auctions if we restrict attention to the equilibrium in weakly dominant strategies.

at a first price auction suppresses ring competition except for the first and second-highest-valuing ring members, but provides no information to these two ring members other than that their values are either highest or second highest, then the unique equilibrium of the auction subgame is for the two ring members and all outside bidders to bid non-cooperatively.

Proof. See the Appendix.

Proposition 3 further highlights the difficulty of finding a profitable *bid coordination mechanism*. It says that as long as the two highest-valuing ring members participate in the auction, each knowing only that they have one of the two highest values, then the equilibrium of the auction subgame involves non-cooperative bidding. So for there to be any gain relative to non-cooperative play, the ring must do more than just reduce the number of bidders attending the auction by suppressing bids of lower-valuing bidders.

At a second price auction, a ring can secure a collusive gain using a *bid coordination mechanism* that merely manipulates the second-highest ring bid, but at a first price auction, a profitable *bid coordination mechanism* must reduce the highest ring bid and manipulate the second-highest ring bid. Thus, the task facing a ring is more difficult at a first price auction than at a second price auction.

Propositions 2 and 3, identify outcomes that cannot be accomplished with a *bid coordination mechanism*. We now characterize a profitable *bid coordination mechanism* for a first price auction. Our characterization result has interesting empirical implications. It says that a *bid coordination mechanism* at a first price auction must sometimes require that ring members other than the highest-valuing ring member bid at the auction. In particular, the mechanism must require that at least one other ring member submit a bid that is close to the highest ring bid.

To see the intuition for this result, first note that if the center recommends a bid to a ring member that is less than that ring member's optimal bid against the outside bidders (assuming no other ring bids), then that ring member can increase his expected payoff by deviating from the recommendation of the center as long as the ring member's information implies positive probability of winning the object at the higher bid. Second, note that if the center always recommends that the highest-valuing ring member bid optimally against the outside bidders and that all other ring members bid something less, then for a positive-measure set of values, some ring member has an incentive to submit a higher bid in an attempt to outbid the highest-valuing ring member. Thus, the center must sometimes recommend a bid greater than the optimal bid for the highest-valuing ring member against the outside bidders. Third, in order for a bid above the highest-valuing ring member's optimal bid against the outside bidders to be incentive compatible, it must be that some other ring member also bids above the optimal bid. Loosely, to prevent deviations from non-highest-valuing ring members, the center must recommend that the highest-valuing ring member bid sufficiently high, but then to prevent deviations from that ring member, the center must recommend that some other ring member submit a bid just below his.

To formalize this result, let $\beta^*(v; b_{k+1}, ..., b_n)$ be the optimal bid for a ring member with value v if all other ring members submit bid $\underline{v}$ and the outside bidders bid according to bid functions $b_{k+1}, ..., b_n$,[32] i.e.,

$$\beta^*(v; b_{k+1}, ..., b_n) \in \arg\max_b E_{\{v_{k+1},...v_n\}} \left((v-b)\, 1_{b \geq \max_{j\in\{k+1,...,n\}} b_j(v_j)} \right).$$

We begin with a lemma.

Lemma 2 *In any* ***bid coordination mechanism*** *at a first price auction, for all but a zero-measure set of ring members' values, the highest ring bid is greater than or equal to* $\beta^*(\max_{j\in\{1,...,k\}} v_j; b_{k+1}, ..., b_n)$*, and strictly greater for a positive-measure set*

[32] A ring member's optimal bid against the outside bidders exists if the conjectured bid functions for the outside bidders are continuous. If the optimal bid is not unique, then Lemma 2 continues to hold if we define β^* to be the minimum of the optimal bids, and then Proposition 4 holds because there can only be multiplicity of optimal bids for a zero measure set of value realizations.

of ring members' values, where $b_{k+1}, ..., b_n$ *are the equilibrium bid functions (assumed continuous) for the outside bidders.*

Proof. See the Appendix.

Lemma 2 says that any *bid coordination mechanism* at a first price auction (almost) always results in a ring bid that is at least as high as what the optimal bid would be for the highest-valuing ring member bidding against the outside bidders, and sometimes strictly greater.

Proposition 4 *In any profitable* ***bid coordination mechanism*** *at a first price auction, for a positive-measure set of value realizations, at least two ring members submit bids at the auction that are greater than or equal to* $\beta^*(\max_{j\in\{1,...,k\}} v_j; b_{k+1}, ..., b_n)$*, where* $b_{k+1}, ..., b_n$*, are the equilibrium bid functions (assumed continuous) for the outside bidders. Furthermore, for any* $\varepsilon > 0$*, there is a positive-measure set of value realizations such that the highest two ring bids are within* ε *of each other.*

Proof. See the Appendix.[33]

Proposition 4 implies that when v^* is the highest value in the ring, we should expect to see multiple ring bids between $\beta^*(v^*; b_{k+1}, ..., b_n)$ and v^*. Furthermore, it follows from Proposition 4 that it must be the highest-valuing ring member who submits one of the bids greater than or equal to $\beta^*(v^*; b_{k+1}, ..., b_n)$.

Corollary 3 *In any profitable* ***bid coordination mechanism*** *at a first price auction, either the highest-valuing ring member bids* $\beta^*(\max_{j\in\{1,...,k\}} v_j; b_{k+1}, ..., b_n)$ *and all other ring members bid less, or the highest-valuing ring member and at least one other ring member bid greater than or equal to* $\beta^*(\max_{j\in\{1,...,k\}} v_j; b_{k+1}, ..., b_n)$ *and all other ring members bid less.*

[33]It is important to note that we do not know if an equilibrium exists when the ring submits two bids. Lemma 1 may not apply.

Proof. Using Proposition 4, if the highest-valuing ring member's recommended bid is less than $\beta^*(\max_{j\in\{1,\ldots,k\}} v_j; b_{k+1}, \ldots, b_n)$, then that ring member believes with probability one that his recommended bid is not highest in the ring and so has zero probability of winning the auction. Because the ring member's value is highest, he must believe that a bid less than his value wins the auction with positive probability. Thus, the ring member has a profitable deviation. Q.E.D.

In environments with discrete bid increments, the empirical implications of Proposition 4 are particularly interesting. Proposition 4 implies that the highest ring bid must sometimes be strictly greater than $\beta^*(\max_{j\in\{1,\ldots,k\}} v_j; b_{k+1}, \ldots, b_n)$, but a bid greater than $\beta^*(\max_{j\in\{1,\ldots,k\}} v_j; b_{k+1}, \ldots, b_n)$ is not optimal for any ring member unless that ring member believes that reducing his bid by one bid increment would decrease the probability with which he outbids the other ring members. Thus, a profitable *bid coordination mechanism* must sometimes require that one ring member submit a bid that is within one bid increment of the highest ring bid.

Corollary 4 *Assume a first price auction with small, discrete bid increments. In any* ***bid coordination mechanism****, for a positive-measure set of value realizations, the two highest ring bids are within one bid increment of each other.*

Corollary 4 says that the two highest ring bids will sometimes be within one bid increment of each other. If the collusive mechanism never allows non-highest-valuing ring members to win the object, then not only must the two highest ring bids be within one bid increment, but they must never be tied.

As an illustration of how a bid coordination mechanism might work, consider a mechanism that recommends that the highest-valuing ring member bid the maximum of $\beta^*(\max_{j\in\{1,\ldots,k\}} v_j; b_{k+1}, \ldots, b_n)$ and the second-highest value in the ring. Furthermore, suppose the mechanism recommends that the second-highest-valuing ring member bid his value minus one bid increment, and that all other ring members bid some lower amount. Then, clearly, no ring member has an incentive deviate from the center's recommended bids. In such a mechanism, there may be a problem satisfying incentive compatibility for reports to the center, but this can be overcome if the center can observe the identity of the winner and penalize ring members who win but who did not have the highest report.[34]

4.2 Leakage

One difficulty in proving general results about the profitability of collusion at first price auctions stems from the fact that at a first price auction, the ring cannot capture the entire collusive gain. Because a profitable ring at a first price auction must reduce the bids submitted by the ring members relative to their non-cooperative bids, some of the collusive gain must go to the bidders outside the ring or, if outside bidders respond to collusion by increasing their bids, then to the auctioneer.

Proposition 5 *At a first price auction, there is no profitable pre-auction mechanism such that the ring captures the entire collusive gain.*

Proof. See the Appendix.

Proposition 5 identifies a reason why collusion may not be sustainable at a first price auction—some of the gains from collusion necessarily spill over to the outside bidders or possibly the auctioneer. For example, in the case of only one outside bidder, he always profits from the presence of a ring because even if the outside bidder does not change from his non-cooperative bid function, his expected payoff is strictly higher when the ring members bid collusively, i.e., reduce their bids, than when they bid non-cooperatively.

[34] The problem in such a mechanism is that a ring member might have an incentive to report an amount less than his value. By under reporting, a ring member might cause the highest bid submitted by any other ring member to be lower, and so allow the deviating ring member to win the object at a price lower than he could have otherwise. To deter under reporting, it is sufficient that the center be able to penalize ring members who win the object but did not have the highest report.

4.3 Bid Submission Mechanisms

Because the definition of an incentive compatible *bid coordination mechanism* requires that a bidder not prefer to submit any bid other than his recommended bid, it is clear that such a mechanism continues to be incentive compatible when we allow shill bidders. More formally, for a *bid coordination mechanism*, the constraint of incentive compatibility for not using a shill is implied by constraint of incentive compatibility for bidding. However, this is not the case for a *bid submission mechanism.* In a *bid submission mechanism*, a ring member's bid affects his payment to the center, which enters the incentive compatibility constraint for bidding. Thus, a ring member may choose one "official" bid because of its effect on his payment to the center, and then use a shill bidder to submit a higher bid at the auction. Ring member i's payment to the center is not affected by a bid submitted on his behalf by a shill bidder, but his expected payoff from the auction depends on the maximum of his "official" bid and the bid submitted by his shill.

As the following proposition shows, although a *bid submission mechanism* may be able to suppress all ring competition, shills strictly reduce the profitability of the ring.

Proposition 6 *For a first price auction, the ability to use shills weakly reduces the expected joint payoff of the ring members from any profitable* ***bid submission mechanism,*** *and strictly for one that suppresses all ring competition.*

Proof. See the Appendix.

If the ring mechanism operates by essentially sending only one ring member to bid at the auction and suppressing the bids of the other ring members, then any ring member not sent to the auction can profitably use a shill as long as there is some probability that the ring member officially sent to the auction submits a bid strictly below his value—the latter being a necessity for collusion to be profitable. Viewed together with Corollary 2, Proposition 6 implies that the presence of shills has a weakly larger impact on the profitability of a ring when the auction is first price than when it is second price.

Proposition 6 implies that shills can reduce the ring's payoff, but the next proposition shows that, in fact, shills mean that a ring at a first price auction may not be able to do any better by using a *bid submission mechanism* than it could by using a *bid coordination mechanism.* Proposition 7 assumes that the center can use a shill to submit a bid—this allows the center for a *bid coordination mechanism* to submit a bid just below the bid of the high-valuing ring member to prevent downward deviations.

Proposition 7 *In a first price auction, if the center can submit a bid, then the maximum expected payoff to a ring from a* ***bid submission mechanism*** *is equal to the maximum expected payoff to a ring from a* ***bid coordination mechanism****.*

Proof. See the Appendix.

Clearly, a ring can do at least as well using a bid submission mechanism as it can using a bid coordination mechanism. But Proposition 7 says the possibility of using shill bidders negates any advantage to the ring from using a *bid submission mechanism.* In particular, Proposition 7 says that if we allow the center to submit a bid, a ring at a first price auction cannot do any better with a *bid submission mechanism* than it could with a *bid coordination mechanism.*

4.4 Environments without shill bidders

In this section we show that there do exist environments in which collusion can be sustained at a first price auction. In particular, we consider *bid submission mechanisms* in an environment without shill bidders.[35] In this environment, the ring mechanism

[35] In this environment, for a second price auction, Corollary 1 implies that there exists a profitable, ex post efficient mechanism that allows the ring to capture the entire collusive gain.

can fix ring members' bids (as a function of the reports) by requiring a large payment from ring members who bid anything other than the center's recommended bids. One way a mechanism like this might be implemented is by requiring that certain ring members not attend the auction or by having the center submit bids on behalf of the ring members.

We construct a mechanism in which ring members report their values and then only the highest-valuing ring member bids at the auction against the outside bidders, implying that all ring competition is suppressed. The center recommends that the highest-valuing ring member bid according to the equilibrium bid function for an auction in which the highest-valuing ring member bids against the $n-k$ outside bidders. Let $\beta^{in}(v)$ be the equilibrium first price bid for a ring member whose value v is the highest in the ring when facing the $n-k$ outside bidders, and let $\beta_i^{out}(v)$ be the equilibrium bid for outside bidder i with value v. Equilibrium bid functions β^{in} and β_i^{out}, which are unique by Lemma 1, are defined by the conditions that for all v,

$$\beta^{in}(v) \in \arg\max_b E_{v_{k+1},...,v_n} \left((v-b) \, 1_{b \geq \max_{j \in \{k+1,...,n\}} \beta_j^{out}(v_j)} \right)$$

and

$$\beta_i^{out}(v) \in \arg\max_b E_{v_{-i}} \left((v-b) \, 1_{b \geq \max\left\{\beta^{in}(\max_{j\in\{1,...,k\}} v_j), \max_{j \in \{k+1,...,n\} \setminus \{i\}} \beta_j^{out}(v_j)\right\}} \right).$$

Note that the equilibrium bid function for the ring member, β^{in}, does not depend on which ring member has the highest value. Note also that β^{in} and β_i^{out} define the equilibrium of the auction subgame for a mechanism that prevents all but the highest-valuing ring member from bidding at the auction, but does not place any restriction on the bid of the highest-valuing ring member. Finally, note that, referring to our definition of β^*, $\beta^{in}(v) = \beta^*(v; \beta_{k+1}^{out}, ..., \beta_n^{out})$.

Consider a payment rule that requires that ring members whose reports are less than the highest report pay $\bar{v}$ to the center if they bid an amount greater than $\underline{v}$, and zero otherwise, and that the ring member with the highest report pay $\hat{p}(r)$ to the center, where r is the second-highest report and

$$\hat{p}(r) \equiv E_{v_{k+1},...,v_n}\left(\left(r - \beta^{in}(r)\right) 1_{\beta^{in}(r) \geq \max_{j \in \{k+1,...,n\}} \beta_j^{out}(v_j)}\right),$$

which can be implemented by having the ring members compete in a second price ex ante auction for the right to be the sole ring member who bids at the main auction (see Graham and Marshall (1987)). The mechanism recommends bids of $\underline{v}$ for ring members who do not have the highest report and recommends a bid of $\beta^{in}(r_i)$ for the ring member i with the highest report. This mechanism induces truthful revelation, and it is a best reply for bidders to follow the recommendations of the ring (see the proof of Proposition 8).

Given this payment rule, ex ante budget balance for the center implies an ex ante non-contingent payment to each ring member of:[36]

$$x \equiv \frac{1}{k} E_{v_1,...,v_k}\left(\hat{p}(v_j) 1_{v_i \geq v_j \geq \max_{\ell \in \{1,...,k\} \setminus \{i,j\}} v_\ell}\right).$$

Letting $\beta_i^{nc}(v)$ be the non-cooperative equilibrium first price bid for bidder i with value v facing $n-1$ other bidders, the interim individual rationality constraint for ring member i with value v_i can be written as the requirement that $g_i(v_i \mid \hat{p}) \geq 0$, where g_i is defined by

$$\begin{aligned} g_i(v_i \mid \hat{p}) \equiv\ & x + E_{v_{-i}}\left(\left(v_i - \beta^{in}(v_i)\right) 1_{v_i \geq \max_{j \in \{1,...,k\} \setminus \{i\}} v_j \text{ and } \beta^{in}(v_i) \geq \max_{j \in \{k+1,...,n\}} \beta_j^{out}(v_j)}\right) \\ & - E_{v_{-i}}\left(\hat{p}\left(\max_{j \in \{1,...,k\} \setminus \{i\}} v_j\right) 1_{v_i \geq \max_{j \in \{1,...,k\} \setminus \{i\}} v_j}\right) \\ & - E_{v_{-i}}\left(\left(v_i - \beta_i^{nc}(v_i)\right) 1_{\beta_i^{nc}(v_i) \geq \max_{j \neq i} \beta_j^{nc}(v_j)}\right), \end{aligned}$$

where the first term is the ex ante non-contingent payment, the second term is the expected surplus from the auction, the third term is the expected payment to the center, and the fourth term is the expected surplus from non-cooperative play.

Proposition 8 *In the absence of shill bidding, there exists a profitable* ***bid submission mechanism*** *for a first price auction that suppresses all ring competition if, for all* $i \in \{1, ..., k\}$, $E_{v_i}(g_i(v_i \mid \hat{p})) \geq 0$, *with a strict inequality for some* i.

[36] Non-symmetric payments could also be made.

Proof. See the Appendix.

Proposition 8 provides conditions under which a profitable collusive mechanism exists for a first price auction, but leaves unanswered the question whether these conditions can be satisfied. Even assuming symmetric bidders, the bid functions β^{in} and β^{out} cannot be represented analytically, so we rely on a numerical calculation to show that the conditions of Proposition 8 can be satisfied.

Proposition 9 *In the absence of shill bidding, there exists a profitable* ***bid submission mechanism*** *for a first price auction that suppresses all ring competition when* $n = 3$, $k = 2$, *and values are drawn from the uniform distribution on* $[0, 1]$.

Proof. See the Appendix.[37]

Proposition 9 shows that for some environments without shills, a *bid submission mechanism* can suppress all ring competition. In contrast, Proposition 2, which holds with or without shills, shows that no bid coordination mechanism can suppress all ring competition. Furthermore, the mechanism constructed to prove Proposition 9 shows that for some environments without shills, a *bid submission mechanism* can achieve the same outcome as a mechanism that prevents all but the highest-valuing ring member from attending the auction. In contrast, Proposition 4, which holds with or without shills, implies that no bid coordination mechanism can achieve this outcome.

The result that collusion at a first price auction can be profitable requires the assumption that the use of shills is not possible. This is because a ring member with less than the highest value, although prevented from bidding at the auction himself because the center can base payments on the ring member's bid, may have

[37] The conditions of Assumption 1 are not satisfied in this example since the distribution of the highest from two uniform random variables is $F(x) = x^2$; however, the conditions provided by Lebrun (2002) are satisfied (see footnote 16).

an incentive to compete against the high-valuing ring member using a shill, which reduces the profitability of the ring.

The proof of Proposition 9 involves the numerical calculation of the ex ante individual rationality constraint using the methods of Marshall et al. (1994) to solve for the equilibrium bid functions. It is interesting that in the example of Proposition 9, although ex ante individual rationality is satisfied, interim individual rationality is *not* satisfied for ring members with values above approximately 0.8.[38] Thus, under interim individual rationality, bidders with sufficiently high values are not willing to join the ring. A potential ring member must weigh whether he captures enough of the collusive gain to justify deviating from non-cooperative play, and bidder with a high value typically gains less from collusive play because his required payment to the center is larger than the ex ante non-contingent payment. In effect, ring members with high values subsidize ring members with low values in satisfying the center's ex ante balanced-budget constraint.

For a one-shot auction, the appropriate individual rationality is interim individual rationality if potential ring members learn their values before deciding whether to join the ring. For example, in Definition 1, the individual rationality constraint for ring member i would need to be modified so that the expectations are taken with respect to v_{-i} rather than with respect to the entire vector of ring members' values.

With this modification, Propositions 1–7 continue to hold. However, when we modify Proposition 8 to require for all $i \in \{1, ..., k\}$, $g_i(v_i \mid \hat{p}) \geq 0$, which is the interim individual rationality constraint for a *bid submission mechanism* at a first price auction, then Proposition 9 no longer holds. In Appendix A, we show that with an all-inclusive ring, the interim individual rationality constraint can be satisfied.

[38]In fact, we have been unable to find any example within the context implicitly defined by Proposition 8 in which interim individual rationality is satisfied for all feasible value realizations.

5 Discussion

We are interested in collusive mechanisms that do not rely on information from the auctioneer, such as the identity of the winner or the amount paid. We refer to these mechanisms as pre-auction mechanisms. Within this class of mechanisms, we identify two types. The first type of mechanism, a *bid coordination mechanism,* gathers information from the ring members regarding their values for the object, arranges for transfers among ring members, and makes recommendations on how they should bid at the auction. The second type of mechanism, a *bid submission mechanism,* gathers the same information from the bidders and arranges for transfers, but instead of merely recommending bids to the ring members, the ring center controls the bids submitted by the ring members. However, the center cannot prevent ring members from using a shill bidders to submit additional bids at the auction. We consider the effectiveness of these two types of pre-auction mechanisms in facilitating collusion at first and second price auctions.

We show that at a second price auction, pre-auction mechanisms allow the ring to suppress all competition among ring members and to capture the entire collusive gain. In contrast, at a first price auction this is not the case—a *bid coordination mechanism* cannot suppress all competition among ring members, and regardless of which type of mechanism is used, a ring cannot capture the entire collusive gain. Bidders at a first price auction are limited in their ability to profitably collude by the facts that (i) a ring may not be able to suppress competition among its members, (ii) the gain to the ring is reduced by leakage to the outside bidders (lower bids by ring members imply increased payoffs for outside bidders), and (iii) ring members may have an incentive to use shill bidders. Despite these limitations, profitable collusion is possible at a first price auction in some cases. We provide a characterization of profitable pre-auction mechanisms at a first price auction, and for some cases provide examples establishing that profitable pre-auction mechanisms exist for first price auctions. However, we have not proven the existence of a profitable *bid coordination mechanism* for a first price auction, although we do provide a characterization. Establishing existence is an important topic for future research.

Our characterization result for *bid coordination mechanisms* for a first price auction provides us with an important empirical implication. In particular, Proposition 4 tells us that the ring using a *bid coordination mechanism* must sometimes require multiple bids from the ring—a high bid and another that is just below it. The observational significance of this result requires further explanation. To begin, this characterization distinguishes a *bid coordination mechanism* from a *bid submission mechanism* since, as shown in Proposition 9, a ring using a *bid submission mechanism* need only submit one bid when the use of shills can be prevented. Further, Appendix A, where we consider an all-inclusive ring, shows that an all-inclusive ring need only submit one bid when the use of shills can be prevented. Thus, with regard to the results of this paper, the clustering of ring bids described in Proposition 4 is unique.

One other auction environment in which one sees clustered bids is a first price auction with *complete information* where bidders act non-cooperatively. In this case, equilibrium behavior involves the highest-valuing bidder's submitting a bid equal to the second-highest value, while the second-highest-valuing bidder aggressively mixes under his value. This implies that the two high bids will be very close to one another. But a key feature differentiates the "close" collusive bids of Proposition 4 from the "close" non-cooperative bids of a complete information environment. The close collusive bids are ring bids and so may or may not be the highest two bids submitted at the auction, depending on the bids of the outside bidders. But the close non-cooperative bids of a complete information environment are always the highest and second-highest bids. In other words, a prediction of Proposition 4 that is entirely unique to collusion is that we will regularly observe pairs of non-winning bids that are very close to one another. This provides a way to detect collusion that requires little information about the bidders or the items being sold.[39]

[39] One might expect to see close bids under non-cooperative bidding when values are close, but then with a discrete bid increment, one would also expect to see ties occasionally, something that

A policy implication of our results seems clear—if collusion is a major concern for auction designers, then a first price auction should be used rather than a second price auction. Another implication for auctioneers (or procurement agents) is to maintain a record of all bids, not just those of winners. As Proposition 4 makes clear, there is potentially a large amount of information in the difference between sequential bids.

would not be expected under collusion.

A Appendix: All-inclusive Ring

When the ring is all-inclusive ($k = n$), collusion can be profitable in a wider set of environments. Our positive results for collusion at a second price auction (Propositions 1 and Corollary 1) continue to hold for an all-inclusive ring, and our negative result for a *bid coordination mechanism* at a first price auction (Proposition 2) continues to hold,[40] but there are some cases in which collusion is profitable at a first price auction when the ring is all-inclusive, but collusion is not profitable when the ring is not all-inclusive.

Consider a *bid submission mechanism* at a first price auction when the use of shills is not possible. Suppose ring members report their values and the center recommends that the ring member with the highest report bid $\underline{v}$ and all others bid less than $\underline{v}$ or not at all. Ring members whose reports are less than the highest report pay $\bar{v}$ to the center if they bid an amount greater than or equal to zero, and the ring member with the highest report pays $r_2 - \underline{v}$ to the center, where r_2 is the second-highest report. This mechanism induces truthful revelation and it is a best reply for bidders to follow the recommendation of the ring.

To show that collusion is profitable, we need only show that individual rationality is satisfied. Ex ante budget balance for the center implies an ex ante non-contingent payment to each ring member of:[41]

$$X^a \equiv \frac{1}{n} E_{v_1,\dots,v_n} \left((v_j - \underline{v}) \, 1_{v_i \geq v_j \geq \max_{\ell \in \{1,\dots,n\} \setminus \{i,j\}} v_\ell} \right).$$

The interim individual rationality for ring member i with value v_i can be written as

[40] The proof follows as before letting $k = n$ and assuming the center recommends that ring members with less than the highest value bid an amount less than $\underline{v}$ or do not bid at all.

[41] Differential payments, reflecting bidders different ex ante marginal contributions to the ring, are also possible (see Graham, Marshall, and Richard (1990), especially Theorem 7).

the requirement that $g_i^a(v_i) \geq 0$, where g_i^a is defined by

$$\begin{aligned} g_i^a(v_i) \equiv & X^a + E_{v_{-i}}\left((v_i - \underline{v})\, 1_{v_i \geq \max_{j\in\{1,\ldots,n\}\setminus\{i\}} v_j}\right) \\ & - E_{v_{-i}}\left(\left(\max_{j\in\{1,\ldots,n\}\setminus\{i\}} v_j - \underline{v}\right) 1_{v_i \geq \max_{j\in\{1,\ldots,n\}\setminus\{i\}} v_j}\right) \\ & - E_{v_{-i}}\left((v_i - \beta_i^{nc}(v_i))\, 1_{\beta_i^{nc}(v_i) \geq \max_{j\neq i} \beta_j^{nc}(v_j)}\right), \end{aligned}$$

where the first term is the ex ante non-contingent payment, the second term is the expected surplus from the auction, the third term is the expected payment to the center, and the fourth term is the expected surplus from non-cooperative play. For symmetric bidders, $\beta^{nc}(v_i) = E_{v_{-i}}\left(\left(\max_{j\in\{1,\ldots,n\}\setminus\{i\}} v_j\right) 1_{v_i \geq \max_{j\in\{1,\ldots,n\}\setminus\{i\}} v_j}\right)$. Thus, $g_i^a(v_i) = X^a$ for all v_i, implying that interim individual rationality is satisfied.[42] This also implies that, for symmetric bidders, ex ante individual rationality is satisfied.

In general, ex ante individual rationality is satisfied whenever the environment is such that

$$\begin{aligned} & E_{\mathbf{v}}\left((v_i)\, 1_{v_i \geq \max_{j\in\{1,\ldots,n\}\setminus\{i\}} v_j}\right) - E_{\mathbf{v}}\left((v_j)\, 1_{v_i \geq v_j \geq \max_{\ell\in\{1,\ldots,n\}\setminus\{i,j\}} v_\ell}\right) \\ \geq \; & E_{\mathbf{v}}\left((v_i - \beta_i^{nc}(v_i))\, 1_{\beta_i^{nc}(v_i) \geq \max_{j\neq i} \beta_j^{nc}(v_j)}\right), \end{aligned}$$

that is, the expected value of the highest from n minus the expected value of the second-highest from n is greater than or equal to the expected surplus from the winner of the auction under non-cooperative play.

This proves the following proposition.

Proposition A.1 *Assuming no shills, symmetric bidders, and an all-inclusive ring, there exists a profitable, ex post efficient* ***bid submission mechanism*** *for first price auction satisfying interim individual rationality that suppresses all ring competition and allows the ring to capture the entire collusive gain.*

[42] To see that interim individual rationality is not necessarily satisfied for this mechanism when bidders are asymmetric, consider the case in which $k = 2$, $F_1(x) = x$, and $F_2(x) = x^{100}$. Then $\beta_1^{nc}(1) = \beta_2^{nc}(1) \cong 0.7391$ (see Marshall et al. (1994)). The cdf for the second highest value is $G(y) = y + y^{100}(1-y)$, so the ex ante non-contingent payment is $\frac{1}{2}\left(\frac{1}{2} + \frac{100}{101} - \frac{101}{102}\right) \cong 0.2500$. If ring member 1 has value 1, his expected payment to the center is $\frac{100}{101}$, so his expected payoff from joining the ring is $0.2500 + 1 - \frac{100}{101} = .2599$, which is less than his expected non-cooperative payoff of 0.2609.

Proposition A.1 contrasts with the result in the text for the case of symmetric bidders with two ring members, one bidder outside the ring, and values drawn from the uniform distribution on $[0, 1]$. In that case, interim individual rationality is not satisfied for ring members with values greater than approximately 0.8. Proposition A.1 shows that when the ring is all-inclusive and bidders are symmetric, interim individual rationality is satisfied for any number of bidders and any distribution of values.

B Appendix: Proofs

Proof of Proposition 1. Consider the following bidding rule: if bidder i's report is not highest, the center recommends a bid of $\underline{v}$, and if bidder i's report is highest, the center recommends a bid equal to the report. Note that if the bidders report truthfully, there is no incentive for any bidder to deviate from the center's recommendation, even with a shill. Consider the following payment rule: if bidder i's report is not highest, bidder i pays zero, but if bidder i's report is highest and r_2 is the second-highest report, then bidder i pays the center $\hat{p}(r_2) = E_{v_{k+1},\dots,v_n}\left(\left(r_2 - \max_{j\in\{k+1,\dots,n\}} v_j\right) 1_{r_2 \geq \max_{j\in\{k+1,\dots,n\}} v_j}\right)$. Note that under this payment rule, the center has positive expected revenue and so can make positive ex ante non-contingent payments to the ring members.

Suppose the other $k-1$ ring members report truthfully. If a ring member with value v_1 reports $v_1+\varepsilon$ (where $\varepsilon > 0$) rather than v_1, his payoff differs only if the highest other value in the ring is $v_2 \in (v_1, v_1+\varepsilon)$. In this case, if the ring member reports truthfully his payoff is zero, and if he reports $v_1+\varepsilon$ and bids v_1 at the auction (his weakly dominant strategy in the continuation game), his expected payoff is

$$\hat{p}(v_1) - E_{v_2}\left(\hat{p}(v_2) \mid v_2 \in (v_1, v_1+\varepsilon)\right) < \hat{p}(v_1) - \hat{p}(v_1) = 0.$$

Thus, the ring member has no incentive to deviate in this way. If a ring member with value v_1 reports $v_1-\varepsilon$ (where $\varepsilon > 0$) rather than v_1, his payoff differs only if the highest other value in the ring is $v_2 \in (v_1-\varepsilon, v_1)$. In this case, if the ring member reports truthfully, his expected payoff is $\hat{p}(v_1) - E_{v_2}\left(\hat{p}(v_2) \mid v_2 \in (v_1-\varepsilon, v_1)\right)$, and if he reports $v_1-\varepsilon$ and bids v_1 at the auction, he makes no payment to the center and has expected payoff

$$
\begin{aligned}
& E_{v_2,v_{k+1},\dots,v_n}\left(\left(v_1-\max_{j\in\{2,k+1,\dots,n\}} v_j\right)1_{v_1\geq\max_{j\in\{2,k+1,\dots,n\}} v_j} \mid v_2\in(v_1-\varepsilon,v_1)\right)\\
= \ & E_{v_2,v_{k+1},\dots,v_n}\left(\begin{array}{c}\left(v_1-\max_{j\in\{k+1,\dots,n\}} v_j\right)1_{v_1\geq\max_{j\in\{k+1,\dots,n\}} v_j\geq v_2}\\ +\left(v_1-v_2\right)1_{v_1\geq v_2\geq\max_{j\in\{k+1,\dots,n\}} v_j} \mid v_2\in(v_1-\varepsilon,v_1)\end{array}\right)\\
= \ & E_{v_2,v_{k+1},\dots,v_n}\left(\begin{array}{c}\left(v_1-\max_{j\in\{k+1,\dots,n\}} v_j\right)1_{v_1\geq\max_{j\in\{k+1,\dots,n\}} v_j\geq v_2}\\ +\left(v_1-\max_{j\in\{k+1,\dots,n\}} v_j\right)1_{v_1\geq v_2\geq\max_{j\in\{k+1,\dots,n\}} v_j}\\ -\left(v_2-\max_{j\in\{k+1,\dots,n\}} v_j\right)1_{v_2\geq\max_{j\in\{k+1,\dots,n\}} v_j} \mid v_2\in(v_1-\varepsilon,v_1)\end{array}\right)\\
= \ & E_{v_2,v_{k+1},\dots,v_n}\left(\begin{array}{c}\left(v_1-\max_{j\in\{k+1,\dots,n\}} v_j\right)1_{v_1\geq\max_{j\in\{k+1,\dots,n\}} v_j}\\ -\left(v_2-\max_{j\in\{k+1,\dots,n\}} v_j\right)1_{v_2\geq\max_{j\in\{k+1,\dots,n\}} v_j} \mid v_2\in(v_1-\varepsilon,v_1)\end{array}\right)\\
= \ & \hat{p}(v_1)-E_{v_2}\left(\hat{p}(v_2)\mid v_2\in(v_1-\varepsilon,v_1)\right),
\end{aligned}
$$

which is the same as his payoff from reporting truthfully. Thus, there is no incentive to deviate in this way.

We have shown that bidders report truthfully. It remains to show that individual rationality is satisfied. If bidder i does not join the ring, play is non-cooperative, and bidder i with value v_i expects payoff $E_{v_{-i}}\left(\left(v_i-\max_{j\neq i} v_j\right)1_{v_i\geq\max_{j\neq i} v_j}\right)$. If bidder $i\in\{1,\dots,k\}$ has value v_i and joins the ring, he expects payoff equal to the ex ante non-contingent payment plus

$$
\begin{array}{l}
E_{v_{-i}}\left(\left(v_i-\max_{j\in\{k+1,\dots,n\}} v_j\right)1_{v_i\geq\max_{j\neq i} v_j}\right)\\
-E_{v_{-i}}\left(\hat{p}\left(\max_{j\in\{1,\dots,k\}\setminus\{i\}} v_j\right)1_{v_i\geq\max_{j\in\{1,\dots,k\}\setminus\{i\}} v_j}\right),
\end{array} \tag{B.1}
$$

where the first term is his expected payoff from the auction and the second term is his expected payment to the center. Note that the payoff from the auction is positive only if i's value is the highest among all n bidders, i.e., $v_i\geq\max_{j\neq i} v_j$, but the payment to the center must be made if i's value is the highest among the k ring members, i.e., $v_i\geq\max_{j\in\{1,\dots,k\}\setminus\{i\}} v_j$. Substituting the definition of $\hat{p}$ and rearranging, (B.1) is equal to

$$
\begin{aligned}
& E_{v_{-i}}\left(\begin{array}{c}
\left(v_i - \max_{j\in\{k+1,\ldots,n\}} v_j\right) 1_{v_i \geq \max_{j\neq i} v_j} \\
-\left(\max_{j\in\{1,\ldots,k\}\backslash\{i\}} v_j - \max_{j\in\{k+1,\ldots,n\}} v_j\right) 1_{v_i \geq \max_{j\in\{1,\ldots,k\}\backslash\{i\}} v_j \geq \max_{j\in\{k+1,\ldots,n\}} v_j}
\end{array}\right) \\
= \; & E_{v_{-i}}\left(\begin{array}{c}
\left(v_i - \max_{j\in\{k+1,\ldots,n\}} v_j\right) 1_{v_i \geq \max_{j\in\{k+1,\ldots,n\}} v_j \geq \max_{j\in\{1,\ldots,k\}\backslash\{i\}} v_j} \\
+\left(v_i - \max_{j\in\{1,\ldots,k\}\backslash\{i\}} v_j\right) 1_{v_i \geq \max_{j\in\{1,\ldots,k\}\backslash\{i\}} v_j \geq \max_{j\in\{k+1,\ldots,n\}} v_j}
\end{array}\right) \\
= \; & E_{v_{-i}}\left(\left(v_i - \max_{j\neq i} v_j\right) 1_{v_i \geq \max_{j\neq i} v_j}\right),
\end{aligned}
$$

which is equal to bidder i's expected payoff if he does not join the ring. Because we have excluded the ex ante payments, interim individual rationality is satisfied strictly. Furthermore, ex ante individual rationality is also satisfied strictly. Because the mechanism does not rely on the bids submitted at the auction or the identity of the winner, it is not affected by the possibility of shills. Q.E.D.

Proof of Proposition 2. Suppose there is an incentive compatible, profitable collusive mechanism in which non-highest-valuing ring members bid $\underline{v}$. Then ring members truthfully report their values and bid according to the recommendations of the center. Let ι be the index of the ring member (randomly selected in the case of a tie) with the highest report. Because ring member ι's payment to the ring does not depend upon his bid at the auction, his recommended bid must be optimal in the auction subgame. In particular, it must be that the center's recommended bid to bidder ι, $\beta_\iota(v_1, ..., v_k)$, and the bids of the outside bidders, $\beta_i(v_i)$ for $i \in \{k+1, ..., n\}$, satisfy

$$
\begin{aligned}
\beta_\iota(v_1, ..., v_k) &\in \arg\max_b E_{v_{-\iota}}\left((v_\iota - b)1_{b \geq \max_{j\in\{k+1,\ldots,n\}} \beta_j(v_j)} \mid v_\iota = \max_{j\in\{1,\ldots,k\}} v_j\right) \\
&= \arg\max_b E_{v_{-\iota}}\left((v_\iota - b)1_{b \geq \max_{j\in\{k+1,\ldots,n\}} \beta_j(v_j)}\right)
\end{aligned}
$$

and for $i \in \{k+1, ..., n\}$,

$$
\beta_i(v_i) \in \arg\max_b E_{v_{-i}}\left((v_i - b)1_{b \geq \max\left\{\beta_\iota(v_1,\ldots,v_k), \max_{j\in\{k+1,\ldots,n\}\backslash\{i\}} \beta_j(v_j)\right\}} \mid v_\iota = \max_{j\in\{1,\ldots,k\}} v_j\right).
$$

Notice that the recommended bid to bidder ι depends only on bidder ι's value and not on the other ring members' values or the identity of the highest-valuing ring member, so we can define function $\hat{\beta}$ such that $\beta_\iota(v_1, ..., v_k) = \hat{\beta}(v_\iota)$. Note also that for all $v > \underline{v}$, $\hat{\beta}(v) < v$.

Suppose ring member i has value $v_i \in (\hat{\beta}(v_\iota), v_\iota)$, which is a positive probability event. Let I_i be bidder i's information, if any, about the values of the other ring members as a result of learning his required payment to the center and his recommended bid. Given this information, bidder i forms beliefs (correct in equilibrium) about the values of the other ring members. Because the center recommends that ring member i bid $\underline{v}$, ring member i's belief must be that the probability that his value is highest is zero. Using the monotonicity of $\hat{\beta}$, this implies that i believes there is zero probability that a bid of $\hat{\beta}(v_i)$ would win the auction. Furthermore, since $v_i \in (\hat{\beta}(v_\iota), v_\iota)$ and beliefs are correct in equilibrium, ring member i must believe that there is positive probability that the center's highest recommended bid is less than i's value, i.e., $E_{v_{-i}}\left(1_{\hat{\beta}(v_\iota)<v_i} \mid I_i\right) > 0$. Thus, ring member i believes he has zero probability of winning the auction with a bid of $\hat{\beta}(v_i)$, but believes there exists some bid less than his value, but above $\hat{\beta}(v_i)$, with positive probability of winning. Thus, ring member i can profitably deviate from his recommended bid, a contradiction. Q.E.D.

Proof of Proposition 3. Let bidders 1 and 2 be the highest and second-highest-valuing ring members, in no particular order. Let $\hat{\beta}^{in}$ be the bid function used by the ring members, and let $\hat{\beta}^{out}$ be the bid function used by the outside bidders. Let

$$\hat{\beta}_i \equiv \begin{cases} \hat{\beta}^{in}, & \text{if } i \in \{1, ..., k\} \\ \hat{\beta}^{out}, & \text{if } i \in \{k+1, ..., n\}. \end{cases}$$

Functions $\hat{\beta}^{in}$ and $\hat{\beta}^{out}$ are defined by the conditions that for all v,

$$\begin{aligned} \hat{\beta}^{in}(v) &\in \arg\max_b E_{v_{-1}}\left((v-b)\,1_{b\geq\max_{j\in\{2,k+1,...,n\}}\hat{\beta}_j(v_j)} \mid \min\{v_1, v_2\} \geq \max_{j\in\{3,...,k\}} v_j\right) \\ &= \arg\max_b E_{v_{-1}}\left((v-b)\,1_{b\geq\max_{j\neq 1}\hat{\beta}_j(v_j)}\right) \end{aligned}$$

and

$$
\begin{aligned}
\hat{\beta}^{out}(v) &\in \arg\max_b E_{v_{-n}} \left((v-b)\, 1_{b \geq \max_{j\in\{1,2,k+1,\ldots,n-1\}} \hat{\beta}_j(v_j)} \mid \min\{v_1, v_2\} \geq \max_{j\in\{3,\ldots,k\}} v_j \right) \\
&= \arg\max_b E_{v_{-n}} \left((v-b)\, 1_{b \geq \max_{j\neq n} \hat{\beta}_j(v_j)} \right).
\end{aligned}
$$

Thus, for all $i \in \{1,...,n\}$, $\hat{\beta}_i(v) \in \arg\max_b E_{v_{-i}} \left((v-b)\, 1_{b \geq \max_{j\neq i} \hat{\beta}_j(v_j)} \right)$, which, using the uniqueness result of Lemma 1, implies $\hat{\beta}_i(v) = \beta_i^{nc}(v)$. Q.E.D.

Proof of Lemma 2. Suppose there exists an incentive compatible bid coordination mechanism. Let $\hat{b}_{k+1}, ..., \hat{b}_n$ be the equilibrium bid functions for the outside bidders.

Case 1. There exists a positive-measure set of ring members' values such that the highest ring bid is less than $\beta^*(\max_{j\in\{1,\ldots,k\}} v_j; \hat{b}_{k+1}, ..., \hat{b}_n)$. Given ring members' values in this set, after the ring announcements, but prior to bidding, there exists a ring member i whose information is such that he believes that there is positive probability that his value is highest, in which case all other ring members have recommended bids less than $\beta^*(v_i; \hat{b}_{k+1}, ..., \hat{b}_n)$. Such a ring member can profitably deviate by bidding $\beta^*(v_i; \hat{b}_{k+1}, ..., \hat{b}_n)$. To see this, suppose that ring member i's recommended bid is highest in the ring. Then by increasing his bid to be an optimal bid against the outside bidders, the ring member increases his expected payoff. If ring member i's recommended bid is not highest in the ring, then by increasing his bid, he at least weakly increases his expected payoff. This provides a contradiction.

Case 2. For all but a zero-measure set of values for ring members, the highest bid from the ring is $\beta^*(\max_{j\in\{1,\ldots,k\}} v_j; \hat{b}_{k+1}, ..., \hat{b}_n)$. Suppose that the highest ring bid is submitted by a ring member whose value is not highest in the ring. Then the highest-valuing ring member receives a recommended bid that is less than $\beta^*(\max_{j\in\{1,\ldots,k\}} v_j; \hat{b}_{k+1}, ..., \hat{b}_n)$. In this case, the highest-valuing ring member's information is such that he believes he will win with his recommended bid with probability zero, but that he will win with a bid greater than $\beta^*(\max_{j\in\{1,\ldots,k\}} v_j; \hat{b}_{k+1}, ..., \hat{b}_n)$ but less than $\max_{j\in\{1,\ldots,k\}} v_j$ (his value) with positive probability, implying that the highest-valuing ring member has a profitable deviation, a contradiction. Thus, for all but a zero-measure set of values for ring members, the highest bid from the ring is $\beta^*(\max_{j\in\{1,...,k\}} v_j; \hat{b}_{k+1}, ..., \hat{b}_n)$ and this bid is submitted by the highest-valuing ring member.

Let β^m and $b^m_{k+1}, ..., b^m_n$ be such that

$$\beta^m(v) \in \arg\max_b E_{v_{k+1},...v_n} \left((v-b)\, 1_{b \geq \max_{j\in\{k+1,...,n\}} b^m_j(v_j)} \right)$$

and for all $i \in \{k+1, ..., n\}$,

$$b^m_i(v_i) \in \arg\max_b E_{v_{-i}} \left((v_i - b) 1_{b \geq \max\left\{\beta^m(\max_{j\in\{1,...,k\}} v_j), \max_{j\in\{k+1,...,n\}\setminus\{i\}} b^m_j(v_j)\right\}} \right).$$

Then β^m and $b^m_{k+1}, ..., b^m_n$ are just the equilibrium bid functions for the case in which $n - k + 1$ bidders compete against one another, where one bidder's value is the highest of $v_1, ..., v_k$ and the other bidders' values are $v_{k+1}, ..., v_n$. By Lemma 1, β^m and $b^m_{k+1}, ..., b^m_n$ exist and are unique, implying that $\beta^* = \beta^m$ and for all $i \in \{k+1, ..., n\}$, $\hat{b}_i = b^m_i$.

Because $\beta^m(v_i)$ is an optimal bid for ring member i if he has no competition from other ring members, ring member i, given his information, never strictly prefers to bid an amount less than $\beta^m(v_i)$. In addition, ring member i never strictly prefers to bid an amount greater than v_i. This allows us to restrict attention to bids in the interval $[\beta^m(v_i), v_i]$. Thus, ring member i chooses his bid to solve:

$$\max_{b\in[\beta^m(v_i), v_i]} E_{v_{-i}} \left((v_i - b)\, 1_{b\geq \max\left\{\max_{j\in\{1,...,k\}\setminus\{i\}} \beta_j(v^k), \max_{j\in\{k+1,...,n\}} b^m_j(v_j)\right\}} \mid I_i \right), \quad \text{(B.2)}$$

where $I_i = \left(v_i, \beta_i(v^k), p_i(v^k)\right)$.

The remainder of the proof shows that for a positive-measure set of ring members' values, there exists a ring member i whose value is not highest in the ring, and whose recommended bid is $\beta^m(v_i)$, but for whom the maximand in (B.2) is increasing in b at $b = \beta^m(v_i)$, which completes our proof.

Note that for $b \geq \beta^m(v_i)$,

$$
\begin{aligned}
& E_{v_{-i}}\left((v_i - b)\, 1_{b \geq \max\left\{\max_{j\in\{1,\ldots,k\}\setminus\{i\}} \beta_j(v^k), \max_{j\in\{k+1,\ldots,n\}} b_j^m(v_j)\right\}} \mid I_i, \beta^m(v_i) > \max_{j\in\{1,\ldots,k\}\setminus\{i\}} \beta_j(v^k)\right) \\
= \; & E_{v_{-i}}\left((v_i - b)\, 1_{b \geq \max_{j\in\{k+1,\ldots,n\}} b_j^m(v_j)}\right),
\end{aligned}
$$

where the equality holds because, conditional on $\beta^m(v_i) > \max_{j\in\{1,\ldots,k\}\setminus\{i\}} \beta_j(v^k)$, ring member i wins the auction if and only if his bid exceeds the bids of the outside bidders. Using this equality and letting

$$
A_i(b) \equiv E_{v_{-i}}\left((v_i - b)\, 1_{b \geq \max_{j\in\{k+1,\ldots,n\}} b_j^m(v_j)}\right)
$$

and

$$
B_i(b; I_i) \equiv E_{v_{-i}}\left(\begin{array}{c} (v_i - b)\, 1_{b \geq \max\left\{\max_{j\in\{1,\ldots,k\}\setminus\{i\}} \beta_j(v^k), \max_{j\in\{k+1,\ldots,n\}} b_j^m(v_j)\right\}} \\ \mid I_i, \beta^m(v_i) \leq \max_{j\in\{1,\ldots,k\}\setminus\{i\}} \beta_j(v^k) \end{array}\right),
$$

the rules for conditional probabilities allow us to rewrite ring member i's problem as:

$$
\begin{aligned}
\max_{b\in[\beta^m(v_i), v_i]} \quad & A_i(b) \Pr\left(\beta^m(v_i) > \max_{j\in\{1,\ldots,k\}\setminus\{i\}} \beta_j(v^k) \mid I_i\right) \\
& + B_i(b; I_i) \Pr\left(\beta^m(v_i) \leq \max_{j\in\{1,\ldots,k\}\setminus\{i\}} \beta_j(v^k) \mid I_i\right),
\end{aligned} \tag{B.3}
$$

where the probabilities are with respect to v_{-i}. Note that by the definition of β^m, $\left.\frac{\partial A_i(b)}{\partial b}\right|_{b=\beta^m(v_i)} = 0$ (Assumption 1 implies differentiability). Note that $B_i(\beta^m(v_i); I_i) = 0$ and for all $b \in [\beta^m(v_i), v_i]$, $B_i(b; I_i) \geq 0$.

The next step in the proof is to consider a ring member i with value v_i and the set of other ring members' values such that the maximum value for the other ring members is greater than or equal to v_i, but the maximum bid for the other ring members using bid function β^m is less than v_i. We can easily show that ring member i's prior distribution must give positive weight to the possibility that this occurs. But then we can show that there must be a positive-measure set of values for ring members other than i such than ring member i's information results in a posterior that gives positive weight to this possibility. For this set of values, $\left.\frac{\partial B_i(b; I_i)}{\partial b}\right|_{b=\beta^m(v_i)} > 0$, which will allow us to complete the proof.

Let $G(\cdot)$ be the joint distribution of the values of the ring members other than i, and let $\bar{G}(\cdot)$ be the distribution of the highest from the values of the ring members other than i. To conserve on notation, let $X_i(v_i)$ be the set of values for ring members other than i such that the maximum value for the other ring members is greater than or equal to v_i, but the maximum bid for the other ring members using bid function β^m is less than v_i, i.e.,

$$X_i(v_i) \equiv \left\{ v_{-i}^k \mid \beta^m(\max_{j\in\{1,\dots,k\}\setminus\{i\}} v_j) \leq v_i \leq \max_{j\in\{1,\dots,k\}\setminus\{i\}} v_j \right\}.$$

Then, using the assumption of independent values, for $b \in [\beta^m(v_i), v_i]$ we can rewrite B_i as

$$\begin{aligned} B_i(b; I_i) &= (v_i - b)\,\bar{G}\left(\beta^{m^{-1}}(b) \mid I_i, \beta^m(v_i) \leq \max_{j\in\{1,\dots,k\}\setminus\{i\}} \beta_j(v^k)\right) \Pr\left(b \geq \max_{j\in\{k+1,\dots,n\}} b_j^m(v_j)\right) \\ &= (v_i - b)\,\bar{G}\left(\beta^{m^{-1}}(b) \mid I_i, v_i \leq \max_{j\in\{1,\dots,k\}\setminus\{i\}} v_j^k\right) \Pr\left(b \geq \max_{j\in\{k+1,\dots,n\}} b_j^m(v_j)\right) \\ &= (v_i - b)\,\bar{G}\left(\beta^{m^{-1}}(b) \mid I_i, v_{-i}^k \in X_i(v_i)\right) \Pr\left(\beta^m(\max_{j\in\{1,\dots,k\}\setminus\{i\}} v_j) \leq v_i \mid I_i\right) \cdot \\ &\quad \Pr\left(b \geq \max_{j\in\{k+1,\dots,n\}} b_j^m(v_j)\right) \end{aligned}$$

where the second equality holds because the highest ring bid is $\beta^m(\max_{j\in\{1,\dots,k\}} v_j)$ and is submitted by the highest-valuing ring member, and the third equality holds because the expected payoff from a bid $b \in [\beta^m(v_i), v_i]$ is zero conditional on $v_i < \beta^m(\max_{j\in\{1,\dots,k\}\setminus\{i\}} v_j)$. Note that $\bar{G}\left(\beta^{m^{-1}}(b) \mid I_i, v_{-i}^k \in X_i(v_i)\right)$ is equal to zero when evaluated at $b = \beta^m(v_i)$. Thus,

$$\begin{aligned} \frac{\partial B_i(\beta^m(v_i); I_i)}{\partial b} &= (v_i - \beta^m(v_i))\,\bar{g}\left(v_i \mid I_i, v_{-i}^k \in X_i(v_i)\right) \cdot \Pr\left(\beta^m(\max_{j\in\{1,\dots,k\}\setminus\{i\}} v_j) \leq v_i \mid I_i\right) \\ &\quad \Pr\left(\beta^m(v_i) \geq \max_{j\in\{k+1,\dots,n\}} b_j(v_j)\right), \end{aligned}$$

which is positive for $v_i > \underline{v}$ if and only if

$$\bar{g}\left(v_i \mid I_i, v_{-i}^k \in X_i(v_i)\right) > 0$$

and

$$\Pr\left(\beta^m(\max_{j\in\{1,\dots,k\}\setminus\{i\}} v_j) \leq v_i \mid I_i\right) > 0. \qquad \text{(B.4)}$$

Note that

$$\bar{G}\left(\beta^{m^{-1}}(b) \mid v_{-i}^k \in X_i\left(v_i\right)\right)=\frac{\int_{v_i}^{\beta^{m^{-1}}(b)} \bar{g}(v) d v}{\bar{G}(\beta^{m^{-1}}\left(v_i\right))-\bar{G}(v_i)},$$

which is increasing in b for $b \geq \beta^m(v_i)$, i.e.,

$$\bar{g}\left(\beta^{m^{-1}}(b) \mid v_{-i}^k \in X_i\left(v_i\right)\right)>0. \tag{B.5}$$

Let $V(v_i, x, y)$ be the set of values for ring members other than i that are consistent with ring member i's receiving a recommended bid of x and required payment of y, i.e.,

$$V(v_i, x, y) \equiv \left\{v_{-i}^k \in \times_{j \in\{1, \ldots, k\} \backslash\{i\}} \mid (v_i, v_{-i}^k) \in \beta_i^{-1}(x) \cap p_i^{-1}(y)\right\}.$$

Then the probability that a ring member with value v_i has information $I_i=(v_i, x, y)$, conditional on $v_{-i}^k \in X_i(v_i)$, is

$$P(x, y) \equiv \int_{V(v_i, x, y)} g(\tilde{v}_{-i}^k \mid v_{-i}^k \in X_i(v_i)) d \tilde{v}_{-i}^k,$$

which is well defined because we assume β_i and p_i are measurable functions. Note that because beliefs are correct in equilibrium, the unconditional beliefs must be equal to the integral of the conditional beliefs over all possible information. Thus, using (B.5) (evaluated at $b=\beta^m(v_i)$),

$$\begin{aligned}
0 &< \bar{g}\left(v_i \mid v_{-i}^k \in X_i(v_i)\right) \\
&= \int_{X_i(v_i)} \bar{g}(v_i \mid I_i=\left(\beta_i(v_i, \tilde{v}_{-i}^k), p_i(v_i, \tilde{v}_{-i}^k)\right), v_{-i}^k \in X_i(v_i)) \cdot \\
&\quad P(\beta_i(v_i, \tilde{v}_{-i}^k), p_i(v_i, \tilde{v}_{-i}^k)) g\left(\tilde{v}_{-i}^k \mid v_{-i}^k \in X_i(v_i)\right) d \tilde{v}_{-i}^k,
\end{aligned}$$

which implies that for a positive measure subset of $X_i(v_i)$, call it $\tilde{X}_i(v_i)$, ring member i's information is such that

$$\bar{g}(v_i \mid I_i, v_{-i}^k \in X_i(v_i))>0. \tag{B.6}$$

Let v^k be such that $v_i>\underline{v}$ and $v_{-i}^k \in \tilde{X}_i(v_i)$ (a positive probability event). In this case, ring member i, given his information, must place positive probability weight on the event that a bid of less than his value wins the auction. If ring member i's recommended bid is not equal to $\beta^m(v_i)$, then ring member i knows with probability one that his recommendation is not highest and that a bid equal to his recommended bid has zero probability of winning the auction, implying that the ring member can profitably deviate by bidding some amount greater than his recommended bid, but less than his value, a contradiction. Thus, it must be that ring member i's recommended bid is equal to $\beta^m(v_i)$. In addition, ring member i's information must be such that (B.4) holds. Then, using (B.6), $\frac{\partial B_i(\beta^m(v_i);I_i)}{\partial b} > 0$. Furthermore, ring member i's information must be such that $\Pr\left(\beta^m(v_i) \leq \max_{j\in\{1,...,k\}\setminus\{i\}} \beta_j(v^k) \mid I_i\right) > 0$. But then, using (B.3), ring member i can profitably deviate by bidding some amount greater than $\beta^m(v_i)$. Since ring member i's recommended bid is $\beta^m(v_i)$, this is a contradiction. Q.E.D.

Proof of Proposition 4. Let $\hat{b}_{k+1}, ..., \hat{b}_n$ be the equilibrium bid functions for the outside bidders. Let $B(v_1, ..., v_k)$ be the equilibrium highest ring bid as a function of $v_1, ..., v_k$. By Lemma 2, for all but a zero-measure set of ring members' values, $B(v_1, ..., v_k) \geq \beta^*(\max_{j\in\{1,...,k\}} v_j; \hat{b}_{k+1}, ..., \hat{b}_n)$, and a strict inequality for a positive-measure set of values.

Suppose that for all but a zero-measure set of value realizations, one ring member bids $B(v_1, ..., v_k)$ and all other ring members bid some amount less than $\beta^*(\max_{j\in\{1,...,k\}} v_j; \hat{b}_{k+1}, ..., \hat{b}_n)$. Suppose that for a given vector of ring members' values, ring member i is the ring member whose recommended bid is $B(v_1, ..., v_k)$, and suppose that $v_i < B(v_1, ..., v_k)$. Then ring member i's beliefs place positive probability on his winning the auction with a bid equal to $B(v_1, ..., v_k)$, and so ring member i can profitably deviate by bidding v_i, a contradiction. Thus, for all ring members' values, if ring member i's recommended bid is $B(v_1, ..., v_k)$ then $B(v_1, ..., v_k) \leq v_i$.

Suppose that the ring member whose recommended bid is $B(v_1, ..., v_k)$ is not the ring member with the highest value. If ring member i is the ring member with the highest value, then ring member i's recommended bid is less than $\beta^*(v_i; \hat{b}_{k+1}, ..., \hat{b}_n)$, and so ring member i's beliefs place probability one on his recommended bid not being the highest among the ring members, and ring member i's beliefs place positive probability on the highest recommended bid from among the ring members being less than his value. Thus, ring member i can profitably deviate by submitting a bid greater than $\beta^*(v_i; \hat{b}_{k+1}, ..., \hat{b}_n)$ but less than v_i, a contradiction. Thus, for all ring members' values, it must be the ring member with the highest value whose recommended bid is $B(v_1, ..., v_k)$.

Let ring member i be a ring member such that with positive probability ring member i's value is highest in the ring and his recommended bid is strictly greater than $\beta^*(\max_{j\in\{1,...,k\}} v_j; \hat{b}_{k+1}, ..., \hat{b}_n)$ (at least one such ring member must exist). In such cases, ring member i has a recommended bid greater than $\beta^*(v_i; \hat{b}_{k+1}, ..., \hat{b}_n)$ and believes that (i) if his value is highest, then all other ring members' recommended bids are less than $\beta^*(v_i; \hat{b}_{k+1}, ..., \hat{b}_n)$, in which case, by the definition of β^*, ring member i strictly prefers to deviate from his recommended bid by bidding $\beta^*(v_i; \hat{b}_{k+1}, ..., \hat{b}_n)$, and (ii) if his value is not highest, then his recommended bid has probability zero of winning, so he is no worse off by bidding $\beta^*(v_i; \hat{b}_{k+1}, ..., \hat{b}_n)$. In such cases, ring member i's information must be such that he believes his value is highest with positive probability, so the deviation to $\beta^*(v_i; \hat{b}_{k+1}, ..., \hat{b}_n)$ is profitable, a contradiction. Thus, at least one ring member in addition to the highest-valuing ring member must bid greater than or equal to $\beta^*(\max_{j\in\{1,...,k\}} v_j; \hat{b}_{k+1}, ..., \hat{b}_n)$.

To complete the proof, suppose that there exists $\varepsilon > 0$ such that there is zero probability that the two highest ring bids are within ε of each other. Suppose the highest recommended bid to any ring member is to ring member i, and suppose the recommendation, β_i, satisfies $\beta_i > \beta^*(v_i; \hat{b}_{k+1}, ..., \hat{b}_n)$, which is a positive probability event. Then ring member i can profitably deviate by bidding $b \in \left(\max\left\{\beta^*(v_i; \hat{b}_{k+1}, ..., \hat{b}_n), \beta_i - \varepsilon\right\}, \beta_i\right)$, a contradiction. Q.E.D.

Proof of Proposition 5. Using the definition of capturing the entire collusive gain, the proposition can be rewritten as the statement that there exists $\mathbf{v}$ such that either (i) $R^{\mu}(\mathbf{v}) < R^{nc}(\mathbf{v})$, (ii) $R^{\mu}(\mathbf{v}) > R^{nc}(\mathbf{v})$ and either $S^{\mu}(\mathbf{v}) > S^{nc}(\mathbf{v})$ or $O^{\mu}(\mathbf{v}) > O^{nc}(\mathbf{v})$, or (iii) $R^{\mu}(\mathbf{v}) = R^{nc}(\mathbf{v})$ and either $S^{\mu}(\mathbf{v}) \neq S^{nc}(\mathbf{v})$ or $O^{\mu}(\mathbf{v}) \neq O^{nc}(\mathbf{v})$. Suppose that for all $\mathbf{v}$, $R^{\mu}(\mathbf{v}) \geq R^{nc}(\mathbf{v})$ and that for all $\mathbf{v}$, $R^{\mu}(\mathbf{v}) = R^{nc}(\mathbf{v})$ implies $S^{\mu}(\mathbf{v}) = S^{nc}(\mathbf{v})$ and $O^{\mu}(\mathbf{v}) = O^{nc}(\mathbf{v})$. Since μ is assumed profitable, there exists non-zero-measure set V such that for all $\mathbf{v} \in V$, $R^{\mu}(\mathbf{v}) > R^{nc}(\mathbf{v})$. To prove the result, we must show that there exists $\mathbf{v}$ such that either $S^{\mu}(\mathbf{v}) > S^{nc}(\mathbf{v})$ or $O^{\mu}(\mathbf{v}) > O^{nc}(\mathbf{v})$. Assume that for all $\mathbf{v} \in V$, $S^{\mu}(\mathbf{v}) \leq S^{nc}(\mathbf{v})$. We now show that there exists $\mathbf{v}$ such that $O^{\mu}(\mathbf{v}) > O^{nc}(\mathbf{v})$.

Let $(\beta_1^{\mu}(v_1, ..., v_k), ..., \beta_k^{\mu}(v_1, ..., v_k), \beta_{k+1}^{\mu}(v_{k+1}), ..., \beta_n^{\mu}(v_n))$ be the equilibrium bids under collusive mechanism μ. For convenience of notation, let

$$\hat{\beta}_i^{\mu}(\mathbf{v}) \equiv \begin{cases} \beta_i^{\mu}(v_1, ..., v_k), & \text{if } i \in \{1, ..., k\} \\ \beta_i^{\mu}(v_i), & \text{otherwise.} \end{cases}$$

Note that for $i \in \{1, ..., k\}$, β_i^{μ} is a function of $v_1, ..., v_k$ rather than just v_i because the center's bid recommendations are a function of the entire vector of reports.

Since for all $\mathbf{v} \in V$, $S^{\mu}(\mathbf{v}) \leq S^{nc}(\mathbf{v})$, it follows that for all $\mathbf{v} \in V$, $\max_{j \in \{1,...,n\}} \hat{\beta}_j^{\mu}(\mathbf{v}) \leq \max_{j \in \{1,...,n\}} \beta_j^{nc}(v_j)$. Because for all $\mathbf{v} \in V$, $R^{\mu}(\mathbf{v}) > R^{nc}(\mathbf{v})$, it follows that for all $\mathbf{v} \in V$, either

$$\max_{j \in \{k+1,...,n\}} \hat{\beta}_j^{\mu}(\mathbf{v}) < \max_{j \in \{1,...,k\}} \hat{\beta}_j^{\mu}(\mathbf{v}) \leq \max_{j \in \{1,...,n\}} \beta_j^{nc}(v_j) = \max_{j \in \{k+1,...,n\}} \beta_j^{nc}(v_j) \tag{B.7}$$

or

$$\max_{j \in \{k+1,...,n\}} \hat{\beta}_j^{\mu}(\mathbf{v}) < \max_{j \in \{1,...,k\}} \hat{\beta}_j^{\mu}(\mathbf{v}) < \max_{j \in \{1,...,n\}} \beta_j^{nc}(v_j) = \max_{j \in \{1,...,k\}} \beta_j^{nc}(v_j). \tag{B.8}$$

To see this, note that the ring's gain from the collusive mechanism must be due to one of two causes: in (B.7), a ring member wins under the collusive mechanism, but not under non-cooperative play, and in (B.8), a ring member wins both under the collusive mechanism and under non-cooperative play, but the high ring bid is lower under the collusive mechanism than under non-cooperative play.

Partition V into V_1 and V_2, where (B.7) holds on V_1 and (B.8) holds on V_2. Suppose V_1 is a positive-measure set of values. Now consider outside bidder $i \in \{k+1, ..., n\}$. If outside bidder i bids according to β_i^{μ}, then his expected payoff is

$$E_{v_{-i}}\left(\left(v_i - \beta_i^{\mu}(v_i)\right)\Pr\left(\beta_i^{\mu}(v_i) \geq \max_{j\neq i}\hat{\beta}_j^{\mu}(v_i, v_{-i})\right) \mid (v_i, v_{-i}) \in V_1\right)\Pr\left((v_i, v_{-i}) \in V_1\right)$$
$$+E_{v_{-i}}\left(\left(v_i - \beta_i^{\mu}(v_i)\right)\Pr\left(\beta_i^{\mu}(v_i) \geq \max_{j\neq i}\hat{\beta}_j^{\mu}(v_i, v_{-i})\right) \mid (v_i, v_{-i}) \in V_2\right)\Pr\left((v_i, v_{-i}) \in V_2\right)$$
$$+E_{v_{-i}}\left(\left(v_i - \beta_i^{\mu}(v_i)\right)\Pr\left(\beta_i^{\mu}(v_i) \geq \max_{j\neq i}\hat{\beta}_j^{\mu}(v_i, v_{-i})\right) \mid (v_i, v_{-i}) \in V^c\right)\Pr\left((v_i, v_{-i}) \in V^c\right).$$

Consider a deviation by bidder i to β_i^{nc}. Then, using (B.7), his payoff conditional on $(v_i, v_{-i}) \in V_1$ strictly increases, and this conditioning event has positive probability. Using (B.8), his payoff conditional on $(v_i, v_{-i}) \in V_2$ weakly increases. Using the definition of V^c, his payoff conditional on $(v_i, v_{-i}) \in V^c$ does not change. Thus, it is a profitable deviation for outside bidder i to use his non-cooperative bid function, a contradiction. Thus, V_1 must be a zero-measure set, implying that the ring's increase in payoff is due entirely to winning at lower bids.

Once again consider outside bidder $i \in \{k+1, ..., n\}$. Using (B.8), for $v \in V_2$, which is a positive-measure set of values,

$$\max_{j\in\{k+1,\ldots,n\}}\hat{\beta}_j^{\mu}(\mathbf{v}) < \max_{j\in\{1,\ldots,k\}}\hat{\beta}_j^{\mu}(\mathbf{v}) < \max_{j\in\{1,\ldots,n\}}\beta_j^{nc}(v_j) = \max_{j\in\{1,\ldots,k\}}\beta_j^{nc}(v_j).$$

For $v \in V^c$, $\max_{j\in\{1,\ldots,k\}}\hat{\beta}_j^{\mu}(\mathbf{v}) = \max_{j\in\{1,\ldots,n\}}\beta_j^{nc}(v_j)$. If outside bidder i bids according to β_i^{nc} when all other bidders bid collusively, then bidder i wins with his non-cooperative bid whenever $\beta_i^{nc}(v_i) \geq \max_{j\neq i}\beta_j^{nc}(v_j)$, i.e., whenever i would have won under non-cooperative play. Thus, by revealed preference, bidder i's expected cooperative payoff must be greater than or equal to his expected non-cooperative payoff. Furthermore, using (B.8) and the fact that V_2 is a positive-measure set, there must exist an outside bidder whose expected payoff from using his non-cooperative bid function against the collusive mechanism is strictly greater than his expected payoff under non-cooperative play, implying $E_{\mathbf{v}}\left(O^{\mu}(\mathbf{v})\right) > E_{\mathbf{v}}\left(O^{nc}(\mathbf{v})\right)$, which implies that there exists $\mathbf{v}$ such that $O^{\mu}(\mathbf{v}) > O^{nc}(\mathbf{v})$. Q.E.D.

Proof of Proposition 6. Because the use of a shill does not affect a ring member's payment to the center, the proof of Proposition 2 implies that for any collusive mechanism for a first price auction in which non-highest-valuing ring members bid $\underline{v}$, there is positive probability that a ring member can increase his expected payoff by using a shill to submit a bid greater than the bid recommended by the center. In particular, a ring member i with value $v_i \in \left(\underline{v}, \max_{j\in\{1,\ldots,k\}\setminus\{i\}} v_j\right)$, whose recommended bid is $\underline{v}$, can profitably deviate by using a shill to bid $v_i - \varepsilon$ for some $\varepsilon \in \left(0, v_i - \underline{v}\right)$. We must show that the use of a shill by ring member i to bid $v_i - \varepsilon$ reduces the expected payoff to the ring. Suppose not. The expected joint payoff to the k ring members if ring member i does not use a shill is (payments to and from the center net to zero in expectation):

$$E_{v_{k+1},\ldots,v_n}\begin{pmatrix} \left(\max_{j\in\{1,\ldots,k\}\setminus\{i\}} v_j - \beta^{in}\left(\max_{j\in\{1,\ldots,k\}\setminus\{i\}} v_j\right)\right)\cdot \\ 1_{\beta^{in}\left(\max_{j\in\{1,\ldots,k\}\setminus\{i\}} v_j\right)\geq\max_{j\in\{k+1,\ldots,n\}}\beta_j^{out}(v_j)} \end{pmatrix}. \tag{B.9}$$

The expected joint payoff to the k ring members if ring member i uses a shill to bid $v_i - \varepsilon$ must take into account the probability that the ring member with the highest value wins and the probability that ring member i wins with his bid of $v_i - \varepsilon$:

$$\begin{aligned} &E_{v_{k+1},\ldots,v_n}\begin{pmatrix} \begin{pmatrix} \max_{j\in\{1,\ldots,k\}\setminus\{i\}} v_j \\ -\beta^{in}\left(\max_{j\in\{1,\ldots,k\}\setminus\{i\}} v_j\right) \end{pmatrix}\cdot \\ 1_{\beta^{in}\left(\max_{j\in\{1,\ldots,k\}\setminus\{i\}} v_j\right)\geq\max\left\{v_i-\varepsilon,\max_{j\in\{k+1,\ldots,n\}}\beta_j^{out}(v_j)\right\}} \end{pmatrix} \\ &+E_{v_{k+1},\ldots,v_n}\begin{pmatrix} \left(v_i-\left(v_i-\varepsilon\right)\right)\cdot \\ 1_{v_i-\varepsilon\geq\max\left\{\beta^{in}\left(\max_{j\in\{1,\ldots,k\}\setminus\{i\}} v_j\right),\max_{j\in\{k+1,\ldots,n\}}\beta_j^{out}(v_j)\right\}} \end{pmatrix}. \end{aligned} \tag{B.10}$$

By our supposition, (B.9) is less than or equal to (B.10). Because $\max_{j\in\{1,\ldots,k\}\setminus\{i\}} v_j > v_i$, (B.10) is less than

$$E_{v_{k+1},\ldots,v_n}\begin{pmatrix} \begin{pmatrix} \max_{j\in\{1,\ldots,k\}\setminus\{i\}} v_j \\ -\max\left\{\beta^{in}\left(\max_{j\in\{1,\ldots,k\}\setminus\{i\}} v_j\right), v_i-\varepsilon\right\} \end{pmatrix}\cdot \\ 1_{\max\left\{\beta^{in}\left(\max_{j\in\{1,\ldots,k\}\setminus\{i\}} v_j\right),v_i-\varepsilon\right\}\geq\max_{j\in\{k+1,\ldots,n\}}\beta_j^{out}(v_j)} \end{pmatrix}. \tag{B.11}$$

But then (B.9) is less than (B.11), which violates the definition of β^{in} because it implies that the ring can increase its expected payoff by having the highest-valuing ring member bid $\max\left\{\beta^{in}\left(\max_{j\in\{1,\ldots,k\}\backslash\{i\}} v_j\right), v_i-\varepsilon\right\}$ rather than $\beta^{in}\left(\max_{j\in\{1,\ldots,k\}\backslash\{i\}} v_j\right)$. Q.E.D.

Proof of Proposition 7. A *bid coordination mechanism* is a more restrictive mechanism than a *bid submission mechanism*, so the maximum expected payoff to a ring from a *bid submission mechanism* is greater than or equal to the maximum expected payoff to a ring from a *bid coordination mechanism*. Thus, to prove the proposition, we need only show that the expected payoff to a ring from any given *bid submission mechanism* is less than or equal to the maximum expected payoff to a ring from a *bid coordination mechanism*. Take a particular profitable *bid submission mechanism,* call it μ, as given. Note that the maximum ring bid must be less than the value of the ring member submitting that bid. Let $\beta^{\mu}(v^k)$ be the recommended bids in mechanism μ, and let $p_i^{\mu}(v^k;b^k)$ be the required payments in mechanism μ (as a function of the ring members' reports and their bids). Consider the following *bid coordination mechanism,* call it μ': let the recommended bids be $\beta^{\mu}(v^k)$, and let the required payment for ring member i be $p_i(v^k)\equiv p_i^{\mu}(v^k,\beta^{\mu}(v^k))$, which is the required payment from mechanism μ evaluated at the recommended bids rather than at the actual bids. In addition, have the ring center submit a bid using a mixed strategy that mixes aggressively just below the highest recommended bid for the ring members (for example, have the center mix according to a distribution that satisfies the conditions of Hirshleifer and Riley (1993, p.374).[43] Then the information I_i available to ring member i at the time he chooses his bid is the same under both mechanism μ and mechanism μ', except under mechanism μ' bidders know that no downward deviation from a recommended bid that is less than their value is profitable. (If the recommended bid were greater than or equal to a ring member's value, then a deviation might be profitable.) Because

[43] We are in an environment in which shill bidding is possible, so it seems reasonable to assume that the ring center could also use a shill to submit a bid at the auction.

a bid of $\beta_i^\mu(v^k)$ is optimal in mechanism μ (i.e., there is no incentive to use a shill), it must be that $\beta_i^\mu(v^k)$ does at least as well in expectation as any $b \geq \beta_i^\mu(v^k)$. Thus, a bid of $\beta_i^\mu(v^k)$ is also optimal in mechanism μ'. Thus, the expected payoff to the ring from *bid submission mechanism* μ is the same as the expected payoff from *bid coordination mechanism* μ', implying that the expected payoff to a ring from *bid submission mechanism* μ is less than or equal to the maximum expected payoff to a ring from a *bid coordination mechanism*. Q.E.D.

Proof of Proposition 8. Let r^1 be the highest report and r^2 the second-highest report in the ring. Let v^1 be the highest value and v^2 the second-highest value in the ring. Consider the following bidding rule: the center recommends that the ring member with the highest report bid $\beta^{in}(r^1)$ and that all others bid $\underline{v}$. Consider the following payment rule: the bidder with the highest report pays the center $\hat{p}(r^2)$, and all others pay zero if their bid is $\underline{v}$ and $\bar{v}$ if their bid is greater than $\underline{v}$. Suppose the bidders join the ring and report truthfully. It is a best reply for bidders with less than the highest value to bid $\underline{v}$ at the auction rather than bid anything else and pay $\bar{v}$ to the center. Because the payment rule faced by the highest-valuing ring member is constant with respect to his bid, the payment rule does not distort the highest-valuing ring member's choice of bid. Thus, in equilibrium the highest-valuing ring member bids $\beta^{in}(v^1)$. Consider whether bidders report truthfully. If all other bidders report truthfully and a bidder with value $\hat{v} < v^1$ reports $\hat{r} > \hat{v}$, causing him to have the highest report, i.e., $\hat{v} < v^1 < \hat{r}$, then his expected payoff from participating in the auction is $p(\hat{v})$, but his payment to the center is $\hat{p}(v^1) > p(\hat{v})$, giving him negative expected payoff. If a bidder with value v^1 reports $\hat{r} < v^1$, causing him not to have the highest report, i.e., $\hat{r} < v^2 < v^1$, then his expected payoff from participating in the auction is negative because the payment rule specifies a payment of $\bar{v}$, but if he reports truthfully his expected payoff from participating in the auction is positive. Because all other deviations have zero expected payoff, this establishes that no ring member has an incentive to misrepresent his value to the center. Given the conditions in the Proposition, individual rationality, either interim or ex ante, is satisfied. Q.E.D.

Proof of Proposition 9. Using the assumption of symmetry,

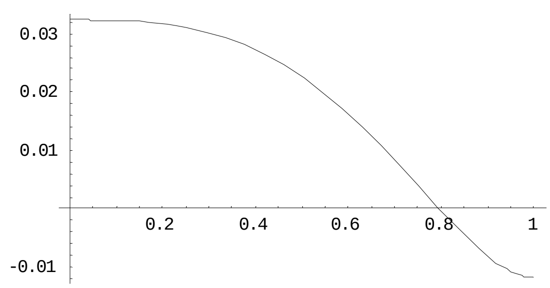
$$\hat{p}(r_2) = (r_2 - \beta^{in}(r_2)) \int_{\underline{v}}^{\beta^{out^{-1}}(\beta^{in}(r_2))} (n-k)F^{n-k-1}(x)f(x)dx,$$

$$X = \frac{1}{k}\int_{\underline{v}}^{\bar{v}} \hat{p}(x)k(k-1)F^{k-2}(x)(1-F(x))f(x)dx,$$

and

$$\begin{aligned}
&g(v_i \mid \hat{p}) = X \\
&+ \textstyle\int_{\underline{v}}^{\beta^{out^{-1}}(\beta^{in}(v_i))} \left(\int_{\underline{v}}^{v_i} \left(v_i - \beta^{in}(v_i)\right)(k-1)F^{k-2}(x)f(x)dx\right)(n-k)F^{n-k-1}(y)f(y)dy \\
&- \textstyle\int_{\underline{v}}^{v_i} \hat{p}(x)(k-1)F^{k-2}(x)f(x)dx - \int_{\underline{v}}^{v_i} \left(v_i - \beta^{nc}(v_i)\right)(n-1)F^{n-2}(x)f(x)dx.
\end{aligned}$$

The ex ante individual rationality constraint is $\int_{\underline{v}}^{\bar{v}} g(x \mid \hat{p})f(x)dx \geq 0$. Substituting in $\underline{v} = 0$, $\bar{v} = 1$, $F(v) = v$, $f(v) = 1$, $n = 3$, $k = 2$, and $\beta^{nc}(v) = \frac{(n-1)v}{n}$, and calculating β^{out} and β^{in} numerically (see Marshall et al. (1994)), we can calculate g numerically as shown in Figure 1.



Graph of $g(v \mid \hat{p})$ for $n = 3$, $k = 2$, and values drawn from $U[0, 1]$.

An additional numerical calculation gives $\int_0^1 g(x \mid \hat{p})dx \approx .017 > 0$. (The code used to calculate this result is available from the authors on request.) Thus, ex ante individual rationality is satisfied. Q.E.D.

References

Athey, Susan (2001), "Single Crossing Properties and the Existence of Pure Strategy Equilibria in Games of Incomplete Information," *Econometrica* 69, 861–890.

Bajari, Patrick (1997), "The First Price Auction with Asymmetric Bidders: Theory and Applications", University of Minnesota Ph.D. Thesis.

Bajari, Patrick (2001), "Comparing Competition and Collusion: A Numerical Approach," *Economic Theory* 18, 187–205.

Blume, Andreas and Paul Heidhues (2001), "Tacit Collusion in Repeated Auctions," Working Paper, University of Pittsburgh.

Cassady, Ralph (1967), *Auctions and Auctioneering,* Berkeley: UC Press.

Chakraborty, Indranil and Georgia Kosmopoulou (2001), "Auctions with Shill Bidding," Working Paper, University of Oklahoma.

Fudenberg, Drew, David Levine, and Eric Maskin (1994), "The Folk Theorem with Imperfect Public Information," *Econometrica* 62, 997–1039.

Graham, Daniel A. and Robert C. Marshall (1987), "Collusive Bidder Behavior at Single-Object second price and English Auctions," *Journal of Political Economy* 95, 1217–1239.

Graham, Daniel A., Robert C. Marshall, and Jean-Francois Richard (1990), "Differential Payments within a Bidder Coalition and the Shapley Value," *American Economic Review* 80, 493–510.

Hidvégi, Zoltán, Wenli Wang, and Andrew B. Whinston (2001), "Sequential Auctions with Shill Bidding," Working Paper, University of Texas at Austin.

Hirshleifer, Jack, and John G. Riley (1993), *The Analytics of Uncertainty and Information*, Cambridge: Cambridge University Press.

Lebrun, Bernard (2002), "Uniqueness of the Equilibrium in First Price Auctions", Working Paper, York University.

Lebrun, Bernard (1999), "First Price Auctions in the Asymmetric N Bidder Case," *International Economic Review* 40, 125–142.

Lebrun, Bernard (1996), "Existence of an Equilibrium in First Price Auctions," *Economic Theory* 7, 421–443.

Lebrun, Bernard (1995a), "First Price Auction in the Asymmetric N Bidder Case: Part I," Working Paper, Université Leval.

Lebrun, Bernard (1995b), "First Price Auction in the Asymmetric N Bidder Case: Part II," Working Paper, Université Leval.

Lyk-Jensen, Peter (1996), "Some Suggestions on How to Cheat the Auctioneer: Collusion in Auctions When Signals Are Affiliated," Working Paper, University of Copenhagen.

Lyk-Jensen, Peter (1997a), "Post-Auction Knock-Outs," Working Paper, University of Copenhagen.

Lyk-Jensen, Peter (1997b), "Collusion at Auctions with Affiliated Signals," Working Paper, Université de Toulouse.

Mailath, George, and Peter Zemsky (1991), "Collusion in Second Price Auctions with Heterogeneous Bidders," *Games and Economic Behavior* 3, 467–486.

Maskin, Eric S. and J. G. Riley (2000a), "Asymmetric Auctions," *Review of Economic Studies* 67, 413–438.

Maskin, Eric S. and J. G. Riley (2000b), "Equilibrium in Sealed High Bid Auctions," *Review of Economic Studies* 67, 439–454.

Maskin, Eric S. and J. G. Riley (1996a), "Uniqueness in Sealed High Bid Auctions," Working Paper, Harvard and UCLA.

Maskin, Eric S. and J. G. Riley (1996b), "Existence in Sealed High Bid Auctions," Working Paper, Harvard and UCLA.

Marshall, Robert C., Michael J. Meurer, Jean-Francois Richard, and Walter Stromquist (1994), "Numerical Analysis of Asymmetric First Price Auctions," *Games and Economic Behavior* 7, 193–220.

Matthews, Steven (1983), "Selling to Risk Averse Buyers with Unobservable Tastes", *Journal of Economic Theory* 30, 370–400.

Matthews, Steven (1987), "Comparing Auctions for Risk Averse Buyers: A Buyer's Point of View," *Econometrica* 55, 633–646.

McAfee, Preston R. and John McMillan (1992), "Bidding Rings," *American Economic Review* 82, 579–599.

Milgrom, Paul R. and Robert J. Weber (1982), "A Theory of Auctions and Competitive Bidding," *Econometrica* 50, 1089–1122.

Murphy, George M. (1960), *Ordinary Differential Equations and Their Solutions* (Princeton: D. Van Nostrand Company, Inc.).

Myerson, Roger B. (1981), "Optimal Auction Design," *Mathematics of Operations Research* 6, 58–73.

Myerson, Roger B. (1983), "Mechanism Design by an Informed Principal," *Econometrica* 51, 1767–97.

Riley, John G. and William Samuelson (1981), "Optimal Auctions," *American Economic Review* 71, 381–392.

Robinson, Marc S. (1985), "Collusion and the Choice of Auction," *RAND Journal of Economics* 16, 141–145.

Skrzypacz, Andrzej and Hugo Hopenhayn (2001), "Tacit Collusion in Repeated Auctions," Stanford GSB Research Paper No. 1698R.

Vickrey, William (1961), "Counterspeculation, Auctions and Competitive Sealed Tenders," *Journal of Finance* 16, 8–37.

Vickrey, William (1962), "Auctions and Bidding Games," in *Recent Advances in Game Theory*, Princeton Conference Series 29 (Princeton: Princeton University Press), 15–27.