Market Price of Risk Specifications for Affine Models: Theory and Evidence

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Abstract

We extend the standard specification of the market price of risk for affine yield models of the term structure of interest rates, and estimate several models using the extended specification. For most models, the extended specification fits US data better than standard specifications, often with extremely high statistical significance. Our specification yields models that are affine under both objective and risk-neutral probability measures, but is never used in financial applications, probably because of the difficulty of applying traditional methods for proving the absence of arbitrage. Using an alternate method, we show that the extended specification does not permit arbitrage opportunities, provided that under both measures the state variables cannot achieve their boundary values. Likelihood ratio tests show our extension is statistically significant for four of the models considered at the conventional 95% confidence level, and at far higher levels for three of the models. The results are particularly strong for affine diffusions with multiple square-root type variables. Although we focus on affine yield models, our extended market price of risk specification also applies to any model in which Feller's square-root process or a multivariate extension is used to model asset prices.

1 Introduction

The square-root process of Feller (1951) has been used widely in financial economics, appearing in term structure models such as Cox, Ingersoll, and Ross (1985) and stochastic volatility models of equity prices such as Heston (1993). The widespread use of this process is undoubtedly due at least in part to its relatively straightforward analytical properties. In the square-root process, a state variable follows a diffusion in which both the drift and the diffusion coefficients are affine functions of the state variable itself. Multivariate extensions of the square-root process have appeared in the term structure literature; see, for example, Duffie and Kan (1996), Dai and Singleton (2000), and Duffee (2002). Of course, a model for asset prices must specify not only the stochastic process followed by a set of underlying factors, but also the attitude of investors towards the risk of those factors; since the pioneering work of Harrison and Kreps (1979) and Harrison and Pliska (1981), this task is often accomplished by specifying the behavior of the state variable(s) under both an objective probability measure and an equivalent martingale measure. A common practice is to have the state variables follow a Feller square-root process under both probability measures, but with different governing parameters.

This latter objective is normally met by assigning to each state variable a market price of risk process that is proportional to the square root of that state variable. Since the instantaneous volatility of each state variable is also proportional to its square root, the product of the market price of risk and volatility is proportional to the state variable itself. Subtraction of this product from the drift under the objective probability measure thus results in a drift under the equivalent martingale measure that is also affine. If a process is within the Feller square-root class under the objective probability measure, this market price of risk specification ensures that it is within the same class under the equivalent martingale measure as well. A market price of risk that is inversely proportional to the square root of the state variable would also retain the affinity of the drift under both measures, but such a market price of risk specification is never used in financial modeling. Cox, Ingersoll, and Ross (1985) discuss this possibility, and point out that it leads to arbitrage opportunities if the boundary value of the process can be achieved. The instantaneous volatility of the state variable is zero at the boundary; however, with this market price of risk specification, the risk premium associated with the state variable does not go to zero as the volatility approaches zero. Ingersoll (1987) imposes the condition that the risk premium goes to zero as volatility goes to zero in a similar setting. Bates (1996), in a stochastic volatility model, also imposes this condition; Chernov and Ghysels (2000), working in a similar setting, discuss the type of market price specification we propose, but leave unresolved the issue of whether it precludes arbitrage opportunities. In a recent term structure application, Duffee (2002) specifically avoids this market price of risk specification. However, whether or not a Feller square-root process can achieve the boundary value depends on the values of the governing parameters. For some parameter values, the instantaneous volatility of the state variable can approach zero arbitrarily closely but never actually achieve this value. The market price of risk can then be arbitrarily large when the state variable takes values near zero, but is always finite. It is not immediately clear whether arbitrage opportunities exist in this case; we show that they do not.

Although the reason for the avoidance of this market price of risk specification in the existent literature is not clear, it may be related to the difficulty of proving that it does not offer arbitrage opportunities. It is quite difficult to determine whether this specification satisfies conventional criteria, e.g., those of Novikov or Kazamaki; however, these criteria are sufficient but not necessary to prove that the Girsanov ratio is a martingale. Using the approach of Cheridito, Filipović, and Yor (2003), we show that this market price of risk specification does not offer arbitrage opportunities, provided certain parameter restrictions are observed. Using the extended market price of risk specification, we estimate several term structure models, and compare the results to those obtained using more traditional market price of risk specifications. We find that, for most models considered, the extended specification results in a significant improvement in the fit of affine yield models to data on US Treasury securities.

The rest of this paper is organized as follows. In Section 2, we describe a class of multivariate term structure models driven by square-root processes, and define the admissible change of measure using our extended market price of risk specification. In Section 3, we show that this specification precludes arbitrage opportunities. In Section 4, we describe the data and estimation procedure used to estimate and test our specification. In Section 5, we present the results and show that the extended market price of risk specification offers significantly better fit to the data than standard specifications for most models, especially those with two or more square-root type state variables. Finally, Section 6 concludes.

2 Models

Throughout, our concern is with affine yield models of the term structure of interest rates, defined as follows.

Definition 1. *An affine yield model is a specification of interest rate and bond price processes such that:*

1. *The instantaneous interest rate r_t is an affine function of an N-vector of state variables denoted by Y_t:*

$$r_t = d_0 + d^T Y_t \tag{2.1}$$

where d_0 is a constant and d is an N-vector. We sometimes refer to individual elements of the vector Y_t, using the notation $Y_t(k)$ for $1 \leq k \leq N$.

2. *The state variables Y_t follow a diffusion process:*

$$dY_t = \mu^P(Y_t)\,dt + \sigma(Y_t)\,dW_t^P \tag{2.2}$$

where $\mu^P(Y_t)$ is an N-vector, $\sigma(Y_t)$ is an $N \times N$ matrix, and W_t^P is an N-dimensional standard Brownian motion under the objective probability measure P.

3. *The instantaneous drift (under the measure P) of each state variable is an affine function of Y_t:*

$$\mu^P(Y_t) = a^P + b^P Y_t \tag{2.3}$$

for some N-vector a^P and some $N \times N$ matrix b^P.

4. *The instantaneous covariance between any pair of state variables is an affine function of* Y_t*:*

$$\left[\sigma\left(Y_t\right)\sigma^T\left(Y_t\right)\right]_{i,j} = \alpha_{i,j} + \beta_{i,j}^T Y_t \tag{2.4}$$

where $\alpha_{i,j}$ *is a constant and* $\beta_{i,j}^T$ *is an* N*-vector for each* $1 \leq i, j \leq N$*, and* $\left[\sigma\left(Y_t\right)\sigma^T\left(Y_t\right)\right]_{i,j}$ *denotes the element in row* i *and column* j *of the product* $\sigma\left(Y_t\right)\sigma^T\left(Y_t\right)$*.*

5. *There exists a probability measure* Q*, equivalent to* P*, such that* Y_t *is a diffusion under* Q*:*

$$dY_t = \mu^Q\left(Y_t\right)dt + \sigma\left(Y_t\right)dW_t^Q \tag{2.5}$$

where $\mu^Q\left(Y_t\right)$ *is an* N*-vector,* W_t^Q *is an* N*-dimensional standard Brownian motion under* Q*, and such that the drift of each state variable is an affine function of the state vector:*

$$\mu^Q\left(Y_t\right) = a^Q + b^Q Y_t \tag{2.6}$$

for some N*-vector* a^Q *and some* $N \times N$ *matrix* b^Q*.*

6. *Prices of zero-coupon bonds are conditional expectations of the discounted payoffs under the measure* Q*:*

$$B\left(t,T\right) = E_t^Q\left[e^{-\int_t^T r_u du}\right] \tag{2.7}$$

Existence of a process satisfying the second, third, and fourth conditions is treated in a univariate setting in Feller (1951), and in a multivariate setting in Duffie and Kan (1996). Duffie, Filipović, and Schachermayer (2003) provide a general mathematical characterization of affine processes, including those with jumps; the diffusions we consider here are special cases. Existence can essentially be characterized as a requirement that the state vector Y_t remain within a region where $\sigma\left(Y_t\right)\sigma^T\left(Y_t\right)$ is positive semidefinite. More formally, there must exist constants $g_1, ..., g_m$ and non-trivial N-vectors $h_1, ..., h_m$ such that $\sigma\left(Y_t\right)\sigma^T\left(Y_t\right)$ is positive definite[1] if and only if:

$$g_i + h_i^T Y_t > 0 \tag{2.8}$$

for each value of $1 \leq i \leq m$. We denote the region where this condition is satisfied (for all i) by D, and the closure of D by $\overline{D}$. Note that $\sigma\left(Y_t\right)\sigma^T\left(Y_t\right)$ is positive definite in D, positive semidefinite in $\overline{D}$, and not positive semidefinite outside $\overline{D}$. Certain conditions must hold on the boundaries of D, to ensure that the state vector cannot leave the region $\overline{D}$. For each value of $Y_t \in \overline{D}$, we must have:

$$\left(g_i + h_i^T Y_t = 0\right) \quad \Rightarrow \quad \left(h_i^T \mu^P\left(Y_t\right) \geq 0\right) \tag{2.9}$$

$$\left(g_i + h_i^T Y_t = 0\right) \quad \Rightarrow \quad \left(h_i^T \sigma\left(Y_t\right)\sigma^T\left(Y_t\right)h_i = 0\right) \tag{2.10}$$

for each value of i. Intuitively, these two requirements are (1) the drift must not pull the state vector Y_t out of the region $\overline{D}$, since $\sigma\left(Y_t\right)\sigma^T\left(Y_t\right)$ then fails to be positive semidefinite, and (2) the volatility must not allow

[1] We assume here a non-degeneracy condition, that the instantaneous covariance matrix of the state variables is full rank for at least some values of the state vector.

Y_t to move stochastically out of $\overline{D}$. Of course, we must also have $Y_0 \in \overline{D}$. It is also possible that $m = 0$, i.e., that D is the entire space $\mathbb{R}^N$, in which case the restrictions of Equations 2.9 and 2.10 are entirely vacuous. There are no separate uniqueness criteria; if a solution to Equation 2.2 exists, it is unique.[2]

In addition to existence and uniqueness, achievement of boundary values is of particular importance when analyzing our market price of risk specification. Intuitively, within the region D, the drift of the state vector must not only satisfy the existence condition of Equation 2.9, but must also pull the state vector back toward the interior of D with sufficient strength to ensure that the boundary cannot be achieved. The univariate case is treated by Feller (1951) and Ikeda and Watanabe (1981); the multivariate case is more complex, and is treated in Duffie and Kan (1996). However, possibly after changing the coordinate system, all the models considered in this paper are such that the region D is of the form $(0, \infty)^M \times \mathbb{R}^{N-M}$, $M = 0, \ldots, N$, in which case it is easy to derive sufficient boundary non-attainment conditions from the one-dimensional case. We will always impose boundary non-attainment conditions, and we will call the first M state variables restricted and the last $N - M$ unrestricted.

As for possible changes of the coordinate system, note that any transformation

$$X_t = A + B \cdot Y_t \tag{2.11}$$

for some N-vector A and some non-singular $N \times N$ matrix B, of a given affine yield model with state variables Y_t, constitutes another affine yield model that can produce exactly the same short rate processes r_t as the original model. To ensure identification of parameters in estimation, we will impose additional restrictions; for example, we require that $\sigma(Y_t)$ be diagonal.[3]

Given a specification of $\mu^P(Y_t)$ and $\sigma(Y_t)$ such that a solution to Equation 2.2 exists, we may consider the existence of an equivalent measure Q by specifying a market price of risk process $\lambda(Y_t)$ which satisfies:

$$\sigma(Y_t)\,\lambda(Y_t) = \mu^P(Y_t) - \mu^Q(Y_t) \tag{2.12}$$

If the P-measure existence conditions, as described in Equations 2.8 through 2.10, and boundary non-attainment conditions are satisfied, then $\sigma(Y_t)$ is full-rank for all values of Y_t that can be achieved. Then there exists a unique $\lambda(Y_t)$ satisfying Equation 2.12. However, existence of such a $\lambda(Y_t)$ is sufficient neither for the existence of the implied probability measure Q, nor for its equivalence to P. However, from Girsanov's Theorem, the following condition is sufficient and necessary for both:

$$E_t^P\left[\exp\left(-\frac{1}{2}\int_t^T \lambda^T(Y_u)\,\lambda(Y_u)\,du - \int_t^T \lambda^T(Y_u)\,dW_u^P\right)\right] = 1 \tag{2.13}$$

Numerous sufficient criteria, such as those of Novikov and Kazamaki (see, for example, Revuz and Yor (1994)) have been developed to show that a given stochastic exponential satisfies Equation 2.13. Dai and Singleton

[2] Throughout, "existence" should be interpreted as the existence of a weak solution, and "uniqueness" refers to uniqueness in distribution.

[3] This normalization is one of several used in Dai and Singleton (2000), although some affine yield models cannot be expressed in this form. However, we consider only models with three or fewer factors, and only affine yield models with four or more factors can fail to have a representation with a diagonal $\sigma(Y_t)$ matrix.

(2000) consider a simple market price of risk specification:

$$\lambda(Y_t) = \sigma^T(Y_t)\lambda \tag{2.14}$$

where λ is a vector of constants. By construction, this specification ensures that $\mu^Q(Y_t)$ is an affine function of Y_t. When $\sigma^T(Y_t)$ does not depend on Y_t, this market price of risk specification satisfies the Novikov criterion for any time interval $[s,t]$. The Novikov criterion may also be satisfied for any time interval even when $\sigma^T(Y_t)$ does depend on Y_t, depending on the values of the model parameters. However, in general, the Dai and Singleton (2000) market price of risk specification only satisfies the Novikov criterion on $[s,t]$ when $t < s + \varepsilon$ for some positive ε. The value of ε depends on the model parameters, but not on s or Y_s. This form of local satisfaction of the Novikov criterion, however, is sufficient for satisfaction of Equation 2.13 (see, for example, Corollary 5.14 in Karatzas and Shreve (1991)).

Duffee (2002) refers to models with the market price of risk specification of Dai and Singleton (2000) as *completely affine*, and introduces the more general class of *essentially affine* models. The only constraint on the market price of risk specification in essentially affine models can be characterized as follows: if a linear combination of state variables is restricted, then the market price of risk of that linear combination must coincide with the completely affine specification. A linear combination of state variables that is unrestricted, by contrast, can have any market price of risk consistent with affine dynamics under both measures. For example, in the univariate model:

$$dY_t = \left(a^P + b^P Y_t\right)dt + \sigma dW_t^P \tag{2.15}$$

the single state variable is unrestricted, so $\lambda(Y_t)$ can be any affine function of Y_t. By contrast, in the univariate model:

$$dY_t = \left(a^P + b^P Y_t\right)dt + \sigma\sqrt{Y_t}dW_t^P \tag{2.16}$$

the single state variable is restricted. Consequently, the essentially affine market price of risk for this model is $\lambda(Y_t) = \lambda\sqrt{Y_t}$ for some constant λ (with $\lambda = 0$ possible). In other words, $\lambda(Y_t)$ is restricted to ensure that, if the volatility of any linear combination of state variables approaches zero, the risk premium of that linear combination also approaches zero. As with the completely affine market price of risk specification, the essentially affine specification satisfies the Novikov criterion for some finite positive time interval (the size of which depends on the model parameters, but not on the initial state vector), thereby ensuring satisfaction of Equation 2.13.

Our market price of risk specification, by contrast, imposes only those restrictions necessary to ensure that the boundary non-attainment conditions are satisfied under both the P and Q measures. In Section 3, we show that this requirement is sufficient to ensure that the market price of risk specification satisfies Equation 2.13. Note that the essentially affine specification nests the completely affine market price of risk, and our specification, which we refer to as the *extended affine* market price of risk, always nests both the completely affine and essentially affine specifications. The completely and essentially affine specifications coincide for some models, as do the the essentially and extended affine specifications. However, the extended affine specification is always more general than the completely affine specification.

Affine yield models are treated in a systematic way by Duffie and Kan (1996), although many other models appearing in the literature, such as Vasicek (1977), Cox, Ingersoll, and Ross (1985), Balduzzi, Das, Foresi, and Sundaram (1996), and Chen (1996), can be viewed as special cases of the general affine model. Dai and Singleton (2000) note that for each integer $N \geq 1$, there are $N+1$ non-nested families of N-factor affine yield models, and develop a classification scheme, which we use below. Each affine yield model can be placed into a family, designated $A_M(N)$, where N is the number of state variables, and M is the number of linearly independent β_{ij}, $1 \leq i, j \leq N$. M necessarily takes on values from 0 to N. The $A_M(N)$ model contains M state variables that are restricted. Each of these state variables follows a process similar to the Feller square-root process, except that the drift of one restricted state variable may depend on the value of another restricted state variable. The remaining $M - N$ state variables are unrestricted. The unrestricted state variables jointly follow a process similar to a multivariate Ornstein-Uhlenbeck process, but with two modifications: both the drift and the variance of an unrestricted state variable may depend on the values of the restricted state variables.

For now, we take as given that our market price of risk specification is free from arbitrage, and examine in detail each of the single-factor, two-factor, and three-factor affine yield models to be estimated. In addition to specifying the dynamics of the state variables under both the P and Q measures and the definition of the interest rate process, we specify any parameter restrictions needed to ensure existence of the specified process, and also restrictions needed to ensure restricted state variables do not achieve their boundary values. We also identify any restrictions needed to make sure that a model has a unique representation.

2.1 Single Factor Models

In a single factor affine yield model, the interest rate process is specified as:

$$r_t = d_0 + d_1 \cdot Y_t(1) \tag{2.17}$$

for some constants d_0 and d_1. However, the state variable $Y_t(1)$ can follow one of two distinct types of diffusions, the $A_0(1)$ and $A_1(1)$ models (as per Dai and Singleton (2000)). In the $A_0(1)$ model, $Y_t(1)$ follows a process:

$$dY_t(1) = \left[b_{11}^P Y_t(1)\right] dt + dW_t^P(1) \tag{2.18}$$

where $W_t^P(1)$ is a standard Brownian motion under the objective measure P, and b_{11}^P is an arbitrary constant. Note that this process has no constant term in the drift, and the diffusion coefficient has been normalized to one. These restrictions are not a loss of generality, but rather a normalization that ensures a unique representation of the model. Any process with an affine drift and constant diffusion can be transformed into a process of this type by an affine transformation of the state variable. An observationally equivalent interest rate model results by making an appropriate change to the d_0 and d_1 coefficients. Under the measure Q, the process $Y_t(1)$ can be written as:

$$dY_t(1) = \left[a_1^Q + b_{11}^Q Y_t(1)\right] dt + dW_t^Q(1) \tag{2.19}$$

where $W_t^Q(1)$ is a standard Brownian motion under Q. The process exists for any value of b_{11}^P; furthermore, there is no boundary value (i.e., the process $Y_t(1)$ can take on any real value). The market price of risk process is defined by:

$$\Lambda_t = [\sigma(Y_t)]^{-1}\left[\mu^P(Y_t) - \mu^Q(Y_t)\right] = -a_1^Q + \left(b_{11}^P - b_{11}^Q\right) Y_t(1) \equiv \lambda_{10} + \lambda_{11} \cdot Y_t(1) \tag{2.20}$$

In the completely affine models of Dai and Singleton (2000), the λ_{11} parameter is restricted to be zero. By contrast, in the essentially affine models of Duffee (2002), the λ_{10} and λ_{11} parameters can take any values. Existence of the measure Q with either the completely affine or essentially affine market price of risk specification follows from satisfaction of the Novikov criterion for a finite positive time interval, as discussed above. For the $A_0(1)$ model, our market price of risk specification coincides with the essentially affine specification, offering no further generality.

The $A_1(1)$ model is based on the square-root process of Feller (1951). Under this specification, the process $Y_t(1)$ can be expressed as:

$$dY_t(1) = \left[a_1^P + b_{11}^P \cdot Y_t(1)\right] dt + \sqrt{Y_t(1)}\, dW_t^P(1) \tag{2.21}$$

where $W_t^P(1)$ is a standard Brownian motion under the objective measure P. Note that the squared diffusion term may be taken to be Y_t itself, rather than some affine function of Y_t, by an appropriate change of variables, as described above. Existence of such a process requires only that $a_1^P \geq 0$. $Y_t(1)$ is bounded below by zero; this state variable cannot achieve its boundary value if $2a_1^P \geq 1$. Under the measure Q, the process $Y_t(1)$ can be written as:

$$dY_t(1) = \left[a_1^Q + b_{11}^Q \cdot Y_t(1)\right] dt + \sqrt{Y_t(1)}\, dW_t^Q(1) \tag{2.22}$$

where $W_t^Q(1)$ is a standard Brownian motion under the measure Q. The market price of risk process is defined as:

$$\Lambda_t = [\sigma(Y_t)]^{-1}\left[\mu^P(Y_t) - \mu^Q(Y_t)\right] = \frac{a_1^P - a_1^Q}{\sqrt{Y_t(1)}} + \left(b_{11}^P - b_{11}^Q\right) \cdot \sqrt{Y_t(1)} \equiv \frac{\lambda_{10}}{\sqrt{Y_t(1)}} + \lambda_{11} \cdot \sqrt{Y_t(1)} \tag{2.23}$$

The completely affine and essentially affine specifications coincide for the $A_1(1)$ model; in both, the λ_{11} parameter can take any arbitrary value, but the λ_{10} parameter is restricted to be zero. For each value of λ_{11}, the Novikov criterion is satisfied for some finite positive time horizon. We permit λ_{10} to take on any value such that boundary non-attainment conditions are satisfied under Q as well as P. This requirement can be expressed as:

$$\lambda_{10} \leq a_1^P - \frac{1}{2} \tag{2.24}$$

It is unclear whether this specification satisfies the traditional Novikov and Kazamaki criteria; in Section 3, we use another method to show that it satisfies Equation 2.13.

2.2 Two Factor Models

Two-factor affine yield models have an interest rate process given by:

$$r_t = d_0 + d_1 \cdot Y_t(1) + d_2 \cdot Y_t(2) \tag{2.25}$$

where the process followed by $Y_t(1)$ and $Y_t(2)$ falls into one of three categories: the $A_0(2)$, $A_1(2)$, and $A_2(2)$ families. The P-measure dynamics for the $A_0(2)$ model are:

$$d\begin{bmatrix} Y_t(1) \\ Y_t(2) \end{bmatrix} = \begin{bmatrix} b_{11}^P & b_{12}^P \\ b_{21}^P & b_{22}^P \end{bmatrix}\begin{bmatrix} Y_t(1) \\ Y_t(2) \end{bmatrix} dt + d\begin{bmatrix} W_t^P(1) \\ W_t^P(2) \end{bmatrix} \tag{2.26}$$

These dynamics reflect any change of variables necessary to ensure that the matrix $\sigma(Y_t)$ is identity, and the constant terms in the drifts of the state variables are zero. Even with these normalizations, however, the $A_0(2)$ representation is not unique, as a new set of state variables can be formed by taking any orthogonal rotation of the old state variables. Dai and Singleton (2000) choose the identification restriction $b_{12}^P = 0$, which guarantees uniqueness whenever the two normalized processes are not independent, i.e., when the normalization does not also cause the b_{21}^P parameter to be zero. If the normalization causes both b_{12}^P and b_{12}^P to be zero, then a reordering of the state variable indices is also possible. This method of normalization also precludes b matrices with eigenvalues that are complex conjugate pairs.[4] Under the measure Q, the process followed by Y_t is given by:

$$d\begin{bmatrix} Y_t(1) \\ Y_t(2) \end{bmatrix} = \left(\begin{bmatrix} a_1^Q \\ a_2^Q \end{bmatrix} + \begin{bmatrix} b_{11}^Q & b_{12}^Q \\ b_{21}^Q & b_{22}^Q \end{bmatrix}\begin{bmatrix} Y_t(1) \\ Y_t(2) \end{bmatrix}\right) dt + d\begin{bmatrix} W_t^Q(1) \\ W_t^Q(2) \end{bmatrix} \tag{2.27}$$

No parameter restrictions are needed to ensure the existence of the process, or of the Q measure. Furthermore, there are no boundaries, and therefore no boundary non-attainment conditions. The market price of risk specification is:

$$\begin{aligned} \Lambda_t &= [\sigma(Y_t)]^{-1}\left[\mu^P(Y_t) - \mu^Q(Y_t)\right] && (2.28) \\ &= \left(-\begin{bmatrix} a_1^Q \\ a_2^Q \end{bmatrix} + \begin{bmatrix} b_{11}^P - b_{11}^Q & b_{12}^P - b_{12}^Q \\ b_{21}^P - b_{21}^Q & b_{22}^P - b_{22}^Q \end{bmatrix}\begin{bmatrix} Y_t(1) \\ Y_t(2) \end{bmatrix}\right) && (2.29) \\ &\equiv \begin{bmatrix} \lambda_{10} \\ \lambda_{20} \end{bmatrix} + \begin{bmatrix} \lambda_{11} & \lambda_{12} \\ \lambda_{21} & \lambda_{22} \end{bmatrix}\begin{bmatrix} Y_t(1) \\ Y_t(2) \end{bmatrix} && (2.30) \end{aligned}$$

[4] Depending on the number and the maturities of the bond yields observed, there may be identification issues when some of the eigenvalues of the slope matrix in the drift are complex. See Beaglehole and Tenney (1991).

The completely affine market price of risk specifications restricts λ_{11}, λ_{12}, λ_{21}, and λ_{22} to be zero. The essentially affine specification relaxes these restrictions, and allows all six market price of risk parameters to take on arbitrary values. Both of these specifications satisfy the Novikov criterion for a finite positive time interval, thereby ensuring that the specified Q measure exists and is equivalent to P. For the $A_0(2)$ model, our specification coincides with the essentially affine market price of risk, offering no further flexibility.

The P measure dynamics of the $A_1(2)$ model are given by:

$$\begin{aligned} d\begin{bmatrix} Y_t(1) \\ Y_t(2) \end{bmatrix} &= \left(\begin{bmatrix} a_1^P \\ 0 \end{bmatrix} + \begin{bmatrix} b_{11}^P & 0 \\ b_{21}^P & b_{22}^P \end{bmatrix} \begin{bmatrix} Y_t(1) \\ Y_t(2) \end{bmatrix} \right) dt \\ &\quad + \begin{bmatrix} \sqrt{Y_t(1)} & 0 \\ 0 & \sqrt{\alpha_2 + \beta_{21} Y_t(1)} \end{bmatrix} d\begin{bmatrix} W_t^P(1) \\ W_t^P(2) \end{bmatrix} \end{aligned} \tag{2.31}$$

where $\alpha_2 \in \{0, 1\}$. Existence of this process requires that $a_1^P \geq 0$ and $\beta_{21} \geq 0$. The process $Y_t(1)$ is bounded below by zero; the additional restriction $2a_1^P \geq 1$ is needed to ensure that $Y_t(1)$ does not achieve the boundary value. The dynamics under the measure Q for the $A_1(2)$ model are given by:

$$\begin{aligned} d\begin{bmatrix} Y_t(1) \\ Y_t(2) \end{bmatrix} &= \left(\begin{bmatrix} a_1^Q \\ a_2^Q \end{bmatrix} + \begin{bmatrix} b_{11}^Q & 0 \\ b_{21}^Q & b_{22}^Q \end{bmatrix} \begin{bmatrix} Y_t(1) \\ Y_t(2) \end{bmatrix} \right) dt \\ &\quad + \begin{bmatrix} \sqrt{Y_t(1)} & 0 \\ 0 & \sqrt{\alpha_2 + \beta_{21} Y_t(1)} \end{bmatrix} d\begin{bmatrix} W_t^Q(1) \\ W_t^Q(2) \end{bmatrix} \end{aligned} \tag{2.32}$$

Note that both b_{12}^P and b_{12}^Q are constrained to be zero. In the $A_0(2)$ model, the constraint on b_{12}^P is to ensure identification, and for the essentially affine market price of risk specifications, there is no corresponding restriction under the Q measure. By contrast, the restriction here is for existence of the process under the P measure, and for the existence of the Q measure. Intuitively, the drift of $Y_t(1)$ cannot depend on $Y_t(2)$, since $Y_t(2)$ can take on any value, positive or negative, whereas $Y_t(1)$ is nonnegative. A non-zero value for b_{12}^P would give the drift of $Y_t(1)$ the wrong sign sometimes, allowing the process to be pulled onto the wrong side of the boundary. This restriction must therefore be imposed under both measures.

The market price of risk process is given by:

$$\Lambda_t = [\sigma(Y_t)]^{-1} \left[\mu^P(Y_t) - \mu^Q(Y_t)\right] \tag{2.33}$$

$$= \begin{bmatrix} \frac{\left(a_1^P - a_1^Q\right)}{\sqrt{Y_t(1)}} + \left(b_{11}^P - b_{11}^Q\right)\sqrt{Y_t(1)} \\ \frac{\left(-a_2^Q\right) + \left(b_{21}^P - b_{21}^Q\right) Y_t(1) + \left(b_{22}^P - b_{22}^Q\right) Y_t(2)}{\sqrt{\alpha_2 + \beta_{21} Y_t(1)}} \end{bmatrix} \tag{2.34}$$

$$\equiv \begin{bmatrix} \frac{\lambda_{10}}{\sqrt{Y_t(1)}} + \lambda_{11}\sqrt{Y_t(1)} \\ \frac{\lambda_{20} + \lambda_{21} Y_t(1) + \lambda_{22} Y_t(2)}{\sqrt{\alpha_2 + \beta_{21} Y_t(1)}} \end{bmatrix} \tag{2.35}$$

Previous studies of affine yield models have all imposed some restrictions on the market price of risk parameters of the $A_1(2)$ model. The completely affine market price of risk allows λ_{11}, λ_{20} and λ_{21} to be non-zero, but requires λ_{20} and λ_{21} to satisfy $\beta_{21}\lambda_{20} = \lambda_{21}\alpha_2$, so only two parameters can be chosen independently. In essentially affine models, all parameters except λ_{10} can be non-zero.[5] Both of these specifications satisfy the Novikov criterion at least for some finite positive time interval. We permit all parameters to be non-zero, requiring only that boundary non-attainment conditions for Y_t are satisfied under the measure Q. This holds if:

$$\lambda_{10} \leq a_1^P - \frac{1}{2} \tag{2.36}$$

When λ_{10} is non-zero, it is unclear whether this specification satisfies the Novikov or the Kazamaki criterion.

The dynamics under the measure P of the $A_2(2)$ model are given by:

$$d\begin{bmatrix} Y_t(1) \\ Y_t(2) \end{bmatrix} = \left(\begin{bmatrix} a_1^P \\ a_2^P \end{bmatrix} + \begin{bmatrix} b_{11}^P & b_{12}^P \\ b_{21}^P & b_{22}^P \end{bmatrix} \begin{bmatrix} Y_t(1) \\ Y_t(2) \end{bmatrix} \right) dt + \begin{bmatrix} \sqrt{Y_t(1)} & 0 \\ 0 & \sqrt{Y_t(2)} \end{bmatrix} d\begin{bmatrix} W_t^P(1) \\ W_t^P(2) \end{bmatrix} \tag{2.37}$$

with existence constraints $a_1^P \geq 0$, $a_2^P \geq 0$, $b_{12}^P \geq 0$, and $b_{21}^P \geq 0$. Both state variables are bounded below by zero; boundary non-attainment conditions are $2a_1^P \geq 1$ and $2a_2^P \geq 1$. The diagonal form of the diffusion matrix is a result of the normalization procedure; apart from a reordering of indices, each $A_2(2)$ model has a unique representation. Dynamics under the measure Q are given by:

$$d\begin{bmatrix} Y_t(1) \\ Y_t(2) \end{bmatrix} = \left(\begin{bmatrix} a_1^Q \\ a_2^Q \end{bmatrix} + \begin{bmatrix} b_{11}^Q & b_{12}^Q \\ b_{21}^Q & b_{22}^Q \end{bmatrix} \begin{bmatrix} Y_t(1) \\ Y_t(2) \end{bmatrix} \right) dt + \begin{bmatrix} \sqrt{Y_t(1)} & 0 \\ 0 & \sqrt{Y_t(2)} \end{bmatrix} d\begin{bmatrix} W_t^Q(1) \\ W_t^Q(2) \end{bmatrix} \tag{2.38}$$

The market price of risk process is defined as:

$$\begin{aligned} \Lambda_t &= [\sigma(Y_t)]^{-1}\left[\mu^P(Y_t) - \mu^Q(Y_t)\right] && (2.39) \\ &= \begin{bmatrix} \frac{\left(a_1^P - a_1^Q\right) + \left(b_{11}^P - b_{11}^Q\right)Y_t(1) + \left(b_{12}^P - b_{12}^Q\right)Y_t(2)}{\sqrt{Y_t(1)}} \\ \frac{\left(a_2^P - a_2^Q\right) + \left(b_{21}^P - b_{21}^Q\right)Y_t(1) + \left(b_{22}^P - b_{22}^Q\right)Y_t(2)}{\sqrt{Y_t(2)}} \end{bmatrix} && (2.40) \\ &\equiv \begin{bmatrix} \frac{\lambda_{10} + \lambda_{11}Y_t(1) + \lambda_{12}Y_t(2)}{\sqrt{Y_t(1)}} \\ \frac{\lambda_{20} + \lambda_{21}Y_t(1) + \lambda_{22}Y_t(2)}{\sqrt{Y_t(2)}} \end{bmatrix} && (2.41) \end{aligned}$$

Completely affine and essentially affine market price of risk specifications coincide for the $A_2(2)$ model. In both, only the λ_{11} and λ_{22} parameters can be non-zero. This specification satisfies the Novikov criterion for a finite positive time interval (which depends on the model parameters). By contrast, our specification permits all six parameters to be non-zero, with only the boundary non-attainment conditions under the measure Q

[5] It should be noted that neither Dai and Singleton (2000) nor Duffee (2002) permit $\alpha_2 = 0$.

restricting their values. These conditions are more complex than in the $A_1(2)$ model:

$$
\begin{aligned}
\lambda_{10} &\leq a_1^P - \frac{1}{2} && (2.42)\\
\lambda_{20} &\leq a_2^P - \frac{1}{2} && (2.43)\\
\lambda_{12} &\leq b_{12}^P && (2.44)\\
\lambda_{21} &\leq b_{21}^P && (2.45)
\end{aligned}
$$

This specification cannot easily be shown to satisfy either the Novikov and Kazamaki criteria for any finite positive time interval.

2.3 Three Factor Models

There are four distinct families of three-factor affine yield models: the $A_0(3)$, $A_1(3)$, $A_2(3)$, and $A_3(3)$ models. In all four, the interest rate process is given by:

$$r_t = d_0 + d_1 \cdot Y_t(1) + d_2 \cdot Y_t(2) + d_3 \cdot Y_t(3) \tag{2.46}$$

Under the $A_0(3)$ model, the state variables follow the process:

$$d \begin{bmatrix} Y_t(1) \\ Y_t(2) \\ Y_t(3) \end{bmatrix} = \begin{bmatrix} b_{11}^P & b_{12}^P & b_{13}^P \\ b_{21}^P & b_{22}^P & b_{23}^P \\ b_{31}^P & b_{32}^P & b_{33}^P \end{bmatrix} \begin{bmatrix} Y_t(1) \\ Y_t(2) \\ Y_t(3) \end{bmatrix} dt + d \begin{bmatrix} W_t^P(1) \\ W_t^P(2) \\ W_t^P(3) \end{bmatrix} \tag{2.47}$$

An $A_0(3)$ model does not have a unique representation unless additional constraints are imposed, since the state variables can be changed through orthogonal rotation. Dai and Singleton (2000) use the identifying restrictions $b_{12}^P = 0$, $b_{13}^P = 0$, and $b_{23}^P = 0$; however, this approach precludes a b matrix with complex eigenvalues. The dynamics of the state variables under the measure Q can be expressed as:

$$d \begin{bmatrix} Y_t(1) \\ Y_t(2) \\ Y_t(3) \end{bmatrix} = \left(\begin{bmatrix} a_1^Q \\ a_2^Q \\ a_3^Q \end{bmatrix} + \begin{bmatrix} b_{11}^Q & b_{12}^Q & b_{13}^Q \\ b_{21}^Q & b_{22}^Q & b_{23}^Q \\ b_{31}^Q & b_{32}^Q & b_{33}^Q \end{bmatrix} \begin{bmatrix} Y_t(1) \\ Y_t(2) \\ Y_t(3) \end{bmatrix} \right) dt + d \begin{bmatrix} W_t^Q(1) \\ W_t^Q(2) \\ W_t^Q(3) \end{bmatrix} \tag{2.48}$$

The market price of risk process is defined as:

$$
\begin{aligned}
\Lambda_t &= [\sigma(Y_t)]^{-1} \left[\mu^P(Y_t) - \mu^Q(Y_t)\right] && (2.49)\\
&= \left(- \begin{bmatrix} a_1^Q \\ a_2^Q \\ a_3^Q \end{bmatrix} + \begin{bmatrix} b_{11}^P - b_{11}^Q & b_{12}^P - b_{12}^Q & b_{13}^P - b_{13}^Q \\ b_{21}^P - b_{21}^Q & b_{22}^P - b_{22}^Q & b_{23}^P - b_{23}^Q \\ b_{31}^P - b_{31}^Q & b_{32}^P - b_{32}^Q & b_{33}^P - b_{33}^Q \end{bmatrix} \begin{bmatrix} Y_t(1) \\ Y_t(2) \\ Y_t(3) \end{bmatrix} \right) && (2.50)\\
&\equiv \begin{bmatrix} \lambda_{10} \\ \lambda_{20} \\ \lambda_{30} \end{bmatrix} + \begin{bmatrix} \lambda_{11} & \lambda_{12} & \lambda_{13} \\ \lambda_{21} & \lambda_{22} & \lambda_{23} \\ \lambda_{31} & \lambda_{32} & \lambda_{33} \end{bmatrix} \begin{bmatrix} Y_t(1) \\ Y_t(2) \\ Y_t(3) \end{bmatrix} && (2.51)
\end{aligned}
$$

As with the $A_0(1)$ and $A_0(2)$ models, the completely affine market price of risk specification restricts the slope coefficients to be zero; only λ_{10}, λ_{20}, and λ_{30} can take on non-zero values. By contrast, the essentially affine specification allows all twelve market price of risk parameters to be non-zero. Both specifications satisfy the Novikov and Kazamaki criteria for some positive finite time interval. Our specification coincides with the essentially affine specification, offering no further generality for the $A_0(3)$ model.

In the $A_1(3)$ model, the state variables follow the process:

$$
\begin{aligned}
d\begin{bmatrix} Y_t(1) \\ Y_t(2) \\ Y_t(3) \end{bmatrix} &= \left(\begin{bmatrix} a_1^P \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} b_{11}^P & 0 & 0 \\ b_{21}^P & b_{22}^P & b_{23}^P \\ b_{31}^P & b_{32}^P & b_{33}^P \end{bmatrix} \begin{bmatrix} Y_t(1) \\ Y_t(2) \\ Y_t(3) \end{bmatrix} \right) dt \\
&\quad + \begin{bmatrix} \sqrt{Y_t(1)} & 0 & 0 \\ 0 & \sqrt{\alpha_2 + \beta_{21} Y_t(1)} & 0 \\ 0 & 0 & \sqrt{\alpha_3 + \beta_{31} Y_t(1)} \end{bmatrix} d \begin{bmatrix} W_t^P(1) \\ W_t^P(2) \\ W_t^P(3) \end{bmatrix}
\end{aligned}
\tag{2.52}
$$

with α_2,$\alpha_3 \in \{0, 1\}$. Existence imposes the restrictions $a_1^P \geq 0$, $\beta_{21} \geq 0$, and $\beta_{31} \geq 0$. The first state variable is bounded below by zero, and non-attainment of the boundary requires $2a_1^P \geq 1$. The dynamics under the measure Q are:

$$
\begin{aligned}
d\begin{bmatrix} Y_t(1) \\ Y_t(2) \\ Y_t(3) \end{bmatrix} &= \left(\begin{bmatrix} a_1^Q \\ a_2^Q \\ a_3^Q \end{bmatrix} + \begin{bmatrix} b_{11}^Q & 0 & 0 \\ b_{21}^Q & b_{22}^Q & b_{23}^Q \\ b_{31}^Q & b_{32}^Q & b_{33}^Q \end{bmatrix} \begin{bmatrix} Y_t(1) \\ Y_t(2) \\ Y_t(3) \end{bmatrix} \right) dt \\
&\quad + \begin{bmatrix} \sqrt{Y_t(1)} & 0 & 0 \\ 0 & \sqrt{\alpha_2 + \beta_{21} Y_t(1)} & 0 \\ 0 & 0 & \sqrt{\alpha_3 + \beta_{31} Y_t(1)} \end{bmatrix} d \begin{bmatrix} W_t^Q(1) \\ W_t^Q(2) \\ W_t^Q(3) \end{bmatrix}
\end{aligned}
\tag{2.53}
$$

The market price of risk process is given by:

$$
\Lambda_t = [\sigma(Y_t)]^{-1} \left[\mu^P(Y_t) - \mu^Q(Y_t) \right] \tag{2.54}
$$

$$
= \begin{bmatrix} \frac{\left(a_1^P - a_1^Q\right)}{\sqrt{Y_t(1)}} + \left(b_{11}^P - b_{11}^Q\right) \cdot \sqrt{Y_t(1)} \\ \frac{\left(-a_2^Q\right) + \left(b_{21}^P - b_{21}^Q\right) \cdot Y_t(1) + \left(b_{22}^P - b_{22}^Q\right) \cdot Y_t(2) + \left(b_{23}^P - b_{23}^Q\right) \cdot Y_t(3)}{\sqrt{\alpha_2 + \beta_{21} Y_t(1)}} \\ \frac{\left(-a_3^Q\right) + \left(b_{31}^P - b_{31}^Q\right) \cdot Y_t(1) + \left(b_{32}^P - b_{32}^Q\right) \cdot Y_t(2) + \left(b_{33}^P - b_{33}^Q\right) \cdot Y_t(3)}{\sqrt{\alpha_3 + \beta_{31} Y_t(1)}} \end{bmatrix} \tag{2.55}
$$

$$
\equiv \begin{bmatrix} \frac{\lambda_{10}}{\sqrt{Y_t(1)}} + \lambda_{11} \cdot \sqrt{Y_t(1)} \\ \frac{\lambda_{20} + \lambda_{21} \cdot Y_t(1) + \lambda_{22} \cdot Y_t(2) + \lambda_{23} \cdot Y_t(3)}{\sqrt{\alpha_2 + \beta_{21} Y_t(1)}} \\ \frac{\lambda_{30} + \lambda_{31} \cdot Y_t(1) + \lambda_{32} \cdot Y_t(2) + \lambda_{33} \cdot Y_t(3)}{\sqrt{\alpha_3 + \beta_{31} Y_t(1)}} \end{bmatrix} \tag{2.56}
$$

Although the λ_{11}, λ_{20}, λ_{21}, λ_{30}, and λ_{31} parameters can be non-zero in the completely affine specification, these parameters must also satisfy the constraints $\alpha_2 \lambda_{21} = \beta_{21} \lambda_{20}$ and $\alpha_3 \lambda_{31} = \beta_{31} \lambda_{30}$. The essentially affine specification relaxes these restrictions, but still requires that the λ_{10} parameter be zero. We relax this constraint also, requiring only that λ_{10} be such that the boundary non-attainment condition is satisfied under the measure Q as well. This condition is satisfied if:

$$\lambda_{10} \leq a_1^P - \frac{1}{2} \tag{2.57}$$

When λ_{10} is not zero, it is unclear whether the Novikov and Kazamaki criteria are satisfied.

The $A_2(3)$ model has dynamics as follows:

$$\begin{aligned} d\begin{bmatrix} Y_t(1) \\ Y_t(2) \\ Y_t(3) \end{bmatrix} &= \left(\begin{bmatrix} a_1^P \\ a_2^P \\ 0 \end{bmatrix} + \begin{bmatrix} b_{11}^P & b_{12}^P & 0 \\ b_{21}^P & b_{22}^P & 0 \\ b_{31}^P & b_{32}^P & b_{33}^P \end{bmatrix} \begin{bmatrix} Y_t(1) \\ Y_t(2) \\ Y_t(3) \end{bmatrix} \right) dt \\ &\quad + \begin{bmatrix} \sqrt{Y_t(1)} & 0 & 0 \\ 0 & \sqrt{Y_t(2)} & 0 \\ 0 & 0 & \sqrt{\alpha_3 + \beta_{31} Y_t(1) + \beta_{32} Y_t(2)} \end{bmatrix} d \begin{bmatrix} W_t^P(1) \\ W_t^P(2) \\ W_t^P(3) \end{bmatrix} \end{aligned} \tag{2.58}$$

with $\alpha_3 \in \{0, 1\}$. Existence considerations require $a_1^P \geq 0$, $a_2^P \geq 0$, $b_{12}^P \geq 0$, $b_{21}^P \geq 0$, $\beta_{31} \geq 0$, and $\beta_{32} \geq 0$. The boundary is not attained if $2a_1^P \geq 1$ and $2a_2^P \geq 1$

The dynamics under the measure Q are given by:

$$\begin{aligned} d\begin{bmatrix} Y_t(1) \\ Y_t(2) \\ Y_t(3) \end{bmatrix} &= \left(\begin{bmatrix} a_1^Q \\ a_2^Q \\ a_3^Q \end{bmatrix} + \begin{bmatrix} b_{11}^Q & b_{12}^Q & 0 \\ b_{21}^Q & b_{22}^Q & 0 \\ b_{31}^Q & b_{32}^Q & b_{33}^Q \end{bmatrix} \begin{bmatrix} Y_t(1) \\ Y_t(2) \\ Y_t(3) \end{bmatrix} \right) dt \\ &\quad + \begin{bmatrix} \sqrt{Y_t(1)} & 0 & 0 \\ 0 & \sqrt{Y_t(2)} & 0 \\ 0 & 0 & \sqrt{\alpha_3 + \beta_{31} Y_t(1) + \beta_{32} Y_t(2)} \end{bmatrix} d \begin{bmatrix} W_t^Q(1) \\ W_t^Q(2) \\ W_t^Q(3) \end{bmatrix} \end{aligned} \tag{2.59}$$

The market price of risk process is given by:

$$\Lambda_t = [\sigma(Y_t)]^{-1} \left[\mu^P(Y_t) - \mu^Q(Y_t) \right] \tag{2.60}$$

$$= \begin{bmatrix} \frac{\left(a_1^P - a_1^Q\right) + \left(b_{11}^P - b_{11}^Q\right) \cdot Y_t(1) + \left(b_{12}^P - b_{12}^Q\right) \cdot Y_t(2)}{\sqrt{Y_t(1)}} \\ \frac{\left(a_2^P - a_2^Q\right) + \left(b_{21}^P - b_{21}^Q\right) \cdot Y_t(1) + \left(b_{22}^P - b_{22}^Q\right) \cdot Y_t(2)}{\sqrt{Y_t(2)}} \\ \frac{\left(-a_3^Q\right) + \left(b_{31}^P - b_{31}^Q\right) \cdot Y_t(1) + \left(b_{32}^P - b_{32}^Q\right) \cdot Y_t(2) + \left(b_{33}^P - b_{33}^Q\right) \cdot Y_t(3)}{\sqrt{\alpha_3 + \beta_{31} Y_t(1) + \beta_{32} Y_t(2)}} \end{bmatrix} \tag{2.61}$$

$$\equiv \begin{bmatrix} \frac{\lambda_{10} + \lambda_{11} \cdot Y_t(1) + \lambda_{12} \cdot Y_t(2)}{\sqrt{Y_t(1)}} \\ \frac{\lambda_{20} + \lambda_{21} \cdot Y_t(1) + \lambda_{22} \cdot Y_t(2)}{\sqrt{Y_t(2)}} \\ \frac{\lambda_{30} + \lambda_{31} \cdot Y_t(1) + \lambda_{32} \cdot Y_t(2) + \lambda_{33} \cdot Y_t(3)}{\sqrt{\alpha_3 + \beta_{31} Y_t(1) + \beta_{32} Y_t(2)}} \end{bmatrix} \tag{2.62}$$

In the completely affine market price of risk specification, five of the parameters (λ_{11}, λ_{22}, λ_{30}, λ_{31}, and λ_{32}) can be non-zero; however, there are only three degrees of freedom, since the restrictions $\beta_{31}\beta_{32}\lambda_{30} = \alpha_3\beta_{32}\lambda_{31} = \alpha_3\beta_{31}\lambda_{32}$ are also imposed. The essentially affine specification relaxes these restrictions, but still requires that λ_{10}, λ_{12}, λ_{20}, and λ_{21} be zero. We further relax these restrictions, and allow all parameters to take any values such that boundary non-attainment conditions are satisfied under both Q as well as P:

$$\lambda_{10} \leq a_1^P - \frac{1}{2} \tag{2.63}$$

$$\lambda_{20} \leq a_2^P - \frac{1}{2} \tag{2.64}$$

$$\lambda_{12} \leq b_{12}^P \tag{2.65}$$

$$\lambda_{21} \leq b_{21}^P \tag{2.66}$$

The $A_3(3)$ model has dynamics:

$$\begin{aligned} d\begin{bmatrix} Y_t(1) \\ Y_t(2) \\ Y_t(3) \end{bmatrix} &= \left(\begin{bmatrix} a_1^P \\ a_2^P \\ a_3^P \end{bmatrix} + \begin{bmatrix} b_{11}^P & b_{12}^P & b_{13}^P \\ b_{21}^P & b_{22}^P & b_{23}^P \\ b_{31}^P & b_{32}^P & b_{33}^P \end{bmatrix} \begin{bmatrix} Y_t(1) \\ Y_t(2) \\ Y_t(3) \end{bmatrix} \right) dt \\ &+ \begin{bmatrix} \sqrt{Y_t(1)} & 0 & 0 \\ 0 & \sqrt{Y_t(2)} & 0 \\ 0 & 0 & \sqrt{Y_t(3)} \end{bmatrix} d \begin{bmatrix} W_t^P(1) \\ W_t^P(2) \\ W_t^P(3) \end{bmatrix} \end{aligned} \tag{2.67}$$

Existence considerations require $a_1^P \geq 0$, $a_2^P \geq 0$, $a_3^P \geq 0$, $b_{12}^P \geq 0$, $b_{13}^P \geq 0$, $b_{21}^P \geq 0$, $b_{23}^P \geq 0$, $b_{31}^P \geq 0$, and $b_{32}^P \geq 0$. All three state variables are bounded below by zero, with boundary non-attainment conditions $2a_1^P \geq 1$, $2a_2^P \geq 1$, and $2a_3^P \geq 1$. Under the measure Q, the state variables follow the dynamics:

$$\begin{aligned} d\begin{bmatrix} Y_t(1) \\ Y_t(2) \\ Y_t(3) \end{bmatrix} &= \left(\begin{bmatrix} a_1^Q \\ a_2^Q \\ a_3^Q \end{bmatrix} + \begin{bmatrix} b_{11}^Q & b_{12}^Q & b_{13}^Q \\ b_{21}^Q & b_{22}^Q & b_{23}^Q \\ b_{31}^Q & b_{32}^Q & b_{33}^Q \end{bmatrix} \begin{bmatrix} Y_t(1) \\ Y_t(2) \\ Y_t(3) \end{bmatrix} \right) dt \\ &+ \begin{bmatrix} \sqrt{Y_t(1)} & 0 & 0 \\ 0 & \sqrt{Y_t(2)} & 0 \\ 0 & 0 & \sqrt{Y_t(3)} \end{bmatrix} d \begin{bmatrix} W_t^Q(1) \\ W_t^Q(2) \\ W_t^Q(3) \end{bmatrix} \end{aligned} \tag{2.68}$$

The market price of risk process is given by:

$$\Lambda_t = [\sigma(Y_t)]^{-1} \left[\mu^P(Y_t) - \mu^Q(Y_t)\right] \tag{2.69}$$

$$= \begin{bmatrix} \frac{(a_1^P - a_1^Q) + (b_{11}^P - b_{11}^Q) \cdot Y_t(1) + (b_{12}^P - b_{12}^Q) \cdot Y_t(2) + (b_{13}^P - b_{13}^Q) \cdot Y_t(3)}{\sqrt{Y_t(1)}} \\ \frac{(a_2^P - a_2^Q) + (b_{21}^P - b_{21}^Q) \cdot Y_t(1) + (b_{22}^P - b_{22}^Q) \cdot Y_t(2) + (b_{23}^P - b_{23}^Q) \cdot Y_t(3)}{\sqrt{Y_t(2)}} \\ \frac{(a_3^P - a_3^Q) + (b_{31}^P - b_{31}^Q) \cdot Y_t(1) + (b_{32}^P - b_{32}^Q) \cdot Y_t(2) + (b_{33}^P - b_{33}^Q) \cdot Y_t(3)}{\sqrt{Y_t(3)}} \end{bmatrix} \tag{2.70}$$

$$\equiv \begin{bmatrix} \frac{\lambda_{10} + \lambda_{11} \cdot Y_t(1) + \lambda_{12} \cdot Y_t(2) + \lambda_{13} \cdot Y_t(3)}{\sqrt{Y_t(1)}} \\ \frac{\lambda_{20} + \lambda_{21} \cdot Y_t(1) + \lambda_{22} \cdot Y_t(2) + \lambda_{23} \cdot Y_t(3)}{\sqrt{Y_t(2)}} \\ \frac{\lambda_{30} + \lambda_{31} \cdot Y_t(1) + \lambda_{32} \cdot Y_t(2) + \lambda_{33} \cdot Y_t(3)}{\sqrt{Y_t(3)}} \end{bmatrix} \tag{2.71}$$

Both the completely affine and essentially affine market price of risk specifications allow only the λ_{11}, λ_{22}, and λ_{33} parameters to be non-zero. By contrast, we allow all twelve market price of risk parameters to be non-zero, requiring only that, as usual, the boundary non-attainment condition is satisfied under the measure Q:

$$\lambda_{10} \leq a_1^P - \frac{1}{2} \tag{2.72}$$
$$\lambda_{20} \leq a_2^P - \frac{1}{2} \tag{2.73}$$
$$\lambda_{30} \leq a_3^P - \frac{1}{2} \tag{2.74}$$
$$\lambda_{12} \leq b_{12}^P \tag{2.75}$$
$$\lambda_{13} \leq b_{13}^P \tag{2.76}$$
$$\lambda_{21} \leq b_{21}^P \tag{2.77}$$
$$\lambda_{23} \leq b_{23}^P \tag{2.78}$$
$$\lambda_{31} \leq b_{31}^P \tag{2.79}$$
$$\lambda_{32} \leq b_{32}^P \tag{2.80}$$

As with the other models in which our specification is more general than traditional specifications, it is unclear whether the Novikov and Kazamaki criteria are satisfied.

3 Absence of Arbitrage

The relation between absence of arbitrage and existence of an equivalent martingale measure is well-known. The foundational work of Harrison and Kreps (1979) and Harrison and Pliska (1981) has been extended by many, such as Delbaen and Schachermayer (1994) and Delbaen and Schachermayer (1998). However, the standard techniques used to demonstrate the existence of an equivalent probability measure do not work well with our extended market price of risk specification. For example, it is not clear whether the Novikov and Kazamaki criteria are satisfied. As a restricted state variable approaches its boundary value, the extended affine specification allows the market price of risk of that state variable to grow (positively or negatively) without bound. Simply being unbounded is not necessarily a problem; for example, the standard market price of risk specification in the model of Cox, Ingersoll, and Ross (1985) grows without bound as the interest rate becomes very large. However, the market price of risk in this model, although unbounded, grows slowly enough with increasing interest rates to allow application of the Novikov and Kazamaki criteria. The extended affine market price of risk grows more quickly near the zero boundary than traditional specifications do near the infinity boundary. We must therefore take another approach, for instance, that of Cheridito, Filipović, and Yor (2003), to demonstrate that our specification precludes arbitrage opportunities.

Theorem 1. *Let $\mu^P(\cdot)$, $\mu^Q(\cdot)$, and $\sigma(\cdot)$ be functions of the form specified in Equations 2.3, 2.6, and 2.4, respectively, such that both pairs (μ^P, σ) and (μ^Q, σ) satisfy the existence conditions 2.8 through 2.10 and boundary non-attainment conditions. Then the following three statements hold:*

(a) There exists a probability space $(\Omega, \mathcal{F}, P)$ *containing a Brownian motion* $(W_t^P)_{t \geq 0}$ *such that for each* $Y_0 \in D$*, there exists a stochastic process* $(Y_t)_{t \geq 0}$ *on* $(\Omega, \mathcal{F}, P)$ *satisfying:*

$$Y_t = Y_0 + \int_0^t \mu^P(Y_s)ds + \int_0^t \sigma(Y_s)dW_s^P, \quad t \geq 0 \tag{3.1}$$

(b) The distribution of $(Y_t)_{t \geq 0}$ *under* P *is unique.*

(c) For each $T > 0$*, there exists a measure* Q *equivalent to* P *such that:*

$$Y_t = Y_0 + \int_0^t \mu^Q(Y_s)\,ds + \int_0^t \sigma(Y_s)\,dW_s^Q, \quad t \in [0, T] \tag{3.2}$$

where $(W_t^Q)_{t \in [0,T]}$ *is a Brownian motion under* Q*.*

Proof: See Appendix.

The term structure literature, from the first use of the square-root process in Cox, Ingersoll, and Ross (1985) until recent work by Duffee (2002), quite explicitly avoids market price of risk specifications that do not go to zero as the volatility of the corresponding state variable goes to zero. Theorem 1 demonstrates that this restriction can be relaxed, provided the parameters of the model do not permit attainment of the boundary under either probability measure. In this case, the market price of risk can become arbitrarily large; however, since the boundary is not achieved, it always remains finite. If the boundary non-attainment conditions are satisfied under one of the P or the Q measures, but not the other, then the two measures cannot (obviously) be equivalent. In this case, the measure under which the boundary cannot be achieved is absolutely continuous with respect to the measure under which the boundary can be achieved. However, absolute continuity is not sufficient to preclude arbitrage opportunities.

From Theorem 1, we can construct arbitrage-free models simply by ensuring that the existence and boundary non-attainment conditions are satisfied under both measures. This result allows considerable flexibility, especially when there are several square-root type state variables in a model. The dynamics of a square-root type variable (we drop the superscript notation indicating the measure for purposes of this example) in a canonical affine diffusion are given by:

$$dY_t = (a_1 + b_{11}Y_t)\,dt + \sqrt{Y_t}dW_t \tag{3.3}$$

Traditional market price of risk specifications permit only the slope coefficient, b_{11}, to differ under the two probability measures. Our specification allows both the slope and constant terms, a_1 and b_{11}, to differ, provided $2a_1 \geq 1$ under both measures. With two square-root type variables, the dynamics are:

$$dY_t = \left(\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} Y_t \right) dt + \begin{bmatrix} \sqrt{Y_t(1)} & 0 \\ 0 & \sqrt{Y_t(2)} \end{bmatrix} d \begin{bmatrix} W_t(1) \\ W_t(2) \end{bmatrix} \tag{3.4}$$

Traditional market price of risk specifications permit only b_{11} and b_{22} to change under the two measures; our specification permits all six drift parameters to change, provided $b_{12} \geq 0$ and $b_{21} \geq 0$ (for existence), and $2a_1 > 1$ and $2a_2 > 1$ (for boundary non-attainment). The extended affine market price of risk specification therefore provides one additional degree of freedom with one square-root type variable, four additional degrees of freedom with two, nine additional degrees of freedom with three, etc.

4 Estimation Procedure

To determine whether our extended market price of risk specification results in a better fit to US data, we estimate the parameters of nine affine yield models (all families of affine yield models with three or fewer state variables) using three different market price of risk specifications: the completely affine specification of Dai and Singleton (2000), the essentially affine specification of Duffee (2002), and our extended affine specification. Although our specification always nests the corresponding essentially affine models, and essentially affine models always nest completely affine models, two of the three specifications sometimes coincide. For any $A_0(N)$ affine yield model, our specification and the essentially affine specification coincide, and for any $A_N(N)$ affine yield model, the essentially affine and completely affine models are the same. Therefore, although there are nine different families of models with three market price of risk specifications for each family, there are only twenty one distinct combinations to be estimated.

Our estimation approach is maximum likelihood, using yields on US Treasury securities from the data set of McCulloch and Kwon (1993). Apart from its efficiency, use of maximum likelihood estimation makes it straightforward to calculate likelihood ratio statistics to test the significance of our extension. However, maximum likelihood estimation in a multifactor setting with a state vector that is not directly observed presents some challenges that must be overcome.

The state variables of the canonical affine diffusion are not observed directly, but must be extracted from the observed term structure of bond prices or yields. We denote by $y(Y_t, t, T)$ the time t continuously-compounded annualized yield of a zero coupon bond maturing at time T, with the value of the state vector equal to Y_t. As per Duffie and Kan (1996), such yields are affine functions of the state vector:

$$\begin{bmatrix} y(Y_t, t, T_1) \\ \vdots \\ y(Y_t, t, T_m) \end{bmatrix} = \begin{bmatrix} A(T_1 - t) \\ \vdots \\ A(T_m - t) \end{bmatrix} + \begin{bmatrix} B_1(T_1 - t) & \cdots & B_N(T_1 - t) \\ \vdots & \ddots & \vdots \\ B_1(T_m - t) & \cdots & B_N(T_m - t) \end{bmatrix} Y_t \tag{4.1}$$

where $y(Y_t, t, T)$ denotes the time t yield of a zero coupon bond maturing at time T, and $A(.)$ and $B_1(.)$ through $B_N(.)$ are deterministic functions that depend on the parameters of the Q-measure dynamics of the state variables, and on the parameters of the interest rate process. One is immediately confronted with a dilemma. If fewer bond prices are observed than state variables in the model, it is not possible to determine the exact value of the state vector at any particular time. Estimation then becomes a filtering problem; the likelihood of the next observation depends not only on the currently observed bond prices, but possibly on the entire history. However, if more bond prices are observed than the number of state variables in the model, the observed prices will generally be inconsistent with any value of the state vector. The values of the state variables can normally be inferred from an equal number of bond prices, and the remaining bond prices are then predicted exactly, without any error. In practice, no dataset ever conforms to a structural model this strictly.

It would seem that the ideal solution would be to use a number of bond prices that is equal to the number of state variables; in this way, for each time series observation of the set of bond yields, the value of the state vector can be uniquely determined. However, in general, not all of the parameters of the model will be identified. To take a simple example, consider the $A_0(1)$ model, which is equivalent to the model of Vasicek (1977). If one observes only the instantaneous interest rate (which we may consider to be the yield on a zero-maturity zero-coupon bond), we find the interest rate follows the process:

$$dr_t = \left(-b_{11}^P d_0 + b_{11}^P r_t\right) dt + d_1 dW_t^P(1) \tag{4.2}$$

The market price of risk parameters (whichever specification we choose) do not affect the observed interest rate process, and are therefore not identified. The situation does not improve if we observe instead a bond with maturity greater than zero; in this case, we may identify d_0 or a single market price of risk parameter, but not both. Similarly, even if the simplest market price of risk restriction is chosen (i.e., the completely affine market price of risk) in an $A_0(N)$ model with $N > 1$, a single parameter is always unidentified.

One way to overcome this difficulty is to collect data on more bonds than state variables, but to assume that some of the bond yields are observed with error; see, for example, Pearson and Sun (1994). We take this approach, assuming that for the $A_M(N)$ model, N yields are observed without error, but some additional bonds are observed with a Gaussian i.i.d. series of errors. The error terms are mean zero, and the error for each maturity is uncorrelated with those of other maturities. An alternate approach, in which all yields are considered observed with error, is described in Brandt and He (2002).

We also have need of the transition density of the state vector Y_t. This density is needed not only to calculate the estimates themselves, but also to calculate standard errors of the estimates, and to perform likelihood ratio tests for the different market price of risk specifications. For four of the nine models we consider (specifically, the $A_0(1)$, the $A_0(2)$, the $A_0(3)$, and the $A_1(1)$ models), the likelihood function is known in closed-form. For the five remaining models (i.e., the $A_1(2)$, $A_2(2)$, $A_1(3)$, $A_2(3)$, and $A_3(3)$ models), the likelihood function is known in closed-form only if additional parameter restrictions are imposed. These restrictions apply under the objective probability measure (i.e., there is no need to calculate likelihoods under the equivalent martingale measure), and can be placed into three categories. First, the β parameters corresponding to the unrestricted state variables in the diffusion matrix must be zero; in other words, the volatility of an unrestricted state variable must be constant. Second, the drift of an unrestricted state variable cannot depend on the values of restricted variables. Finally, the drift of one restricted state variable cannot depend on the value of another restricted state variable. We impose these restrictions in order to allow estimation, and calculation of likelihood ratio statistics, using the exact closed-form likelihood function. The estimated models are therefore less general than those that can be estimated using other methods, such as the method of moments of Dai and Singleton (2000), the quasi-maximum likelihood approach of Duffee (2002), or the approximate maximum likelihood approach of Aït-Sahalia (2001), as implemented in Aït-Sahalia and Kimmel (2002). However, the same restrictions are imposed for all market price of risk specifications; since our purpose is to test different specifications with the data, the likelihood ratio tests are still fair comparisons.

Parameter restrictions needed to ensure a closed-form likelihood function are imposed under the P measure. Analogous restrictions under the Q measure would also ensure closed-form bond prices. With the completely affine market price of risk specification, the one implies the other. However, for the more general market price of risk specifications we consider, this is not necessarily the case. Consequently, we cannot rely on the existence of closed-form bond prices. However, one of the main advantages of affine yield models is that, even when bond prices cannot be found in closed-form, they can be found numerically through very fast algorithms. Bond prices are solutions to the Feynman-Kac partial differential equation; provided a diffusion is affine under the Q measure and the interest rate is an affine function of the state variables, this partial differential equation can be decomposed into a system of ordinary differential equations, which can be solved far more rapidly than a general parabolic partial differential equation of the same dimensionality.[6] We calculate bond prices numerically, even when the market price of risk specification is sufficiently constrained to allow closed-form bond prices. Since our purpose is to compare different market price of risk specifications, use of the same method to calculate bond prices ensures that any differences found are due to the specification itself, and not the computational method used in the estimation procedure.

As discussed in Duffie and Kan (1996) and as shown in Equation 4.1, bond yields in affine yield models are affine functions of the state variables; this is the case for all three market price of risk specifications we consider. Our estimation procedure for an $A_M(N)$ model is then as follows. For a particular value of the parameter vector (in addition to the parameters of the $A_M(N)$ model, this vector includes standard deviations of observation errors for any extra bonds, σ_{N+1} through σ_K), we numerically calculate the coefficients of bond yields from Equation 4.1 for N maturities, $y(Y_t, t, T_1)$ through $y(Y_t, t, T_N)$. We use rolling maturities throughout, i.e., the value of $T_i - t$ is held fixed, not the value of T_i itself. The bond pricing formula, being affine in Y_t, is easily inverted to find the value of the state variables for each time series observation of the N bond yields. Holding the model parameters fixed, the state variables are given by:

$$Y_t = \begin{bmatrix} B_1(T_1 - t) & \cdots & B_N(T_1 - t) \\ \vdots & \ddots & \vdots \\ B_1(T_N - t) & \cdots & B_N(T_N - t) \end{bmatrix}^{-1} \begin{bmatrix} y(Y_t, t, T_1) - A(T_1 - t) \\ \vdots \\ y(Y_t, t, T_N) - A(T_N - t) \end{bmatrix} \tag{4.3}$$

The time series values of Y_t (conditional on the current choice of the parameter vector) in hand, we calculate the joint likelihood of the implied time series of observations of the state vector, using the closed-form likelihood expressions. If any of the implied values of the restricted components of Y_t (i.e., the first M elements in the $A_M(N)$ model) are on the wrong side of the boundary, the joint likelihood of the entire time series is set to zero.[7] Using the change of variables formula, we then calculate the joint likelihood of the time series of observations of the N bond yields themselves (note that, for a given value of the parameter vector, the

[6] The numeric tractability of bond pricing depends only on affinity under the measure Q, continuing to hold even if the state variable dynamics are not affine under P.

[7] Use of maximum likelihood ensures that the estimated parameter values are consistent with the observed data. Duffee (2002) points out that not all estimation techniques have this property; the estimated parameter vector for such techniques may imply that the observed time series of bond yields could not have occurred.

determinant of the Jacobian matrix does not depend on the values of the state variables):

$$L_y\left(\begin{bmatrix} y(Y_t,t,T_1) \\ \vdots \\ y(Y_t,t,T_N) \end{bmatrix} \mid \begin{bmatrix} y(Y_{t-\Delta},t-\Delta,T_1-\Delta) \\ \vdots \\ y(Y_{t-\Delta},t-\Delta,T_N-\Delta) \end{bmatrix}\right) = \frac{L_Y\left(Y_t \mid Y_{t-\Delta}\right)}{\left\|\begin{bmatrix} B_1(T_1-t) & \cdots & B_N(T_1-t) \\ \vdots & \ddots & \vdots \\ B_1(T_m-t) & \cdots & B_N(T_m-t) \end{bmatrix}\right\|} \tag{4.4}$$

where $L_y(\cdot)$ and $L_Y(\cdot)$ denote the transition likelihoods for the yield vector and the vector of state variables Y_t, respectively. Finally, we calculate the implied observation errors for the additional bond yields $y(Y_t,t,T_{N+1}),...,y(Y_t,t,T_K)$:

$$\begin{bmatrix} \varepsilon(t,T_{N+1}) \\ \vdots \\ \varepsilon(t,T_K) \end{bmatrix} = \begin{bmatrix} y(Y_t,t,T_{N+1}) \\ \vdots \\ y(Y_t,t,T_K) \end{bmatrix} - \tag{4.5}$$

$$\left(\begin{bmatrix} A(T_{N+1}-t) \\ \vdots \\ A(T_K-t) \end{bmatrix} - \begin{bmatrix} B_1(T_{N+1}-t) & \cdots & B_N(T_{N+1}-t) \\ \vdots & \ddots & \vdots \\ B_1(T_K-t) & \cdots & B_N(T_K-t) \end{bmatrix} Y_t\right) \tag{4.6}$$

and multiply the likelihood of the time series of the first N bond yields by the likelihood function for these observation errors (which, as per the previous discussion, are assumed to be Gaussian mean zero and i.i.d.). The result is the joint likelihood of the panel of bond data, including the maturities assumed to be observed with error. We repeat this procedure for many values of the parameter vector, until the parameter vector that maximizes the value of the likelihood function is discovered. Our search procedure is the Nelder-Mead simplex search.

Many search algorithms perform poorly when there are hard parameter constraints. Particularly troublesome in estimation of affine yield models is the boundary non-attainment condition for the restricted state variables (which are, of course, our primary interest). As shown in Feller (1951), the conditional likelihood of the square root process (conditional on a past observation) goes to zero near the boundary when the boundary non-attainment condition is satisfied. When the boundary non-attainment inequality is not satisfied, the likelihood either goes to positive infinity near the boundary, or to a finite non-zero value. This strong sensitivity of the likelihood to small changes in model parameters confuses many search algorithms. Consequently, we employ several normalizations to the model parameters to make the likelihood depend on them more smoothly. For example, in the $A_1(1)$ model, we replace a_1^P by:

$$c_1^P = \sqrt{a_1^P - 0.5} \tag{4.7}$$

Maximum likelihood estimation is invariant to the particular parameterization chosen, so this change of parameters does not affect the estimated model. However, despite this convenient normalization, all parameter estimates, standard errors, etc. are reported in terms of the original model parameters.

5 Results

The estimated parameters of the nine affine yield models considered are shown in Tables 1 through 9. As discussed, the extended affine specification is more general than the essentially affine specification of Duffee (2002) in six of the nine models, but all nine are shown for completeness. For each $A_M(N)$ model, we use $N+4$ zero coupon bonds maturing at two year intervals, e.g., for the $A_1(2)$ model, we use the instantaneous interest rate (i.e., the limiting value of the yield of a zero-coupon bond as the time-to-maturity decreases to zero) and zero-coupon bond yields with maturities of 2, 4, 6, 8, and 10 years. Each model is estimated with the completely affine, essentially affine, and extended affine market price of risk specifications. Likelihood ratio tests comparing the different market price of risk specifications are shown in Table 10.

For the $A_0(1)$, $A_0(2)$, and $A_0(3)$ models, the extended affine market price of risk coincides with the essentially affine specification. The strong likelihood ratio statistic for the essentially affine specification, relative to the completely affine specification, for each of these three models confirms the improved fit of the essentially affine specification found by Duffee (2002). However, our primary interest is in those cases where the extended specification is strictly more flexible than the essentially affine specification, which is the case for $A_M(N)$ models with $M > 0$.

When $M = 1$, the likelihood ratio statistics indicate that the extended specification is statistically significant at the 95% level only for the $A_1(1)$ model. As shown in Table 1, the completely affine and essentially affine specifications coincide, and both estimate the a_1^P parameter (i.e., the constant term in the drift of the state variable under the P-measure) very close to its limiting value of 0.5. (Recall that the boundary cannot be attained if $a_1^P \geq 0.5$.) Since these two specifications do not allow the a_1^P and a_1^Q parameters to differ, the state variable dynamics are very close to boundary non-attainment under both measures. By contrast, the extended affine market price of risk specification allows these two parameters to differ. The estimated value of a_1^P is well above the limiting value of 0.5, whereas a_1^Q remains very close to 0.5. This finding suggests that there is some tension between modeling the time series behaviour of the interest rate process and the cross-sectional shape of the yield curve. The time series behaviour of the interest rate is governed by the P-measure parameters, and the estimated value of a_1^P shows that the drift of the interest rate process is strong enough near its boundary value to keep the probability of being near the boundary very low. By contrast, the interest rate will have much greater probability of being near the boundary value under the Q-measure, which is what is used to price bonds. If only the instantaneous interest rate were used in the estimation, the a_1^Q parameter would be unidentified; the difference between a_1^P and a_1^Q is therefore at least partly driven by the need to match the cross-sectional shape of the yield curve, due to the extra bonds (observed with error) included in the estimation. The likelihood ratio statistic shows that the extended market price of risk specification is statistically significant (relative to either the completely affine or essentially affine specification) at the conventional 95% level; the p-value for the statistic is slightly less than 99%.

The $A_1(2)$ model hardly benefits at all from the extended market price of risk specification. The a_1^P parameter estimated very close to the limiting value of 0.5 with both the completely affine and essentially affine market prices of risk, which constrain a_1^Q to have the same value. The extended market price of risk allows the a_1^P and a_1^Q parameters to have different values, but both still estimate very close to the limiting value of 0.5. The likelihood ratio of 0.66 from Table 10 shows a lack of statistical significance, and is the result of very small deviations in the values of a_1^P and a_1^Q from their essentially affine values (the deviations are too small to show up in the four decimal places shown in Table 4). The $A_1\,(3)$ model benefits more from our extension; with the completely affine and essentially affine specifications, a_1^P and a_1^Q estimate very close to 0.5. With the extended affine specification, the a_1^Q parameter remains stubbornly at 0.5, but the a_1^P parameter estimates well away from this limiting value, at 1.67. Most of the other parameters are estimated at very similar values under the essentially and extended affine specifications. As with the $A_1\,(1)$ model, the need to model the time series behaviour of the interest rate process and the cross-sectional shape of the yield curve appear to conflict. When the a_1^P and a_1^Q parameters are constrained to have the same value, the need to capture the cross-sectional shape of the yield curve wins the struggle. Allowing the two to be different improves the fit of the model, but not by very much; note the statistically insignificant likelihood ratio statistic from Table 10 of 2.27.

When a model includes two or more square-root type variables (i.e., an $A_M\,(N)$ model with $M \geq 2$), the story changes dramatically. In all three such models ($A_2\,(2)$, $A_2\,(3)$, and $A_3\,(3)$), the extended affine specification has an extremely strong likelihood ratio statistic, relative to either the completely or essentially affine specifications. Considering only the essentially affine model as a comparison (note that it coincides with the completely affine model in two of the three cases), the 95% cutoff values are 9.49, 9.49, and 16.92, respectively, but the likelihood ratio statistics from Table 10 are 379.33, 178.21, and 355.92, respectively. The $A_2\,(2)$ model appears to benefit from two effects. First, the a_2^P and a_2^Q parameters are estimated at different values; these two parameters are constrained to be the same under the other market price of risk specifications. Second, the cross-terms in the drift are permitted to be different under the P and Q measures, and are indeed estimated at different values. Of particular interest is that the d_2 parameter estimates very close to zero with the extended affine specification. Since the two state variables are independent under the P measure, the second state variable has no influence whatsoever on the P-measure dynamics of the interest rate. In other words, the interest rate is a Markov process under the P measure. The sole impact of the second state variable is felt through the Q measure dynamics; here, the second state variable does not affect the interest rate directly, but rather through its influence on the drift of the first state variable (which, at the estimated parameter vector, is a affine transform of the interest rate). Often, the canonical state variables of an affine diffusion do not admit simple interpretations; here, the first is essentially the interest rate, and the second is a stochastic central tendency variable.

In the $A_2\,(3)$ model, it is clear that the extended affine specification makes a difference in the estimation. Under the other specifications, the a_1^P and a_1^Q parameters are estimated at 0.5, and the a_2^P and a_2^Q parameters are estimated at a much higher value. Under the extended affine specification, the a_1^Q and a_2^Q parameters are estimated at 0.5, but the a_1^P and a_2^P parameters are estimated at higher values. The cross-terms in the drift between square-root state variables (i.e., b_{12}^Q and b_{21}^Q) are also estimated at non-zero variables. The net effect is a large likelihood ratio statistic, more than 18 times as large as the 95% cutoff value. Similarly, in the $A_3\,(3)$ model, all three of the constant terms in the drift are estimated at very different values under the two measures; furthermore, several of the cross-terms in the drift are estimated at non-zero values. The net effect is a likelihood ratio statistic more than 21 times as large as the 95% cutoff value.

Thus, among the six models for which the extended market price of risk specification is more general than the essentially affine specification, a few patterns appear to emerge. In $A_M(N)$ models with $M = 1$, the extended affine parameter estimates are sometimes different from the essentially affine estimates, but the differences result in a statistically significant likelihood ratio in only one of the three models. The statistical significance occurs in the $A_1(1)$ model, which has no unrestricted state variables. The extended affine specification is always significant relative to the completely affine specification, but the incremental contribution of the extended specification over the essential specification is much smaller. In neither of the $A_M(N)$ models with $M = 1$ and $N > M$ is the extended specification significant.

However, for any $A_M(N)$ model with $M > 1$, the extended affine specification is always highly significant. The parameter estimates suggest that the ability to have differences between the a_i^P and a_i^Q parameters, $1 \leq i \leq M$, is important, but allowing the b_{ij}^P and b_{ij}^Q parameters, $1 \leq i, j \leq M$ and $i \neq j$, also appears to be important. The essentially affine specification of Duffee (2002) improves the fit of affine yield models to the data by allowing unrestricted state variables to have more flexible market price of risk specifications. Our extended affine specification does the same thing for restricted state variables. As shown, the extension is significant for most of the models, and highly significant for those models with two or more restricted state variables.

6 Conclusion

We have introduced a new market price of risk specification for affine diffusions, shown that this specification does not offer arbitrage opportunities, and demonstrated that the new specification provides a better fit to US term structure data than standard specifications for most affine yield models. Our specification is particularly important for models with two or more restricted state variables, where likelihood ratio statistics for the extended specification are typically many times the 95% cutoff values. Although each model is different, it seems that the additional flexibility offered by our specification helps relieve the tension between matching the time series behaviour of the interest rate process and matching the cross-sectional shape of the yield curve. The former is determined by the parameters of the interest rate process under the objective probability measure; the latter is determined by the parameters under an equivalent martingale measure. Traditional market price of risk specifications for affine diffusions constrain many of the parameters to be the same under both measures, so that the same parameters must capture both aspects of interest rate and term structure behaviour. By contrast, our specification allows the parameters under the two measures to differ essentially arbitrarily, subject only to existence and boundary non-attainment considerations. Rather than having one set of parameters do two jobs, we have a separate set of parameters for each task. The increased flexibility seems to result in a dramatically better fit for some models. Note that our results compare different market price of risk specifications for the same model (e.g., completely, essentially, and extended affine for the $A_2(3)$ model), but make no comparisons across families of affine yield models (e.g., $A_1(2)$ vs. $A_2(3)$ model). If the two models are nested, the likelihood ratio tests we apply could also be applied in this manner, provided the data set used is the same for both models (i.e., the same zero coupon bond maturities are used). Such a comparison is necessarily a test of both the underlying models and the observation error specification.

Our technique is limited neither to term structure applications nor to affine models. Stochastic volatility models of equity prices, such as Heston (1993), often have a volatility state variable that follows a square-root type process; our specification can readily be applied to such models, allowing a more flexible treatment of volatility risk. Similarly, multiple country models of interest rates and exchange rates, such as Brandt and Santa-Clara (2002), have used square-root type processes, and may also benefit from a more flexible market price of risk. Furthermore, the proof of absence of arbitrage does not depend in any essential way on the affinity of the drifts, variances, and covariances of the state variables. What is needed is the existence and uniqueness in distribution of a process with risk-neutral dynamics implied by the market price of risk specification, and that the state variables do not achieve their boundary values under either measure. Our technique might therefore be applied to some non-affine models as well.

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7 Appendix

7.1 Proof of Theorem 1

Theorem 1 is a consequence of Theorem 2.7 in Duffie, Filipović, and Schachermayer (2003) and Theorem 2.3 in Cheridito, Filipović, and Yor (2003). We present a version of the proof adapted to the affine diffusions considered in this paper.

Parts (a) and (b) follow from Theorem 2.7 in Duffie, Filipović, and Schachermayer (2003). To show (c), we fix $Y_0 \in D$ and $T > 0$. Since the pair $(\mu^P(.), \sigma(.))$ satisfies the existence and boundary non-attainment conditions, the market price of risk:

$$\lambda(Y_t) = \sigma(Y_t)^{-1}\left[\mu^P(Y_t) - \mu^Q(Y_t)\right], \quad t \geq 0 \tag{7.1}$$

is a well-defined continuous process. Therefore,

$$Z_t = \exp\left(-\int_0^t \lambda(Y_s)^T dW_s^P - \frac{1}{2}\int_0^t \lambda(Y_s)^T \lambda(Y_s)\, ds\right), \quad t \in [0, T] \tag{7.2}$$

is a well-defined, positive local martingale with respect to P, and thus also a P-supermartingale. Hence, if we can show that

$$E^P[Z_T] = 1 \tag{7.3}$$

then $(Z_t)_{t\in[0,T]}$ is a P-martingale, $Q = Z_T \cdot P$ is a probability measure equivalent to P, and by Girsanov's theorem, the process:

$$W_t^Q = W_t^P + \int_0^t \lambda(Y_s)\, ds, \quad t \in [0, T] \tag{7.4}$$

is a Brownian motion under Q. Moreover:

$$Y_t = Y_0 + \int_0^t \mu^Q(Y_s)\, ds + \int_0^t \sigma(Y_s)\, dW_s^Q, \quad t \in [0, T] \tag{7.5}$$

and (c) is proved.

It remains to show (7.3). By (a), there exists a stochastic process $(\tilde{Y}_t)_{t\geq 0}$ on $(\Omega, \mathcal{F}, P)$ that satisfies

$$\tilde{Y}_t = Y_0 + \int_0^t \mu^Q(\tilde{Y}_s)ds + \int_0^t \sigma(\tilde{Y}_s)dW_s^P, \quad t \geq 0 \tag{7.6}$$

and by (b), the distribution of $(\tilde{Y}_t)_{t\geq 0}$ is unique. Since the pair $(\mu^Q(.), \sigma(.))$ also satisfies the existence and boundary non-attainment conditions,

$$\lambda(\tilde{Y}_t) = \sigma(\tilde{Y}_t)^{-1}\left[\mu^P\left(\tilde{Y}_t\right) - \mu^Q\left(\tilde{Y}_t\right)\right], \quad t \geq 0 \tag{7.7}$$

is a well-defined continuous process. For each $n \geq 1$, we define the stopping times:

$$\tau_n = \inf\left\{t > 0 \mid \|\lambda(Y_t)\|_2 \geq n\right\} \wedge T \tag{7.8}$$

and

$$\tilde{\tau}_n = \inf\left\{t > 0 \mid \left\|\lambda\left(\tilde{Y}_t\right)\right\|_2 \geq n\right\} \wedge T \tag{7.9}$$

where $\|\lambda(Y_t)\|_2$ denotes the Euclidean norm of the vector $\lambda(Y_t)$. These stopping times satisfy:

$$\lim_{n\to\infty} P[\tau_n = T] = \lim_{n\to\infty} P[\tilde{\tau}_n = T] = 1 \quad \text{as} \quad n \to \infty \tag{7.10}$$

For each $n \geq 1$, we define the process:

$$\lambda_t^n = \lambda(Y_t) 1_{\{t \leq \tau_n\}}, \quad t \in [0, T] \tag{7.11}$$

Note that, by construction, $\left(\int_0^t (\lambda_t^n)^T \lambda_t^n ds\right)$ is bounded by $n^2 \cdot t$. For each n, the process satisfies the Novikov criterion (under the P-measure):

$$E^P\left[\exp\left(\frac{1}{2}\int_0^t (\lambda_s^n)^T \lambda_s^n ds\right)\right] \leq \exp\left(\frac{n^2 \cdot t}{2}\right) < +\infty \tag{7.12}$$

It follows that, for each $n \geq 1$, the process defined by:

$$Z_t^n = \exp\left(-\int_0^t (\lambda_s^n)^T dW_s^P - \frac{1}{2}\int_0^t (\lambda_s^n)^T \lambda_s^n ds\right), \quad t \in [0, T] \tag{7.13}$$

is a P-martingale, and by (7.10), $Z_T^n 1_{\{\tau_n = T\}} = Z_T 1_{\{\tau_n = T\}} \to Z_T$, P-almost surely, as $n \to \infty$. For all $n \geq 1$, $Q^n = Z_T^n \cdot P$ is a probability measure equivalent to P, and it follows from Girsanov's theorem that

$$W_t^n = W_t^P + \int_0^t \lambda_s^n ds\,, \quad t \geq 0 \tag{7.14}$$

is a Brownian motion under Q^n. It is easy to see that:

$$Y_{t\wedge\tau_n} = Y_0 + \int_0^{t\wedge\tau_n} \mu^Q(Y_s) ds + \int_0^{t\wedge\tau_n} \sigma(Y_s) dW_s^n, \quad t \in [0, T] \tag{7.15}$$

and it can be deduced from (a), (b) (7.6) and (7.15) that under Q^n, the stopped process $(Y_{t\wedge\tau_n})_{t\geq 0}$ has the same distribution as the stopped process $(\tilde{Y}_{t\wedge\tilde{\tau}_n})_{t\geq 0}$ under P. Therefore:

$$E^P[Z_T] = \lim_{n\to\infty} E^P\left[Z_T^n 1_{\{\tau_n = T\}}\right] = \lim_{n\to\infty} Q^n[\tau_n = T] = \lim_{n\to\infty} P[\tilde{\tau}_n = T] = 1 \tag{7.16}$$

The first step in this chain of equalities holds because $Z_T^n = Z_T \geq 0$ on the set $\{\tau_n = T\}$ and the sets $\{\tau_n = T\}$ increase to Ω as $n \to \infty$, P-almost surely. The second step holds by applying the definition of the measures Q^n; note that Z_T^n is the Radon-Nikodym derivative of Q^n with respect to P. The third step follows because the distribution of $(Y_{t\wedge\tau_n})_{t\geq 0}$ under Q^n is the same as the distribution of $(\tilde{Y}_{t\wedge\tilde{\tau}_n})_{t\geq 0}$ under P. The last step follows from (7.10).

Parameter	Completely Affine		Essentially Affine		Extended Affine	
	Estimate	Std. Err.	Estimate	Std. Err.	Estimate	Std. Err.
b_{11}^P	-0.008290	0.0020	-0.6295	0.1914	-0.6295	0.1914
a_1^Q	0.1878	0.1969	0.2196	0.0047	0.2196	0.0047
b_{11}^Q	-0.008290	0.0020	-0.007952	0.0022	-0.007952	0.0022
d_0	0.1834	0.8051	0.04739	0.0138	0.04739	0.0138
d_1	0.03388	0.0008	0.03457	0.0009	0.03457	0.0009
σ_1	0.0105	0.0013	0.0105	0.0013	0.0105	0.0013
σ_2	0.0109	0.0043	0.0109	0.0043	0.0109	0.0043
σ_3	0.0117	0.0078	0.0117	0.0079	0.0117	0.0079
σ_4	0.0123	0.0055	0.0123	0.0056	0.0123	0.0056

Table 1: $A_0(1)$ Model Estimates

This table shows the parameter estimates and standard errors for the $A_0(1)$ model parameters, for the completely affine, essentially affine, and extended affine market price of risk specifications. The instantaneous interest rate is assumed to be observed without error; zero-coupon bond yields with maturities 2, 4, 6, and 8 years are assumed to be observed with error. The essentially affine and extended affine specifications coincide for this model. Note that, for the completely affine market price of risk specification, the b_{11}^P and b_{11}^Q parameters must coincide. For the other two market price of risk specifications, all parameters can vary independently.

Parameter	**Completely Affine**		**Essentially Affine**		**Extended Affine**	
	Estimate	Std. Err.	Estimate	Std. Err.	Estimate	Std. Err.
a_1^P	0.5000	0.0020	0.5000	0.0020	1.5675	0.4699
b_{11}^P	-0.09740	0.0718	-0.09740	0.0718	-0.3485	0.1183
a_1^Q	0.5000	0.0020	0.5000	0.0020	0.5000	0.0143
b_{11}^Q	0.008011	0.0018	0.008011	0.0018	0.008168	0.0015
d_0	-0.005785	0.0014	-0.005785	0.0014	-0.006401	0.0015
d_1	0.01229	0.0004	0.01229	0.0004	0.01228	0.0005
σ_1	0.0107	0.0015	0.0107	0.0015	0.0107	0.0014
σ_2	0.0110	0.0043	0.0110	0.0043	0.0110	0.0043
σ_3	0.0117	0.0071	0.0117	0.0071	0.0117	0.0071
σ_4	0.0127	0.0053	0.0127	0.0053	0.0127	0.0053

Table 2: $A_1(1)$ **Model Estimates**

This table shows the parameter estimates and standard errors for the $A_1(1)$ model parameters, for the completely affine, essentially affine, and extended affine market price of risk specifications. The instantaneous interest rate is assumed to be observed without error; zero-coupon bond yields with maturities 2, 4, 6, and 8 years are assumed to be observed with error. The completely affine and essentially affine specifications coincide for this model. Note that, for the completely affine and essentially affine market price of risk specifications, the a_1^P and a_1^Q parameters must coincide. For the extended affine market price of risk specification, all parameters can vary independently.

Parameter	Completely Affine		Essentially Affine		Extended Affine	
	Estimate	Std. Err.	Estimate	Std. Err.	Estimate	Std. Err.
b_{11}^P	-0.02620	0.0010	-0.1270	0.0724	-0.1270	0.0724
b_{21}^P	1.4373	0.0862	-0.1497	0.4486	-0.1497	0.4486
b_{22}^P	-2.0809	0.0539	-6.0648	0.5001	-6.0648	0.5001
a_1^Q	0.001899	0.1618	0.6524	0.1242	0.6524	0.1242
a_2^Q	0.9379	0.1652	0.7854	0.2090	0.7854	0.2090
b_{11}^Q	-0.02620	0.0010	0.02777	0.1065	0.02777	0.1065
b_{12}^Q	0.0000	0.0000	-1.5078	0.1735	-1.5078	0.1735
b_{21}^Q	1.4373	0.0862	0.07417	0.1558	0.07417	0.1558
b_{22}^Q	-2.0809	0.0539	-2.0804	0.1086	-2.0804	0.1086
d_0	0.1017	0.1175	0.05520	0.0223	0.05520	0.0223
d_1	-0.1903	0.0010	0.01454	0.0022	0.01454	0.0022
d_2	0.03010	0.0006	0.02877	0.0013	0.02877	0.0013
σ_1	0.0026	0.0002	0.0026	0.0002	0.0026	0.0002
σ_2	0.0040	0.0009	0.0040	0.0009	0.0040	0.0009
σ_3	0.0048	0.0012	0.0047	0.0012	0.0047	0.0012
σ_4	0.0053	0.0008	0.0053	0.0008	0.0053	0.0008

Table 3: $A_0\,(2)$ **Model Estimates**

This table shows the parameter estimates and standard errors for the $A_0\,(2)$ model parameters, for the completely affine, essentially affine, and extended affine market price of risk specifications. The instantaneous interest rate and a zero-coupon bond yield with maturity of 2 years are assumed to be observed without error; zero-coupon bond yields with maturities 4, 6, 8, and 10 years are assumed to be observed with error. The essentially affine and extended affine specifications coincide for this model. Note that, for the completely affine market price of risk specification, the slope coefficient parameters in the drift must coincide (i.e., b_{11}^P and b_{11}^Q are the same, b_{21}^P and b_{21}^Q are the same, and b_{22}^P and b_{22}^Q are the same). Furthermore, for the completely affine market price of risk specification, the b_{12}^Q parameter is held fixed at zero. For the other two market price of risk specifications, all parameters can vary independently.

Parameter	Completely Affine		Essentially Affine		Extended Affine	
	Estimate	Std. Err.	Estimate	Std. Err.	Estimate	Std. Err.
a_1^P	0.5000	0.0132	0.5000	0.0159	0.5000	0.1313
b_{11}^P	-0.04895	0.0185	-0.006282	0.0364	-0.01579	0.0402
b_{22}^P	-3.1846	0.1062	-5.1792	0.4927	-5.5356	0.5146
a_1^Q	0.5000	0.0132	0.5000	0.0159	0.5000	0.0472
a_2^Q	1.1235	0.1261	0.006282	0.0749	0.1008	0.1018
b_{11}^Q	-0.009747	0.0007	-0.01221	0.0008	-0.01209	0.0011
b_{21}^Q	0.0000	0.0000	0.08122	0.0065	0.07361	0.0069
b_{22}^Q	-3.1846	0.1062	-2.7386	0.0914	-2.7574	0.0943
d_0	-0.003683	0.0010	0.008901	0.0013	0.007483	0.0017
d_1	0.004171	0.0001	0.003147	0.0001	0.003244	0.0001
d_2	0.03429	0.0006	0.03536	0.0009	0.03575	0.0009
σ_1	0.0027	0.0002	0.0026	0.0002	0.0026	0.0002
σ_2	0.0041	0.0009	0.0040	0.0009	0.0040	0.0009
σ_3	0.0049	0.0013	0.0048	0.0013	0.0048	0.0013
σ_4	0.0054	0.0009	0.0054	0.0009	0.0054	0.0009

Table 4: $A_1(2)$ Model Estimates

This table shows the parameter estimates and standard errors for the $A_1(2)$ model parameters, for the completely affine, essentially affine, and extended affine market price of risk specifications. The instantaneous interest rate and a zero-coupon bond yield with maturity of 2 years are assumed to be observed without error; zero-coupon bond yields with maturities 4, 6, 8, and 10 years are assumed to be observed with error. The b_{21}^P and β_{21} parameters (not shown) are held fixed at zero, to ensure that the likelihood function is known in closed-form. Note that, for the completely affine and essentially affine market price of risk specifications, the a_1^P and a_1^Q parameters must coincide. For the completely affine market price of risk specification, the b_{21}^Q parameter must be zero, and the b_{22}^P and b_{22}^Q parameters must be the same. For the extended affine market price of risk specification, all parameters can vary independently.

Parameter	**Completely Affine**		**Essentially Affine**		**Extended Affine**	
	Estimate	Std. Err.	Estimate	Std. Err.	Estimate	Std. Err.
a_1^P	0.5000	0.0134	0.5000	0.0134	0.5000	0.0205
a_2^P	23.8020	2.5332	23.8020	2.5332	1.0596	0.5037
b_{11}^P	−0.01582	0.0283	−0.01582	0.0283	−0.03024	0.0364
b_{22}^P	−3.3520	0.0955	−3.3520	0.0955	−0.3384	0.1119
a_1^Q	0.5000	0.0134	0.5000	0.0134	0.5000	0.0264
a_2^Q	23.8020	2.5332	23.8020	2.5332	1.5794	0.3229
b_{11}^Q	−0.01055	0.0007	−0.01055	0.0007	−0.01638	0.0013
b_{12}^Q	0.0000	0.0000	0.0000	0.0000	0.0000	0.0030
b_{21}^Q	0.0000	0.0000	0.0000	0.0000	0.6141	0.0828
b_{22}^Q	−2.9839	0.0935	−2.9839	0.0935	−2.3233	0.0632
d_0	−0.1045	0.0032	−0.1045	0.0032	−0.001148	0.0004
d_1	0.004179	0.0001	0.004179	0.0001	0.0000	0.0002
d_2	0.01416	0.0005	0.01416	0.0005	0.01600	0.0014
σ_1	0.0026	0.0002	0.0026	0.0002	0.0026	0.0002
σ_2	0.0041	0.0009	0.0041	0.0009	0.0040	0.0009
σ_3	0.0049	0.0013	0.0049	0.0013	0.0048	0.0013
σ_4	0.0054	0.0009	0.0054	0.0009	0.0054	0.0009

Table 5: $A_2(2)$ **Model Estimates**

This table shows the parameter estimates and standard errors for the $A_2(2)$ model parameters, for the completely affine, essentially affine, and extended affine market price of risk specifications. The instantaneous interest rate and a zero-coupon bond yield with maturity of 2 years are assumed to be observed without error; zero-coupon bond yields with maturities 4, 6, 8, and 10 years are assumed to be observed with error. The completely affine and essentially affine specifications coincide for this model. The b_{12}^P and b_{21}^P parameters (not shown) are held fixed at zero, to ensure that the likelihood function is known in closed-form. Note that, for the completely affine and essentially affine market price of risk specifications, the a_1^P and a_1^Q parameters must be the same, the a_2^P and a_2^Q must be the same, and the b_{12}^Q and b_{21}^Q parameters must be zero. For the extended affine market price of risk specification, all parameters can vary independently.

Parameter	Completely Affine		Essentially Affine		Extended Affine	
	Estimate	Std. Err.	Estimate	Std. Err.	Estimate	Std. Err.
b_{11}^P	−0.01011	0.0005	−0.1039	0.0751	−0.1039	0.0751
b_{21}^P	0.1439	0.0257	−0.6327	0.2245	−0.6327	0.2245
b_{22}^P	−0.7529	0.0098	−1.3658	0.2133	−1.3658	0.2133
b_{31}^P	0.4950	0.2829	0.5795	0.8522	0.5795	0.8522
b_{32}^P	3.9866	0.2834	2.4831	0.9190	2.4831	0.9190
b_{33}^P	−5.7304	0.4626	−10.0865	0.8606	−10.0865	0.8606
a_1^Q	−0.01129	0.1645	0.8649	0.2566	0.8649	0.2566
a_2^Q	0.08620	0.1763	0.4579	0.2492	0.4579	0.2492
a_3^Q	1.8387	0.2660	1.6581	0.5755	1.6581	0.5755
b_{11}^Q	−0.01011	0.0005	0.1227	0.1373	0.1227	0.1373
b_{12}^Q	0.0000	0.0000	0.3250	0.2301	0.3250	0.2301
b_{13}^Q	0.0000	0.0000	−2.2892	0.5540	−2.2892	0.5540
b_{21}^Q	0.1439	0.0257	−0.04854	0.1109	−0.04854	0.1109
b_{22}^Q	−0.7529	0.0098	−0.1039	0.1124	−0.1039	0.1124
b_{23}^Q	0.0000	0.0000	−1.2278	0.6599	−1.2278	0.6599
b_{31}^Q	0.4950	0.2829	0.7670	0.5363	0.7670	0.5363
b_{32}^Q	3.9866	0.2834	1.7919	0.5438	1.7919	0.5438
b_{33}^Q	−5.7304	0.4626	−5.7782	0.6560	−5.7782	0.6560
d_0	0.1921	1.0999	0.05687	0.9800	0.05687	0.9800
d_1	0.005479	0.0012	0.01517	0.0022	0.01517	0.0022
d_2	0.002673	0.0006	0.003323	0.0034	0.003323	0.0034
d_3	0.03178	0.0006	0.02922	0.0015	0.02922	0.0015
σ_1	0.0009	0.0001	0.0009	0.0001	0.0009	0.0001
σ_2	0.0015	0.0003	0.0015	0.0003	0.0015	0.0003
σ_3	0.0021	0.0006	0.0021	0.0005	0.0021	0.0005
σ_4	0.0027	0.0004	0.0027	0.0001	0.0027	0.0001

Table 6: $A_0(3)$ Model Estimates

This table shows the parameter estimates and standard errors for the $A_0(3)$ model parameters, for the completely affine, essentially affine, and extended affine market price of risk specifications. The instantaneous interest rate and zero-coupon bond yields with maturities of 2 and 4 years are assumed to be observed without error; zero-coupon bond yields with maturities 6, 8, 10, and 12 years are assumed to be observed with error. The essentially affine and extended affine specifications coincide for this model. Note that, for the completely affine market price of risk specification, the slope coefficient parameters in the drift must coincide (i.e., b_{11}^P and b_{11}^Q are the same, b_{21}^P and b_{21}^Q are the same, b_{22}^P and b_{22}^Q are the same, b_{31}^P and b_{31}^Q are the same, b_{32}^P and b_{32}^Q are the same, and b_{33}^P and b_{33}^Q are the same). Furthermore, for the completely affine market price of risk specification, the b_{12}^Q, b_{13}^Q, and b_{23}^Q parameters are held fixed at zero. For the other two market price of risk specifications, all parameters can vary independently.

Parameter	Completely Affine		Essentially Affine		Extended Affine	
	Estimate	Std. Err.	Estimate	Std. Err.	Estimate	Std. Err.
a_1^P	0.5000	0.0760	0.5000	0.0016	1.6996	1.7843
b_{11}^P	0.006502	0.0290	0.007257	0.0293	-0.04448	0.0443
b_{22}^P	-0.8101	0.0115	-0.9360	0.1582	-0.9365	0.1590
b_{32}^P	3.7680	0.2418	2.3237	0.9206	2.3240	0.9182
b_{33}^P	-5.3422	0.4280	-10.1202	0.8247	-10.1214	0.8276
a_1^Q	0.5000	0.0760	0.5000	0.0016	0.5000	0.0037
a_2^Q	0.1029	0.1747	-0.006028	0.1781	-0.006462	0.1780
a_3^Q	1.6953	0.2429	0.6346	0.2762	0.6333	0.2764
b_{11}^Q	0.001355	0.0009	0.0007466	0.0009	0.0007563	0.0004
b_{21}^Q	0.0000	0.0000	0.03244	0.0056	0.03243	0.0052
b_{22}^Q	-0.8101	0.0115	0.09181	0.1012	0.09229	0.1016
b_{23}^Q	0.0000	0.0000	-2.3721	0.5571	-2.3737	0.5569
b_{31}^Q	0.0000	0.0000	0.02886	0.0090	0.02887	0.0089
b_{32}^Q	3.7680	0.2418	1.7566	0.5066	1.7568	0.5066
b_{33}^Q	-5.3422	0.4280	-5.4517	0.5008	-5.4530	0.5015
d_0	-0.006662	0.0057	0.01285	0.0058	0.01280	0.0057
d_1	0.002203	0.0001	0.001401	0.0002	0.001401	0.0001
d_2	0.002878	0.0012	0.01212	0.0028	0.01212	0.0028
d_3	0.03220	0.0006	0.03049	0.0015	0.03049	0.0015
σ_1	0.0009	0.0001	0.0009	0.0001	0.0009	0.0001
σ_2	0.0016	0.0003	0.0016	0.0003	0.0016	0.0003
σ_3	0.0021	0.0006	0.0021	0.0006	0.0021	0.0006
σ_4	0.0027	0.0004	0.0027	0.0005	0.0027	0.0005

Table 7: $A_1(3)$ Model Estimates

This table shows the parameter estimates and standard errors for the $A_1(3)$ model parameters, for the completely affine, essentially affine, and extended affine market price of risk specifications. The instantaneous interest rate and zero-coupon bond yields with maturities of 2 and 4 years are assumed to be observed without error; zero-coupon bond yields with maturities 6, 8, 10, and 12 years are assumed to be observed with error. The b_{21}^P, b_{31}^P, β_{21}, and β_{31} parameters (not shown) are held fixed at zero, to ensure that the likelihood function is known in closed-form. When these restrictions are imposed, the model becomes unidentified; consequently, we also impose the identifying restriction that b_{23}^P is held at zero. Note that, for the completely affine and essentially affine market price of risk specifications, the a_1^P and a_1^Q parameters must coincide. For the completely affine market price of risk specification, the b_{21}^Q, b_{23}^Q, and b_{31}^Q parameters must be zero. Furthermore, b_{22}^P and b_{22}^Q must be the same, b_{32}^P and b_{32}^Q must be the same, and b_{33}^P and b_{33}^Q must be the same. For the extended affine market price of risk specification, all parameters can vary independently.

Parameter	Completely Affine		Essentially Affine		Extended Affine	
	Estimate	Std. Err.	Estimate	Std. Err.	Estimate	Std. Err.
a_1^P	0.5000	0.0216	0.5000	0.0854	1.8105	2.1392
a_2^P	4.7456	1.2803	5.8672	1.6413	1.1125	0.7500
b_{11}^P	0.006696	0.0277	0.006640	0.0279	−0.04556	0.0468
b_{22}^P	−0.7885	0.0756	−0.8637	0.0635	−0.1449	0.0754
b_{33}^P	−10.6479	0.7426	−9.4848	0.7239	−9.5140	0.7271
a_1^Q	0.5000	0.0216	0.5000	0.0854	0.5000	0.1469
a_2^Q	4.7456	1.2803	5.8672	1.6413	0.5000	0.4848
a_3^Q	2.1224	0.2926	−122.9010	615.5	43.1382	1093.2
b_{11}^Q	0.0006241	0.0004	0.0003567	0.0009	−0.007533	0.0218
b_{12}^Q	0.0000	0.0000	0.0000	0.0000	0.01383	0.0776
b_{21}^Q	0.0000	0.0000	0.0000	0.0000	0.1899	0.0326
b_{22}^Q	−0.7258	0.0092	−0.7842	0.0014	−0.6989	0.0233
b_{31}^Q	0.0000	0.0000	1.7500	8.8798	−11.7418	296.95
b_{32}^Q	0.0000	0.0000	18.8626	94.707	73.2124	1853.8
b_{33}^Q	−10.6479	0.7426	−210.0525	1061.8	−982.7415	2491.9
d_0	−0.06960	0.0048	−0.04242	0.0045	−0.0005266	0.0021
d_1	0.002186	0.0001	0.001890	0.0001	0.0002631	0.0001
d_2	0.01026	0.0007	0.006159	0.0005	0.006020	0.0005
d_3	0.04344	0.0011	0.03333	0.0012	0.03327	0.0013
σ_1	0.0009	0.0001	0.0009	0.0001	0.0009	0.0001
σ_2	0.0016	0.0003	0.0016	0.0002	0.0016	0.0003
σ_3	0.0021	0.0006	0.0021	0.0005	0.0021	0.0006
σ_4	0.0027	0.0004	0.0027	0.0004	0.0027	0.0005

Table 8: $A_2(3)$ Model Estimates

This table shows the parameter estimates and standard errors for the $A_2(3)$ model parameters, for the completely affine, essentially affine, and extended affine market price of risk specifications. The instantaneous interest rate and zero-coupon bond yields with maturities of 2 and 4 years are assumed to be observed without error; zero-coupon bond yields with maturities 6, 8, 10, and 12 years are assumed to be observed with error. The b_{12}^P, b_{21}^P, b_{31}^P, b_{32}^P, β_{31}, and β_{32} parameters (not shown) are held fixed at zero, to ensure that the likelihood function is known in closed-form. Note that, for the completely affine and essentially affine market price of risk specifications, the a_1^P and a_1^Q parameters must coincide, as must the a_2^P and a_2^Q parameters; furthermore, the b_{12}^Q and b_{21}^Q parameters must be equal to their counterparts under the P measure (which, as noted above, are held fixed at zero). For the completely affine market price of risk specification, the b_{31}^Q, b_{32}^Q, and b_{33}^Q parameters must be equal to their counterparts under the P measure, b_{31}^P, b_{32}^P, and b_{33}^P. For the extended affine market price of risk specification, all parameters can vary independently.

Parameter	Completely Affine		Essentially Affine		Extended Affine	
	Estimate	Std. Err.	Estimate	Std. Err.	Estimate	Std. Err.
a_1^P	0.5000	0.1360	0.5000	0.1360	1.7844	2.2275
a_2^P	1464.3	98.049	1464.3	98.049	712.3582	159.75
a_3^P	4.7175	1.2308	4.7175	1.2308	1.2243	0.8465
b_{11}^P	0.02147	0.0042	0.02147	0.0042	−0.04316	0.0469
b_{22}^P	−10.4489	0.2046	−10.4489	0.2046	−8.9894	0.7412
b_{33}^P	−0.7886	0.0758	−0.7886	0.0758	−0.1973	0.0880
a_1^Q	0.5000	0.1360	0.5000	0.1360	0.5000	226.97
a_2^Q	1464.3	98.049	1464.3	98.049	3624.5	351.30
a_3^Q	4.7175	1.2308	4.7175	1.2308	0.5213	131.85
b_{11}^Q	0.0006452	0.0012	0.0006452	0.0012	−0.002986	0.0198
b_{12}^Q	0.0000	0.0000	0.0000	0.0000	0.0000014	1.4400
b_{13}^Q	0.0000	0.0000	0.0000	0.0000	0.00000013	0.6456
b_{21}^Q	0.0000	0.0000	0.0000	0.0000	0.0006549	0.8943
b_{22}^Q	−10.2775	0.1894	−10.2775	0.1894	−45.9946	5.3708
b_{23}^Q	0.0000	0.0000	0.0000	0.0000	20.9368	4.3427
b_{31}^Q	0.0000	0.0000	0.0000	0.0000	0.1507	0.0288
b_{32}^Q	0.0000	0.0000	0.0000	0.0000	0.001383	0.8366
b_{33}^Q	−0.7259	0.0093	−0.7259	0.0093	−0.6902	0.3884
d_0	−0.5910	0.0196	−0.5910	0.0196	−0.3045	0.0207
d_1	0.002186	0.0002	0.002186	0.0002	0.0001764	0.0002
d_2	0.003720	0.000005	0.003720	0.000005	0.003839	0.0002
d_3	0.01032	0.0007	0.01032	0.0007	0.007424	0.0007
σ_1	0.0009	0.0001	0.0009	0.0001	0.0009	0.0001
σ_2	0.0016	0.0003	0.0016	0.0003	0.0016	0.0003
σ_3	0.0021	0.0006	0.0021	0.0006	0.0021	0.0006
σ_4	0.0027	0.0005	0.0027	0.0005	0.0027	0.0005

Table 9: $A_3(3)$ Model Estimates

This table shows the parameter estimates and standard errors for the $A_3(3)$ model parameters, for the completely affine, essentially affine, and extended affine market price of risk specifications. The instantaneous interest rate and zero-coupon bond yields with maturities of 2 and 4 years are assumed to be observed without error; zero-coupon bond yields with maturities 6, 8, 10, and 12 years are assumed to be observed with error. The completely affine and essentially affine specifications coincide for this model. The b_{12}^P, b_{13}^P, b_{21}^P, b_{23}^P, b_{31}^P, and b_{32}^P parameters (not shown) are held fixed at zero, to ensure that the likelihood function is known in closed-form. Note that, for the completely affine and essentially affine market price of risk specifications, the a_1^Q, a_2^Q, and a_3^Q parameters must be equal to their P-measure counterparts, a_1^P, a_3^P, and a_3^P. Furthermore, the b_{12}^Q, b_{13}^Q, b_{21}^Q, b_{23}^Q, b_{31}^Q and b_{32}^Q parameters must be equal to their counterparts under the P measure (which, as noted above, are held fixed at zero). For the extended affine market price of risk specification, all parameters can vary independently.

	Ess. Aff. vs. Comp. Aff.			Ext. Aff. vs. Comp. Aff.			Ext. Aff. vs. Ess. Aff.		
Model	**DF**	**Cutoff**	**LR**	**DF**	**Cutoff**	**LR**	**DF**	**Cutoff**	**LR**
$A_0\,(1)$	1	3.84	13.76	1	3.84	13.76	0	–	–
$A_1\,(1)$	0	–	–	1	3.84	6.55	1	3.84	6.55
$A_0\,(2)$	4	9.49	63.13	4	9.49	63.13	0	–	–
$A_1\,(2)$	2	5.99	84.69	3	7.82	85.35	1	3.84	0.66
$A_2\,(2)$	0	–	–	4	9.49	379.33	4	9.49	379.33
$A_0\,(3)$	9	16.92	85.79	9	16.92	85.79	0	–	–
$A_1\,(3)$	6	12.59	99.02	7	14.07	101.29	1	3.84	2.27
$A_2\,(3)$	3	7.82	199.64	7	14.07	377.85	4	9.49	178.21
$A_3\,(3)$	0	–	–	9	16.92	355.92	9	16.92	355.92

Table 10: Likelihood Ratio Statistics

This table shows likelihood ratio statistics for the different nested market price of risk specifications within each of the nine affine yield models considered. The first column lists the model under consideration. The next three columns contain information on the likelihood ratio of the completely affine yield market price of risk specification, relative to the essentially affine specification, which nests the completely affine specification. The following three columns contain analogous information for the completely affine specification relative to the extended affine specification, which nests both the other specifications. The last three columns compare the essentially affine specification to the nesting extended affine specification. For each comparison, the column labeled DF lists the additional degrees of freedom contained in the nesting model. The column labeled Cutoff contains the 95% chi-squared cutoff value for a likelihood ratio statistic with degrees of freedom corresponding to the number in the DF column. The column labeled LR contains the actual likelihood ratio statistic. The hypothesis that the restrictions included in the less flexible model are valid is rejected if the quantity in the LR column is greater than the quantity in the Cutoff column. Six of the 27 comparisons considered are degenerate, in that the restricted and nesting models coincide. In these six cases, the DF column contains the value 0, and the Cutoff and LR columns are not filled in.