On the Feasibility of Debt Ponzi Schemes - A Bond Portfolio Approach

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Abstract

This paper provides a general approach in the framework of a complete markets stochastic overlapping generations model to assess whether debt Ponzi schemes are feasible and Pareto-improving. We derive conditions in terms of bond interest rates of different maturities which can be used to assess different roll over strategies. A main result of this paper is that the feasibility of roll over strategies is unrelated to intergenerational insurance considerations that have received much attention in the literature. In fact, under the empirically relevant scenario of countercyclical real interest rates a Pareto improving roll over strategy of debt typically implies an increased variability of old age consumption. We develop a detailed intuition for our results along a series of examples.

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1 Introduction

Economists today believe that households are not overaccumulating capital.[1] So does this imply that successful, Pareto-improving Ponzi schemes are a theoretical possibility but not of practical relevance for the US economy or other economies around the world? The recent literature on public debt has discovered a potential insurance aspect of roll over schemes that is orthogonal to the issue of capital overaccumulation.[2] Even if markets are complete, households face risk, namely non-diversifiable aggregate risk like for example the risk of the business cycle. Issuing debt entitles households who hold this debt to safe claims on future endowments and thus insulates their consumption possibilities from aggregate fluctuations. But since aggregate risk is not diversifiable who will be willing to carry this risk?

An intriguing idea is to spread aggregate risk over time. A Ponzi scheme could be an instrument to achieve this kind of intergenerational risk sharing. The idea is to exploit the law of large numbers by spreading the risk of the cycle over many, in fact infinitely many generations [Gale (1990)]. The most straightforward application of this idea is a "simple Ponzi scheme" where safe one-period debt is issued, interest payments are financed by issuing even more debt, and this roll over strategy is pursued indefinitely [see Ball, Elmendorf and Mankiw (1998) for an empirical assessment of such a scheme].

But more sophisticated roll over strategies are possible by making use of bonds with longer maturities. Suppose e.g. that the government issues two-period bonds to a young (first) generation and transfers the receipts to the old generation (generation zero). When the first generation is old, the government buys these one-period bonds, of course at the prevalent market price, and finances this by issuing again two-period bonds. This state-contingency is beneficial for the first generation whenever the price of one period bonds is high in economic downturns and low otherwise, i.e., whenever bond prices are countercyclical or equivalently when interest rates are procyclical. In this case, issuing two-period bonds and the implicit state contingency of the resale price helps smoothing consumption in old age. It is important that in this scheme the young generation provides insurance against a risk for the old generation that is not diversifiable on markets. The young generation in turn obtains insurance during old age, i.e., a decreased variability of old age consumption, from the newly born young generation [see Manuelli (1990), Bertocchi (1991, 1994)]. Apparently, much more sophisticated schemes of debt seem to

[1] See Bohn (1995) and Elmendorf and Mankiw (1999), section 4.2.3, for a concise survey of the literature.

[2] see Blanchard and Weil (2001) for a survey of the recent literature.

be possible by making use of different maturities of bonds and time-varying compositions of the outstanding bond portfolio [Barbie, Hagedorn and Kaul (2001)].

This paper reconsiders the role of debt in stochastic OLG model of capital accumulation. The paper makes two main contributions. First, we reveal the economic structure necessary to understand when and why debt Ponzi schemes may be feasible and Pareto improving in economies that do not oversave in capital. The key point of this paper is that the feasibility of debt Ponzi schemes has nothing to do with intergenerational risk sharing. The payoff structure induced by a certain Ponzi scheme may or may not - depending on the cyclicality of interest rates - decrease the variability of consumption of some age cohort. But this is of no significance for assessing whether such a scheme is feasible. In fact, under the empirically plausible scenario of countercyclical real interest rates [Fama (1990), King and Watson (1996), Stock and Watson (1999)] a feasible and Pareto improving Ponzi scheme will even increases the difference between old age consumption in a boom and a recession.

Second, we derive a single criterion that allows to check the feasibility of any candidate debt Ponzi scheme. The information that is needed are the interest rates of bonds of different maturities issued over time as well as the composition of this bond portfolio outstanding at every point of time. It is necessary but not sufficient to face low average interest rates (relative to the growth rate of the economy) in order to perpetually roll over bonds of a certain maturity. The necessary condition already provides feasibility on a representative path (and most paths are in fact representative). But extreme paths may still exhibit explosive behavior of the debt-GDP ratio. Sufficient conditions for feasibility require information about the tails of the distribution of the relevant interest rate process. Loosely speaking, extreme paths of the product of interest rates over long horizons must not deviate too much from a representative path.

The main message of this paper is that the term structure of interest rates matters for assessing the feasibility of debt Ponzi schemes. If some kind of interest rate (or average interest rate of a bond portfolio consisting of bonds of different maturities) is low along all paths then a Ponzi scheme will be feasible. Essentially, a roll over strategy has to identify cheap ways of providing transfers to the old (no matter whether in booms or recessions). Whether a path is cheap has nothing to do with the occurrence of many recessions along this path but solely with the price of bonds on those paths. In general roll over is not cheaper if it insures against risky old age consumption. In particular, given the interest rates an optimal roll over scheme is not more likely to be feasible if interest rates are procyclical and thus provide intergenerational risk sharing.

We embed the results and intuition of previous contributions to the literature in our general framework. The reason why previous papers concluded that roll over of debt may be facilitated by providing better risk sharing is that they all considered the case of procyclical interest rates. In that case (and only in that case, as we will argue) the feasibility of rolling over safe debt is coincidentally related to the possibility of providing insurance for the old generation against economic downturns. By "coincidentally" we mean that the insurance arrangement induced by procyclical interest rates does not facilitate the roll over strategy as long as it does not lower the relevant interest rates, namely the interest rates in the debt portfolio. The mechanics behind the insurance argument was also used above in our suggestive example of rolling over two period bonds.

The paper is organized as follows. Section 2 introduces the model, a stochastic version of the Diamond (1965) OLG model. Section 3 presents some preliminary facts about improving transfer schemes in inefficient economies. In that section we essentially justify the use of our terminology "a feasible and Pareto improving Ponzi scheme exists". Section 4 presents two stylized examples which unveil the economic structure behind the question of the existence of Pareto improving roll over schemes. The first example is one with procyclical interest rates. The example replicates and sheds new light on previous results and interpretations in the literature. The second example with countercyclical interest rates shows that it is not intergenerational risk-sharing that facilitates Ponzi schemes. Section 5 presents a first general result about whether and which Ponzi schemes are feasible if shocks are i.i.d. The answer here is extremely simple: Ponzi schemes are feasible if and only if two-period bonds can be rolled over. Other schemes may be possible but they are dominated by the roll over of two-period bonds. The answer here is independent of the nature of the underlying endowment process or shocks and therefore independent of the cyclicality of interest rates. Section 6 then presents our main theoretical result, a complete characterization of when Ponzi schemes are feasible. Our condition to assess the feasibility of a given scheme only uses information about interest rates of different maturities and requires to specify weights that are attached to the different kinds of debt (in terms of their maturity) over time. In particular, any kind of candidate Ponzi scheme can be assessed by our criterion. In section 7 we extend our framework to allow for testing candidate schemes by using data from the term structure of interest rates and growth rates of the economy. We present empirical evidence that is consistent with the claim that the US economy is likely to be inefficient and thus indicate scope for "sophisticated Ponzi schemes". We also discuss the contribution of Abel, Mankiw, Summers and Zeckhauser (1989) and how it can be interpreted with our framework. Section 8 concludes.

2 The Model

We consider a straightforward extension of the Diamond (1965) OLG model to aggregate uncertainty.

2.1 Time and Uncertainty

Time is discrete and extends from $t = 0, 1, ...,$ to infinity. Aggregate uncertainty is represented by a date-event tree. The root of the tree is given by some fixed event z_0 at $t = 0$. A node of the tree at time t can be identified with a history of shocks until t, $z^t = (z_0, z_1, z_2, ..., z_t)$, where the z_s's are the shocks at each point of time $s = 1, 2, ...$ We assume that the shocks are drawn from some finite set Z. Let $q(z^t)$ denote the probability that node z^t is reached and $q\left(z_t \middle| z^{t-1}\right)$ the conditional probability of shock z_t given that history z^{t-1} has occurred. Furthermore, if no confusion arises, we simply write z for a history z^t. We also denote by z' an arbitrary successor node of z, i.e. we write z' for (z_{t+1}, z) for some $z_{t+1} \in Z$.

2.2 Households

The economy is populated by overlapping generations of agents. At each node in the tree one agent is born who lives for two periods.[3] Hence agents are distinguished according to date and state of nature in which they are born. Therefore, agents can be identified with the node at which they are born, so that in the rest of the paper the agent born in node z will be called agent z or generation z.

Since time starts at $t = 0$ we have one initially old agent (born in period -1 so to speak). This agent has preferences which are strictly monotone in the single consumption good in period 0. His consumption in period 0 is denoted by $c^o(z_0)$.

Consider now all other agents. Their von Neumann-Morgenstern preferences are described by a function $E_z\, u(c(z)) = \sum_{z_{t+1}} q_{t+1}\left(z_{t+1} \middle| z\right) \cdot u\left(c^y(z), c^o((z, z_{t+1}))\right)$ for an agent born at z at time t. Here we denote by $c(z) = \left(c^y(z), (c^o(z'))_{z'}\right)$ the consumption vector of agent z, $c^y(z)$ is his consumption in his youth, $(c^o(z'))_{z' \in Z^{t+1}}$ is his consumption in the different states of nature in his old age. The function $u : \mathbb{R}^2_+ \to \mathbb{R}$ is smooth, strictly increasing and strictly concave.

[3] Thus, there is no population growth, $L_t = L = 1$ for all t. This assumption is only made for the simplicity of exposition. It can easily, at the cost of some additional notation, be dispensed with. We will consider a growing economy in section 7.4.

In his youth, agent z receives a wage income $w(z)$ from inelastically supplying one unit of labor. This wage income is allocated to savings and consumption. In old age, an agent receives interest payments from his savings. Given wage income $w(z)$ and returns $r(z')$ to savings $s(z)$ the household born in z (at time t) solves:

$$\max_{\left(c^y(z), s(z), (c^o(z'))_{z' \in Z^{t+1}}\right)} E_z \; u(c(z)) \tag{1}$$

$$\begin{aligned} s.t. \quad c^y(z) + s(z) &= w(z) \\ c^o(z') &= r(z') \cdot s(z). \end{aligned}$$

Since there is only one individual born per node, an agent in any node z faces complete markets. If there were several (heterogeneous) agents born per node, we would simply introduce complete Arrow-Debreu markets explicitly. All our results would still hold. Note that market completeness implies that the only risk an old agent might face is non-diversifiable (aggregate) risk.

2.3 Production

There is a representative firm which uses labor L and capital K to produce a single consumption-capital good according to a constant returns to scale technology. The production function $F(K_t, L_t, z_t)$ includes the current shock. $F(.)$ has the usual properties. For simplicity, the depreciation rate is assumed to be 1. More specifically, the production function satisfies:

- $F(K_t, L_t, z_t)$ is homogeneous of degree 1 in K_t, L_t, strictly increasing, strictly concave and twice continuously differentiable in K_t, L_t. Further, $F(0, L_t, z_t) = F(K_t, 0, z_t) = 0$. It also satisfies the Inada conditions $\lim_{K_t \to 0} F_K(K_t, L_t, z_t) = \infty$ and $\lim_{K_t \to \infty} F_K(K_t, L_t, z_t) = 0$. As usual, define $f(k_t, z_t) = F\left(\frac{K_t}{L_t}, 1, z_t\right)$, the per capita production function with $k = K/L$.

The firm's problem is to decide after the shock realization at each node z how much capital to invest and how much labor to employ. There is uncertainty about future prices of the output good so that the firm maximizes expected profits. The prices for the firm are given by the stochastic discount factor of the representative agent:

$$m(z, z') = \frac{u_2(c^y(z), c^o(z'))}{\sum_{z_{t+1}} q_{t+1}(z_{t+1}| z) \cdot u_1(c^y(z), c^o(z'))} \tag{2}$$

Maximization with these discount factors over $k(z)$ yields immediately to the standard asset pricing Euler equation:.

$$E_z[m(z, z') \cdot f'(k(z), z_{t+1})] = 1 \tag{3}$$

2.4 Feasibility, Market Structure, Equilibrium and Welfare Criterion

Next, we define feasible allocations in this economy, the notion of interim Pareto optimality and a competitive equilibrium. For notational convenience allocations will sometimes be simply denoted by (c, k) in the rest of the paper.

Definition 1 *A feasible allocation (given initial capital k_{-1}) is a tuple (c, k) such that*

1. $c^o(z_0) + c^y(z_0) + k(z_0) = f(k_{-1}, z_0)$,
2. *For any* $z : c^o(z') + c^y(z') + k(z') = f(k(z), z_{t+1}) \qquad \forall z' = (z, z_{t+1})$.

The concept of Pareto optimality adopted in our paper is now introduced [Muench (1977), Peled (1982)].[4] This definition of optimality is also used in AMSZ and Kubler and Kruger (2002).

Definition 2 *A feasible allocation (c, k) is called interim Pareto optimal if there exists no other feasible allocation $\left(\widehat{c}, \widehat{k}\right)$ such that $\widehat{c}^o(z_0) \geq c^o(z_0)$ and $E_z u(\widehat{c}(z)) \geq E_z u(c(z))$ for all z, with at least one strict inequality.*

Note that this definition considers agents born in different *states* as distinct agents. Then the usual concept of Pareto optimality is applied to this set of agents. Together with the fact that markets are complete once an individual is born we may conclude that *complete markets are an important maintained feature of our setup throughout the paper.*[5]

[4]The term interim Pareto optimal is not used consistently in the literature. We follow the terminology of Demange and Laroque (2000).

[5]Markets are called *sequentially complete* if they are complete once an individual is born [see for example Dutta and Polemarchakis (1990)]. Since we consider agents born in different states as distinct agents sequentially complete markets obviously imply complete markets. Markets are complete if equilibria in the sense of definition 3 below can be replicated with a full set of contingent claims markets. Kubler and Kruger (2002) analyze the welfare implications for the case where markets are incomplete.

This feature of our model distinguishes our analysis from Gordon and Varian (1988), Blanchard and Weil (1992, 2001) and Bohn (1998b) who adopt ex ante Pareto-optimality as welfare criterion. The latter welfare concept considers agents born at one *date* (but in different states) as *one* representative agent (who has ex ante, i.e. being unborn, well-defined preferences over states of nature at the date at which he will be born). Since an unborn agent obviously cannot buy insurance against the state in which he will be born, markets are naturally incomplete in a stochastic OLG model when ex ante Pareto optimality is used as a welfare concept. This opens a source of welfare improvements for a *government* by providing insurance on behalf of an individual against the state in which the individual is born. We do not consider this kind of improvements by the use of interim Pareto optimality as a welfare criterion.

Now, we introduce the concept of a competitive equilibrium for the economy. We denote by r the vector of returns and by w the vector of household incomes in the different states of the world and by z_{-1} the predecessor node of a node z.

Definition 3 *A competitive equilibrium is given by*

1. *a feasible allocation* (c^*, k^*).

2. *by wages* w^* *and returns* r^* *and savings* s^*, *such that*(c^*, s^*) *solves (1) for every household,*

3. *Firms maximize profits, i.e., they satisfy the Euler equation* $E_z\left[m^*\left(z, z'\right) \cdot f'\left(k^*\left(z, z_{t+1}\right)\right)\right] = 1$ *for all* $z \in Z^t$ *and* $z' = (z, z_{t+1})$ $(t \geq 0)$ *given the stochastic discount factors* $m^*\left(z, z'\right)$.

4. $k^*\left(z\right) = s^*\left(z\right) \forall z \in Z^t$, $r^*\left(z'\right) = f'(k^*\left(z\right), z_{t+1})$, $w^*\left(z\right) = f\left(k^*\left(z_{-1}\right), z_t\right) - f'\left(k^*\left(z_{-1}\right), z_t\right) \cdot k^*\left(z_{-1}\right)$ *for all* $z \in Z^t$, $t \geq 0$ *and* $m^*\left(z, z'\right)$ *given by* (2) *evaluated at* c^*.

All conditions in this definition are standard: feasibility, utility maximization, profit maximization, market clearing and marginal productivity factor prices.

3 Preliminaries

In this section we provide a theoretical basis for the link between interim Pareto suboptimality of competitive equilibria and government debt policy. We start with a result that is a second welfare theorem for the OLG economies we consider here.

Proposition 4 *Any interior interim Pareto optimal allocation can be supported as a competitive equilibrium by issuing government debt.*

To prove this result, one first establishes that any optimal allocation can be supported as a competitive equilibrium with lump sum transfers. In a second step one shows that these transfers can be replicated with an appropriate government debt policy [see Barbie, Hagedorn and Kaul (2001) for details].

Now, we want to apply this second welfare theorem in order to answer the question whether Pareto-*improving* debt policies exist. Therefore, we need the following proposition which shows that for each interim Pareto suboptimal allocation a Pareto-superior *and* interim Pareto optimal allocation exists. If this allocation is interior then, by the second welfare theorem, it can be decentralized with an appropriate debt policy.

Proposition 5 *Let* (c, k) *be a given interim Pareto suboptimal allocation. Then there exists another feasible allocation* $\left(\widetilde{c}, \widetilde{k}\right)$ *such that*

- $\left(\widetilde{c}, \widetilde{k}\right)$ *interim Pareto-dominates* (c, k) *and,*
- $\left(\widetilde{c}, \widetilde{k}\right)$ *is interim Pareto optimal.*

The proof of this result is also contained in Barbie, Hagedorn and Kaul (2001) who generalize a well-known result for finite economies [see e.g. Aliprantis and Border (1994)]. The results presented here indicate the full scope for Pareto-improvements through a dynamic debt policy under uncertainty. If interim Pareto optimality is violated, a well-designed debt policy can improve the allocation of risk relative to the pure market outcome (or to some initially given government debt structure).

Both results together justify the terminology that we will use in the rest of the paper: whenever a competitive equilibrium is interim suboptimal, we will simply say that an improving debt policy or scheme exists.

4 Examples

In this section we present two examples that highlight the important mechanisms when government debt of possibly different maturities is rolled over. In contrast to the setup introduced so far we neglect the issue of capital accumulation in the examples by restricting attention to pure exchange economies. We will justify this assumption in section 7.4 where we will show that the overaccumulation of capital is empirically not a relevant issue. This does not imply, however, that there is no role for a government or that debt Ponzi schemes are infeasible, as was first pointed out by Blanchard and Weil (1992) and Bertocchi (1991, 1994). So the examples should be thought of describing economies that do not overaccumulate capital. This will be true if the risky rate of return (the net of depreciation marginal productivity of capital) will be on average high enough (roughly speaking higher than the average growth rate of the economy, see section 7.4).

The first example is in the spirit of Blanchard and Weil (1992, 2001) and Manuelli (1990).[6] The key feature of the example is that bond interest rates are procyclical. In this case (and only in this case, as we will argue) the feasibility of rolling over safe debt is coincidently related to the possibility of providing insurance for the old generation against economic downturns. To be more precise safe debt is used to smooth the state-dependent old-age consumption across states by sharing their endowment when old with the endowment of the young of the successive age cohort.
The second example is the mirror image of the first example. It considers the empirically relevant case of countercyclical interest rates. This example highlights that the feasibility of debt Ponzi schemes has really nothing to do with missing insurance opportunities on private markets as was conjectured in previous contributions to the literature. In fact, Ponzi schemes are feasible in this example if and only if they increase the variability of old age consumption.

Both examples have one feature in common: shocks are persistent. The assumption is not crucial for our results in the examples. But it allows to develop a simple intuition related to a standard deterministic OLG model. There, two types of equilibria are possible; one inefficient (no trade) equilibrium where the economy is characterized by high endowments of the young and low endowment of the old ("Allais-Samuelson equilibrium"); and an efficient equilibrium which is more likely if endowments are relatively equal across the life cycle ("Ramsey equilibirum"). The latter has essentially the same properties as equilibria in a standard infinite horizon representative consumer economy. Our stochastic economy will fluctuate between these two equilibria.

The insights from the examples will be further developed in the next sections when we analyze a more general setup. Details of the derivations in the examples can be found in an appendix.

[6]see example 4 in Blanchard and Weil (1992, 2002) and example 2 in Manuelli (1990)

4.1 First Example: Procyclical Bond Interest Rates and Persistent Shocks

4.1.1 The Basic Setup

We consider a stationary pure exchange economy with two states of the world, a boom b and a recession r. Utility is time separable and logarithmic with a discount factor $0 < \beta < 1$. That means for an individual born in a recession r

$$u(c^y(r), c^o(r), c^o(b)) = \ln c^y(r) + \beta \cdot (q(r,r) \ln c^o(r) + q(r,b) \ln c^o(b))$$

where $q(r,b)$ is the transition probability of going from a recession in period t to a boom in period $t+1$. Thus, $q(r,r) + q(r,b) = 1$. Analogous expressions hold for an individual born in a boom.

The stationary aggregate endowment structure is as follows:

$$e(b) = 2 \text{ and } e(r) = 1$$

with $e^y(b) = 1$, $e^o(b) = 1$ as endowment in a boom of young and old agents respectively and with $e^y(r) = \frac{2}{3}$, $e^o(r) = \frac{1}{3}$ as endowment in a recession of young and old agents respectively. The crucial assumption here is that the old are hit stronger by a recession than the young. This will turn out to be an important prerequisite for obtaining procyclical interest rates as well as a intergenerational insurance arrangement that smoothes old age consumption. The example could have also been constructed in a way that the young have a fixed, non-stochastic endowment, no matter whether boom or recession. This is the assumption made in several examples given in the previous literature [see example 4 in Blanchard and Weil (1992, 2001) and example 2 in Manuelli (1990) as well as Gale (1990)].

It will be helpful for our interpretations to introduce the price for state-contingent consumption in a state z' in terms of one unit of consumption in the predecessor node z, the relative Arrow-Debreu price. In general it can be defined as:

$$p(z,z') = q(z,z')\beta \frac{\frac{\partial u}{\partial c^o(b)}}{\frac{\partial u}{\partial c^y(r)}} = q(z,z')m(z,z')$$

where z is either the boom state b or the recession state r and z' is the successor of z and can also be either b or r and $m(z,z')$ is the stochastic discount factor. The inverse of $m(z,z')$ is the marginal rate of substitution between consumption in state z and consumption one period later in state z'. This marginal rate will play an important role when assessing the feasibility of all kinds of rollover schemes since it answers the question how much compensation a household requires in a particular state during old age for a unit of consumption he has to give up in his young age.

Due to logarithmic utility the state-contingent prices of consumption, i.e. the relative Arrow-Debreu price, in the pure endowment case are particulary simple to derive:

$$p(z, z') = q(z, z')\beta \frac{e^y(z)}{e^o(z')}, \tag{4}$$

The stochastic discount factor is $m(z, z') = \beta \frac{e^y(z)}{e^o(z')}$. Now we make some assumptions about the transition probabilities between boom and recession:

Assumption (Persistence of Shocks): $q := q(b, b) = q(r, r) \geqslant 0.5$ and large enough.

We will formalize "large enough" in the course of the example. A prerequisite for the analysis in this example is the assumption that our economy is inefficient so that there is indeed scope for an improving debt issue. A sufficient condition for this is that:

$$q > \frac{1}{2\beta}$$

Intuitively, a low value of the discount factor β implies efficiency because then due to strong discounting the economy will essentially be like a infinite horizon representative consumer economy [see for example Santos and Woodford (1997)].

Clearly, since $q < 1$, in this example a necessary condition for inefficiency is that $\beta > 1/2$. We will assume that $\beta = 0.8$ which corresponds to $q > 0.625$ as a sufficient condition for inefficiency, i.e. strong persistence of shocks. Intuitively, our assumptions about the initial endowments imply that in a recession the young have a low marginal rate of substitution between consumption today and postponing consumption to their old age in the recession state. This is so because the recession hits them strong and thus they would like to shift consumption to the recession state during old age. If a shock is very persistent, then a path with many recessions (and thus long histories of low marginal rates of substitution) is very likely and thus on such a path the households will demand low compensation in order to postpone consumption. This facilitates constructing an improving transfer scheme in the Allais-Samuelson style (redistribute from the young who have a lot to the old who have very little endowment) and thus obtaining an inefficient economy.

4.1.2 Procyclical Interest Rates

Now let us derive the one period interest rates and verify conditions to obtain procyclical interest rates. We have

$$R_1(b) = \frac{1}{\beta\,(3-2q)}$$

and

$$R_1(r) = \frac{1}{\beta\,(2/3+4/3q)}$$

It can be shown:

$$R_1(b) > R_1(r) \Leftrightarrow q > 0.7.$$

This condition is stronger than the sufficient condition for inefficiency we already imposed. Thus in this example we assume that interest rates are procyclical by requiring q to be "sufficiently large", i.e., larger than 0.7. Let us briefly mention the role of the endowments. Recall that we assumed that the recession hits the old more than the young. This assumption actually facilitates obtaining procyclical interest rates. If we had perfect persistence ($q = 1$), we would have two economies, one in which consumers face a Allais-Samuelson type endowment structure (the "recession economy"), and another which would represent a stationary equilibrium in a Ramsey type infinite horizon economy (the "boom economy"). In the Allais-Samuelson economy, interest rate is below 1 (thus rolling over one period debt is possible as in the Allais-Samuelson equilibrium of a deterministic OLG model), in the Ramsey economy the interest rate is of course above 1 [as in the standard infinite horizon representative consumer economy]. If we decrease persistence slightly, the interest rate in the Allais-Samuelson type economy (i.e. in recessions) remains below 1 and in the Ramsey economy (i.e. in booms) above 1. Thus, we have procyclical interest rates for q large enough.It is important to note that whether we have a Allais-Samuelson type economy (with high endowment in young age and small endowment in old age) or Ramsey type economy (with equal endowment across life cycle) is not related to whether we start in a boom or a recession. If we have high persistence, all what matters is the endowment structure between young and old age in the same states, independent of whether the Allais-Samuelson or Ramsey endowment structure occurs in a recession or a boom.

Indeed, example 2 below will have an endowment pattern with high young age and low old age consumption in a boom and equal young and old age consumption during recessions. In that example a feasible roll over strategy will increase the variability of old age consumption. This indicates that roll over possibilities have nothing to do with intergenerational insurance considerations.

The high persistence assumptions may seem extreme. But the goal of the examples is to highlight important mechanism at the cost of realism. In section 7.4 we will be more careful about the numerical values of our parameters.

4.1.3 Rolling over one period debt

We first show that rolling over safe one period debt does not work in this economy if the persistence of shocks is sufficiently strong. Roll over of one period debt does not work if $R_1(b) > 1$. To see this suppose we roll over one period debt and consider paths that contain the state boom sufficiently often. Here, the product of one period interest rates diverges. Thus, the amount of real transfer that has to be paid along such a path explodes. Given that the economy is bounded, this shows the infeasibility of such a transfer scheme. We have $R_1(b) > 1$ if and only if $q > \left(3 - \frac{1}{\beta}\right)/2$. For the case of $\beta = 0.8$ we thus obtain $q > 0.875$ as a condition that rules one-period debt Ponzi schemes. For $0.7 < q < 0.875$ it will be feasible to roll over one-period debt. To see this recall that in this case, due to procyclicality, $R_1(r) < R_1(b) < 1$ and thus rolling over is possible independent of the history of shocks.

High persistence drives up the interest rate during booms. This is so because the higher the persistence, the economy starting in a boom has a higher probability of staying in a boom and thus resembles more and more the Ramsey economy in which the interest rate is above 1, namely $1/\beta$.

4.1.4 Rolling over two period bonds

Our next step is to show under what conditions issuing two period bonds every period which is sold after one period for the then prevailing state contingent price of one period bonds provides improving transfers. The state contingent return of a of two period bond issued in a boom and sold one period later in boom is given by

$$\frac{R_2(b)}{R_1(b)} = \frac{(3 - 2q)}{q\beta(3 - 2q + 2\beta + 2q\beta - 4q\beta^2)}$$

Suppose $\frac{R_2(b)}{R_1(b)} > 1$ and consider a path along which there are sufficiently many booms. Then a debt scheme that rolls over two-period debt will explode and will thus not be feasible. This condition holds for q that satisfy:

$$6\beta q^2 - (3\beta + 2)\, q + 3\beta - 1 > 0$$

The solution is :

$$q > \frac{1}{12\beta}\left(2 + 5\beta + \sqrt{(4 - 52\beta + 73\beta^2)}\right)$$

For $\beta = 0.8$ the relevant range for the transition probabilities is $q > 0.9396$. We can see that for $0.875 < q < 0.9396$ rolling over one period debt is not feasible but rolling over two-period debt is feasible. For this it suffices to check $\frac{R_2(r)}{R_1(r)} < 1$. Since then both $\frac{R_2(b)}{R_1(b)} < 1$ and $\frac{R_2(r)}{R_1(r)} < 1$, so that on paths with both booms and recessions, the "cross terms" $\frac{R_2(r)}{R_1(b)} \cdot ... \cdot \frac{R_2(b)}{R_1(r)} = \frac{R_2(b)}{R_1(b)} \cdot ... \cdot \frac{R_2(r)}{R_1(r)} < 1$ are also smaller than one.

We have

$$\frac{R_2(r)}{R_1(r)} = \frac{1 + 2q}{3\beta\left(-q + 2q^2 + 1\right)}$$

If $\beta = 0.8$, the condition $\frac{R_2(r)}{R_1(r)} < 1$ is satisfied for all q.

What is the intuition why a sufficiently high q makes a roll over schme using two period bonds impossible? The reason for this is that $R_2(b)/R_1(b)$ converges to steady state one period interest rate of the Ramsey economy, since for q close enough to one the difference between holding a two-period bond and holding two one period bonds consecutively vanishes.

This constellation generates an improving transfer scheme as in example 4 of Blanchard and Weil (1992, 2001). Interestingly and related to the literature in stochastic OLG models, such a debt scheme provides insurance against recessions, as we will show below. This means that the payoff structure is such that payments during recessions are higher than during booms, thus smoothing old age consumption. This insurance is provided by sharing endowment risk with future age cohorts [see also Manuelli (1990), p.278].

This possibility has lead to the widespread belief that under dynamic efficiency the role for a government is to provide better intergenerational risk sharing. We will show in the next example that the risk sharing issue coincides with the issue feasibility of rolling over debt only if real interest rates are procyclical, but that in general these are two distinct issues. This is an important observation since empirical evidence tends to be in favor of countercyclical interest rates.

Note that another roll over strategy using two-period bonds is to issue them and have them outstanding until maturity (two periods later). Thus, new two period bonds are issued every second period. The relevant interest rate for the feasibility of such a strategy is $R_2(b)$. This strategy is dominated here since $R_2(b)/R_1(b) > 1$ and $R_1(b) > 1$ directly imply $R_2(b) > 1$. Thus, if neither rolling over one period bonds nor rolling over two period bonds period-wise works then rolling over two period bonds every second period does not work either. In this sense, this criterion has no value added if the others have been checked. So we will not further discuss this strategy.

4.1.5 The optimal policy: Issuing and rolling over two period bonds and holding one period bonds

We will now show that using short sales of one period bonds a government can always achieve an improvement if the economy is inefficient. Intuitively, rolling over two period bonds period-wise is a strategy that goes in the "right direction". However, as explained above, if q becomes close to 1, the economy in the boom state converges to a stationary equilibrium of a Ramsey economy, in which roll over is impossible. Thus, it makes sense to increase transfers in a recession and decrease transfers in a boom. To achieve this, additionally holding one period bonds is a good idea for a government.

To see the reason for this, suppose we start in a boom and issue a two period bond. Given that one period interest rates are procylical, i.e. $R_1(b) > R_1(r)$, the return in a recession of this two period bond (which is $\frac{R_2(b)}{R_1(r)}$), is higher than the return of the same bond in a boom, $\frac{R_2(b)}{R_1(b)}$. If the government additionally holds (or in other words sells short) one period bond (which costs tomorrow $R_1(b)$ per unit of short sales), the relative return of our mixed portfolio between boom and recession becomes:

$$\frac{\frac{R_2(b)}{R_1(b)} - R_1(b)}{\frac{R_2(b)}{R_1(r)} - R_1(b)}.$$

Given that $\frac{R_2(b)}{R_1(b)} < \frac{R_2(b)}{R_1(r)}$, the relative return boom to recession with short sales is smaller than without short sales, which is $\frac{R_2(b)}{R_1(b)} / \frac{R_2(b)}{R_1(r)} = \frac{R_1(r)}{R_1(b)}$. This shows that short sales shifts the relative returns toward recessions, in which it is easier to roll over since the recession economy is the Samuelson economy. An analogous argument can be made for a bond issued in a recession. Also in that case short sales of one-period bonds raise the return of the mixed portfolio in recessions relative to the return in a boom.

A stationary improvement is a transfer $v(b)$ in booms from young to old and transfer $v(r)$ in recessions paid from young to old. For a stationary improvement $(v(b), v(r))$ to work, the following conditions must be satisfied:

$$p(r,b) \cdot v(b) + p(r,r) \cdot v(r) - v(r) > 0 \tag{5}$$

$$p(b,b) \cdot v(b) + p(b,r) \cdot v(r) - v(b) > 0$$

The conditions can be interpreted as follows. The first equation states that for a consumer born in state r, who has to pay $v(r)$ during young age, the present value of transfers (discounted at the state contingent prices) must exceed his payments during young age. An analogous interpretation holds for the second equation. The equations must hold for an improvement since transfers are supposed to be strictly improving. Thus, the allocation after transfers cannot be in the budget set over which the consumer optimizes in the competitive equilibrium without transfers. Basically, the condition requires that consumption after receiving the transfers must be outside the competitive equilibrium budget set. Otherwise the allocation could have been chosen before and by revealed preferences cannot be improving. See the appendix for a rigorous argument.

If we define $y := \frac{v(b)}{v(r)}$ as the relative transfer the optimal scheme is described by (see the Appendix):

$$y = \frac{1}{\frac{4}{3}(1-q)} \cdot \left(-q + \sqrt{q^2 + 8(1-q)^2}\right)$$

This is a decreasing function of q. In our example we can calculate y to be smaller than 1 over the whole range of $q > 0.7$. Thus transfers in recessions are higher than transfers in booms, so that we have a kind of insurance scheme implemented by issuing debt.

If we want to implement the efficient transfer scheme by issuing one and two period debt, we have to find a portfolio of one and two period that yield one unit of consumption in a recession tomorrow and y units of consumption in a boom tomorrow. Thus, if we take the stationary transfer scheme $(v(b), v(r)) = (y, 1)$, the quantity of one period bonds μ_1 and two period bonds μ_2 has to solve if we start in a recession:

$$\begin{pmatrix} R_1(r) & \frac{R_2(r)}{R_1(r)} \\ R_1(r) & \frac{R_2(r)}{R_1(b)} \end{pmatrix} \cdot \begin{pmatrix} \mu_1(r) \\ \mu_2(r) \end{pmatrix} = \begin{pmatrix} 1 \\ y \end{pmatrix} \tag{6}$$

A similar equation holds if we start in a boom. The solution is:

$$\mu_1(r) = -\frac{-\frac{R_2(r)}{R_1(b)} + \frac{R_2(r)}{R_1(r)} \cdot y}{R_1(r) \cdot \left(\frac{R_2(r)}{R_1(b)} - \frac{R_2(r)}{R_1(r)}\right)} \quad \text{and} \quad \mu_2(r) = \frac{y-1}{\frac{R_2(r)}{R_1(b)} - \frac{R_2(r)}{R_1(r)}}.$$

Note that $\mu_1(r) < 0$ and $\mu_2(r) > 0$ iff $y < R_1(r)/R_1(b) < 1$, i.e., if interest rates are sufficiently procyclical we will have short sales in one period bonds and a long position in two period bonds In this example with $\beta = 0.8$ we have $y < R_1(r)/R_1(b) = \frac{9-6q}{2+4q}$ iff

$q > 0.7$. The same result holds for $\mu_1(b)$ and $\mu_2(b)$ if we start in a boom.

So for $q > 0.9396$ rolling over two period bonds alone will not improve. However a scheme that uses short sales of one period bonds additionally will do the job.

4.1.6 Summary

We can summarize the example as follows. First, we derived conditions under which two particularly simple roll over schemes (using one and two period bonds respectively) Pareto improve an inefficient market equilibrium. We then derive an optimal transfer scheme and show that this optimal scheme involves short sales of one period debt and issuing two period debt by the government. This transfer scheme provides insurance against old age consumption risk by inducing a payoff structure that decreases the variability of old age consumption. This example underlines and replicates the intuition stressed in much of the stochastic OLG about the role of a government in providing insurance against macroeconomic risk, or to be more precise, in providing a risk sharing arrangement over time that shares aggregate risk during old age with future cohorts [see Blanchard and Weil (2001), sections 5 and 6 for a concise summary of the literature].

The feasibility of the different rolling over schemes was related to the persistence of the shocks and the endowment structure in booms versus recessions. We stress that the role of a debt scheme to provide insurance (more transfers in recessions than in booms) is closely related to the interest rate being procyclical.

4.2 Second Example: Countercyclical Bond Interest Rates and Persistent Shocks

The Basic Setup This example is a mirror image of the previous one. The set up is the same as in the previous example, with the exception of transition probabilities and individual endowments. The assumptions we make in this section ensure that interest rates will be countercyclical.

Take $e^y(b) = \frac{5}{4}$, $e^o(b) = \frac{3}{4}$ as endowment of young resp. old agents in a boom and with $e^y(r) = \frac{1}{2}$, $e^o(r) = \frac{1}{2}$ as endowment of young resp. old agents in a recession. Again we assume sufficient persistence of shocks. The key assumption in this example is the fact that both old and young have the same endowment during recessions. Thus, in this example the recession economy is the Ramsey economy whereas the boom economy is the Allais-Samuelson economy.

Again we restrict attention to inefficient economies. A sufficient condition for this is that:

$$q > \frac{3}{5\beta}$$

For $\beta = 0.8$ this requires $q > 0.75$.

Countercyclical Interest Rates Furthermore, the interest rates are countercyclical here, i.e. $R_1(r) > R_1(b)$, for all choices of transition probabilities. Intuitively, since the recession economy is of the Ramsey type, the interest rate corresponding to the economy without uncertainty when going from a recession to a recession is larger than one. But when going from a recession to a boom, the interest is even larger. The one period interest rate faced by a consumer here is a convex combination of the two, and thus larger than one, too. Further, the interest when we start in a boom and stay there is smaller than the one in the recession economy. The interest rate when going from a boom to a recession is even smaller. Thus the one period interest rate in a boom is for all transition probabilities lower than the one period interest rate in a recession.

Roll over of one-period bonds With these parameters we have $R_1(r) > 1 \iff 2 + q < \frac{3}{\beta}$. This holds for all values of q since $\beta < 1$. Thus by the arguments given in the previous example, improving with rolling over one period bonds is impossible for all $\beta < 1$.

Roll over of two-period bonds Also rolling over debt in two period bonds, i.e. issuing two period bonds in every period and buying it back one period later at the prevailing market price of one period bonds, is impossible here since $\frac{R_2(r)}{R_1(r)} > 1$ if and only if $q > \frac{1}{16\beta}[3 + 14\beta - \sqrt{(9 + 276\beta - 284\beta^2)}]$. For $\beta = 0.8$ this reduces to $q > 0.568$. Since we assumed $q > 0.75$ rolling over two-period bonds will not work here.

The Optimal Policy Now we will again solve for the optimal stationary transfer scheme. The result is:

$$y = \frac{3}{2} \cdot \frac{1}{1-q} \cdot \left(\frac{2}{3}q + \sqrt{\frac{4}{9}q^2 + \frac{20}{3}(1-q)^2} \right)$$

We then can again solve the equation system (11) and obtain again:

$$\mu_1(r) = -\frac{-\frac{R_2(r)}{R_1(b)} + \frac{R_2(r)}{R_1(r)} \cdot y}{R_1(r) \cdot \left(\frac{R_2(r)}{R_1(b)} - \frac{R_2(r)}{R_1(r)} \right)} \quad \text{and} \quad \mu_2(r) = \frac{y-1}{\frac{R_2(r)}{R_1(b)} - \frac{R_2(r)}{R_1(r)}}.$$

However now the interest rates are countercyclical, $R_1(r) > R_1(b)$. Then we obtain for $q > 0.6$ that $y > 1$. This implies that $\mu_2(r) > 0$. Furthermore, if $y > R_1(r)/R_1(b)$ then we obtain $\mu_1(r) < 0$. This is true for all values of q (and β) (see the Appendix). Thus, the improving transfer is again generated by short sales of one period bonds and issuing two period bonds.

Summary Note that, although the portfolio has the same pattern as in the preceding example, the payoff structure generated by it is different. Because interest rates are countercyclical, going issuing two period bonds and holding one period bonds implies a high payoff in a boom and a low payoff in a recession. Thus the improving transfer scheme, in contrast to the previous example, does not provide insurance against variability in old age consumption. To the contrary, it even increases the difference between old age consumption in a boom and a recession that results from a equilibrium without transfers.

This might seem counterintuitive at first glance. Would it not be better to smooth old age consumption by providing this insurance for consumers reduce the transfer payments? The answer is no. Smoothing old age consumption would require to reduce transfers in a boom and increase the transfers in a recession. Because consumers were willing to pay a risk premium such a transfer would in fact be cheaper measured in the amount of young age consumption consumers were willing to give up to receive the transfers in old age. But what matters for the feasibility of a transfer scheme is not the size of the *average* transfer payment, but the amount of transfers paid in each state. A scheme which provides insurance has a higher payoff in a recession than the "destabilizing" scheme proposed above. But the stochastic discount factors are lower in a recession than in a boom. This implies that it is more difficult to roll over a given amount of debt if we start in a recession. Thus, if we insure the old generation (born at $t-1$) at time t in a recession by paying a high transfer, then the young generation born at time t (in a recession) has to carry this burden. So to make the young willing to give up a large amount of their income in a recession (in order to finance the high transfer to the old), they have to be promised a high compensation in old age at time $t+1$. If we continue in that way, the compensation necessary to Pareto improve becomes larger and larger. Thus under countercyclical interest rates rolling over debt in recessions, i.e., providing a smoother old age consumption by intergenerational risk sharing, is particularly expensive and typically not feasible. In that case, if rolling over debt is feasible at all, it has to take the counterintuitive form of a scheme that pays relatively high transfers in booms.

5 A first general result: The Case of i.i.d Shocks

In this section we consider a stationary pure exchange economy in which shocks have no persistence, that is shocks are i.i.d. over S states of the world. It turns out that the i.i.d. case allows for strong and general conclusions about the structure of an improving bond portfolio. These conclusions are independent of specific assumptions about preferences and endowments. The intuition underlying the results in this section can be generalized to a production economy where allocations far in the future are stochastic but independent of the state today.

There are $s = 1, ..., S$ states of the world which are drawn each with probability $q(s)$ independently over time. The preferences are additively separable over time and take the following form for an agent born in state s:

$$u\left(c^y(s)\right) + \sum_{s' \in S} q(s') \cdot w\left(c^o(s')\right)$$

where the functions u and w satisfy standard assumptions (monotonicity, concavity etc.). Let arbitrarily positive stationary endowments $(\omega^y(s), \omega^o(s))$, $s = 1, ..., S$ be given. We can compute the matrix of stochastic discount factors denoted by A as:

$$A = \begin{pmatrix} \frac{q(1)\cdot w'(c^o(1))}{u'(c^y(1))} & \frac{q(2)\cdot w'(c^o(2))}{u'(c^y(1))} & \cdots & \frac{q(S)\cdot w'(c^o(S))}{u'(c^y(1))} \\ \frac{q(1)\cdot w'(c^o(1))}{u'(c^y(2))} & \frac{q(2)\cdot w'(c^o(2))}{u'(c^y(2))} & \cdots & \frac{q(S)\cdot w'(c^o(S))}{u'(c^y(2))} \\ . & . & . & . \\ . & . & . & . \\ . & . & . & . \\ \frac{q(1)\cdot w'(c^o(1))}{u'(c^y(S))} & \frac{q(2)\cdot w'(c^o(2))}{u'(c^y(S))} & \cdots & \frac{q(S)\cdot w'(c^o(S))}{u'(c^y(S))} \end{pmatrix}$$

An important observation is that since all rows are a linear multiples of the first row, we have $rank(A) = 1$. In particular the rows are linearly dependent. Similar as in the previous examples a stationary improvement exists if and only if rolling over two period bonds is feasible[7]. The condition for a stationary improvement is

$$Av > v$$

where $v = (v(1), ..., v(S))$ is the transfer paid from the young to the old in each state.

[7] Restricting attention to stationary improving transfer schemes is not restrictive. It is shown in Demange and Laroque (1999) and Chattopadhyhay and Gottardi (1999) that if an improving transfer exists, it can always be chosen to be stationary.

The interpretation of this inequality is again that transfers paid during young age have to be lower than the discounted value of transfers received during old age, where the discounting is according to the stochastic discount factor matrix A.

Proposition 6 *In the case of i.i.d. shocks, any inefficient market allocation can be improved if and only if perpetual roll over of two period bonds is feasible. The improving return vector is given by* $v = \left(\frac{R_2(s)}{R_1(1)}, \frac{R_2(s)}{R_1(2)}, ..., \frac{R_2(s)}{R_1(S)}\right)$ *for a consumer born in state* s*.*

A proof is given in the appendix.

Why is it in the i.i.d. case sufficient to consider only two period bonds in order to to assess the efficiency of a competitive equilibrium? In particular, why can we omit the idea of short sales developed in the previous examples for the case of persistent shocks? The best roll over strategy is to increase transfers if the compensation per unit of foregone young age consumption is low. Put differently, what matters for roll over is a comparison of the utility gain during old age and the utility loss during young age. With i.i.d. shocks the (expected) utility gains in old age are independent of the state in which an agent was born. Therefore, to assess a roll over strategy, the costs of roll over in young age are the only relevant factor. It is thus intuitive to choose the transfers inversely proportional to the utility costs per unit of foregone young age consumption in order to equalize the total utility costs per individual across individuals born in different states (at one point of time). This means that the ratio of transfers in state s and s' equals $\frac{u'(c^y(s'))}{u'(c^y(s))}$, i.e. the ratio of the inverse of marginal costs $(1/u')$. A transfer pattern that has precisely this property is the one described in the proposition. To see this, consider the relative transfer implied by a roll over of two period bonds for two arbitrary states s and s' starting from s''. It will be:

$$\frac{R_2\left(s''\right)}{R_1(s)} / \frac{R_2\left(s''\right)}{R_1(s')} = \frac{R_1(s')}{R_1(s)} = \frac{u'\left(c^y\left(s'\right)\right)}{u'\left(c^y\left(s\right)\right)}$$

An important feature of this transfer scheme is that any other transfer scheme increases utility costs in same states and decreaes them in some other states. Why is this not a good idea? The reason for this is that compensation in old age has to be increased in states following high utility costs in young age. Of course the compensation in other states then falls. But a given roll over strategy has to work on all paths so neither "cheap" paths nor "average paths" are crucial to assess the feasibility but rather the "expensive" paths. Spreading costs unequally increases the roll over costs on the most expensive path in comparison to the strategy suggested in our proposition.

Summary This example has shown that in the case of i.i.d. shocks the critical strategy for assessing the feasibility of debt Ponzi schemes is the roll over of two-period bonds. The main insight of this example is the fact that changing the underlying endowments or transition probabilities (of course preserving the i.i.d. nature of shocks) may change the cyclicality of interest rates and thus the payoff structure of a roll over scheme in booms versus recessions. Again, the procyclical case implies smoothing of old age consumption and the countercyclical case implies increased variability of old age consumption. But in order to assess feasibility only the effective interest rate R_2/R_1 has to be checked. If it is low roll over is possible no matter what payoff structure results. This underlines our previous point: the feasibility of a roll over strategy has nothing to do with risk sharing!

6 A Portfolio Characterization of Suboptimality

In this section we will derive a general result about the feasibility of Ponzi games. We will provide a single necessary and sufficient condition for the feasibility of debt Ponzi games. The condition will be stated in terms of a weighted portfolio of bonds of different maturities and different issuing dates where the weights can be arbitrarily chosen in order to assess whether a certain roll over strategy can be successful. In this and the next section we will operationalize the criterion in order to check whether some simple schemes that have received attention in the literature are feasible.

One of the main insights of this section is the simplicity of the condition. Neither is it necessary to know anything about transition probabilities between different states nor does one need any knowledge about initial endowments. These details will of course influence the allocation and equilibrium prices and (cyclicality of) interest rates. But the necessary information is contained in the interest rates of different maturities alone; no further information is needed about the underlying economy which is not contained or does not affect the interest rates.

The main message of this section is that low interest rates are the key determinant of whether debt Ponzi schemes are feasible. If some kind of interest rate is low along all paths then a Ponzi scheme will be feasible. Essentially, a roll over strategy has to identify cheap ways of providing transfers to the old (no matter whether in booms or recessions). Whether a path is cheap has nothing to do with the occurrence of many recessions along this path but solely with the price of bonds on those paths. In general roll over is not cheaper if it insures against risky old age consumption.

We first give the definition of a dynamic portfolio. Let the maximal number of states be n. A dynamic portfolio is nothing else than a normalized portfolio of government bonds of different maturities which generate nonnegative payoffs in each state. That is, given a portfolio weight for k period bonds of $\mu_k(z^t)$, we require for each state z^{t+1} that aggregate payoff generated by selling all k period bonds issued at z^t in z^{t+1}, is nonnegative. This leads to the following definition.

Definition 7 *A dynamic portfolio is a function of date-events* $\mu(z^t) = [\mu_1(z^t), \ldots, \mu_n(z^t))]$ *such that:* $\sum_k \mu_k(z^t) = 1$ *and* $\mathbf{R}(z^{t+1}, z^t) \bullet \mu(z^t) \geq 0$, *where* $\mathbf{R}(z^{t+1}, z^t) = [R_1(z^t), R_2(z^t)/R_1(z^{t+1}), \ldots,$ *and* $\bullet$ *is the inner product.*

$\mu_k(z^t) > 0$ means that the government sells $\mu_k(z^t)$ units of $k-$period bonds at date event z^t to the young generation. The household then holds a portfolio where k-period bonds have a weight of $\mu_k(z^t)$. If $\mu_k(z^t) < 0$, then the government sells short k-period bonds in z^t. It turns out that this normalized portfolio suffices to derive a Cass (1972) type characterization for the existence of an welfare improving debt policy.

Theorem 8 *A generically necessary and sufficient condition for the existence of an welfare improving debt policy is that there exists a dynamic portfolio* $\mu = (\mu_1, \ldots, \mu_n)$ *and a finite positive number* M *such that for every path* $(z^0, z^1, \ldots)$ *in the date event tree*

$$\sum_{t=0}^{\infty} \prod_{s=1}^{t} \mu(z^{s-1}) \bullet \mathbf{R}(z^s, z^{s-1}) \leq M < \infty,$$

The theorem is a generalization of the arguments used in the examples to non-stationary environments. If the economy under consideration is non-stationary, it may become necessary to change the portfolio which generates the improving transfers over time. For example, it may be necessary to start with rolling over one period bonds, than 4 period bonds and then 5 period bonds. Then start again with 1 period bonds etc. In this case, we have $\mu(z^0) = (1, 0, \ldots, 0)$, $\mu(z^1) = (0, 0, 0, 1, 0, \ldots, 0)$ and $\mu(z^2) = (0, 0, 0, 0, 1, 0, \ldots, 0)$ and get $R_1(z^0) \cdot R_4(z^1)/R_3(z^2) \cdot R_5(z^2)/R_4(z^3) \ldots$. This is a cyclical example. Of course, different permutations result in different combinations of our instruments and in the extreme case the transfer can be completely non-stationary. As in the stationary case, short sales may be necessary in constructing improving portfolios.

The portfolios considered in the examples are of course also in the non-stationary environment possible. We now state a number of conditions that we checked in the previous examples as sufficient conditions for the existence of an improving debt policy.

Corollary 9 *1. If there exists an n so that the risk-free gross rate of return for n-period bonds $R_n(z^t)$ is smaller than some constant $c < 1$ at every date event z^t in a competitive equilibrium, an improving debt policy exists.*

2. If for every path $(z^0, z^1, z^2, ...)$ the expression $\sum_{t=1}^{\infty}\prod_{j=0}^{t} R_k\left(z^{jk}\right)$ is uniformly (over all paths) bounded for some $1 \leq k \leq n$, then an improving debt policy exists.

3. If there exists an k so that for every path $(z^0, z^1, z^2, ...)$ the expression $\sum_{t=1}^{\infty}\prod_{j=1}^{t}[(1+\alpha)\frac{R_2\left(z^{j-1}\right)}{R_1(z^j)} - \alpha R_1\left(z^{j-1}\right)]$ is uniformly (over all paths) bounded for some $\alpha > 0$, then an improving debt policy exists.

4. If for every path the expression $\sum_{t=1}^{\infty}\prod_{j=1}^{t}\frac{R_k\left(z^{j-1}\right)}{R_{k-1}(z^j)}$ is uniformly (over all paths) bounded for $1 \leq k \leq n$, then an improving debt policy exists.

The first criterion generalizes the strong sufficient result of Abel, Mankiw, Summers and Zeckhauser (abbreviated AMSZ) (1989) for inefficiency. For $n = 1$ it corresponds to the case in AMSZ (1989) where an asset having a return uniformly lower than 1 exists. Their result uses a specific asset, namely the return from investing in risky capital. Our result shows that what drives their result is the fact that the return to the asset is uniformly, i.e. for all realizations of shocks, lower than one (or in general lower than the growth rate of the economy). Analogously, sufficient criteria for efficiency can be obtained if the return to an asset is uniformly larger than the growth rate.

The second is just roll over of safe n-period debt $(\mu_1(z^t) = 1)$. The third criterion corresponds to a roll over strategy considered in our examples, namely issuing two period bonds and holding one period bonds $(\mu_2(z^t) = 1 + \alpha$ and $\mu_1(z^t) = -\alpha)$. The fourth criterion is for $k = 2$ the roll over of two period bonds considered in the examples. In general it considers the roll over of $k-$period bonds $(\mu_{k2}(z^t) = 1)$.

7 How to Test for Inefficiency

In this section we develop a procedure how a particular roll over scheme can be tested for feasibility and thus how a particular scheme could be used to assess Pareto improving policies. Our procedure starts from the conditions for improving schemes stated in the previous corollary. Note first, that what matters for the feasibility of roll-over schemes is the comparison of interest rates and growth rates. This intuition carries over from the standard Diamond model under certainty to the present model with uncertainty. Thus whenever we refer to an interest rate or return process we actually mean the "normalized interest rate",i.e., the interest rate divided by the growth rate of the economy. We make a regularity assumption about the stochastic processes of the log o f the real interest rates. We assume that they follow an ergodic process. The procedure we suggest consists of two steps. In the first step a simple necessary condition for feasibility is tested. If this condition is met then it ensures that almost all paths improve upon the original allocation. The schemes that pass the first test then have to be examined further in a second step. In that step the remaining potentially "pathological" (or extreme) paths have to be considered. In this section, we also clarify the relationship of the conditions for efficiency and the test in AMSZ and how it can be interpreted in our framework.

7.1 A Simple Necessary Condition for Inefficiency: Checking a Representative Path

The first step in order to assess a certain roll over scheme is to check the potentially explosive behavior of debt on a representative (or roughly "average") path under this candidate scheme. Note that the existence of such a representative path is guaranteed by the assumption that the underlying stochastic processes are ergodic. This allows us to apply a Law of Large Numbers to rule out schemes that cannot be sustained on most paths and are thus definetely not sustainable. This step can be viewed as a formalization and qualification of the ideas developed in Gale (1990) who argued that by spreading the risk of the cycle over many, in fact infinitely many generations debt can provide insurance. First, our argument shows that a LLN may indeed be applicable. However, it cannot be used to insure generations (see our examples) but only provides bounds on the explosive behavior of debt on a representative path. Second, there will in general be paths that are nor representative, thus no LLN can be applied on those paths, so that a priori not much can be said about the behavior of debt on those paths (more on this in step 2).

We use the folowing mathematical fact which follows from the Birkhoff Ergodic Theorem [see Billingsley (1995)]:

Remark 10 *If the stochastic process* $\log X_t$ *is ergodic then* $E[\log X_t] > 0$ *implies* $\sum_{t=1}^{\infty} \prod_{j=1}^{t} X_j$ *diverges a.s.*

If the stochastic process $\log X_t$ *is ergodic then* $E[\log X_t] < 0$ *implies* $\sum_{t=1}^{\infty} \prod_{j=1}^{t} X_j$ *converges a.s.*

The random vaiable X in the remark corresponds to a particular bond interest rate or ratio of bond interest rates depending on the roll over scheme under consideration.

For example X could be R_1or R_2/R_1 if we want to test the feasibility of rolling over one-period or two-period bonds respectively. The first step of our procedure is thus particularly simple: check the empirical mean of the log of the relevant interst rate series. If the mean is above zero then this scheme is not feasible. Roll over schemes that pass this test are now further examined. Interestingly, schemes that satisfy $E[\log X_t] < 0$ already provide us with a strong presumption for inefficiency in the following sense. If a Ponzi scheme fails in that case, i.e., if debt diverges to infinity on some paths, then this will only happen on a set of measure zero, i.e. with probability zero. Roughly speaking, explosive behavior with low interest rates is a singular event that will only happen on "isolated" paths. Howeveer, this leaves open the possibility that the debt-GDP ratio hits a prespecified upper bound (for example a debt-GDP ratio of 60% as in the Maastricht Treaty) with strictly positive probablity in finite time.[8] The reason for this is that debt along a path may converge eventually but will hit the given bound relatively early in time. Thus, focusing attention on representative paths is not innocuous from a policy point of view since it may not be enough to convince policy makers or the public to ignore an increasing and seemingly exploding debt-GDP ratio by pointing to a LLN that applies "in the long run". In the words of Ball, Elmendorf and Mankiw (1998) debt is a gamble. Our analysis shows that a debt scheme has to pass our first test. Otherwise debt is not gamble but explodes with probabilty one. But if interest rates are low relative to growth rates and the initial debt-GDP ratio is a choice variable of the government then the probability of an unsuccessful gamble can be made arbitrarily small if the debt-GDP ratio is chosen small.

7.2 Sufficient Conditions for Inefficiency: Checking the Behavior of Extreme Paths

So what do we have to test in order to turn a Ponzi Gamble [see Ball, Elmendorf and Mankiw (1998) for this terminology] into a feasible Ponzi scheme? Is a low enough mean sufficient? Is there a relationship of the following kind: If a scheme X and Y have $E[\log X] < E[\log Y] < 0$ then the X scheme is superior to the Y scheme in the sense that it is more likely to be feasible or less likley to explode or hit a given bound? The answer is no and the reason for this can be related to our examples.

As we saw in the last section, checking moments as the mean of the log interest rates

[8] Blanchard and Weil (1992, 2002) present an example where the expected interest rate is below one and still the debt GDP ratio explodes on some paths and reaches any given bound with positve probability.

is not enough to judge the feasibility of Ponzi schemes. The behavior of every single path is important. Thus we have also to check the behavior of "extreme" paths, on which interest rates will have more "high" realizations than one would expect according to their statistical mean. In particular, it can happen that for two schemes X and Y we have that $E[\log X] < E[\log Y] < 0$, but nevertheless scheme X will fail with strictly positive probability, while the transfer scheme Y is feasible on every path. This kind of behavior occurs also in our examples above. In the case of countercyclical real interest rates, the mean of R_2/R_1 is higher than the mean of R_1, but still roll over of one period bonds may be infeasible whereas two-period bonds can be rolled over. The intuitive reason for this is that in the case of countercyclical interest rates, as discussed in the examples, the transfer scheme induced by roll over of two period bonds disinsurances the consumer, while roll over of safe debt leads of course to equal payoffs across states. Thus in equilibrium, consumers have to be promised an on average higher payoff from holding two period bonds than from holding one period bonds. As shown in the examples, this does not mean that roll over of one period bonds is simpler. In fact, whenever roll over of one period bonds is possible, also roll over of two period bonds works, but the converse does not hold. This shows that the behavior of extreme paths is crucial for determining the feasibility of Ponzi schemes.

So what do we have to test then? We need information about the tails of the distribution of a stochastic process to determine whether Ponzi schemes are feasible and at which point in time with which probability a Ponzi scheme will fail. For example, when interest rates follow an i.i.d. process, the extreme path we have to consider is the path with the highest realization for the interest rate in the support. Of course, in such a case, the whole support of the interest rate has to be below one if we want to conclude that the economy is inefficient. Another example, which indicates inefficiency and is nevertheless compatible with a highest realization of interest rate above zero, is given by a strong case of mean reversion with two possible realization of interest rates, in which after each high realization of interest rates a low one follows with probability one and vice versa. If the product of high and low interest rate is smaller than one, a Ponzi scheme is feasible in that case. Generally, if there are cycles (of possibly different length) on every path such that the product of interest rates along these cycles are smaller than one then Ponzi schemes will be feasible. Essentially this ensures that extreme paths do not differ too much from representative paths if long enough time intervals are considered.

Overall, to judge the feasibility of Ponzi schemes and whether the economy is efficient, information about simple statiscal properties of the relevant interest rate time series are not sufficent. We need information about the extreme value distribution of the relevant stochastic process. In particular, (strong forms of) mean reversion are helpful for obtaining feasible and improving Ponzi schemes. The important insight is that for true feasibility no LLN can be readily applied. In particular any kind of test will heavily depend and be sensitive with respect to the tails of the disribution of the interest rate process.

7.3 How to interpret the AMSZ Condition and Test in our Framework

In an influential paper Abel, Mankiw, Summers and Zeckhauser (1989) [henceforth AMSZ], provide a strong sufficient condition for efficiency and presented evidence that this efficiency criterion is fullfilled for the US economy and most other OECD countries. The criterion of AMSZ states that if the net dividend is stritly positive at each point in time and each state of the world, the economy is interim Pareto efficient. The net dividend is defined as (net) income for the factor capital minus investment. In our model this reads as:

$$f'\left(k\left(z^{t-1}\right), z_t\right) \cdot k\left(z^{t-1}\right) > k\left(z^t\right) \qquad \text{for all } z^t.$$

If we identify f' with the rate of return of the market portfolio R^M, assume that capital is bounded away from zero and grows at rate $g\left(z^t\right)$, iterating this equation yields:

$$\prod_{s=1}^{t}\left(R^M\left(z^s\right)/\left(1+g\left(z^s\right)\right)\right) > 1 \qquad \text{for all } z^t$$

AMSZ show that along the actually observed path, this criterion is satisfied. But to check the validity of this criterion, one has to assess the behavior on *all* potential paths, not just on the randomly observed one. This requires to know the tail distribution of the growth adjusted rate of return of the market portfolio. As in our tests for efficiency with bond interest rates above, one needs in their test information about the behavior of "extreme" values, which determine the behavior on the most unfavorable path, i.e., in this case a path with low realizations of the return of the market portfolio. It seems to be plausible to assume that there may be histories where the average return is lower than the average growth rate of the economy. In that case the AMSZ test would not be conclusive. But if we assume again that the $R^M/(1+g)$ is ergodic we can again make some inference. Then the test presented by AMSZ translates to $E\left[\ln\left(R^M/\left(1+g\right)\right)\right] > 0$. This condition coincides with one presented by Zilcha (1991) which implies dynamic efficiency of the economy, i.e. that there is no overaccumulation of capital.

For checking whether the economy is additionally also interim Pareto optimal, more eleborate procedures involving distributions of the return process are necessary. We stress the parallell to our test procedure as far as informational requirements are concerned. In any case of assessing feasibility or inefficiency we need assumptions about the tails of the relevant distributions.

7.4 Empirical Evidence

In this section we will present some empirical evidence to assess roll over strategies. We will restrict attention to our first step above. Carrying out the second step is beyond the scope of this paper. We will use the interest rates R_1 on U.S government bonds with one year to maturity from 1871 to 1999 as one data source. This bond interest rate series is the only one that is available over such a long period of time. Our data source is the same as in Ball, Elmendorf and Mankiw (1998).[9] We extended their data set from 1993 until 1999 by using information from the National Income and Product Accounts (growth rates) and the yield on government bonds with one year to maturity as reported by the Fed.We will be brief about this since it is identical to the analysis in Ball, Elmendorf and Mankiw (1998). Details about the estimation procedure can be found there. The most important summary statistic for our purposes is that $E[R_1]$ is 0.989 and thus below one for our sample. The autoregressive coeffcient ranges from 0.21 to 0.47 depending on the inclusion of dummies for the war periods. These different specifications leave our main conclusion unaffected. A Dickey-Fuller test clearly rejects the null hypothesis of a unit root. Furthermore, a Breusch-Godfrey test shows no significant serial correlation of the residuals.

In order to apply the proposition in order to detect suboptimality, the only thing we have to show in our data is that the mean of the $\log R_1$ is below 0.[10] For our data this is indeed clearly and significantly the case [see also the summary statistics in Ball, Elmendorf and Mankiw (1998)]. We obtain a mean of -0.014. A standard hypothesis test indicates that the mean is significantly smaller than zero at a p-value of 0.053, i.e. almost at the 5% level. Thus we can conclude that the US economy is likely not to be interim Pareto optimal.

Let us briefly mention that we neglected capital taxes in the US in our analysis since

[9] We would like to thank Doug Elmendorf for providing us with the data set.

[10] While our theoretical condition from proposition 6 requires uniform convergence, ergodicity gives only pathwise convergence. However, this implies uniform convergence on set of paths with measure arbitrarily close to 1, which we regard as sufficent to conclude suboptimality in an empirical analysis.

we did not have access to reliable data. Including capital taxes would lower the net rate of return households earn by holding bonds. This would make our result on suboptimality even stronger. Thus our comparison of the growth rate and the pre-tax rate of return on bonds underestimates the scope for welfare improvements of a debt policy.

For the case of dynamic efficiency a similar criterion was derived by Zilcha (1991). It says that $E[log(R^M)/(1+g)] > 0$ implies dynamic efficiency. The only difference to our criterion is that the risk-free rate of return is replaced by the risky rate of return. We use the capital rental rate in the US from 1929-1997 as derived by Mulligan (2001) as the risky rate of return.[11] The average rental rate for that period was 0.083 with a standard deviation of 0.0178. When compared to the real growth rate over the same period, the data clearly indicate that the US economy is dynamically efficient. Alternative measures like stock returns do not change our conclusions.

The fact that we can conclude that overaccumulation of capital is not an issue when considering the optimal issue of debt can also be seen as a justification of our analysis of pure exchange economies in the examples above. In work in progress we consider other criteria discussed above like R_2/R_1.

8 Conclusion

This paper provides a general approach in the framework of a complete markets stochastic overlapping generations model to assess whether debt Ponzi schemes are feasible and Pareto-improving. We derive conditions in terms of bond interest rates of different maturities which can be used to assess different roll over strategies. Furthermore, we clarify how a dynamic roll over strategy of debt interacts with considerations in providing insurance against aggregate shocks by intergenerational risk sharing. A main result of this paper is that the feasibility of roll over strategies is unrelated to intergenerational insurance considerations that have received much attention in the literature. In fact, under the empirically relevant scenario of countercyclical real interest rates a Pareto improving roll over strategy of debt typically implies an increased variability of old age consumption. We develop a detailed intuition for our results along a series of examples. We underline our analysis with empirical evidence that is consistent with the claim that the US economy is likely to be inefficient.

[11] We would like to thank Casey B. Mulligan for providing us with his data set.

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9 Appendix

9.1 Example 1

Here is a more detailed and more general description of the first example. The economy is a stationary pure exchange economy with two states of the world, a boom b and a recession r. Utility is time separable and logarithmic with a discount factor $0 < \beta < 1$. That means for an individual born in a recession r

$$u(c^y(r), c^o(r), c^o(b)) = \ln c^y(r) + \beta \cdot (q(r,r) \ln c^o(r) + q(r,b) \ln c^o(b))$$

where $q(r,b)$ is the transition probability of going from a recession in period t to a boom in period $t+1$. Thus, $q(r,r) + q(r,b) = 1$. Analogous expressions hold for an individual born in a boom.

The stationary aggregate endowment structure is as follows:

$$e(b) = 2 \text{ and } e(r) = 1$$

with $e^y(b) = 1$, $e^o(b) = 1$ as endowment in a boom of young and old agents respectively and with $e^y(r) = \frac{2}{3}$, $e^o(r) = \frac{1}{3}$ as endowment in a recession of young and old agents respectively. The crucial assumption here is that the old are hit stronger by a recession then the young. This will turn out to be an important prerequisite for obtaining procyclical interest rates as well as a intergenerational insurance arrangement.

In will be helpful for our interpretations to introduce the price for state-contingent consumption in a state z' in terms of one unit of consumption in the predecessor node z, the relative Arrow-Debreu price. In general it can be defined as:

$$p(z,z') = q(z,z')\beta \frac{\frac{\partial u}{\partial c^o(b)}}{\frac{\partial u}{\partial c^y(r)}} = q(z,z')m(z,z')$$

where z is either the boom state b or the recession state r and z' is the successor of z and can also be either b or r and $m(z,z')$ is the stochastic discount factor. The inverse of $m(z,z')$ is the marginal rate of substitution between consumption in state z and consumption one period later in state z'. This marginal rate will play an important role when assessing the feasibility of all kinds of rollover schemes since it answers the question how much compensation a household requires in a particular state during old age for a unit of consumption he has to give up in his young age.

Due to logarithmic utility the state-contingent prices of consumption, i.e. the relative Arrow-Debreu price, in the pure endowment case are particulary simple to derive:

$$p(z,z') = q(z,z')\beta\frac{e^y(z)}{e^o(z')}, \tag{7}$$

The stochastic discount factor is $m(z,z') = \beta\frac{e^y(z)}{e^o(z')}$.

Prices are derived as in the text. Now we allow for more general transition probabilities.

Now we make some assumptions about the transition probabilities between boom and recession which a more general than the ones made in the text and include those as a special case.

Assumption 1 (Persistence of Booms): $q(b,b) \geqslant 0.5$

Assumption 2 (Persistence of Recessions): $q(r,r) \geqslant 0.5$

Assumption 3 (Booms more Persistent than Recessions): $q(b,b) \geqslant q(r,r)$

The relative Arrow-Debreu prices in our example are:

$$\begin{aligned} p(r,b) &= \frac{\frac{\partial u}{\partial c^o(b)}}{\frac{\partial u}{\partial c^y(r)}} = q\left(r,b\right)\cdot\beta\cdot\tfrac{2}{3} \quad & p(r,r) &= \frac{\frac{\partial u}{\partial c^o(r)}}{\frac{\partial u}{\partial c^y(r)}} = q(r,r)\cdot\beta\cdot 2 \\ p(b,b) &= \frac{\frac{\partial u}{\partial c^o(b)}}{\frac{\partial u}{\partial c^y(b)}} = q\left(b,b\right)\cdot\beta & p(b,r) &= \frac{\frac{\partial u}{\partial c^o(r)}}{\frac{\partial u}{\partial c^y(b)}} = 3\cdot\beta\cdot q\left(b,r\right) \end{aligned}$$

In the stationary setup, the one ad two period interest rates on safe debt issued in a boom b and a recession r respectively are:

$$R_1(b) = \frac{1}{p(b,b)+p(b,r)},\ R_1(r) = \frac{1}{p(r,b)+p(r,r)}$$

$$R_2(b) = \frac{1}{p(b,b)\left(p(b,b)+p(b,r)\right)+p(b,r)\left(p(r,b)+p(r,r)\right)}$$

$$R_2(r) = \frac{1}{p(r,r)\left(p(r,b)+p(r,r)\right)+p(r,b)(p(b,b)+p(b,r))}$$

9.1.1 Procyclical Interest Rates

The condition for procyclicality of interest rates is:

$$R_1(b) > R_1(r) \Leftrightarrow 2/3 + 4/3q(r,r) > 3 - 2q(b,b)$$

$$\Longleftrightarrow \frac{4}{3} \cdot q\left(r,r\right) + 2 \cdot q\left(b,b\right) > \frac{7}{3}$$

If $q\left(r,r\right) = q\left(b,b\right)$, this condition reduces to $q\left(r,r\right) = q\left(b,b\right) > 0.7$. Further, because $2 \cdot q\left(b,b\right) \leq 1$, then a necessary condition for the interest rates to be procyclical is $q\left(r,r\right) \geq \frac{1}{4}$. If $q\left(r,r\right) = 0.5$, interest rates are only procyclical if $q\left(b,b\right) > \frac{5}{6}$. In what follows, we will assume procyclical interest rates, as we did in the text.

9.1.2 Rolling over one period debt

We first show that rolling over safe one period debt does not work in this economy if discounting is sufficiently strong.

We have $R_1(b) > 1$ if and only if $3 - 2q(b,b) < \frac{1}{\beta} \Leftrightarrow \left(3 - \frac{1}{\beta}\right) \cdot \frac{1}{2} < q\left(b,b\right)$. Thus, if we have almost no discounting, i.e. if $\frac{1}{\beta}$ is close to 1, $q\left(b,b\right)$ must be close to 1. If one choses a small β of 0.5 as one could justify by taking a time period to be something like 30 years then persistent booms, i.e. $q\left(b,b\right) > 0.5$, imply $R_1(b) > 1$.

If we try to roll over one period debt, we see that on paths that contain the state boom sufficiently often, the product of one period interest rates diverges. Thus, the amount of real transfer that has to be paid along such a path explodes. Given that the economy is bounded, this shows the infeasibility of such a transfer scheme.

9.1.3 Rolling over two period bonds period-wise

Our next step is to show under what conditions issuing two period bonds every period which are sold after one period for the state contingent price of one period bonds is sufficient to provide improving transfers. The state contingent return of a two period bond issued in a boom and sold one period later in a boom is given by

$$\begin{aligned}
\frac{R_2(b)}{R_1(b)} &= \frac{1}{\beta} \cdot \frac{3 - 2q(b,b)}{q(b,b) \cdot (3 - 2q(b,b)) + 3 \cdot (1 - q\left(b,b\right)) \cdot (2/3 + 4/3q(r,r))} \\
&= \frac{1}{\beta} \cdot \frac{3 - 2q(b,b)}{q(b,b) \cdot (3 - 2q(b,b)) + (1 - q\left(b,b\right)) \cdot (2 + 4q(r,r))}
\end{aligned}$$

For $\frac{R_2(b)}{R_1(b)} > 1$ to hold, we must have

$$3 - 2q(b,b) > \beta \cdot [q(b,b) \cdot (3 - 2q(b,b)) + (1 - q(b,b)) \cdot (2 + 4q(r,r))]$$

$$\Longleftrightarrow (1 - \beta \cdot q(b,b)) \cdot (3 - 2q(b,b)) > \beta \cdot (1 - q(b,b)) \cdot (2 + 4q(r,r))$$
$$\Longleftrightarrow \frac{(1 - \beta \cdot q(b,b))}{\beta \cdot (1 - q(b,b))} \cdot (3 - 2q(b,b)) > (2 + 4q(r,r))$$

This condition holds for fixed β for $q(b,b)$ sufficiently close to 1, since in this case $\frac{(1-\beta \cdot q(b,b))}{\beta \cdot (1-q(b,b))} \to \infty$.

In particular in the case where $q(b,b) = q(r,r) =: q$ this will hold iff

$$q > \frac{1}{12\beta}\left(2 + 5\beta + \sqrt{(4 - 52\beta + 73\beta^2)}\right)^{12}$$

$$\begin{aligned}\frac{R_2(r)}{R_1(r)} &= \frac{1}{\beta} \cdot \frac{2/3 + 4/3q(r,r)}{q(r,r) \cdot 2 \cdot (2/3 + 4/3q(r,r)) + 2/3 \cdot (1 - q(r,r)) \cdot (3 - 2 \cdot q(b,b))} \\ &= \frac{1}{\beta} \cdot \frac{1 + 2q(r,r)}{q(r,r) \cdot (2 + 4q(r,r)) + (1 - q(r,r)) \cdot (3 - 2 \cdot q(b,b))}\end{aligned}$$

Thus $\frac{R_2(r)}{R_1(r)} < 1$ iff

$$3 - 2 \cdot q(b,b) + q(r,r) \cdot (-1 + 4q(r,r) + 2q(b,b)) > \frac{1}{\beta} \cdot (1 + 2q(r,r))$$

We have $3 - 2 \cdot q(b,b) + q(r,r) \cdot (-1 + 4q(r,r) + 2q(b,b)) > 3 - 2 \cdot q(b,b) + q(r,r) \cdot (1 + 2q(b,b))$ since $q(r,r) \geq \frac{1}{2}$ by assumption. If $q(b,b) < \frac{1}{\beta} - \frac{1}{2}$, we see that

$$3 - 2 \cdot q(b,b) + q(r,r) \cdot (1 + 2q(b,b)) > \frac{1}{\beta} \cdot (1 + 2q(r,r))$$

holds for all $\frac{1}{\beta} - \frac{1}{2} > q(r,r) \geq \frac{1}{2}$ if $\beta > \frac{3}{4}$.

If $q(r,r) = q(b,b)$, we have $\frac{R_2(r)}{R_1(r)} < 1$ iff

$$6\beta q^2 - (3\beta + 2)q + 3\beta - 1 > 0$$

which has no real solution and thus holds for all q.

[12] If $\sqrt{4 - 52\beta + 73\beta^2}$ is negative, the roots of the quadratic equation are complex and the condition is satisfied for all q.

9.1.4 Rolling over two period bonds every second period

Another roll over strategy might consist of selling two period bonds at time t, buying them back at the prevailing market price in $t+1$ and selling them again as one period bonds in $t+1$. In $t+2$ the same procedure as in t starts. The return from t to $t+1$ is given by $\frac{R_2(z_t)}{R_1(z_{t+1})}$, the return from $t+1$ to $t+2$ is given by $R_1(z_{t+1})$. Thus, overall we have

$$\frac{R_2(z_t)}{R_1(z_{t+1})} \cdot R_1(z_{t+1}) \cdot \frac{R_2(z_{t+2})}{R_1(z_{t+3})} \cdot R_1(z_{t+3}) \cdot ... = R_2(z_t) \cdot R_2(z_{t+2}) \cdot ...,$$

i.e. two period bonds are rolled over every second period.

This strategy is not possible here since $R_2(b)/R_1(b) > 1$ and $R_1(b) > 1$ directly imply $R_2(b) = R_2(b)/R_1(b) \cdot R_1(b) > 1$.

Thus, if neither rolling over one period bonds nor rolling over two period bonds periodwise works then rolling over two period bonds every second period does not work either. In this sense, this criterion has no value added if the others have been checked.

9.1.5 The optimal policy: Issuing and Rolling over two period bonds and holding one period bonds

A stationary improvement is a transfer $v(b)$ in booms and transfer $v(r)$ in recessions. For a stationary improvement $(v(b), v(r))$ to work, the following conditions are necessary and sufficient[13]:

$$p(r,b) \cdot v(b) + p(r,r) \cdot v(r) > v(r) \tag{8}$$

$$p(b,b) \cdot v(b) + p(b,r) \cdot v(r) > v(b)$$

That these conditions are necessary for a stationary improvement is easy to see: the first equation states that for a consumer born in state r, the consumption bundle after transfers are paid, evaluated at the competive equilibrium prices, must have a strictly higher value than the competitive equilibrium allocation. Since transfers are supposed to be strictly improving, they cannot be in the budget set over which the consumer optimizes in the competitive equilibrium.

The sufficiency follows by taking a second order Taylor approximation of the consumers utility around the competitive equilibrium allocation. (8) says that the effect from the first order term is always strictly positive. But if we take the vector $(v(b), v(r))$ sufficiently

[13] We restrict here attention to stationary improving transfers. Demange and Laroque (1999) and Chattopadhyay and Gottardi (1999) have shown that if a stationary economy is inefficient, a stationary improving transfer scheme always exists.

small (by scaling it with a scalar), the first order effect always dominates the second order effect. Thus, the strict inequality in (8) implies also sufficiency for the existence of a stationary improving transfer[14].

If we define $y := \frac{v(b)}{v(r)}$ as the relative transfer, we can rewrite the equations as[15]

$$p(r,b) \cdot y + p(r,r) > 1 \tag{9}$$

$$p(b,b) + p(b,r) \cdot \frac{1}{y} > 1$$

We see that the left hand side expression in the first equation is increasing in y, while it is decreasing in y in the second equation. Thus, whenever a (positive) stationary improvement exists, then we can take ratio y of these transfers as the solution of the equation

$$p(r,b) \cdot y + p(r,r) = p(b,b) + p(b,r) \cdot \frac{1}{y} \tag{10}$$

$$\Longleftrightarrow y = \frac{1}{2p(r,b)} \cdot \left(p(b,b) - p(r,r) + \sqrt{\left(p(b,b) - p(r,r)\right)^2 + 4p(r,b)p(b,r)} \right)$$

This is the optimal relative stationary transfer (if it exists).

To check whether these relative transfers are really improving, we have to check that the y computed above satisfies indeed both of the inequalities in (9).

Recall that under the persistence assumption $q\,(r,r) \geq \frac{1}{2}$, we have that $p(r,r) = q(r,r) \cdot \beta \cdot 2 > q\,(b,b) \cdot \beta = p(b,b)$.

Using the result on y, we get, using $p(r,r) > p(b,b)$:

$$p(r,b) \cdot \frac{p(b,b) - p(r,r) + \sqrt{\left(p(b,b) - p(r,r)\right)^2 + 4p(r,b)p(b,r)}}{2p(r,b)} + p(r,r)$$

$$\begin{aligned} &= \frac{1}{2} \cdot \left(p(b,b) - p\,(r,r) + \sqrt{\left(p(b,b) - p(r,r)\right)^2 + 4p(r,b)p(b,r)} \right) + p\,(r,r) \\ &\geq \frac{1}{2} \cdot \left(p(r,r) + p(b,b) + \left(p(r,r) - p(b,b)\right)\right) = p\,(r,r) \end{aligned}$$

[14] We assume here that $v\,(b)$ and $v\,(r)$ are positive, i.e. the transfer is paid from the young to the old generation. If only one of the transfers is negative, say e.g. $v\,(b)$, it is easy to see that (8) still holds with $v\,(b) = 0$. The case that both $v\,(b)$ and $v\,(r)$ are negative is ruled out later.

[15] Here we assume that both transfers are strictly positive.

Thus, $p(r,r) > 1$ is a sufficient condition for inefficiency, i.e. for an improving stationary transfer to exist. Given the definition of $p(r,r)$ this amounts to

$$q(r,r) \cdot \beta \cdot 2 > 1$$

$$\Leftrightarrow q(r,r) > \frac{1}{2\beta}$$

When will this condition be satisfied? For $\beta = 0.8$, the condition is satisfied for $q(r,r) > 0.625$.

If we take the stationary transfer scheme $(v(b), v(r)) = (y, 1)$, the quantity of one period bonds μ_1 and two period bonds μ_2 has to solve

$$\begin{pmatrix} R_1(r) & \frac{R_2(r)}{R_1(r)} \\ R_1(r) & \frac{R_2(r)}{R_1(b)} \end{pmatrix} \cdot \begin{pmatrix} \mu_1(r) \\ \mu_2(r) \end{pmatrix} = \begin{pmatrix} 1 \\ y \end{pmatrix} \tag{11}$$

if we start in a recession and

$$\begin{pmatrix} R_1(b) & \frac{R_2(b)}{R_1(r)} \\ R_1(b) & \frac{R_2(b)}{R_1(b)} \end{pmatrix} \cdot \begin{pmatrix} \mu_1(b) \\ \mu_2(b) \end{pmatrix} = \begin{pmatrix} 1 \\ y \end{pmatrix}$$

if we start in a boom.

The solution is $\mu_1(r) = -\frac{-\frac{R_2(r)}{R_1(b)} + \frac{R_2(r)}{R_1(r)} \cdot y}{R_1(r) \cdot \left(\frac{R_2(r)}{R_1(b)} - \frac{R_2(r)}{R_1(r)}\right)}$ and $\mu_2(r) = \frac{y-1}{\frac{R_2(r)}{R_1(b)} - \frac{R_2(r)}{R_1(r)}}$ for the first equation system. Note that:

$$R_2(b) = \frac{1}{q\beta^2(3 - 2q + 2\beta + 2q\beta - 4q\beta^2)}$$

$$R_2(r) = \frac{1}{2\beta^2(-q + 2q^2 + 1)}$$

Using procyclicality $R_1(b) > R_1(r)$, we see that the nominators are negative. Further, we can choose y smaller than $R_1(r)/R_1(b)$, because $y \leq \frac{2p(b,r)}{p(r,r) - p(b,b)} = \frac{6 \cdot (1 - q(b,b))}{2q(r,r) - q(b,b)}$ and this expression goes to zero as $q(b,b)$ becomes close to 1. This implies indeed that $\mu_1(r) < 0$ and $\mu_2(r) > 0$ can occur. Thus, in this case an improving transfer involves short sales of one period bonds and is long in two period bonds. The same result holds for $\mu_1(b)$ and $\mu_2(b)$.

Note that $\mu_1(r) < 0$ iff $y < R_1(r)/R_1(b) < 1$, i.e. if interest rates are sufficiently procyclical we will have short sales.

For $q(r,r) = q(b,b)$ we have $y = \frac{1}{\frac{4}{3}(1-q)} \cdot \left(-q + \sqrt{q^2 + 8(1-q)^2}\right)$.

In this case we have $y < R_1(r)/R_1(b) = \frac{9-6q}{2+4q}$ iff $q > 0.7$. Thus, for $q(r,r) = q(b,b)$, if an improving transfer exists, we have short sales in one period bonds and positive sales in two period bonds iff $q > 0.7$.

10 Example 2

This example is a mirror image of the previous one. The set up is the same as in the previous example, with the exception of transition probabilities and individual endowments. The assumptions we make in this section ensure that interest rates will be countercyclical.

Take $e^y(b) = \frac{5}{4}$, $e^o(b) = \frac{3}{4}$ as endowment of young resp. old agents in a boom and with $e^y(r) = \frac{1}{2}$, $e^o(r) = \frac{1}{2}$ as endowment of young resp. old agents in a recession.

The relative Arrow Debreu prices are now

$$p(r,b) = \frac{\frac{\partial u}{\partial c^o(b)}}{\frac{\partial u}{\partial c^y(r)}} = q(r,b)\cdot\beta\cdot\frac{2}{3} \qquad p(b,b) = \frac{\frac{\partial u}{\partial c^o(r)}}{\frac{\partial u}{\partial c^y(r)}} = q(r,r)\cdot\beta$$

$$p(b,b) = \frac{\frac{\partial u}{\partial c^o(b)}}{\frac{\partial u}{\partial c^y(b)}} = q(b,b)\cdot\beta\cdot\frac{5}{3} \quad p(b,r) = \frac{\frac{\partial u}{\partial c^o(r)}}{\frac{\partial u}{\partial c^y(b)}} = \frac{5}{2}\cdot\beta\cdot q(b,r)$$

With these parameters, the one period interest rates are given as

$$R_1(r) = \frac{1}{q(r,b)\cdot\beta\cdot\frac{2}{3} + q(r,r)\cdot\beta}$$

$$R_1(b) = \frac{1}{q(b,b)\cdot\beta\cdot\frac{5}{3} + q(b,r)\cdot\beta\cdot\frac{5}{2}}$$

$R_1(r) > 1 \Longleftrightarrow \beta\cdot\left(\frac{2}{3}+\frac{1}{3}\cdot q(r,r)\right) < 1 \Longleftrightarrow 2+q(r,r) < \frac{3}{\beta}$. This holds for all values of $q(r,r)$ since $\beta < 1$. Thus by the arguments given in the previous example, improving with rolling over one period bonds is impossible. Note that interest rates are countercyclical, i.e. $R_1(r) > R_1(b)$, here since

$$R_1(r) = \frac{1}{q(r,b)\cdot\beta\cdot\frac{2}{3} + q(r,r)\cdot\beta} > \frac{1}{q(b,b)\cdot\beta\cdot\frac{5}{3} + q(b,r)\cdot\beta\cdot\frac{5}{2}} = R_1(b)$$

for all choices of transition probabilities.

Also rolling over debt in two period bonds, i.e. issuing two period bonds in every period and buying it back one period later at the prevailing market price of one period bonds, is impossible here since

$$\frac{R_2(r)}{R_1(r)} = \frac{\beta\cdot\left(\frac{2}{3}+\frac{1}{3}\cdot q(r,r)\right)}{q(r,b)\cdot\beta^2\cdot\frac{2}{3}\cdot\left(\left(q(b,b)\cdot\frac{5}{3}+q(b,r)\cdot\frac{5}{2}\right)\right) + q(r,r)\cdot\beta^2\cdot\left(\frac{2}{3}+\frac{1}{3}\cdot q(r,r)\right)}$$

Again, for $q(r,b)$ sufficiently small, we have $\frac{R_2(r)}{R_1(r)} > 1$ since $\frac{R_2(r)}{R_1(r)} \to \frac{1}{\beta}$ as $q(r,b) \to 0$.

$$\begin{aligned} \frac{R_2(r)}{R_1(r)} &= \frac{1}{\beta} \cdot \frac{2+q(r,r)}{(1-q(r,r)) \cdot 2 \cdot \left(\frac{5}{2} - q(b,b) \cdot \frac{5}{6}\right) + q(r,r) \cdot (2+q(r,r))} \\ &= \frac{1}{\beta} \cdot \frac{2+q(r,r)}{(1-q(r,r)) \cdot \left(5 - q(b,b) \cdot \frac{5}{3}\right) + q(r,r) \cdot (2+q(r,r))} \end{aligned}$$

Nevertheless, the competitive equilibrium is not Pareto optimal. To see this, consider again possible stationary improvements given by equations (8). With y defined in (10), we to show that given our parameter values, the inequality $p(r,b) \cdot y + p(r,r) > 1$ (and by construction of y also the inequality $p(b,b) + p(b,r) \cdot \frac{1}{y} > 1$) holds. We have

$$p(r,b) \cdot y + p(r,r) = \frac{1}{2} \cdot \left(p(b,b) - p(r,r) + \sqrt{(p(b,b) - p(r,r))^2 + 4p(r,b)p(b,r)} \right) + p(r,r)$$

$$\geq \frac{1}{2} \cdot 2 \cdot (p(b,b) - p(r,r)) + p(r,r) = p(b,b)$$

The inequality holds if $p(b,b) > p(b,b)$. Thus, if $q(b,b)$ is sufficiently large, we get the inequality and also $p(b,b) > 1$. This implies the desired condition (8) to hold.

Thus, we can again solve the equation system (11) and get $\mu_1(r) = -\frac{-\frac{R_2(r)}{R_1(b)} + \frac{R_2(r)}{R_1(r)} \cdot y}{R_1(r) \cdot \left(\frac{R_2(r)}{R_1(b)} - \frac{R_2(r)}{R_1(r)}\right)}$ and $\mu_2(r) = \frac{y-1}{\frac{R_2(r)}{R_1(b)} - \frac{R_2(r)}{R_1(r)}}$. If $q(r,b)$ is sufficiently close to zero, we see that y becomes larger than 1. Further, as discussed above $R_1(r) > R_1(b)$. This implies that $\mu_2(r) > 0$ and $\mu_1(r) < 0$. Thus, the improving transfer is generated by short sales of one period bonds (safe debt) and issuing two period bonds.

11 Proof of Proposition 6

As argued in the text a transfer vector x is an stationary improving if and only if $Ax > x$.

Note first that for the transfers generated by roll over of two period bonds, i.e for $v = \left(\frac{R_2(s)}{R_1(1)}, \frac{R_2(s)}{R_1(2)}, ..., \frac{R_2(s)}{R_1(S)}\right)$, the expected present value of tomorrows pay-off, Av is proportional to v, that is

$$Av = \lambda v$$

for some positive λ. A crucial feature of v is that $v(i)/v(j) = (Av)(i)/(Av)(j)$, this means relative transfers are equal to relative expected payoffs. A proof, available from the authors upon request, shows that this generalizes to arbitrary stochastic processes.

Furthermore, recall that the rank of A is equal to 1. This implies that any dynamic portfolio x has a unique decomposition such that $x = \mu v + w$, where w is a portfolio with zero payoff, $Aw = 0$ and μ some scalar. If such a portfolio is improving, we must have

$$Ax > x \Longleftrightarrow A(\mu v + w) = \mu\lambda v > \mu v + w$$

Note that since by definition a dynamic portfolio has strictly positive payoff, we must have $\mu > 0$. Since the matrix A has only strictly positive entries, a vector $w \neq 0$ from its nullspace must have some strictly positive entry, e.g. $w(i)$. For $\mu\lambda v(i) > \mu v(i) + w(i)$ to hold, we must have $\lambda > 1$. This implies

$$Av = \lambda v > v$$

which shows that v itself is an improving stationary transfer.

The set of inequalities

$$Av > v$$

can be rewritten as

$$\sum_{j=1}^{S} \underbrace{\frac{q(j) \cdot w'(c^o(j))}{u'(c^y(i))}}_{=:a_{ji}} \cdot \frac{v(j)}{v(i)} > 1 \qquad \text{for all } i = 1, ..., S \tag{12}$$

We will first prove if some strictly positive solution $\widetilde{v}$ to (12) exists, there exists also a solution such that

$$\sum_{j=1}^{S} a_{ji_1} \cdot \frac{v(j)}{v(i_1)} = \sum_{j=1}^{S} a_{ji_2} \cdot \frac{v(j)}{v(i_2)}. \tag{13}$$

To see this, note first that because the inequality system is homogenous of degree 0 in v, we can restrict attention to solutions from the perturbed $S-1$ dimensional unit simplices $\Delta_n := \left\{ v \in \mathbb{R}^S_{++} \middle| \sum_{s=1}^{S} v_s = 1,\ v_s \geq \frac{1}{n}\ s = 1, ..., S \right\}$. Each Δ_n is compact and for n sufficiently large it contains the assumed strictly positive solution $\widetilde{v}$ to (12). Since there are only finitely many inequalities, there is some $\varepsilon > 0$ such that

$$\sum_{j=1}^{S} a_{ji} \cdot \frac{\widetilde{v}(j)}{\widetilde{v}(i)} \geq 1 + \varepsilon \qquad \text{for all } i = 1, ..., S$$

So $\widetilde{v} \in B_n := \left\{ v \in \mathbb{R}^S_{++} \middle| \sum_{j=1}^S a_{ji} \cdot \frac{v(j)}{v(i)} \geq 1 + \frac{\varepsilon}{n} \; s = 1, ..., S \right\}, n \in \mathbb{N}$. Note that $C_n := \Delta_n \cap B_n$ is also compact and is nonempty for n sufficiently large. Consinder for those n

$$\max_{v \in C_n} h(v) \tag{14}$$

where $h(v) := \min_{i=1,...,S} \left[\sum_{j=1}^S a_{ji} \cdot \frac{v(j)}{v(i)} - 1 \right]$ is a continuous function.

Let v_n be the solution to (14).

For n sufficiently larger it must be that $v_n \notin \partial \Delta_n$. If this were not the case, we can be going to a subsequence w.l.o.g. assume that $v_n(1) = \frac{1}{n}$. Since for n sufficiently large, take the $s \in S$ such that $s \in \arg\min_{i=1,...,S} \left[\sum_{j=1}^S a_{ji} \cdot \frac{v_n(j)}{v_n(i)} - 1 \right]$. Note that for n sufficiently large, $1 \notin \arg\min_{i=1,...,S} \left[\sum_{j=1}^S a_{ji} \cdot \frac{v_n(j)}{v_n(i)} - 1 \right]$. If we multiply all $v_n(s)$ by a factor $\gamma < 1$ sufficiently close to 1 and increase $v_n(1)$ so that the vector is still contained in C_n, we see that for the so obtained new vector $\widetilde{v}_n$ that $h(\widetilde{v}_n) > h(v_n)$, which contradicts the optimality of v_n.

Thus we have $v_n \in int(C_n)$. To see this, note that since $h(\widetilde{v}) \geq 1 + \varepsilon$, v_n will also be in the interior of B_n. If we had $\sum_{j=1}^S a_{ji_1} \cdot \frac{v_n(j)}{v_n(i_1)} > \sum_{j=1}^S a_{ji_2} \cdot \frac{v_n(j)}{v_n(i_2)}$, by similar arguments as used in the previous paragraph, we would have $v_n \notin \arg\max_{v \in C_n} h(v)$, contradicting the definition of v_n.

So, for n sufficiently large, $\sum_{j=1}^S a_{ji_1} \cdot \frac{v_n(j)}{v_n(i_1)} = \sum_{j=1}^S a_{ji_2} \cdot \frac{v_n(j)}{v_n(i_2)}$ for all $i_1, i_2 \in S$. We thus know that always solutions with this property exist.

Given that $a_{ji} = \mu_i \cdot a_{j1}$ for all $i, j \in S$, with $\mu_i = \frac{u'(c^y(1))}{u'(c^y(i))}$, we see from $\sum_{j=1}^S a_{ji} \cdot \frac{v(j)}{v(i)} = \frac{\mu_i}{v(i)} \cdot \sum_{j=1}^S a_{j1} \cdot v(j)$ that for equality (13) to hold that

$$\frac{\mu_{i_1}}{\mu_{i_2}} = \frac{v(i_1)}{v(i_2)}$$

for all $i_1, i_2 \in S$ has to hold. Note that $\frac{v(i_1)}{v(i_2)} = \frac{\mu_{i_1}}{\mu_{i_2}} = \frac{u'(c^y(i_2))}{u'(c^y(i_1))} = \frac{R_2(s)/R(i_1)}{R_2(s)/R(i_2)}$ for all $s \in S$. Thus shows that a portfolio which holds two period bonds, which are sold one period before maturity, generates the desired payoff structure.

12 Proof of Theorem 8

First, we present the characterization of interim Pareto optrimality of competitive equilibria in Barbie, Hagedorn and Kaul (2001) for a stochastic Diamond model. If $p(z^t)$ denotes the state contingent price of date event z^t, the economoy is not interim Pareto optimal iff for every path $(z^0, z^1, ...)$ in the date event tree

$$\sum_{t=0}^{\infty} \prod_{s=1}^{t} \lambda(z^s) \cdot \frac{1}{q\left(z_s, z^{s-1}\right) \cdot m\left(z^{s-1}, z^s\right)} \leq M < \infty,$$

for some constant M. The λ are nonnegative and satisfy

$$\sum_{z_{t+1} \in Z} \lambda\left(z, z_{t+1}\right) = 1 \tag{15}$$

For fixed z^{s-1}, consider the $(n \times n)$ return matrix

$$\mathbf{R}\left(z^{s-1}\right) = \begin{pmatrix} R_1(z^{s-1}) & R_2(z^{s-1})/R_1(z^{s-1}, z1) & ... & R_n(z^{s-1})/R_{n-1}(z^{s-1}, z1) \\ . & . & . & . \\ . & . & . & . \\ . & . & . & . \\ R_1(z^{s-1}) & R_2(z^{s-1})/R_1(z^{s-1}, zn) & ... & R_n(z^{s-1})/R_{n-1}(z^{s-1}, zn) \end{pmatrix}$$

Generically, this matrix has full rank. Therefore there exists a vector $\mu(z^{s-1})$ such that

$$\mathbf{R}\left(z^{s-1}\right) \cdot \mu(z^{s-1}) = \left(\lambda(z^s) \cdot \frac{1}{q\left(z_s, z^{s-1}\right) \cdot m\left(z^{s-1}, z^s\right)}\right)_{z_s \in Z} \tag{16}$$

Note that the $\mu(z^{s-1})$ is a dynamic portfolio since

$$\begin{aligned} \sum_{k=1}^{n} \mu_k(z^{s-1}) &= \sum_{k=1}^{n} \mu_k(z^{s-1}) \cdot \underbrace{E_{z^{s-1}}\left[\frac{R_k\left(z^{s-1}\right)}{R_{k-1}(z^s)} \cdot m\left(z^{s-1}, z^s\right)\right]}_{=1} \\ &= \sum_{z_s \in Z} \underbrace{\sum_{k=1}^{n} \frac{R_k\left(z^{s-1}\right)}{R_{k-1}(z^s)} \cdot \mu_k(z^{s-1})}_{=\lambda(z^s) \bullet \frac{1}{q\left(z_s, z^{s-1}\right) \cdot m\left(z^{s-1}, z^s\right)}} \cdot q\left(z_s, z^{s-1}\right) \cdot m\left(z^{s-1}, z^s\right) \\ &= \sum_{z_s \in Z} \lambda(z^s) = 1 \end{aligned}$$

where$E_{z^{s-1}}\left[\frac{R_k\left(z^{s-1}\right)}{R_{k-1}(z^s)} \cdot m\left(z^{s-1}, z^s\right)\right] = 1 \iff E_{z^{s-1}}\left[\frac{1}{R_{k-1}(z^s)} \cdot m\left(z^{s-1}, z^s\right)\right] = \frac{1}{R_k(z^{s-1})}$ follows from no arbitrage, and $\mathbf{R}\left(z^{s-1}\right) \cdot \mu(z^{s-1})$ is nonnegative since the λ are nonnegative.

If a dynamic portfolio $\mu(z^{s-1})$ is given, we can define the λ by (16). These $\lambda(z^s)$ are nonnegative since $\mu(z^{s-1})$ has nonnegative payoffs in each state, and by the same

arguments as above, equation (15) holds.
So with this transfer pattern, the convergence in the condition of the main theorem occurs if and only if

$$\sum_{t=0}^{\infty}\prod_{s=1}^{t}\mu(z^{s-1})\cdot\mathbf{R}(z^{s},z^{s-1}) \qquad \leq M<\infty.$$

is uniformly bounded for every path by the definition of $\mathbf{R}(z^{s},z^{s-1})$ as the z_s-th row of the matrix $\mathbf{R}\left(z^{s-1}\right)$.