Interest Rate Caps "Smile" Too! But Can the LIBOR Market Models Capture It?

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ABSTRACT

Using more than two years of daily interest rate cap price data, this paper provides a systematic documentation of a volatility smile in cap prices. We find that Black (1976) implied volatilities exhibit an asymmetric smile (sometimes called a sneer) with a stronger skew for in-the-money caps than out-of-the-money caps. The volatility smile is time varying and is more pronounced after September 11, 2001. We also study the ability of generalized LIBOR market models to capture this smile. We show that the best performing model has constant elasticity of variance combined with uncorrelated stochastic volatility or upward jumps. However, this model still has a bias for short- and medium-term caps. In addition, it appears that large negative jumps are needed after September 11, 2001. We conclude that the existing class of LIBOR market models can not fully capture the volatility smile.

JEL Classification: C4, C5, G1

Interest rate caps and swaptions are widely used by banks and corporations for managing interest rate risk. They are the most liquid over-the-counter interest rate derivatives traded. Indeed, according to the Bank of International Settlement, by the end of 2001 the combined notional values of interest rate caps and swaptions was well over 10 trillion dollars. This notional value is many times larger than that of comparable exchange traded interest rate derivatives. Consequently, accurate and efficient pricing of caps is an important topic for academic research. As pointed out by Dai and Singleton (2002a), there is also an "enormous potential for new insights from using derivatives data in (dynamic term structure) model estimations."

Despite the fact that these markets are so voluminous, the majority of the existing literature uses only at-the-money (ATM) caps and swaptions. The current caps and swaptions pricing literature has here-to-fore primarily focused on two issues.[1] The first issue is the so-called "unspanned stochastic volatility" puzzle documented by Collin-Dufresne and Goldstein (2002) and Heidari and Wu (2001), see also Fan, Gupta, Ritchken (2002). The "unspanned stochastic volatility" puzzle is that there appear to be risk factors that drive caps and swaptions prices not spanned by the factors explaining LIBOR or swap rates. The second issue is the relative pricing between caps and swaptions. A number of recent papers, including Longstaff, Santa-Clara, Schwartz (2001) and Jagannathan, Kaplin, Sun (2001), show that there is significant and systematic mispricing between caps and swaptions using various multi-factor term structure models, see also Collin-Dufresne and Goldstein (2001).

There are almost no studies documenting the relative pricing of caps with different strike prices.[2] This is due to the absence of a comprehensive cap price data base. Caps and swaptions are traded over-the-counter and the common data sources, such as DataStream, only supply ATM option prices. Nonetheless, several studies have provided anecdotal evidence for the existence of a volatility smile in interest rate caps, and they have even developed theoretical models to capture this phenomenon.[3] In contrast, the attempt to capture the volatility smile in equity option markets is voluminous and it has been the driving force behind the development of the equity option pricing literature for the past quarter of a century (see Bakshi, Cao, and Chen and references therein).[4] Analogously, it is our hope that studying caps and swaptions with different strike prices will provide new insights about existing term structure models that are not available from ATM options.

To remedy this omission in the literature, using more than two years of daily cap price data

[1] For a review of the current term structure literature, see Dai and Singleton (2002 a, b).

[2] The only study that considers caps with different strike prices is Gupta and Subrahmanyam (2001). As shown below, however, their data is more limited than that used herein. They also test different term structure models.

[3] See Hull and White (2000), Andersen and Andreasen (2000), Andersen and Brotherton-Ratcliffe (2001), and Glasserman and Kou (2002).

[4] For reviews of the equity option literature, see Duffie (2002) and Campbell, Lo and MacKinlay (1997).

with different strikes (from August 1, 2000 to November 2, 2002) from SwapPX,[5] we document the existence of a volatility smile in the interest rate cap markets. Our data set contains rich cross-sectional information. For example, we have deep ITM and OTM caps with ten different strike prices and fifteen different maturities ranging from six months to ten years. The changes in macroeconomic conditions during our sample period also enable us to study cap prices under different economic environments. We show that the implied volatilities from Black's model (1976) exhibit an asymmetric volatility smile (sometimes called a "sneer") that is similar to those observed in equity option markets. ITM caps are shown to have a stronger skew than do OTM caps. The volatility smile is also time varying and more pronounced after September 11, 2001. This is similar (except in the date) to a shift in equity option market smiles documented by Rubinstein (1994) after the stock market crash of 1987.

Capturing the volatility smile in caps offers an interesting challenge to existing term structure models, and it provides an alternative perspective for examining model performance. In our analysis of these issues, we focus on a subclass of the Heath, Jarrow and Morton (HJM, 1992) models known as the LIBOR market models, developed by Brace, Gatarek and Musiela (BGM, 1997) and Miltersen, Sandmann and Sondermann (MSS, 1997). Compared to either the spot rate models of Vasicek (1977) and Cox, Ingersoll and Ross (1985), or the affine models of Duffie and Kan (1996), the LIBOR market models have several advantages. First, the LIBOR market models are consistent with a market convention of quoting cap prices using Black's formula.[6] This makes calibration of LIBOR market models very simple, because the quoted implied Black volatility can be directly inserted into the model, thereby avoiding complicated numerical fitting procedures. Second, LIBOR market models are based on observable market rates, not their continuously compounded counterparts. These advantages explain the popularity of the LIBOR market model in the financial industry. From an academic perspective, LIBOR models price caps similar to equity options. Therefore, by focusing on LIBOR models, we can also utilize the equity option pricing literature to help interpret our empirical findings.

In analyzing the LIBOR market model, we reach the following conclusions. First, the standard LIBOR market model has large pricing errors and performs especially poorly after September 11, 2001, a period with a more pronounced volatility smile. Second, we also consider some generalized

[5] Jointly developed by GovPX and Garban-ICAP, SwapPX is the first widely distributed service delivering 24 hour real-time rates, data and analytics for the world-wide interest rate swaps market. GovPX was established in early 1990s by the major U.S. fixed-income dealers as a response to regulators' demands to increase the transparency of the fixed-income markets. It aggregates quotes from most of the largest fixed-income dealers in the world. Garban-ICAP is the world's leading swap broker specializing in trades between dealers and between dealers and large customers. According to Harris (2003), "Its securities, derivatives, and money brokerage businesses have daily transaction volumes in excess of 200 billion dollars".

[6] BGM (1997) and MSS (1997) show that market practice is consistent with arbitrage-free pricing if the LIBOR rates follow a log-normal distribution under the appropriate forward measure.

LIBOR models capable of generating a volatility smile. These include the stochastic volatility model of Andersen and Brotherton-Ratcliffe (2001), the jump diffusion model of Glasserman and Kou (2002), and a combined stochastic volatility and jump model that nests the previous two. We introduce a new/different approach for model calibration, and we compare relative model performance by measuring the difference in their pricing errors. It is shown that the generalized LIBOR model reduces the pricing error relative to the standard model, especially after September 11, 2001. The constant elasticity variance model combined with uncorrelated stochastic volatility or upward jumps performs best, except for short- and medium-term caps. Even for this model, however, significant downward jumps are needed to capture the volatility skew after September 11. Although an improvement over the standard LIBOR market model, we show that these generalized LIBOR models are incapable of capturing the entire smile.

The rest of this paper is organized as follows. In Section I, we introduce the data and document the volatility smile in cap markets. In Section II, we introduce the generalized LIBOR market models, our new calibration procedure, and a statistic for measuring relative model performance. In Section III, we study the performance of these generalized LIBOR models using both in-sample and out-of-sample criterion. Section IV concludes.

I. A Volatility Smile in the Interest Rate Cap Markets

Interest rate caps and floors are portfolios of call and put options on LIBOR rates. Specifically a cap gives its holder a series of European call options, called caplets, on LIBOR forward rates. Each caplet has the same strike price as the others, but with different expiration dates. For many currencies, caps are typically written on three-month LIBOR. The expiration dates for the caplets are on the same cycle as the frequency of the underlying LIBOR rate. For example, a five-year cap on three-month LIBOR struck at six percent represents a portfolio of 19 separately exercisable caplets with quarterly maturities ranging from 6 month to 5 years, where each caplet has a strike price of 6%.

Formally, let $L(t,T)$ be the LIBOR forward rate at $t \leq T$, for the interval from T to $T+\delta$, where δ is the fixed accrual period expressed as a fraction of a year (for three-month rates, $\delta = 1/4$). Thus, a party entering into a contract at time t to borrow \$1 over the interval $[T, T+\delta]$ would receive \$1 at time T and return to the lender $\$(1+\delta L(t,T))$ at time $T+\delta$.

Denoting by $B(t,\tau)$ the time-t price of a zero coupon bond maturing at τ, the LIBOR forward rate satisfies

$$L(t,T) = \frac{1}{\delta}\left(\frac{B(t,T)}{B(t,T+\delta)} - 1\right), \tag{1}$$

and conversely, for any $k = 1, 2, ...,$

$$B(t,t+k\delta) = \prod_{i=0}^{k-1} \frac{1}{1+\delta L(t,t+i\delta)}. \tag{2}$$

A caplet for the period $[T, T+\delta]$ struck at K pays $\delta\left(L\left(T,T\right)-K\right)^{+}$ at $T+\delta$. Note that while the cash flow on this caplet is received at time $T+\delta$, the LIBOR rate is determined at time T. This means that there is no uncertainty about the caplet's cash flow after the LIBOR rate is set at time T.

Analogously, floorlets are put options on the LIBOR rate. The cash flow from an individual floorlet with expiration date $T+\delta$ is $\delta\left(K-L\left(T,T\right)\right)^{+}$. Thus, floors are essentially a series of European put options on the LIBOR rate. The market for interest-rate caps and floors is generally termed the caps market.

Standard industry practice is to use Black's formula to price the caplet, which yields at time $t<T$,

$$B\left(t,T+\delta\right)\delta\left[L\left(t,T\right)\Phi\left(d_{+}\right)-K\Phi\left(d_{-}\right)\right] \tag{3}$$

where Φ is the cumulative normal distribution function and $d_{\pm}=\frac{\log(L(t,T)/K)\pm\frac{1}{2}\sigma_T^2(T-t)}{\sigma_T\sqrt{T-t}}$. The above formula is consistent with a LIBOR forward rate $L\left(t,T\right)$ that follows a log-normal distribution

$$\frac{dL\left(t,T\right)}{L\left(t,T\right)}=\sigma_T dW_t. \tag{4}$$

Several recent studies, such as Hull and White (2000), Andersen and Andreasen (2000), Andersen and Brotherton-Ratcliffe (2001), and Glasserman and Kou (2002), point out that the Black implied volatilities of caps with different strikes exhibit a volatility smile. The evidence provided in these studies, although interesting, is only anecdotal. This is because the evidence is based on only limited quotes from specific investment banks.

One contribution of our paper is that we provide a comprehensive empirical documentation of volatility smiles in caps based on more than two years of data collected from SwapPX.[7] Our data contains different strike daily cap prices (between 3:30 and 4:00 pm) from August 1, 2000 to November 1, 2002, and the underlying spot and forward swap rates. The caps are written on 3-month LIBOR and the swap rates are 6-month fixed-rate in exchange for LIBOR. Our data covers a wide range of strike prices and maturities. For example, every day for each maturity, there are ten different strike prices, which are 4.0, 4.5, 5.0, 5.5, 6.0, 6.5, 7.0, 8.0, 9.0, and 10.0 percent between August 1, 2000 and October 17, 2001, and 2.5, 3.0, 3.5, 4.0, 4.5, 5.0, 5.5, 6.0, 6.5, and 7.0 percent between November 2, 2001 and November 1, 2002.[8] Throughout the whole sample, caps have fifteen different maturities, which are 0.5, 1.0, 1.5, 2.0, 2.5, 3.0, 3.5, 4.0, 4.5, 5.0, 6.0, 7.0, 8.0, 9.0, and 10.0 years.

[7] The descriptions of the data supplied by SwapPX can be found at: www.govpx.com/mkting/start_swappx.html. There are twenty two pages of data on LIBOR and swap rates, and prices of interest rate caps, floors and swaptions. In our study, the forward rates are constructed from the forward swap rates on page 263-4 and the caps prices from page 290, which according to SwapPX represent the mid point of the bid and ask prices.

[8] The strike prices are lowered to 1.0, 1.5, 2.0, 2.5, 3.0, 3.5, 4.0, 4.5, 5.0 and 5.5 percent between October 18 and November 1, 2001.

To our knowledge, the only existing study that considers caps with different strikes is Gupta and Subrahmanyam (2001). Their data, obtained from Tullett and Tokoyo Liberty, covers a shorter time period (March 1 to October 31, 1998), has a narrower spectrum of strikes and maturities (four choices for each), and the maximum maturity is only five years.[9] Their sample also covers the turbulent periods of the Russian financial crisis and the collapse of LTCM in the summer and fall of 1998, making their data less reliable. In contrast, our data comes from the world's leading swap broker Garban-ICAP, covering longer time periods, with more strikes and maturities. Even though our sample includes September 11, 2001, we have sufficient observations to exclude September and October 2001 from our analysis. For this reason, we focus on the data from September 1, 2000 to August 31, 2001 and from November 2, 2001 to November 1, 2002.

Figure 1 contains the three-dimensional plot of the LIBOR forward rate curve constructed from the forward swap rates. In the last four months of year 2000, the yield curve is relatively flat. Starting with 2001, short-term forward rates have steadily declined. As long-term forward rates have remained relatively constant, the forward rate curve is upward sloping in 2001 and 2002. According to NBER, the economy peaked in March, 2001.[10] Thus, the steady declining of the short-term interest rates in 2001 may be a result of the economic recession. As indicated, the general macroeconomic conditions and term structure of interest rates experience rich variations in our sample, making it possible to study the pricing of caps under different economic environments.

Next we consider the caps' Black implied volatilities across ten different strike prices. For brevity, we focus on caps with 1.5, 3, 5, 7, and 10 year maturities, providing balanced coverage over the 10 year horizon.[11] Ideally we should use caplet prices to back out the implied volatilities. Unfortunately, we only observe caps prices. To facilitate the computation, we consider the differences between the prices of the 1.5, 3, 5, 7, and 10 year caps and the 1, 2.5, 4.5, 6, and 9 year caps, respectively. Thus, our analysis deals with only the sum of the few caplets between the two neighboring maturities. For example, for the rest of the paper, 1.5 year caps represent the sum of the 1.25 and 1.5 year caplets. We eliminate all observations that violate various arbitrage restrictions.

To examine the time-varying behavior of implied volatilities, we divide our two-year sample into four sub-samples: September 1, 2000 to March 2, 2001, March 5, 2001 to August 31, 2001, November 1, 2001 to May 3, 2002, and May 6, 2002 to November 1, 2002. Figure 2 plots the time series averages of the implied volatilities across moneyness for the four sub-samples.[12] The figure

[9] We would like to thank Anurag Gupta for detailed explanations of the data used in his paper.

[10] The following NBER site contains the business cycle dates: www.nber.org/cycles.html.

[11] We obtain similar evidence using caps with other maturities.

[12] Moneyness is defined as the ratio between the strike price and the forward rates underlying the caplets corresponding to each of the five maturities. On each day, for each specific maturity, we infer the Black implied volatilities from cap prices with different strikes. We interpolate the implied volatilities linearly between different

reveals several interesting facts.

First, short-term caps generally have higher implied volatilities than long-term caps. Previous studies, such as Leippold and Wu (2001), show that between 1995 and 2000, the implied volatilities for ATM caps are hump-shaped: the implied volatility first increases with horizon, reaches a plateau, then decreases. We find similar results in the first sub-sample. However, for the three other sub-samples, the implied volatility decreases monotonically with horizon. Second, before September 11, 2001, there is a relatively balanced distribution of ITM and OTM caps. However, after September 11, except for 1.5 year caps, most maturities are in-the-money, which is a result of the steep forward curve during this period. Third, the implied volatilities increase after September 11.

As indicated, there is a volatility smile in interest rate cap prices. Across all four sub-samples, we see a strong left skew for ITM caps and a mild right smile for OTM caps. This smile pattern is similar to that observed in equity options. Finally, the shape of the volatility smile is time varying. The smile is less pronounced and flattens out for deep ITM caps before September 11, but becomes more skewed after September 11. This could be the result of changing market expectations with respect to lower interest rates, due to the economic recession caused by the tragic event on September 11. Indeed, short-term interest rates declined and the forward curve steepened after September 11.

We also explored the dependence of implied volatilities on various factors through multivariate linear regressions in Table I. The independent variables include moneyness, time-to-maturity, and the level, slope and curvature factors of the term structure of interest rates. To construct the three term structure factors, we first convert the LIBOR forward curves into zero-coupon yield curves and follow the conventions in the literature: we measure the level factor with the 6 month yield, the slope factor with the difference between the 10 year and 6 month yields, and the curvature factor using the 10 year yield minus twice the 2 year yield plus the 6 month yield. We also measured the level factor by the 10 year yield and obtained similar results.

The regression results generally confirm the observations from Figure 2. Consistent with the volatility smile, we find that there is a significant negative relation between implied volatility and moneyness, with the regression coefficient becoming more negative after September 11, 2001. The level of implied volatility depends negatively on time-to-maturity, which is consistent with long-term caps having lower implied volatilities than short-term caps. The implied volatility also depends significantly on the three term structure factors, with negative loadings on the level and the slope factor, and a positive loading on the curvature factor. Again, the dependence becomes

strikes and compute the time series average of the implied volatilities for different moneyness over each of the four sub-samples. In results not reported here, we also divide the data into finer sub-samples and find that the implied volatilities in the above four periods are quite stable.

stronger after September 11, 2001. Consistent with the "unspanned stochastic volatility" puzzle, we find that the three term structure factors explain only 50 to 70% of the variations in implied cap volatilities (even when combined with moneyness and time-to-maturity).

II. LIBOR Market Models and the Calibration Procedure

The volatility smile observed in the cap markets raises an interesting challenge to existing term structure models and provides an opportunity for examining model performance. In this section, we focus on the LIBOR market models, a subclass of HJM models that have been widely used in the financial industry. We introduce a new approach for model calibration that is different from the procedure used in many equity option studies. We also compare the relative performance of different models using a statistic developed in the time series forecasting literature.

A. LIBOR Market Models

As previously mentioned, common industry practice is to quote caps based on the implied volatilities from Black's model. This essentially assumes that LIBOR rates follow a log-normal distribution. However, it is well known that LIBOR rates of different maturities cannot simultaneously follow the same log-normal distribution and still be arbitrage-free. In other words, there is no equivalent probability measure under which forward rates for all maturities simultaneously evolve according to the log-normal model. To overcome this inconsistency, BGM (1997) and MSS (1997) present an arbitrage-free interest rate model, the LIBOR market models, in which forward LIBOR rates follow log-normal processes under their corresponding forward measures, leading to Black's formula for caplets. Thus, the LIBOR models justify market practice.

There are several advantages of the LIBOR market models. First, the LIBOR market models are consistent with the market convention of quoting caps using Black's formula. This makes calibration very simple, because the quoted implied Black volatility can be directly inserted into the model, avoiding complex numerical procedures. Second, the market models are based on observable market rates. Simple compounding of this type is characteristic of three-month or six-month LIBOR rates, in contrast to the continuously compounded alternative. Third, caps are priced in LIBOR models similarly to equity options. Therefore, by focusing on LIBOR models, we can utilize the insights from the equity option pricing literature to help interpret our empirical findings.

Since the log-normal assumption of the standard LIBOR market model is not consistent with a volatility smile, several generalized models have been developed including the constant elasticity variance (CEV) model of Andersen and Andreasen (2000), the stochastic volatility model of Andersen and Brotherton-Ratcliffe (2001), and the jump diffusion model of Glasserman and Kou (2001)). The empirical performance of the analogous equity option pricing models has been extensively studied. However, for interest rate caps, the performance of these models remains an open question. Our empirical analysis attempts to answer this question.

Throughout our analysis, we follow the notation of Glasserman and Kou (2002) and restrict the cap maturity T to a finite set of dates $0 = T_0 < T_1 < ... < T_M < T_{M+1}$. We assume that the intervals $T_{i+1} - T_i$ are equally spaced by δ, a quarter of a year. Let $L_n(t) = L(t, T_n)$, so L_n is the forward rate for the actual period $[T_n, T_{n+1}]$. Similarly, let $B_n(t) = B(t, T_n)$ denote the price of a zero-coupon bond maturing on T_n. In LIBOR models, it is much easier to consider derivative pricing under the forward measure rather than the risk-neutral measure. Therefore, throughout our discussion of various models, we focus on the dynamics of the LIBOR forward rates $L_n(t)$ under the forward measure P_{n+1}, under which the discounted prices using $B_{n+1}(t)$ as the numeraire, is a martingale.

The first model considered in our paper is the stochastic volatility model of Andersen and Brotherton-Ratcliffe (2001), which can generate fat tails to help explain the volatility smile.[13] In this model, the LIBOR rate $L_n(t)$ satisfies:

$$\frac{dL_n(t)}{L_n(t)} = \frac{\varphi(L_n(t))}{L_n(t)} \sqrt{V(t)} dW_{n+1}(t), \tag{5}$$

$$dV(t) = \kappa(\theta - V(t)) dt + \eta \sqrt{V(t)} dZ_{n+1}(t), \tag{6}$$

where W_{n+1} and Z_{n+1} are independent Brownian motions under P_{n+1}. This model is very similar to the stochastic volatility model of Hull and White (1987), but quite different from that of Heston (1993) where stock prices are negatively correlated with their instantaneous volatility. In Heston's (1993) model, the negative correlation between stock price and volatility is essential for explaining the asymmetric smile in the equity options market.

As pointed out by Andersen and Brotherton-Ratcliffe (2001), the assumption of independence between the LIBOR rate and its volatility is consistent with the evidence that in all major fixed-income markets, the correlations between short-dated forward rates and their volatilities are indistinguishable from 0 (see e.g., Chen and Scott 1991). Throughout this paper, we choose the functional form of $\varphi(x) = x^\gamma$, where $0 < \gamma < 1$, which can generate a downward-sloping volatility skew. The smaller the γ, the stronger the skew. The uncorrelated stochastic volatility process $V(t)$ helps generate a symmetric smile in implied volatilities. Thus, the combination of both of these model features makes it possible to capture the asymmetric smile in cap markets. If $V(t)$ is constant, then the above stochastic volatility model reduces to the simpler CEV model of Andersen and Andreasen (2000).

Under the forward measure P_{n+1}, $C_n(t)/B(t, T_{n+1})$ is a martingale and thus in the absence

[13] There is strong empirical support for stochastic volatility in interest rates. The GARCH literature has shown that most financial time series, including interest rates, exhibit conditional heteroskedasticity. Andersen and Lund (1997) and Brenner, Harjes, and Kroner (1996) show that stochastic volatility or GARCH significantly improve the performance of pure diffusion models for spot interest rates.

of arbitrage, the time t price of a caplet maturing at T_{n+1} is

$$C_n(t) = \delta B(t, T_{n+1}) E_{T_{n+1}} \left[(L_n(T_n) - K)^+ | \mathcal{F}_t \right], \tag{7}$$

where $E_{T_{n+1}}$ is taken with respect to P_{n+1} and $\mathcal{F}_t$ is the information set at t.

Andersen and Brotherton-Ratcliffe (2001) provide a closed-form solution for a caplet's price using the Taylor series expansion technique of Hull and White (1987). Specifically they show that caplet's price equals

$$C_n(t) = \begin{cases} \delta B(t, T_{n+1}) g(t, L_n(t); c), & \text{if } V(t) \text{ is constant,} \\ \delta B(t, T_{n+1}) g(t, L_n(t); c^*(t, V)), & \text{if } V(t) \text{ follows (6),} \end{cases} \tag{8}$$

where $g(t, L_n(t); c) = L_n(t) \Phi(d_+) - K \Phi(d_-)$, $d_\pm = \frac{\ln(L_n(t)/K) \pm \frac{1}{2} \Omega(t, L_n(t), c)^2}{\Omega(t, L_n(t), c)}$, and c, $\Omega(t, L, c)$ and $c^*(t, V)$ are given in the appendix. If $V(t)$ is a constant, we normalize it to be 1.

The second model considered is the jump diffusion model of Glasserman and Kou (2002), extended to have a constant elasticity of variance diffusion.[14] Jumps provide a convenient way to generate fat-tailed distributions that help to explain the volatility smile. The original model of Glasserman and Kou (2002) is very similar to the jump diffusion model of Merton (1976) for equity options. In our extension, under P_{n+1},

$$\frac{dL_n(t)}{L_n(t-)} = -\widehat{\lambda}_n m_n dt + \frac{\varphi(L_n(t))}{L_n(-)} dW_{n+1}(t) + d\left(\sum_{j=1}^{N_t} (Y_j - 1) \right), \tag{9}$$

where W_{n+1} is a standard Brownian motion under P_{n+1}, N_t is a Poisson process with rate $\widehat{\lambda}_n$, W_{n+1} and N_t are independent, and $Y_j - 1$ are independent and distributed with $N\left(m_n, s_n^2\right)$. For $t < T_n$, the cap price equals

$$\begin{aligned} C_n(t) &= \delta B_{n+1}(t) E_{T_{n+1}} \left[(L_n(T_n) - K)^+ | \mathcal{F}_t \right] \\ &= \delta B_{n+1}(t) \sum_{j=0}^{\infty} e^{-\widehat{\lambda}_n (T_n - t)} \frac{\left(\widehat{\lambda}_n (T_n - t) \right)^j}{j!} g\left(t, L_n^{(j)}(t); c, j\right) \end{aligned} \tag{10}$$

where $L_n^{(j)}(t) = L_n(t) e^{-\widehat{\lambda}_n m_n (T_n - t)} (1 + m_n)^j$ and $g(t, L; c, j)$ equals $g(t, L; c)$ with $\Omega(t, L, c)^2$ replaced by $\Omega(t, L, c)^2 + j s_n^2$.

Finally, we consider a combined model that allows for both stochastic volatility and jumps. We assume that under P_{n+1} the LIBOR rate follows

$$\frac{dL_n(t)}{L_n(t-)} = -\widehat{\lambda}_n m_n dt + \frac{\varphi(L_n(t))}{L_n(-)} \sqrt{V(t)} dW_{n+1}(t) + d\left(\sum_{j=1}^{N_t} (Y_j - 1) \right), \tag{11}$$

[14] Many studies show that interest rates experience large discontinuous jumps due to various economic shocks or news announcements. For example, Johannes (2000) and Das (2002) show that stochastic volatility diffusion models cannot capture the excess kurtosis in spot interest rates, and that jumps significantly improve the performance of pure diffusion models.

where $V(t)$ follows the process in (6), and Z_{n+1}, W_{n+1}, and N_t are independent from each other. The price of a caplet in this generalized model equals

$$C_n(t) = \delta B_{n+1}(t) \sum_{j=0}^{\infty} e^{-\widehat{\lambda}_n (T_n - t)} \frac{\left(\widehat{\lambda}_n (T_n - t)\right)^j}{j!} G\left(t, L_n^{(j)}(t), V(t), j\right), \tag{12}$$

where $G(t, L, V, j) = g(t, L; c^*(t, V), j)$. Again in the above formula, $L_n(t)$ is replaced by $L_n^{(j)}(t)$ and $\Omega(t, L, c)^2$ by $\Omega(t, L, c)^2 + j s_n^2$ for the j-th jump, which account for the extra j argument in the functions $g(\cdot)$ and $G(\cdot)$.

Figure 2 suggests that to have a good empirical fit, the model has to generate an asymmetric smile. The above three models, which we will refer to as the SV, JD and SVJ models, all have this potential.[15] For equity options, Bakshi, Cao and Chen (1997) show that stochastic volatility provides first order improvements in both pricing and hedging performance. In our empirical analysis, we hope to identify similar robust and stylized empirical facts about the performance of various LIBOR models in capturing the volatility smile in interest rate caps. Unfortunately, we can not study hedging performance because the caps in our data are all constant maturity contracts. Therefore, we necessarily focus only on pricing.

B. Calibration Procedure

When implementing the above LIBOR market models, we have to estimate the model parameters and the unobservable instantaneous volatility. The common practice (see Bakshi, Cao and Chen 1997 and references therein) is to estimate the model parameters and instantaneous volatility from a cross-section of option prices on each date. This approach essentially allows time varying model parameters, and it has the disadvantage of not testing the true implications of a given model. As a result, this approach to calibration ignores the information in the time-series of option prices and it risks overfitting noise in the data. In our implementation of the LIBOR models, we adopt a different calibration procedure that keeps the model parameters constant throughout the entire sample period.

For models that do not have stochastic volatility, we estimate the parameters by minimizing the squared pricing errors of cap prices summed across different strikes and dates. For models that have stochastic volatility, the instantaneous volatility is not observable. Following an approach in the term structure literature that (for a given set of model parameters) backs out latent state variables from observed bond yields (see Duffee, 2002), we solve for the instantaneous volatility that prices ATM caps perfectly.[16] The parameters are obtained by minimizing the sum of squared

[15] In results not reported here, we also study the pure CEV model of Andersen and Andreasen (2000) and the pure jump diffusion model of Glasserman and Kou (2002). We find that they significantly underperform the three models considered in this paper.

[16] In our analysis, ATM caps have moneyness between 0.93 to 1.07.

pricing errors of caps over different strike prices and dates. Assuming ATM caps are perfectly priced makes it easy to solve for the instantaneous implied volatility.

Our calibration procedure can be described formally as follows. Let Θ represent the model parameters which remain constant over the whole sample period. Suppose we have time series observations of the prices of N caplets with the same time to maturity but different strike prices, $C(t, \tau, K_i)$, where $i = 1, 2, ..., N$, $t = 1, 2, ..., T$, τ is time to maturity and K_i is strike price. Let $\hat{C}(t, \tau, K_i, V(t, \Theta), \Theta)$ be the corresponding theoretical price under a given model, where $V(t, \Theta)$ is solved from ATM cap prices given Θ for models with stochastic volatility. For each i and t, denote the pricing error as

$$\varepsilon_{i,t}[\Theta] = C(t, \tau, K_i) - \hat{C}(t, \tau, K_i, V(t, \Theta), \Theta). \tag{13}$$

Then, the sum of squared pricing errors (SSE) on date t is $SSE(t) = \frac{1}{N}\sum_{i=1}^{N} |\varepsilon_{i,t}[\Theta]|^2$.

To minimize the impact of outliers, we choose the parameter vector Θ to minimize the average of log daily SSE over the whole sample period

$$SSE \equiv \min_{\Theta} \frac{1}{T} \sum_{t=1}^{T} \log[SSE(t)]. \tag{14}$$

In contrast, what has been done in the equity option literature, such as Bakshi, Cao, and Chen (1997) and others, is to estimate $V(t)$ and Θ each day from N options on day t, i.e.,

$$\min_{\Theta, V(t)} \sum_{i=1}^{N} \left| C(t, \tau, K_i) - \hat{C}(t, \tau, K_i, V(t), \Theta) \right|^2. \tag{15}$$

This approach will produce time varying parameters which are inconsistent with the assumed models.

The pricing errors of a good model should be close to zero. While previous studies have compared relative model performance in terms of pricing errors, they have not rigorously tested whether the difference in pricing errors between different models is statistically significant. For this purpose, we use a statistic developed by Diebold and Mariano (1995) in the time-series forecast literature. Consider two models whose associated daily sum of squared pricing errors are $\{SSE_1(t)\}_{t=1}^{T}$ and $\{SSE_2(t)\}_{t=1}^{T}$. The null hypothesis that the two models have the same pricing errors is $E[SSE_1(t)] = E[SSE_2(t)]$, or $E[d_t] = 0$, where $d_t = SSE_1(t) - SSE_2(t)$. Diebold and Mariano (1995) show that if $\{d_t\}_{t=1}^{T}$ is covariance stationary and short memory, then

$$\sqrt{T}\left(\overline{d} - \mu\right) \sim N(0, 2\pi f_d(0)), \tag{16}$$

where $\overline{d} = \frac{1}{T}\sum_{t=1}^{T}[SSE_1(t) - SSE_2(t)]$, $f_d(0) = \frac{1}{2\pi}\sum_{q=-\infty}^{\infty}\gamma_d(q)$ and $\gamma_d(q) = E[(d_t - \mu)(d_{t-q} - \mu)]$. In large samples, $\overline{d}$ is approximately normally distributed with mean μ and variance $2\pi f_d(0)/T$.

Thus under the null hypothesis of equal pricing errors, the following statistic

$$S = \frac{\overline{d}}{\sqrt{2\pi \widehat{f}_d(0)/T}} \tag{17}$$

is distributed asymptotically as $N(0,1)$, where $\widehat{f}_d(0)$ is a consistent estimator of $f_d(0)$.[17] We use the above statistic to measure whether one model has significantly smaller pricing errors than another.

Before moving on to the empirical section, we briefly discuss two important issues that could affect the interpretation of our results. First, part of the volatility smile could be due to liquidity, i.e., ITM and OTM caps are less liquid than ATM caps. Unfortunately, our data does not contain any measures of liquidity, such as a bid-ask spread for ITM and OTM caps. Although it is unlikely that all of the pricing errors we document are due to liquidity, we acknowledge that by ignoring liquidity we could understate the performance of these models. Second, we observe only the sums of caplet prices, rather than individual caplet prices themselves. If we allow the LIBOR rates underlying each of the caplets to follow different processes, we would end up with too many parameters. For example, we need to estimate 28 parameters for SVJ using 10 year caps which consist of 9.25, 9.5, 9.75, and 10 year caplets. The parameters we obtain can be considered as averages of the individual LIBOR rates included in the cap. While the restrictions we impose might worsen in-sample performance, they might also help out-of-sample performance. This is because an overparameterized model could over fit the data. Although these simplifications may affect our *exact* estimates of the pricing errors or parameters, they are unlikely to affect our findings concerning the different modeling mechanisms for capturing the volatility smile.

III. Empirical Results

In this section, we study the empirical performance of alternative LIBOR models in capturing the cap volatility smile. We examine both the in-sample and out-of-sample performance of the model pricing errors, and we compare the performance of the different models with the Diebold-Mariano statistic.

A. Implied Parameters and In-Sample Performance

As suggested by Figure 2 and Table I, the pattern of the volatility smile is quite different before and after September 11, 2001, suggesting that there may be a structural break in the data. Therefore, we estimate model parameters separately using the samples before and after September 11. Tables II and III report model performance for the pre- and post-September 11 samples, respectively. Four different measures are included: (i) time series averages of daily log SSE; (ii) relative model performance as measured by the Diebold-Mariano statistic; (iii) pricing

[17] Asymptotically parameter estimation uncertainty has no impact on the test statistic.

errors across strike prices and dependence of the pricing errors on various economic variables; and (iv) the calibrated model parameters.

Panel A of Table II contains the time-series averages of the daily log SSE for the four LIBOR models for the pre-September 11 sample.[18] All three models improve upon Black's model.[19] The most comprehensive model, SVJ, has the smallest pricing errors for caps with all maturities. The SVJ model reduces the pricing errors of the Black model by about 40% (1.5 year), 30% (5 year), and 20% (all other maturities). JD has smaller pricing errors than SV for short-term (1.5 and 3 year) caps, but bigger pricing errors for medium- and long-term (5, 7, and 10 year) caps. The Diebold-Mariano statistics in Panel B of Table II allow us to measure whether the models in the second column have significantly higher pricing errors than the models in the first row.[20] We see that the only model that significantly outperforms Black's model across five different maturities is JD. The SVJ model has significantly smaller pricing errors than Black's model only for 5 year caps. Even though SVJ has smaller pricing errors than JD, the difference is not statistically significant for all five maturities.

Despite the improvements, it is clear that all three models are still misspecified and cannot fully capture the volatility smile in interest rate caps. Figure 3 presents average squared pricing errors as a function of moneyness.[21] Consistent with the time-series averages in Panel A of Table II, Black's model has large pricing errors, which, similar to the Black implied volatilities, also exhibit an asymmetric smile. All three models reduce the pricing errors of Black's model across different strikes. While performing reasonably well for OTM caps, they still have significant pricing errors for ITM caps. SVJ generally has the smallest pricing errors, and SV has smaller pricing errors than JD for medium- and long-term caps.

The inadequacies of the above models can also be seen from a multivariate regression of the pricing errors of caps with different strikes and maturities on moneyness, time-to-maturity, and the three term structure factors in Panel C of Table II. The three models have significant moneyness and maturity biases, although they are generally weaker than Black's model. Consistent with the results in Table I, ITM and long-term caps have higher pricing errors. The three models reduce

[18] Given calibrated parameters and implied instantaneous volatility, we calculate the $\log[SSE(t)]$ of each day and its time series average $\frac{1}{T}\sum_{t=1}^{T}\log[SSE(t)]$. The cap prices we observe represent the percentage of principal amount which equals \$10,000 in our analysis.

[19] It is obvious that the constant volatility Black's model would not be able to fit the data well. Therefore, in our implementation we allow the volatility in Black's model to be time varying. We estimate it each day by minimizing the daily sum of squared pricing errors.

[20] The statistics are calculated using $\{\log[SSE(t)]\}_{t=1}^{T}$ of each model.

[21] Each day we compute the squared pricing errors of caps with different moneyness based on the calibrated parameters and the inferred volatilities from ATM caps. We linearly interpolate the pricing errors across moneyness and calculate the time series average of pricing errors at the same moneyness over the sample period. We also consider absolute pricing errors and obtain similar results.

the dependence of the Black model's pricing errors on the slope and curvature factors, but they increase the dependence on the level factor.

The R^2 of the regression measures the percentage of pricing errors of a given model explained by the five factors. A perfect model should have a R^2 close to zero. Interestingly, JD, although having a bigger moneyness bias, has the smallest R^2. While SV and JD have similar pricing errors, SV has a higher R^2.

Calibrated model parameters in Panel D of Table II reveal insights about the performance of the three models in explaining the asymmetric volatility smile.[22] The estimated CEV parameter γ is significantly smaller than 1 in all models, which according to Andersen and Brotherton-Ratcliffe (2001), can generate a strong downward-sloping volatility skew. In contrast, the CEV is the only mechanism that generates volatility skew in JD as the estimated jump sizes are positive with small variances. The CEV parameters in SV and SVJ are generally higher than that in JD. However, combining SV and JD does not significantly improve model performance. Parameter estimates for SVJ, especially the variances of the jump sizes seem to be unreasonably high. This indicate that although SVJ has smaller pricing errors than JD, it may be misspecified. Therefore, the CEV combined with upward jumps in JD or uncorrelated stochastic volatility in SV provides a good characterization of the asymmetric smile observed in the data. These characteristics cannot be significantly improved upon by the more complicated model SVJ.

The results for the post-September 11 sample generally confirm the above findings, although with certain modifications. Figure 2 and Table I indicate that after September 11, the implied volatilities increase dramatically and the volatility smile becomes more pronounced, especially for ITM but also for OTM caps. Consistent with this evidence, the average pricing errors of Black's model in Panel A of Table III increase dramatically. This indicates that the log-normal assumption becomes much more problematic after September 11. The advantages of the generalized LIBOR models also become more apparent as they provide a more significant reduction in pricing errors after September 11. For example, SVJ reduces the pricing errors of Black's model by 50 to 60 % for all caps, and the resulting pricing errors are smaller than those of the pre-September 11 sample. The Diebold-Mariano statistics in Panel B of Table III show that the three models significantly

[22] We do not report standard errors of model parameters because our models are calibrated rather than estimated using rigorous econometric methods such as the method of moments or maximum likelihood. Although our calibration procedure can be regarded as a nonlinear regression, it is definitely non-standard. The data used in our study has a panel structure with both cross section and time series components. However, the number of observations on each day differs because some observations are deleted due to violations of arbitrage resctrictions. Our calibration also involves inferring unobserved latent stochastic volatility from ATM caps. Some of our parameter estimates are also on the boundaries of the possible values making it difficult to compute standard errors even in standard settings. Currently we are not aware of any existing econometric procedure that *rigorously* addresses all the above issues.

outperform Black's model. Unlike the previous findings, no single model clearly dominates the other models for all maturities. The three models have similar performance for 1.5 and 7 year caps, and SVJ has the best performance for 3 and 5 year caps, while JD has the best performance for 10 year caps.

Figure 4 shows that the pricing errors of Black's model increase dramatically across different strikes after September 11, which is consistent with the pronounced volatility smile observed in this period. The three models also significantly reduce the pricing errors of the Black model across strikes and their pricing errors are generally smaller than that in the pre-September 11 sample. However, all the models still have significant pricing errors, especially for ITM caps.

The regression results also show that the three models perform much better in the post-September 11 sample. After September 11, although Black's model has stronger moneyness bias and a higher R^2, the three models have weaker moneyness bias and smaller R^2s than in the pre-September 11 sample. The three models still have significant maturity bias, with smaller pricing errors for long-term caps versus short-term caps. The three models also weaken the dependence of the pricing errors on the term structure factors. In summary, it seems that the three models capture caps prices much better in the post-September 11 era.

Parameter estimates in Panel D of Table III show that similar to the pre-September 11 sample, the estimated CEV parameters for the three models (except for JD at 1.5 year) are significantly smaller than one. Therefore, the CEV model again plays an important role in explaining the volatility skew in the data for all three models. The only difference is that after September 11, the implied volatility from ITM caps become so skewed that CEV alone is not adequate. Strong negative jumps are also needed to capture the smile, especially for short- and medium-term caps. For example, for caps with maturities less than 5 years, in JD and SVJ, the estimated CEV parameters are significantly less than one and jump sizes are also significantly negative with extremely high variances. However, for long-term (7 and 10 years) caps, CEV with upward jumps or uncorrelated stochastic volatility again have the best performance.

The empirical evidence in Table II and III shows that the most important modeling feature in capturing the asymmetric volatility smile is the constant elasticity of variance, which helps to generate the downward-sloping volatility skew. Combined with uncorrelated stochastic volatility or upward jumps, the CEV model provides good performance for all caps in the pre-September 11 sample and long-term (7 and 10 years) caps in the post- September 11 sample. These cannot be significantly improved by the more complicated SVJ model. However, after September 11, for short- and medium-term (less than 5 year) caps, the strong left skew in implied volatilities require both CEV and strong negative jumps to capture the smile. Therefore, SVJ has the best performance for 3 and 5 year caps because uncorrelated stochastic volatility is needed to capture the mild smile for OTM caps.

B. Out-of-Sample Performance

The in-sample analysis shows that the more complex models significantly improve the Black model in capturing the volatility smile in caps. To check the robustness of our results and avoid any in-sample overfitting, we study the out-of-sample performance of the different models. We divide the data before and after September 11, 2001 into two sub-samples, and we calculate pricing errors in the second sub-sample based on parameters estimated during the first sub-sample. The out-of-sample performance of alternative LIBOR models using data before and after September 11 are reported in Tables IV and V respectively. Again, we examine the out-of-sample performance with the four measures used in the in-sample performance.

Panel A and B of Table IV report in-sample and out-of-sample pricing errors based on the first and second sub-samples before September 11, respectively. It is obvious that the out-of-sample pricing errors are dramatically higher than the in-sample pricing errors. Diebold-Mariano statistics for in-sample (not reported) and out-of-sample (Panel C of Table IV) pricing errors show that JD has the best in-sample and out-of-sample performance, although the percentages of reduction in pricing errors decline from around 20% for in-sample to less than 10% for out-of-sample results.

Figure 5 shows that the three models only provide marginal improvements in pricing errors across strikes. JD has better performance than the other models, especially for 3 and 5 year caps. Regressions in Panel D of Table IV show that the three models reduce the dependence of the pricing errors of the Black model on all factors, although they have bigger moneyness bias and higher R^2s than in the regressions using the whole sample. Therefore, before September 11, the three models have much worse out-of-sample performance and less significant improvements over Black's model. The deterioration of model performance is mainly due to the steepening of the volatility smile in the second sub-sample (see Figure 2 (a) and (b)). Nonetheless, the same modeling mechanism, i.e., CEV combined with upward jumps, still has the best in-sample and out-of-sample performance.

The generalized LIBOR models have much better out-of-sample performance after September 11. Panel A and B of Table V show that the out-of-sample pricing errors, except for 1.5 and 7 year caps, are comparable or even smaller than the in-sample pricing errors, and all models significantly improve upon Black's model. In the second sub-sample, the volatility smile becomes less (more) pronounced for 1.5 year (5, 7 and 10 year) caps, and the smile for OTM caps become more significant for 3 year caps. Because of the changes, the out-of-sample performance of JD becomes significantly worse, especially for 3 and 5 year caps. On the other hand, it is interesting to see that SVJ has both a good in-sample and out-of-sample performance, and its out-of-sample pricing errors are smaller than the in-sample pricing errors. For 1.5 year caps, SV has the best performance, and for 3, 5, 7, and 10 year caps, SVJ has the best performance (although some

Diebold-Mariano statistics are significant only at 10% level). Parameter estimates in Panel E of Table V show that the estimated jump sizes for 5, 7, and 10 year caps in SVJ are positive (which help capture the smile for OTM options). In spite of the improved out-of-sample performance, the three models still have significant pricing errors for ITM caps as shown in Figure 6 and in the regressions in Panel D of Table V. Consistent with the in-sample results, we find that the three models also have much smaller regression coefficients on moneyness in the post-September 11 sample.

Our empirical analysis reveals some stylized facts about the performance of the generalized LIBOR models in capturing the cap volatility smile. We find that one modeling mechanism that is instrumental for all models, all sample periods, and both in-sample and out-of-sample performance, is the constant elasticity of variance model. Combined with uncorrelated stochastic volatility in SV or upward jumps in JD, CEV has good performance for most cases, except for short- and medium-term caps. These caps also require strong downward jumps to explain the data after September 11. Our out-of-sample analysis shows that although the shape of volatility smile seems to be time varying, some LIBOR models have reasonably good in-sample and out-of-sample performance, especially after September 11. Our findings are similar to the equity option pricing literature. As shown by Bakshi, Cao and Chen (1997) for equity options, stochastic volatility is of first order importance in improving upon the Black-Scholes model. In their stochastic volatility model, the negative correlation between stock price and its volatility generates the downward-sloping skew in implied volatilities. In contrast, in the three LIBOR models considered in our paper, the downward-sloping skew is generated by the CEV model and the uncorrelated stochastic volatility or upward jumps helps to generate the mild smile for OTM caps.

IV. Conclusion

In this paper, we have made two contributions to a growing literature on pricing LIBOR and swap-based interest rate derivatives. First, we provide a systematic documentation of volatility smiles in interest rate caps. Our data, obtained from the world leading swap broker Garban-ICAP, contains rich cross-sectional information on interest rate caps with different strike prices and maturities. We show that Black implied volatilities exhibit an asymmetric smile which becomes more pronounced after September 11, 2001. Second, we study the empirical performance of generalized LIBOR market models, which have been widely used in the financial industry. We find that a constant elasticity variance model combined with uncorrelated stochastic volatility or upward jumps has the best performance, except for short- and medium-term caps. For these caps, strong negative jumps are also needed after September 11. However, even the most sophisticated LIBOR market models cannot fully capture the volatility smile. In this paper we focus on a subclass of the HJM models—the LIBOR market models. However, one could also study other subclasses of HJM models such as the affine class of Dai and Singleton (2000), the quadratic class of Ahn, Dittmar and Gallant (2002), and the random field models of Santa-Clara and Sornette (2001), Goldstein (2000) and Collin-Dufresne and Goldstein (2001). The volatility smiles documented herein hopefully will provide new opportunities for understanding the performance of these various term structure models.

Mathematical Appendix

Proposition 1 of Andersen and Brotherton-Ratcliffe (2001) shows that $g(t, L, c)$ equals

$$g(t, L, c) = L\Phi(d_+) - K\Phi(d_-), \; d_{\pm} = \frac{\ln(L/K) \pm \frac{1}{2}\Omega(t, L, c)^2}{\Omega(t, L, c)},$$

where Φ is the cumulative normal distribution function, $\tau = T - t$, and

$$\begin{aligned}
\Omega(t, L, c) &= \Omega_0(L)\, c^{1/2}\tau^{1/2} + \Omega_1(L)\, c^{3/2}\tau^{3/2} + O\left(\tau^{5/2}\right), \\
\Omega_0(L) &= \frac{\ln(L/K)}{\int_K^L \varphi(u)^{-1}\, du}, \\
\Omega_1(L) &= -\frac{\Omega_0(L)}{\left(\int_K^L \varphi(u)^{-1}\, du\right)^2} \ln\left(\Omega_0(L)\left(\frac{LK}{\varphi(L)\,\varphi(K)}\right)^{1/2}\right).
\end{aligned}$$

Proposition 3 of Andersen and Brotherton-Ratcliffe (2001) shows that $G(t, L, V) = g(t, L, c^*(t, V))$, where

$$c^*(t, V) = \overline{c}(t, V) + \alpha_0\eta^2 + \alpha_1\eta^2 Y^2 + O\left(\eta^4\right),$$

where $Y = \ln(L/K)$, $\overline{c}(t, V) = \tau^{-1}\mu_U(t, V)$, $\mu_U(t, V) = \int_t^T \left(\theta + (V(t) - \theta)\, e^{-\kappa(u-t)}\right) du$, and the coefficients αs are given in Andersen and Brotherton-Ratcliffe (2001). The authors have shown that the approximation works very well for realistic model parameters with maturities up to 10 years. In our setting, the constant $c = 1$.

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Table I

Multivariate Regression of Black Implied Volatilities

This table reports multivariate regressions of Black implied volatilities for caps with different strike prices and maturities on moneyness, time to maturity, and level, slope, and curvature factors of the term structure of interest rates. The pre-September 11 sample is from September 1, 2000 to August 31, 2001 and the post-September 11 sample is from November 2, 2001 to November 1, 2002. To construct the three term structure factors, we first convert LIBOR forward curves into zero-coupon yield curves and measure the level factor by 6 month yield, the slope factor by the difference between 10 year and 6 month yields, and the curvature factor by 10 year yield minus twice of 2 year yield plus 6 month yield. We also measure the level factor by 10 year yield and obtain similar results. Standard errors are shown in the parentheses.

	Pre-September 11 Sample	**Post-September 11 Sample**
	Parameter Estimates	Parameter Estimates
Intercept	0.435 (0.011)	0.871 0.027)
Moneyness	-0.058 (0.001)	-0.132 (0.002)
Maturity	-0.004 (0.000)	-0.033 (0.000)
Level	-2.915 (0.146)	-7.254 (0.938)
Slope	-1.587 (0.148)	-4.815 (0.276)
Curvature	1.152 (0.304)	3.750 (0.452)
R^2	0.470	0.692

Table II

In-Sample Performance for the Pre-September 11 Sample

This table reports the in-sample performance of four LIBOR market models calibrated using the data from September 1, 2000 to August 31, 2001. We consider average daily log SSE, relative model performance measured by Diebold-Mariano statistics, dependence of pricing errors on various economic variables, and calibrated model parameters.

Panel A: Average Daily Log SSE.

	Maturities (year)				
Model	1.5	3	5	7	10
Black	2.504	3.827	4.399	4.226	4.231
SV	2.011	3.307	3.389	3.255	3.436
JD	1.696	3.072	3.467	3.483	3.565
SVJ	1.511	2.995	3.080	3.252	3.433

Panel B: Relative Model Performance. Diebold-Mariano statistics measure whether the models in the second column have significantly higher pricing errors than that in the first row. The statistics are calculated according to equation (17) using daily log SSE. The lag order q equals 40. The statistics follow an asymptotic standard Normal distribution.

		SV	JD	SVJ
1.5 year	Black	1.14	2.86	1.23
	SV	-	1.86	1.30
	JD	-	-	0.34
3.0 year	Black	0.70	3.12	1.91
	SV	-	0.46	1.00
	JD	-	-	0.35
5 year	Black	1.85	7.29	3.07
	SV	-	-0.17	1.38
	JD	-	-	1.11
7 year	Black	1.07	3.76	1.07
	SV	-	-0.31	0.49
	JD	-	-	0.31
10 year	Black	0.99	3.42	0.99
	SV	-	-0.19	0.49
	JD	-	-	0.20

Panel C: Dependence of Pricing Errors on Various Economic Variables. We regress the pricing errors of caps with different strike prices and maturities for the pre-September 11 sample, on moneyness, time to maturity, the level, slope, and curvature factors of the term structure of interest rates. We measure the level factor by 6 month yield, the slope factor by the difference between 10 year and 6 month yields, and the curvature factor by 10 year yield minus twice of 2 year yield plus 6 month yield. We also measure the level factor by 10 year yield and obtain similar results. Standard errors are shown in the parentheses.

	Black		**SV**		**JD**		**SVJ**	
Intercept	0.013	(0.008)	-0.015	(0.003)	-0.006	(0.004)	-0.008	(0.003)
Moneyness	-0.033	(0.001)	-0.010	(0.000)	-0.013	(0.000)	-0.009	(0.000)
Maturity	6.1E-04	(8.3E-05)	5.4E-04	(2.9E-05)	4.0E-04	(4.1E-05)	6.1E-04	(3.0E-05)
Level	0.114	(0.116)	0.415	(0.041)	0.197	(0.057)	0.253	(0.041)
Slope	0.260	(0.117)	0.115	(0.041)	0.167	(0.058)	-0.018	(0.042)
Curvature	3.279	(0.240)	0.917	(0.084)	1.659	(0.119)	0.830	(0.086)
R^2	0.213		0.221		0.150		0.168	

Panel D: Calibrated Model Parameters. Model parameters are obtained by minimizing the objective function in equation (14) using the pre-September 11 sample.

		Parameter Estimates						
Model	Maturity	γ	κ	θ	η	λ_n	m_n	s_n
SV	1.5	0.0012	0.0007	0.0831	2.1E-05			
	3	0.0000	2.1E-05	4.3378	0.0146			
	5	0.2321	0.0158	0.0138	0.0297			
	7	0.0643	7.3E-06	8.4337	0.0152			
	10	0.2308	3.9E-05	2.1213	0.0187			
JD	1.5	5.5E-04				0.3237	0.1507	3.1E-07
	3	9.0E-06				0.0948	0.3860	0.0016
	5	1.5E-04				0.0177	0.8225	0.3566
	7	6.8E-04				0.0259	0.6030	0.3518
	10	0.0015				0.0132	1.1853	0.0277
SVJ	1.5	0.0031	0.0009	0.0635	0.0114	0.0079	-0.6698	3.5175
	3	0.0024	0.0007	0.1191	0.0117	0.0012	1.2459	4.6650
	5	0.0403	0.0003	0.2758	0.0154	0.0017	0.9935	4.6088
	7	0.0619	8.4E-05	0.7178	0.0152	3.6E-06	-0.3801	7.4638
	10	0.2342	0.0001	1.3418	0.0185	0.0002	-0.4811	4.5110

Table III

In-Sample Performance for the Post-September 11 Sample

This table reports the in-sample performance of four LIBOR market models calibrated using the data from November 2, 2001 to November 1, 2002. We consider average daily log SSE, relative model performance measured by Diebold-Mariano statistics, dependence of pricing errors on various economic variables, and calibrated model parameters.

Panel A: Average Daily Log SSE.

	Maturities (year)				
Model	1.5	3	5	7	10
Black	3.966	6.197	5.930	5.736	4.783
SV	1.825	2.980	2.547	2.649	2.544
JD	1.614	3.316	3.250	2.723	2.439
SVJ	1.506	2.605	2.542	2.342	2.439

Panel B: Relative Model Performance. Diebold-Mariano statistics measure whether the models in the second column have significantly higher pricing errors than that in the first row. The statistics are calculated according to equation (17)) using daily log SSE. The lag order q equals 40. The statistics follow an asymptotic standard Normal distribution.

		SV	JD	SVJ
1.5 year	Black	8.62	4.93	4.05
	SV	-	0.38	0.52
	JD	-	-	0.14
3.0 year	Black	18.61	11.80	15.71
	SV	-	-1.21	2.45
	JD	-	-	3.52
5 year	Black	17.23	23.54	17.30
	SV	-	-3.53	3.60
	JD	-	-	3.56
7 year	Black	19.02	12.32	8.75
	SV	-	-0.56	0.76
	JD	-	-	0.72
10 year	Black	9.16	9.51	9.56
	SV	-	1.97	1.97
	JD	-	-	-

Panel C: Dependence of Pricing Errors on Various Economic Variables. We regress the pricing errors of caps with different strike prices and maturities for the post-September 11 sample, on moneyness, time to maturity, the level, slope, and curvature factors of the term structure of interest rates. We measure the level factor by 6 month yield, the slope factor by the difference between 10 year and 6 month yields, and the curvature factor by 10 year yield minus twice of 2 year yield plus 6 month yield. We also measure the level factor by 10 year yield and obtain similar results. Standard errors are shown in the parentheses.

	Black		**SV**		**JD**		**SVJ**	
Intercept	0.257	(0.019)	0.010	(0.003)	0.016	(0.004)	-0.008	(0.026)
Moneyness	-0.097	(0.001)	-0.005	(0.000)	-0.008	(0.000)	-0.004	(0.002)
Maturity	-5.9E-03	(1.5E-04)	-1.6E-04	(2.1E-05)	-5.2E-04	(3.0E-05)	-4.4E-04	(2.1E-04)
Level	-4.094	(0.650)	-0.135	(0.087)	-0.247	(0.125)	-0.690	(0.908)
Slope	-1.042	(0.191)	-0.007	(0.026)	-0.024	(0.037)	0.901	(0.267)
Curvature	0.984	(0.313)	0.021	(0.042)	0.310	(0.060)	0.160	(0.438)
R^2	0.392		0.076		0.123		0.002	

Panel D: Calibrated Model Parameters. Model parameters are obtained by minimizing the objective function in equation (14) using the post-September 11 sample.

		Parameter Estimates						
Model	Maturity	γ	κ	θ	η	λ_n	m_n	s_n
SV	1.5	0.2754	8.2E-04	1.8501	0.0623			
	3	0.0000	5.0E-04	0.2804	0.0259			
	5	0.0755	6.3E-05	1.6569	0.0200			
	7	0.0971	8.9E-06	8.7446	0.0111			
	10	0.2473	2.0E-05	4.7228	0.0125			
JD	1.5	1				0.0562	-0.9267	5.2997
	3	0.0000				0.0439	-0.5655	8.7079
	5	0.0000				0.0173	-0.2916	17.7680
	7	0.0000				0.0066	-0.0662	8.6766
	10	0.0000				0.0059	1.1517	0.3947
SVJ	1.5	0.4363	0.0014	2.8204	0.0933	0.0424	-0.7932	10.5940
	3	0.0007	0.0008	0.1411	0.0232	0.0160	-0.8655	11.3900
	5	0.0772	1.5E-05	7.1236	0.0202	2.0E-06	-0.0982	9.9815
	7	0.0003	6.9E-05	0.4338	0.0113	0.0157	0.6624	0.0003
	10	0.0000	0	0	0	0.0059	1.1517	0.3947

Table IV

Out-of-Sample Performance for the Pre-September 11 Sample

This table reports the out-of-sample performance of four LIBOR market models calibrated using the first half of the pre-September 11 sample. Based on parameters estimated using the data between September 1, 2000 and March 2, 2001, we calculate the in-sample and out-of-sample pricing errors using the first and second half of the pre-September 11 sample respectively. We consider in-sample and out-of-sample average daily log SSE, relative model performance measured by Diebold-Mariano statistics, dependence of out-of-sample pricing errors on various economic variables, and calibrated model parameters.

Panel A: In-Sample Average Daily Log SSE

	Maturities (year)				
Model	1.5	3	5	7	10
Black	1.538	2.593	3.454	3.079	3.209
SV	1.498	2.473	2.506	2.240	2.935
JD	1.234	2.238	2.643	2.183	2.640
SVJ	1.234	2.238	2.504	2.169	2.588

Panel B: Out-of-Sample Average Daily Log SSE

	Maturities (year)				
Model	1.5	3	5	7	10
Black	3.477	5.031	5.246	5.372	5.010
SV	3.435	4.794	4.780	5.011	4.585
JD	3.183	3.880	4.247	5.050	4.576
SVJ	3.183	3.880	4.787	5.058	4.598

Panel C: Relative Model Performance. Diebold-Mariano statistics measure whether the models in the second column have significantly higher out-of-sample pricing errors than that in the first row. The statistics are calculated according to equation (17)) using daily log SSE. The lag order q equals 40. The statistics follow an asymptotic standard Normal distribution.

		SV	JD	SVJ
1.5 year	Black	7.56	11.58	11.64
	SV	-	12.19	12.16
	JD	-	-	-
3.0 year	Black	34.67	32.22	32.24
	SV	-	25.64	25.66
	JD	-	-	-
5 year	Black	6.90	7.30	6.91
	SV	-	5.52	-5.96
	JD	-	-	-5.55
7 year	Black	9.48	9.45	10.07
	SV	-	-1.63	-2.45
	JD	-	-	-1.36
10 year	Black	8.20	19.22	25.00
	SV	-	0.19	-0.20
	JD	-	-	-1.09

Panel D: Dependence of Out-of-Sample Pricing Errors on Various Economic Variables. We regress the out-of-sample pricing errors of caps with different strike prices and maturities for the pre-September 11 sample, on moneyness, time to maturity, the level, slope, and curvature factors of the term structure of interest rates. We measure the level factor by 6 month yield, the slope factor by the difference between 10 year and 6 month yields, and the curvature factor by 10 year yield minus twice of 2 year yield plus 6 month yield. We also measure the level factor by 10 year yield and obtain similar results. Standard errors are shown in the parentheses.

	Black		**SV**		**JD**		**SVJ**	
Intercept	-0.042	(0.018)	-0.034	(0.014)	-0.030	(0.012)	-0.033	(0.013)
Moneyness	-0.061	0.001)	-0.042	(0.001)	-0.035	(0.001)	-0.037	(0.001)
Maturity	1.1E-03	(1.4E-04)	5.5E-04	(1.1E-04)	1.2E-03	(9.6E-05)	1.2E-03	(1.0E-04)
Level	2.170	(0.281)	1.641	(0.213)	1.358	(0.191)	1.473	(0.202)
Slope	0.941	(0.265)	0.643	(0.201)	0.418	(0.180)	0.455	(0.191)
Curvature	1.922	(0.405)	1.509	(0.307)	1.153	(0.274)	1.362	(0.291)
R^2	0.307		0.270		0.265		0.268	

Panel E: Calibrated Model Parameters. Model parameters are obtained by minimizing the objective function in equation (14) using the first half of the pre-September 11 sample.

		Parameter Estimates						
Model	Maturity	γ	κ	θ	η	λ_n	m_n	s_n
SV	1.5	1	2.8376	0.0146	0.0051			
	3	0.8391	0.0694	0.1316	0.0425			
	5	0.8010	1.0199	0.0024	0.2786			
	7	0.8209	0.0178	0.1476	0.0594			
	10	0.6970	0.0534	0.0207	0			
JD	1.5	0.7344				1.1711	0.0906	1.7E-08
	3	2.8E-04				1.0269	0.1436	1.2E-06
	5	5.4E-05				0.0362	0.5163	0.3444
	7	0.9999				0.0043	-0.6529	7.5368
	10	0.2120				0.2991	0.2752	1.0E-05
SVJ	1.5	0.7344	0	0	0	1.1711	0.0906	1.7E-08
	3	2.8E-04	0	0	0	1.0269	0.1436	1.2E-06
	5	0.8035	1.0374	0.0020	0.2808	5.8E-07	-1	2.0E-05
	7	0.9430	0.0294	0.0958	0.0544	0.0022	-0.8116	15.6870
	10	0.3412	0.0067	0.0093	0.0057	0.0385	0.6690	0.0551

Table V
Out-of-Sample Performance for the Post-September 11 Sample

This table reports the out-of-sample performance of four LIBOR market models calibrated using the first half of the post-September 11 sample. Based on parameters estimated using the data between November 2, 2001 and May 3, 2001, we calculate the in-sample and out-of-sample pricing errors using the first and second half of the post-September 11 sample respectively. We consider in-sample and out-of-sample average daily log SSE, relative model performance measured by Diebold-Mariano statistics, dependence of pricing errors on various economic variables, and calibrated model parameters.

Panel A: In-Sample Average Daily Log SSE

	Maturities (year)				
Model	1.5	3	5	7	10
Black	3.638	6.234	5.759	5.319	4.311
SV	1.485	3.225	2.742	1.797	2.425
JD	0.541	3.257	3.032	1.815	2.388
SVJ	0.517	3.124	2.675	1.785	2.327

Panel B: Out-of-Sample Average Daily Log SSE

	Maturities (year)				
Model	1.5	3	5	7	10
Black	4.367	6.159	6.098	6.148	5.251
SV	2.406	2.890	2.531	3.656	2.951
JD	3.001	4.106	3.500	3.657	2.926
SVJ	3.004	2.294	2.251	3.395	2.255

Panel C: Relative Model Performance. Diebold-Mariano statistics measure whether the models in the second column have significantly higher out-of-sample pricing errors than that in the first row. The statistics are calculated according to equation (17) using daily log SSE. The lag order q equals 40. The statistics follow an asymptotic standard Normal distribution.

		SV	JD	SVJ
1.5 year	Black	10.13	2.94	3.06
	SV	-	-0.96	-0.96
	JD	-	-	0.61
3.0 year	Black	18.94	3.85	44.33
	SV	-	-1.79	5.09
	JD	-	-	3.01
5 year	Black	21.71	14.06	21.92
	SV	-	-5.79	4.85
	JD	-	-	5.84
7 year	Black	21.85	18.67	30.86
	SV	-	-0.01	2.00
	JD	-	-	1.83
10 year	Black	16.67	27.29	8.78
	SV	-	0.29	1.50
	JD	-	-	1.66

Panel D: Dependence of Out-of-Sample Pricing Errors on Various Economic Variables. We regress the out-of-sample pricing errors of caps with different strike prices and maturities for the post-September 11 sample, on moneyness, time to maturity, the level, slope, and curvature factors of the term structure of interest rates. We measure the level factor by 6 month yield, the slope factor by the difference between 10 year and 6 month yie lds, and the curvature factor by 10 year yield minus twice of 2 year yield plus 6 month yield. We also measure the level factor by 10 year yield and obtain similar results. Standard errors are shown in the parentheses.

	Black		**SV**		**JD**		**SVJ**	
Intercept	0.177	(0.034)	0.011	(0.005)	0.032	(0.008)	0.096	(0.324)
Moneyness	-0.095	(0.002)	-0.006	(0.000)	-0.014	(0.000)	-0.005	(0.015)
Maturity	-0.006	(0.000)	-5.7E-05	(3.3E-05)	-1.4E-03	(5.6E-05)	0.009	(0.002)
Level	1.404	(1.727)	0.127	(0.252)	0.319	(0.424)	-4.443	(16.240)
Slope	-2.392	(0.362)	-0.201	(0.053)	-0.623	(0.089)	1.697	(3.405)
Curvature	2.421	(0.519)	0.140	(0.076)	0.623	(0.128)	-6.500	(4.880)
R^2	0.407		0.129		0.242		0.007	

Panel E: Calibrated Model Parameters. Model parameters are obtained by minimizing the objective function in equation (14) using the first half of the post-September 11 sample.

		Parameter Estimates						
Model	Maturity	γ	κ	θ	η	λ_n	m_n	s_n
SV	1.5	0.3539	5.2E-04	7.3304	0.0910			
	3	0.0000	2.3E-04	0.6061	0.0274			
	5	0.1377	9.6E-05	1.6421	0.0246			
	7	0.2421	3.8E-05	4.6173	0.0200			
	10	0.3053	1.2E-06	2.0476	0.0071			
JD	1.5	1				0.0522	-0.9379	8.0377
	3	1				0.0441	-0.8535	8.6448
	5	0.0000				0.0161	-0.2090	9.5571
	7	1.2E-05				0.0073	0.0977	8.9971
	10	7.3E-04				0.0364	0.0391	0.6969
SVJ	1.5	1.0000	0.0586	1.9392	0.1957	0.0501	-1.0000	7.7562
	3	3.9E-05	0.0003	0.4057	0.0240	0.0129	-0.8337	0.8167
	5	0.0045	0.0017	0.0383	0.0154	0.0179	0.3559	0.3955
	7	0.0382	1.6E-04	0.2865	0.0090	0.0175	0.3586	0.5810
	10	0.0514	2.3E-07	0.0009	0.0064	0.1019	0.1997	0.1659

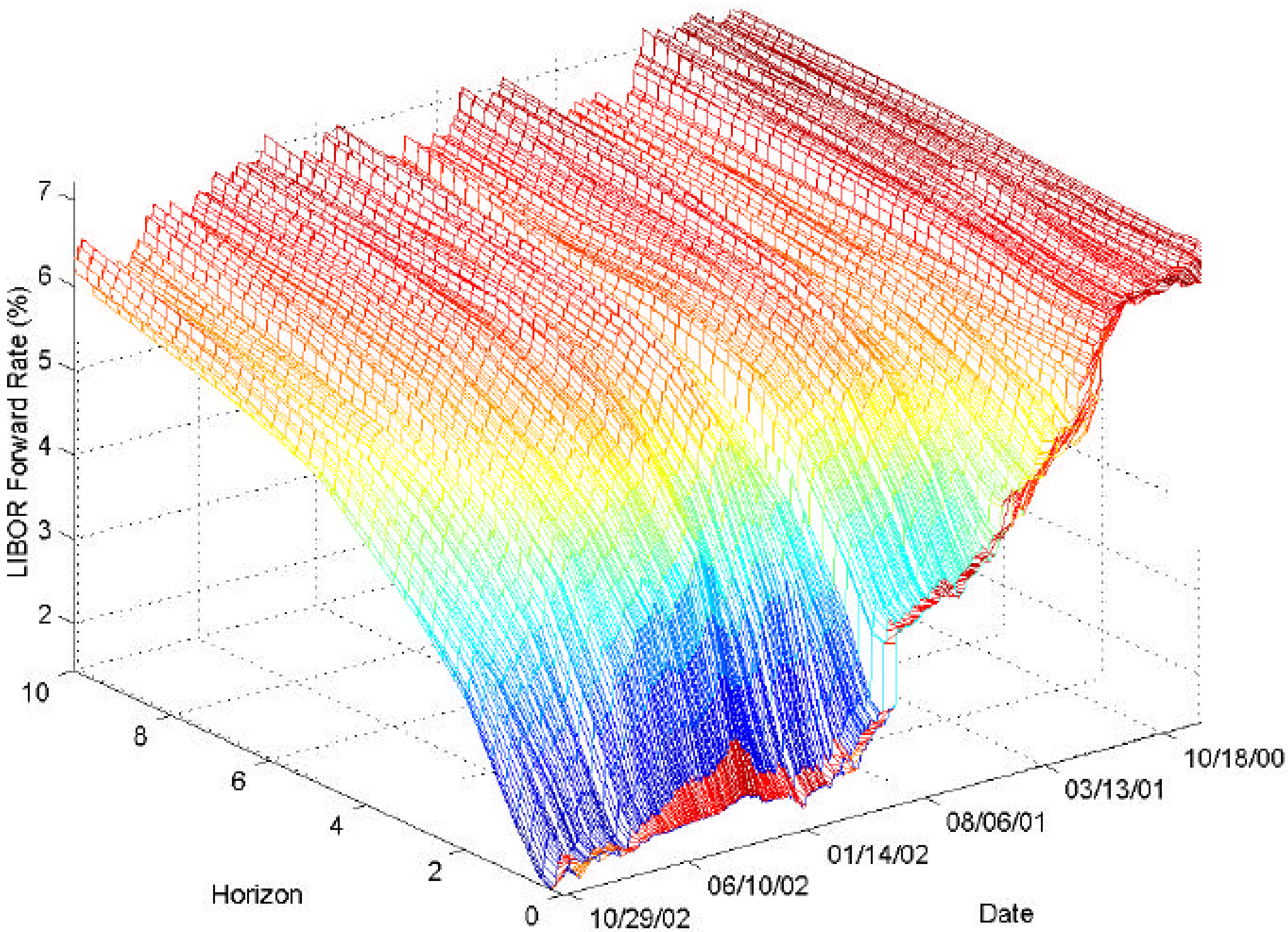


Figure 1: LIBOR Forward Rate Curves between 09/01/00 and 11/01/02

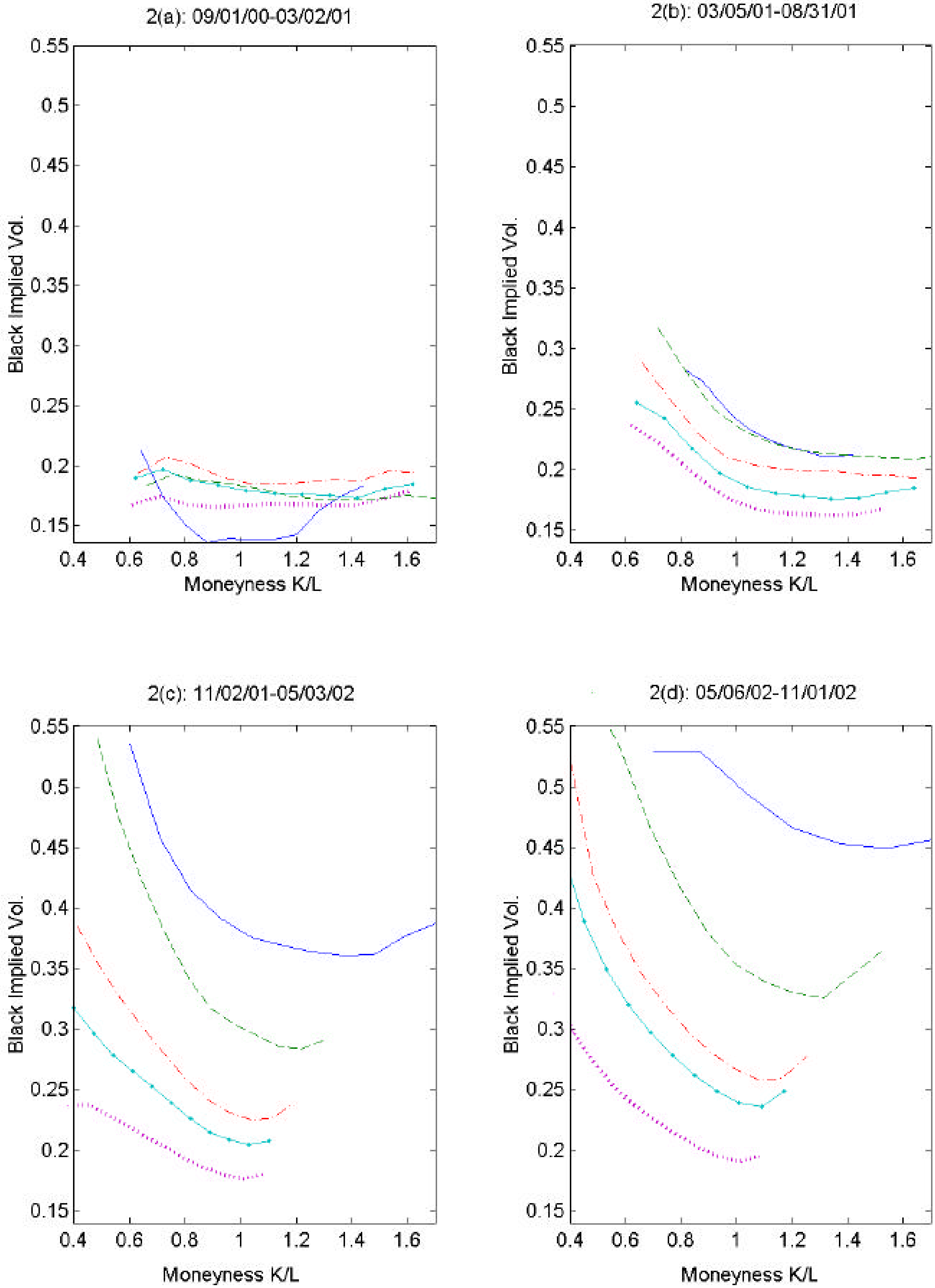


Figure 2: Average Black Implied Volatility of Interest Rate Caps between 09/01/00 and 11/01/00
(solid: 1.5 yr; dash: 3 yr; dash-dot: 5 yr; solid-dot: 7 yr; dot: 10 yr.)

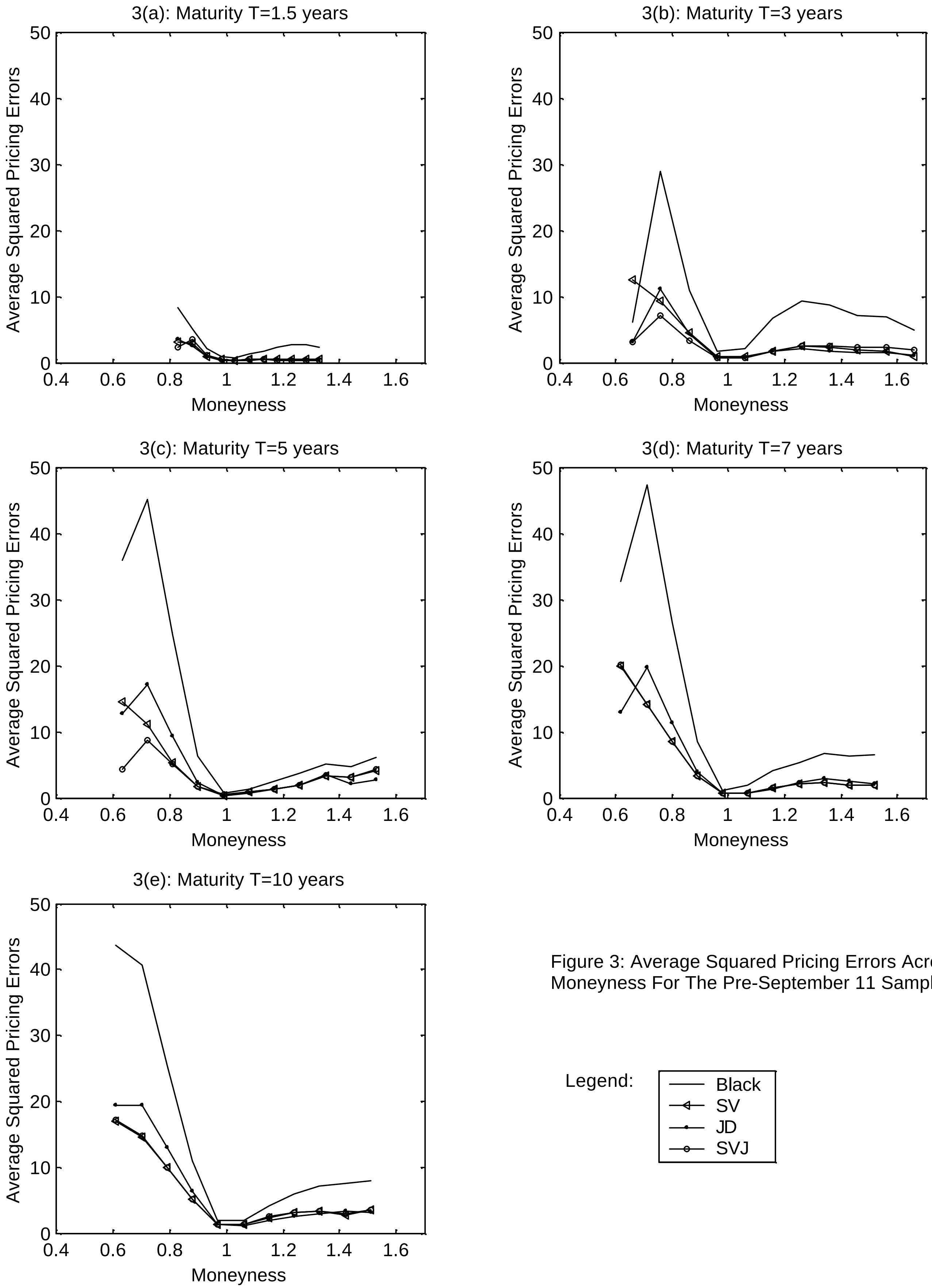
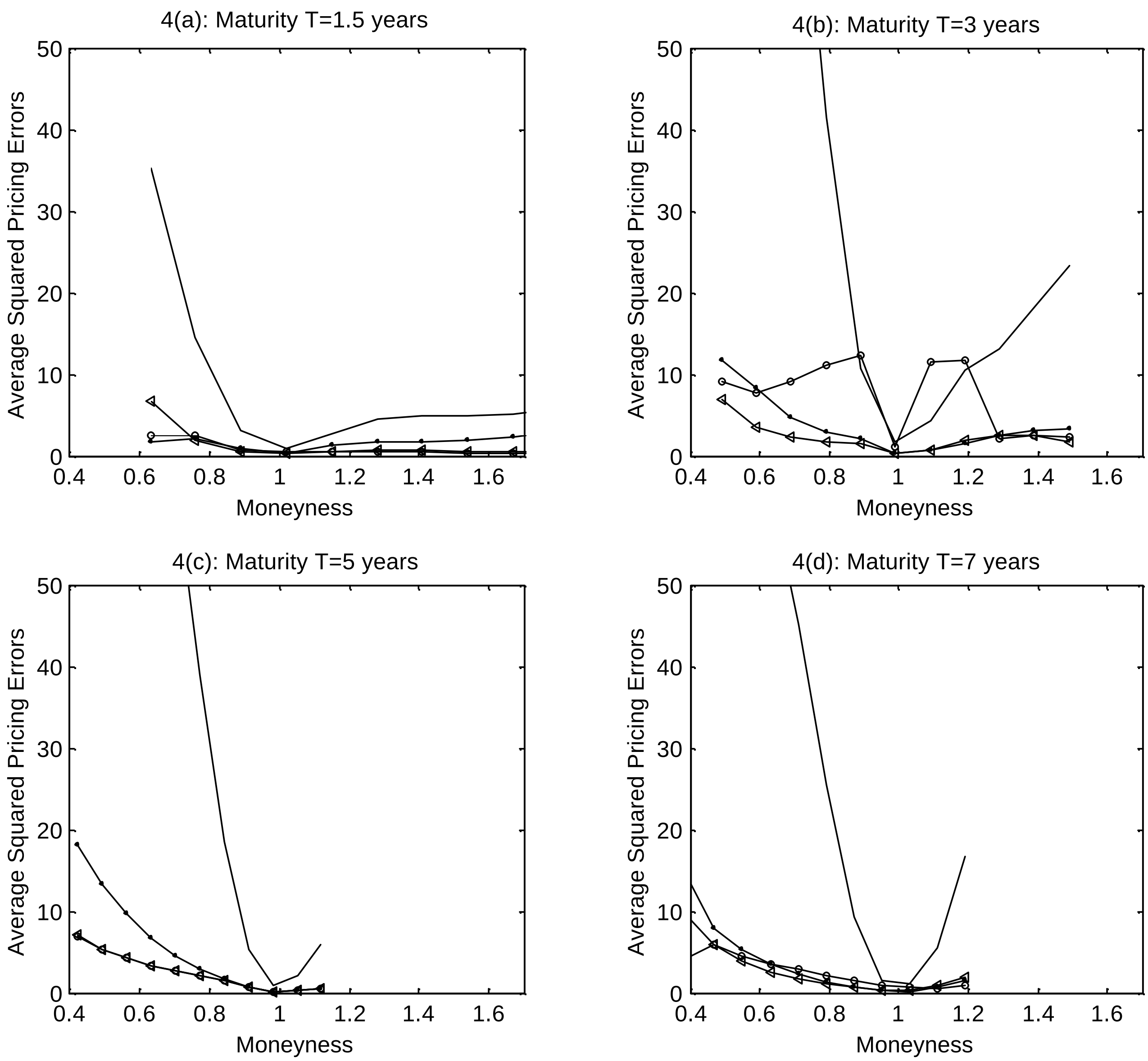


Figure 3: Average Squared Pricing Errors Across Moneyness For The Pre-September 11 Sample.



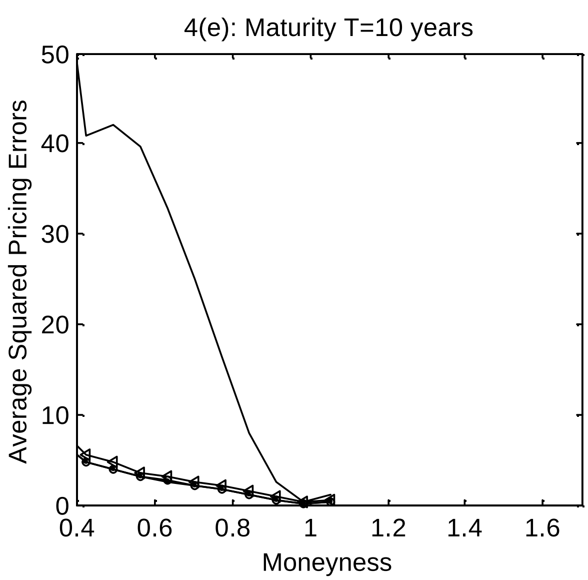
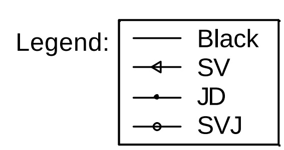
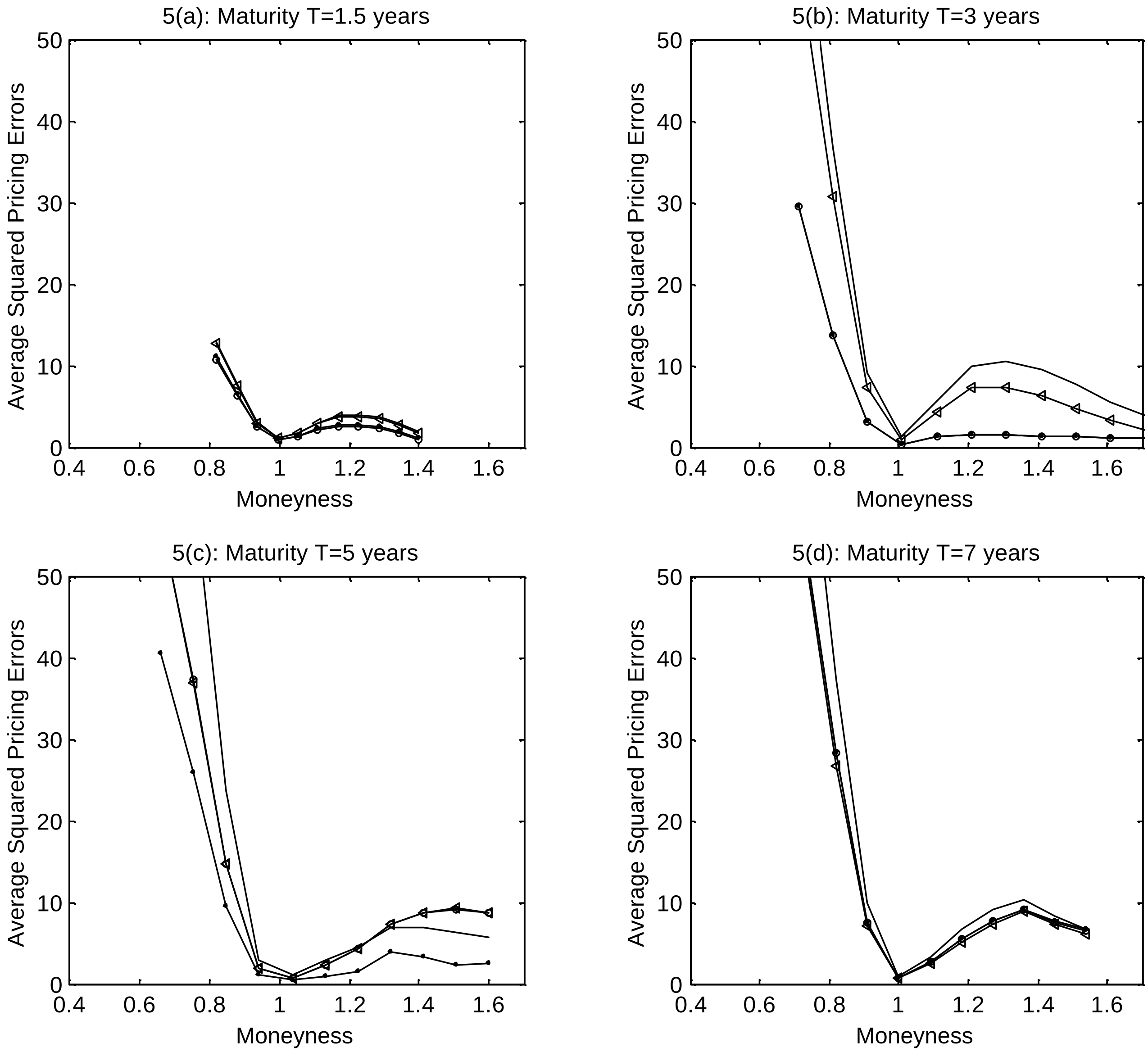


Figure 4: Average Squared Pricing Errors Across Moneyness For The Post-September 11 Sample.





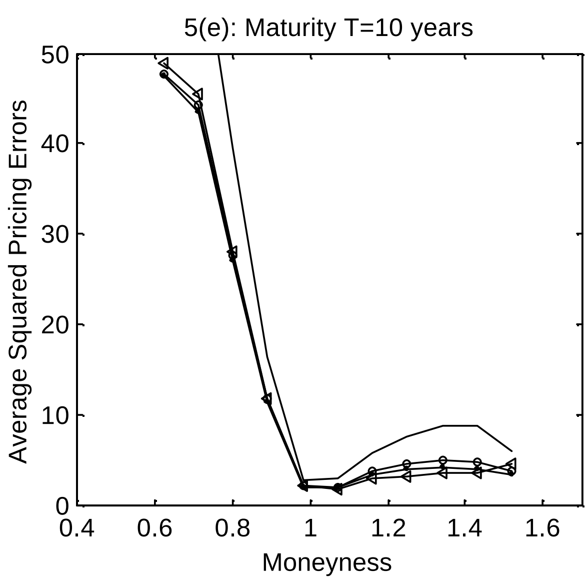
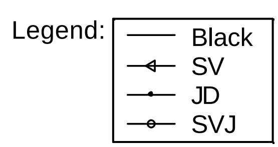


Figure 5: Average Out-of-Sample Squared Pricing Errors Across Moneyness For The Pre-September 11 Sample.



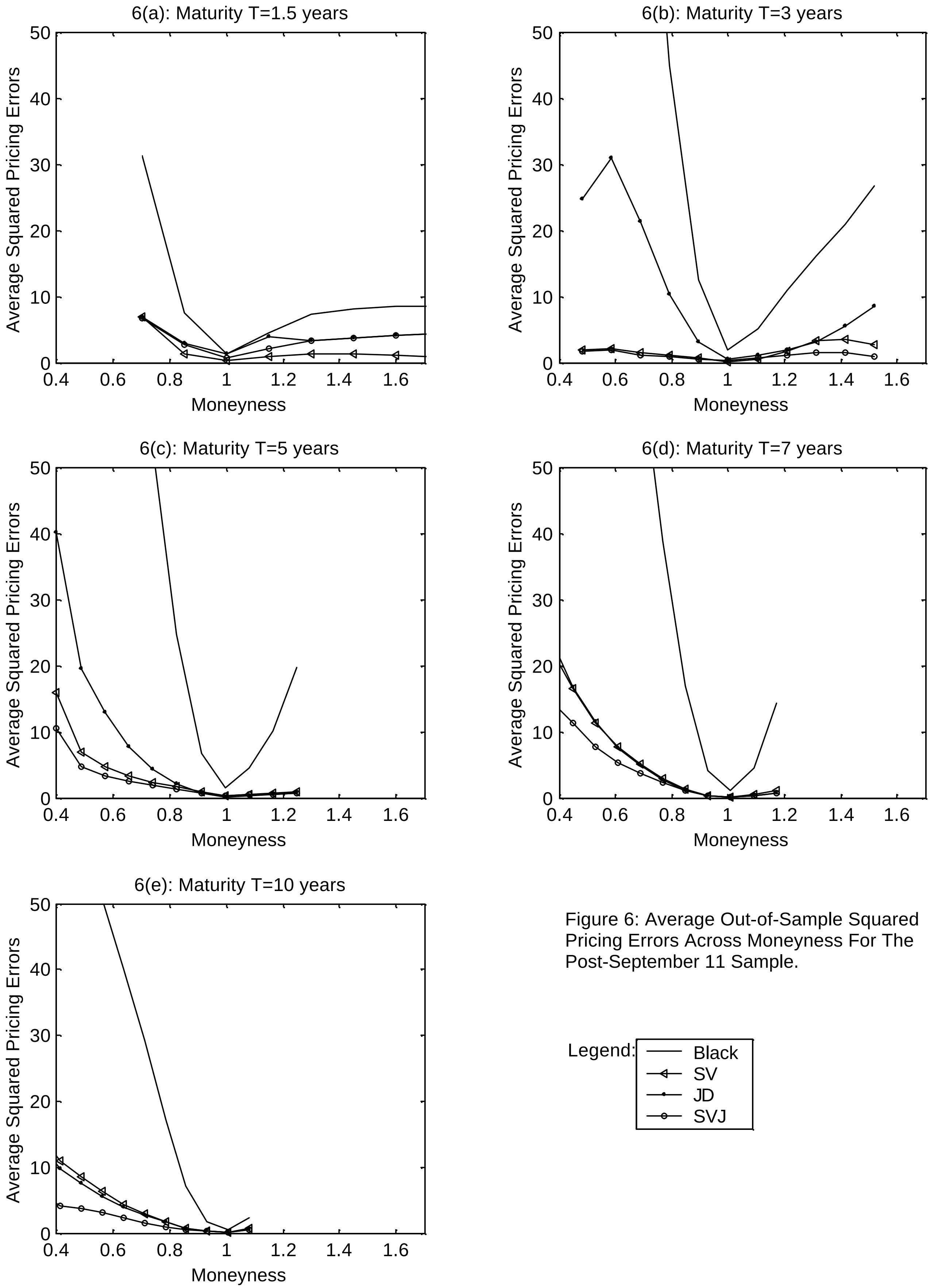


Figure 6: Average Out-of-Sample Squared Pricing Errors Across Moneyness For The Post-September 11 Sample.