Pricing Derivatives on Two Lévy-driven Stocks

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Abstract

The aim of this work is to study the pricing problem for derivatives depending on two stocks driven by a bidimensional Lévy process. The main idea is to apply Girsanov's Theorem for Lévy processes, in order to reduce the posed problem to the pricing of a one Lévy driven stock in an auxiliary market, baptized as "dual market". In this way, we extend the results obtained by Gerber and Shiu (1996) for two dimensional Brownian motion. Also we examine an existing relation between prices of put and call options, of both the European and the American type. This relation, based on a change of numeraire corresponding to a change of the probability measure through Girsanov's Theorem, is called *put–call duality.* It includes as a particular case, the relation known as put–call symmetry. Necessary and sufficient conditions for put–call symmetry to hold are obtained, in terms of the triplet of predictable characteristic of the Lévy process.

Key Words: Lévy processes, Optimal stopping, Girsanov's Theorem, Dual Market Method, Derivative pricing, Symmetry.
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1 Introduction

Since Margrabe's (1978) paper, many important extensions have been carrying on to study derivatives written on two stocks. Margrabe studied the pricing of European options for the case of two non-dividend-paying stocks driven by geometric Brownian motions, to be more exactly, the pricing of the right to change one asset for another at the end of some fixed period of time obtaining closed form formulas for this problem, extending in this way the Black and Scholes pricing model.

The American option pricing problem leads to the solution of an optimal stopping problem, that in general does not admit closed form solutions (see Jacka (1991)). In the perpetual case, i.e. the option has no expiration date, Gerber and Shiu (1996) obtain a closed form formula using the optional sampling theorem, assuming that stock prices are driven by geometric Brownian motions and stocks pay constant rate continuous dividends. They also study the pricing of the *Perpetual Maximum Option*, it is an option whose payoff is the maximum between two or more stocks and has no expiration date, and finally they study American perpetual options with more general payoffs which are homogeneous of degree one.

In the present paper we consider the problem of pricing European and American type derivatives written on a two dimensional stock driven by a two dimensional Lévy processes (it can be said that the stock follows a *two dimensional geometric Lévy process*), with a payoff function homogeneous of an arbitrary degree.

In the second part of the paper we study an existing relation between prices of put and call options, of both the European and the American type. This relation is called *put–call duality*. It includes as a particular case, the relation known as put–call symmetry. We suppose that the underlying stock in the market model is driven by a general Lévy processes, i.e. a stochastic process with independent and homogeneous increments, possible with discontinuous paths. In this market model, called a *Lévy market*, necessary and sufficient conditions for put–call symmetry to hold are obtained, in terms of the drift, the volatility, and the jump structure of the underlying log–stock price (i.e. in terms of the triplet of predictable characteristic of the Lévy process) As particular cases, we obtain the known conditions for symmetry in the lognormal jump–diffusion model introduced by Merton (1976), and examine other models of asset returns proposed in the literature. The corresponding results for stochastic volatility models, and for diffusion with jumps were obtained in Schroder (1999).

The paper is organized as follows: in section 2 we describe the market model and introduce the pricing problem, illustrating with several important examples of traded derivatives. In section 3 we describe the *Dual Market Method*, a method which allows to reduce the two stock problem into a one stock problem. In section 4 we derive some closed form formulas based on the proposed method and known results for one-dimensional problems, In section 5 we study the put–call relation and finally we have the conclusions and an appendix.

2 Market Model

2.1 Multidimensional Lévy processes

$\mathbf{X} = (X^1, \dots, X^d)$ be a d-dimensional Lévy process defined on a stochastic basis $\mathcal{B} = (\Omega, \mathcal{F}, \{\mathcal{F}\}_{t\geq 0}, P)$. This means that $\mathbf{X}$ is a stochastically continuous stochastic process with independent increments, such that the distribution of $\mathbf{X}_{t+s} - \mathbf{X}_s$ does not depend on s, with $P(\mathbf{X}_0 = 0) = 1$ and trajectories continuous from the left with limits from the right. The basis $\mathcal{B}$ is supposed to satisfy the usual assumptions, i.e. continuity from the right and $\mathcal{F}_0$ is P complete. For $z = (z_1, \dots, z_d)$ in C^d, when the integral is convergent (and this is always the case if $z = i\lambda$ with λ in R^d, Lévy-Khinchine formula states, that $Ee^{zX_t} = \exp(t\Psi(z))$ where the function Ψ is the *characteristic exponent* of the process, and is given by

$$\Psi(z) = (a, z) + \frac{1}{2}(z, \Sigma z) + \int_{R^d} \left(e^{(z,y)} - 1 - (z, y)\mathbf{1}_{\{|y|\leq 1\}}\right)\Pi(dy), \qquad (1)$$

where $a = (a_1, \dots, a_d)$ is a vector in $I\!R^d$, Π is a positive measure defined on $R^d \setminus \{0\}$ such that $\int_{R^d}(|y|^2 \wedge 1)\Pi(dy)$ is finite, and $\Sigma = ((s_{ij}))$ is a symmetric nonnegative definite matrix, that can always be written as $\Sigma = A'A$ (where $'$ denotes transposition) for some matrix A.

The triplet (a, Σ, Π) completely determines the law of the process X. Particular interest has the case when $\alpha = \int_{R^d} \Pi(dy)$ is finite, i.e. X is a diffusion with jumps. Introducing F by $\Pi(dy) = \alpha F(dy)$, Lévy-Khinchine formula is (changing the value of a if necessary)

$$\Psi(z) = (a, z) + \frac{1}{2}(z, \Sigma z) + \int_{R^d} \left(e^{(z,y)} - 1\right)\Pi(dy), \tag{2}$$

and the process $X = \{X_t\}_{t\geq 0}$ can be represented by

$$X_t = at + AW_t + \sum_{k=1}^{N_t} Y_k,$$

where W is a standard d-dimensional Brownian motion, $N = \{N_t\}_{t\geq 0}$ is a Poisson process with parameter α, and $\{Y_k\}_{k\geq 1}$ is a sequence of independent d-dimensional random vectors with identical distribution $F(dy)$.
Another important case is when the coordinates of X are independent processes. This happens if and only if Σ is a diagonal matrix (and A can be chosen to be diagonal also) and the measure Π has support on the set $\{x \in R^d \colon \prod_{k=1}^d x_k = 0\}$, (i.e. it is concentrated on the union of the coordinate axes, see E 12.10 in Sato (1999)). In this case $\Psi(z) = \sum_{k=1}^d \Psi_k(z_k)$, where Ψ_k is the characteristic exponent of the k-coordinate of X, given by

$$\Psi_k(z_k) = a_k z_k + \frac{1}{2} s_{kk} z_k^2 + \int_R \left(e^{z_k y} - 1 - z_k y \mathbf{1}_{\{|y|\leq 1\}}\right)\Pi_k(dy),$$

where $\Pi_k(A) = \int_{\{x\in R^d \colon x_k \in A\}} \Pi(dx)$. For general reference on the subject see [13], [26], [3] and [22].

2.2 Market and Problem

Consider a market model with three assets (S^1, S^2, S^3) given by

$$S_t^1 = e^{X_t^1}, \quad S_t^2 = S_0^2 e^{X_t^2}, \quad S_t^3 = S_0^3 e^{X_t^3} \tag{3}$$

where (X^1, X^2, X^3) is a three dimensional Lévy process, and for simplicity, and without loss of generality we take $S_0^1 = 1$. The first asset is the bond and is usually deterministic. Randomness in the bond $\{S_t^1\}_{t\geq 0}$ allows to consider more general situations, as for example the pricing problem of a derivative written in a foreign currency, referred as *Quanto option*.
Consider a function:

$$f \colon (0,\infty) \times (0,\infty) \to \mathbb{R}$$

homogenous of an arbitrary degree α; i.e. for any $\lambda > 0$ and for all positive x, y

$$f(\lambda x, \lambda y) = \lambda^\alpha f(x, y).$$

In the above market a derivative contract with payoff given by

$$\Phi_t = f(S_t^2, S_t^3)$$

is introduced.

Assuming that we are under a risk neutral martingale measure, thats to say, $\frac{S^k}{S^1}$ $(k = 2, 3)$ are P-martingales, i.e. P is an equivalent martingale measure (EMM)[1], we want to price the derivative contract just introduced. In the European case, the problem reduces to the computation of

$$E_T = E(S_0^2, S_0^3, T) = \mathsf{E}\left[e^{-X_T^1} f(S_0^2 e^{X_T^2}, S_0^3 e^{X_T^3})\right] \tag{4}$$

In the American case, if $\mathcal{M}_T$ denotes the class of stopping times up to time T, i.e:

$$\mathcal{M}_T = \{\tau :\ 0 \leq \tau \leq T,\ \tau \text{ stopping time}\}$$

for the finite horizon case, putting $T = \infty$ for the perpetual case, the problem of pricing the American type derivative introduced consists in solving an optimal stopping problem, more precisely, in finding the value function A_T and an optimal stopping time τ^* in $\mathcal{M}_T$ such that

$$\begin{aligned} A_T = A(S_0^2, S_0^3, T) &= \sup_{\tau \in \mathcal{M}_T} \mathsf{E}\left[e^{-X_\tau^1} f(S_0^2 e^{X_\tau^2}, S_0^3 e^{X_{\tau^3}^3})\right] \\ &= \mathsf{E}\left[e^{-X_{\tau^*}^1} f(S_0^2 e^{X_{\tau^*}^2}, S_0^3 e^{X_{\tau^*}^3})\right]. \end{aligned}$$

2.3 Examples of Derivatives

In what follows we introduce some relevant derivatives as particular cases of the problem described.

2.3.1 *Option to Default.* Consider the derivative which has the payoff

$$f(x, y) = \min\{x, y\}$$

[1] See appendix

if $X^1 = rt$, then the value of the *Option to Default* a promise S_T^1 backed by a collateral guarantee S_T^2, at the time T would be:

$$D = \mathsf{E}\left[e^{-rT}\min\{S_T^1, S_T^2\}\right]$$

2.3.2 *Margrabe's Options.* Consider the following cases:

a) $f(x,y) = \max\{x,y\}$, called the *Maximum Option*,

b) $f(x,y) = |x-y|$, the *Symmetric Option*,

c) $f(x,y) = \min\{(x-y)^+, ky\}$, the *Option with Proportional Cap.*

2.3.3 *Swap Options.* Consider

$$f(x,y) = (x-y)^+,$$

obtaining the option to exchange one risky asset for another.
2.3.1 *Quanto Options.* Consider

$$f(x,y) = (x-ky)^+,$$

and take $S_t^2 = 1$, then

$$E_T = \mathsf{E}^{\mathcal{Q}} e^{X_T^0}(S_T^1 - k)^+$$

where $e^{X_T^1}$ is the spot exchange rate (foreign units/domestic units) and S_T^1 is the foreign stock in foreign currency. Then we have the price of an option to exchange one foreign currency for another.
2.3.4 *Equity-Linked Foreign Exchange Option (ELF-X Option).* Take

$$S = S^1: \text{ foreign stock in foreign currency}$$

and Q is the spot exchange rate. We use foreign market risk measure, then an ELF-X is an investment that combines a currency option with an equity forward. The owner has the option to buy S_t with domestic currency which can be converted from foreign currency using a previously stipulated strike exchange rate R (domestic currency/foreign currency).
The payoff is:

$$\Phi_t = S_t(1 - RQ_T)^+$$

Then take $S^2 = 0$ and $f(x,y) = (y - Rx)^+$.

2.3.5 *Vanilla Options.* Take

$$X^1 = rt, \quad X^1 = x$$

then in the call case we have

$$f(x, y) = (x - ky)^+$$

and

$$f(x, y) = (ky - x)^+$$

in the put case with $S^1 = S_0^1 e^{X_t}$ and $S^2 = 1$.

3 Dual Market method

The main idea to solve the posed problems is the following: make a change of measure through Girsanov's Theorem for Lévy processes, in order to reduce the original problems to a pricing problems for an auxiliary derivative written on one Lévy driven stock in an auxiliary market with deterministic interest rate. This method was introduced in Shepp and Shiryaev (1994) and Kramkov and Mordecki (1994) with the purpose of pricing American perpetual options with path dependent payoffs. It was employed by Aloisio and De Deus (1997) to consider the pricing of swaps, and is strongly related with the election of the *numéraire* (see Geman et al. (1995)). This auxiliary market will be called the *Dual Market*.
More precisely, observe that

$$e^{-X_t^1} f(S_0^2 e^{X_t^2}, S_0^3 e^{X_t^3}) = e^{-X_t^1 + \alpha X_t^3} f(S_0^2 e^{X_t^2 - X_t^3}, S_0^3),$$

let $\rho = -\log \mathsf{E} e^{-X_1^1 + \alpha X_1^3}$, that we assume finite. The process

$$Z_t = e^{-X_t^1 + \alpha X_t^3 + \rho t}$$

is a density process (i.e. a positive martingale starting at $Z_0 = 1$) that allow us to introduce a new measure $\tilde{P}$ by its restrictions to each $\mathcal{F}_t$ by the formula

$$\frac{d\tilde{P}_t}{dP_t} = Z_t.$$

Denote now by $X_t = X_t^2 - X_t^3$, and $S_t = S_0^2 e^{X_t}$. Finally, let

$$F(x) = f(x, S_0^3).$$

With the introduced notations, under the change of measure we obtain

$$E_T = \tilde{\mathsf{E}}\left[e^{-\rho T}F(S_T)\right]$$

$$A_T = \sup_{\tau\in\mathcal{M}_T}\tilde{\mathsf{E}}\left[e^{-\rho\tau}F(S_\tau)\right]$$

The following step is to determine the law of the process X under the auxiliar probability measure $\tilde{P}$.

Lemma 3.1. *Let X be a Lévy process on R^d with characteristic exponent given in (1). Let u and v be vectors in R^d. Assume that $\mathsf{E}e^{(u,X_1)}$ is finite, and denote $\rho = -\log \mathsf{E}e^{(u,X_1)} = \Psi(u)$. In this conditions, introduce the probability measure $\tilde{P}$ by its restrictions $\tilde{P}_t$ to each $\mathcal{F}_t$ by*

$$\frac{d\tilde{P}_t}{dP_t} = \exp[(u, X_t) + \rho t].$$

Then
(a) the law of the unidimensional Lévy process $\{(v, X_t)\}_{t\geq 0}$ under $\tilde{P}$ is given by the triplet

$$\begin{cases} \tilde{a} = (a, v) + \frac{1}{2}[(v,\Sigma u) + (u,\Sigma v)] + \int_{R^d} e^{(u,y)}(v,y)\mathbf{1}_{\{|(v,y)|\leq 1, |x|>1\}}\Pi(dx) \\ \tilde{\sigma}^2 = (v,\Sigma v) \\ \tilde{\pi}(A) = \int_{R^d}\mathbf{1}_{\{(v,y)\in A\}}e^{(u,y)}\Pi(dy). \end{cases} \tag{5}$$

(b) In the particular case when X is a diffusion with jumps which characteristic exponent given in (2) the law of the unidimensional Lévy process $\{(v, X_t)\}_{t\geq 0}$ under $\tilde{P}$ is given by the triplet

$$\begin{cases} \tilde{a} = (a, v) + \frac{1}{2}[(v,\Sigma u) + (u,\Sigma v)] \\ \tilde{\sigma}^2 = (v,\Sigma v) \\ \tilde{\pi}(A) = \int_{R^d}\mathbf{1}_{\{(v,y)\in A\}}e^{(u,y)}\Pi(dy). \end{cases} \tag{6}$$

Furthermore, the intensity of the Poisson process under $\tilde{P}$ is given by

$$\tilde{\alpha} = \int_{R^d} e^{(u,y)}\Pi(dy) = \alpha\int_{R^d} e^{(u,y)}F(dy)$$

(c) Assume (b), and let $\Pi(dy) = \alpha F(dy)$ where F is the common distribution of the random variables $\{Y_k\}_{k\geq 1}$, and has characteristic function (under P) given by

$$\phi(z) = \int_{R^d} e^{(z,y)}F(dy).$$

Then, the characteristic function of the same random variables under $\tilde{P}$ is given by

$$\tilde{\phi}(\theta) = \frac{\phi(\theta v + u)}{\phi(u)}. \tag{7}$$

Remark: Consider a diffusion with gaussian jumps, in what can be considered as a multidimensional extensions of the jump-diffusion model proposed by Merton (1976). then, that the characteristic function corresponding to the distribution of the jumps is given by

$$\phi(z) = \exp[(z,\nu) + \frac{1}{2}(z,\Delta z)],$$

where the d-dimensional vector ν is the drift of the jumps, and the nonnegative definite matrix Δ is the covariance. According to (7), the characteristic exponent of the jumps of the process $\{(v,X_t)\}_{t\geq 0}$ under the probability measure $\tilde{P}$ in the Lemma 3.1 is given by

$$\tilde{\phi}(\theta) = \frac{\phi(\theta v + u)}{\phi(u)} = \exp\Big\{\theta((v,\nu) + \frac{1}{2}[(v,\Delta u) + (u,\Delta v)] + \frac{1}{2}\theta^2(v,\Delta v)\Big\}. \tag{8}$$

In conclusion, jumps under $\tilde{P}$ are also gaussian, with mean and variance obtained in (8)

Proof of the Lemma. First compute the expectation under $\tilde{P}$ as an expectation under P.

$$\tilde{\mathsf{E}}e^{\theta(v,X_t)} = \mathsf{E}e^{(u+\theta v,X_t)+\rho t} = \exp\{t[\Psi(u+\theta v,X_t) - \Psi(u)]\}.$$

Now, compute the characteristic exponent of (v,X),

$$\begin{aligned}\Psi(u+\theta v) - \Psi(u) &= (a,u+\theta v) - (a,u) + \frac{1}{2}[(u+\theta v,\Sigma u + \theta v) \\ -(u,\Sigma u) &+ \int_{R^d}\Big(e^{(u+\theta v,y)} - 1 - (u+\theta v,y)\mathbf{1}_{\{|y|\leq 1\}}\Big)\Pi(dy) \\ &- \int_{R^d}\Big(e^{(u,y)} - 1 - (u,y)\mathbf{1}_{\{|y|\leq 1\}}\Big)\Pi(dy) \\ &= \theta\{(a,v) + \frac{1}{2}[(v,\Sigma u) + (u,\Sigma v)]\} + \frac{1}{2}(v,\Sigma v) \\ &+ \int_{R^d}\Big(e^{(u+\theta v,y)} - e^{(u,y)} - (\theta v,y)\mathbf{1}_{\{|y|\leq 1\}}\Big)\Pi(dy)\end{aligned}$$

$$= \theta\{(a,v) + \frac{1}{2}[(v,\Sigma u) + (u,\Sigma v)] + \int_{R^d} e^{(u,y)}(v,y)\mathbf{1}_{\{|(v,y)|\leq 1, |x|>1\}}\Pi(dx)\}$$
$$+\frac{1}{2}(v,\Sigma v) + \int_{R^d}\left(e^{(\theta v,y)} - 1 - (\theta v, y)\mathbf{1}_{\{|y|\leq 1\}}\right)e^{(u,y)}\Pi(dy)$$

giving (5).
In what concerns (6), similar calculations give the result.
Let us see (c). As the distribution of the jumps under $\tilde{P}$ is given by $\frac{1}{\tilde{\alpha}}\tilde{\pi}(dy)$,

$$\tilde{\phi}(\theta) = \frac{1}{\tilde{\alpha}}\int_R e^{\theta x}\tilde{\pi}(dx)$$
$$= \frac{\alpha}{\tilde{\alpha}}\int_{R^d} e^{(\theta v+u,y)}F(dy) = \frac{\phi(\theta v+u)}{\phi(u)}.$$

4 Examples

European derivative

Let $X_t^1 = rt$ and (X_t^2, X_t^3) be a bidimensional Lévy Process. We can choose an EMM ($\mathcal{Q}^\theta$, $\theta = (\theta_2,\theta_3)$) using the Gerber and Shiu (1994) approach, i.e. the density of the EMM is given by the Esscher transform:

$$d\mathcal{Q}^\theta = \frac{e^{\theta_2 X_T^2 + \theta_3 X_T^3}}{Ee^{\theta_2 X_T^2 + \theta_3 X_T^3}}dP$$

where θ is such that $\mathcal{Q}^\theta$ is an EMM, for more details see the appendix.

Now consider a defaultable contingent promise S_T^2 backed by a collateral guarantee S_T^3, then it's price D would be:

$$D = \mathsf{E}^\theta\left[e^{-rT}\min\{S_T^2, S_T^3\}\right] = \mathsf{E}^\theta\left[e^{-rT}S_T^2\right] - \mathsf{E}^\theta\left[e^{-rT}(S_T^2 - S_T^3)^+\right].$$

$$= S_0^2 e^{-rT}\underbrace{\int_{-\infty}^{\infty} e^{X_T^2}d\mathcal{Q}^\theta}_{I_1} - \underbrace{\int_{\mathcal{A}} e^{-rT}(S_0^2 e^{X_T^2} - S_0^3 e^{X_T^3})d\mathcal{Q}^\theta}_{I_2}$$

where $\mathcal{A} = \{\omega \in \Omega : S_0^2 X_T^2(\omega) > S_0^3 X_T^3(\omega)\}$. We proceed to compute I_1 and I_2:

$$I_1 = \int_{-\infty}^{\infty} e^{X_T^2} \frac{e^{\theta_2 X_T^2 + \theta_3 X_T^3}}{Ee^{\theta_2 X_T^2 + \theta_3 X_T^3}} dP(x)$$

$$= \frac{Ee^{(\theta_2+1)X_T^2 + \theta_3 X_T^3}}{Ee^{\theta_2 X_T^2 + \theta_3 X_T^3}}$$

and assuming for simplicity $S_0^2 = S_0^3 = 1$ we have

$$\begin{aligned} I_2 &= \int_{\mathcal{A}} e^{-rT}(e^{X_T^2} - e^{X_T^3}) d\mathcal{Q}^\theta \\ &= \int_{\{S_T > 1\}} e^{-rT} e^{X_T^3}(S_T - 1) d\mathcal{Q}^\theta \\ &= \int_{\{S_T > 1\}} e^{-rT} e^{X_T^3}(S_T - 1) d\mathcal{Q}^\theta \\ &= e^{-\rho T} \int_{\{S_T > 1\}} (S_T - 1) d\widetilde{\mathcal{Q}} \end{aligned}$$

where $\rho = -logEe^{-r+X_1^3} = r - logEe^{X_1^3}$ and

$$d\widetilde{\mathcal{Q}} = \frac{e^{X_T^3}}{Ee^{X_T^3}} d\mathcal{Q}^\theta$$

since $S_T = e^X$ and $X = X^2 - X^3$, then I_2 can be computed as

$$I_2 = e^{-\rho T} \int_{\{S_T > 1\}} S_T d\widetilde{\mathcal{Q}} - e^{-\rho T} \int_{\{S_T > 1\}} d\widetilde{\mathcal{Q}}$$

$$= e^{-\rho T} \int_{\{S_T > 1\}} e^{X_T^2 - X_T^3} \frac{e^{X_T^3}}{Ee^{X_T^3}} d\mathcal{Q}^\theta - e^{-\rho T} \widetilde{\mathcal{Q}}(S_T > 1)$$

$$= e^{-\rho T} \frac{Ee^{X_T^2}}{Ee^{X_T^3}} \widehat{\mathcal{Q}}(S_T > 1) - e^{-\rho T} \widetilde{\mathcal{Q}}(S_T > 1)$$

where $d\widehat{\mathcal{Q}} = \frac{e^{X_T^2}}{Ee^{X_T^2}} d\mathcal{Q}$.

American derivative

Now consider an American perpetual swap, it is a derivative with the payoff function at any time t given by

$$f(S_t^2, S_t^3) = (S_t^2 - S_t^3)^+$$

then using the Dual market method, the pricing problem of this derivative would be :

$$A_T = \sup_{\tau \in \mathcal{M}_T} \tilde{\mathsf{E}} \left[e^{-\rho\tau}(S_\tau - S_0^3)^+ \right] = \tilde{\mathsf{E}} \left[e^{-\rho\tau^*}(S_{\tau^*} - S_0^3)^+ \right],$$

and this problem can be solved using the following proposition

Proposition 4.1. *Let* $M = \sup_{o \le t \le \tau} X_t$ *with* τ *an independent exponential random variable with parameter* ρ*, then* $\tilde{\mathsf{E}} e^M < \infty$ *and*

$$A(S_0^2, S_0^3) = \frac{\tilde{\mathsf{E}} \left[S_0^2 e^M - S_0^3 \tilde{\mathsf{E}}(e^M) \right]}{\tilde{\mathsf{E}}(e^M)}$$

the optimal stopping time is

$$\tau_c^* = \inf\{t \ge 0, S_t \ge S_0^3 \tilde{\mathsf{E}}(e^M)\}$$

Proof:
See Mordecki (2002). □

5 Put-Call Duality

Now Consider a real valued unidimensional Lévy process $X = \{X_t\}_{t \ge 0}$. As we have seen in chapter 2 we can characterize the law of X under $\mathbf{P}$, consider, for $q \in I\!R$ the Lévy-Khinchine formula, that states

$$\mathbf{E}\, e^{iqX_t} = \exp\left\{ t \Big[iaq - \frac{1}{2}\sigma^2 q^2 + \int_{I\!R} \big(e^{iqy} - 1 - iqh(y) \big) \Pi(dy) \Big] \right\}, \qquad (9)$$

with

$$h(y) = y \mathbf{1}_{\{|y|<1\}}$$

a fixed truncation function, a and $\sigma \geq 0$ real constants, and Π a positive measure on $I\!R \setminus \{0\}$ such that $\int (1 \wedge y^2)\Pi(dy) < +\infty$, called the *Lévy measure*. The triplet (a, σ^2, Π) is the *characteristic triplet* of the process, and completely determines its law.
It is useful to consider, in the same formula (9), the possibility of having

$$h(y) = 0, \quad \text{for all } y \in I\!R, \tag{10}$$

in the particular case, when $\int |y|\Pi(dy) < +\infty$. This condition corresponds to the subclass of Lévy processes with finite variation, including the diffusions with jumps.
Consider the set

$$\mathbb{C}_0 = \Big\{ z = p + iq \in \mathbb{C} \colon \int_{\{|y|>1\}} e^{py}\Pi(dy) < \infty \Big\}. \tag{11}$$

The set $\mathbb{C}_0$ is a vertical strip in the complex plane, contains the line $z = iq$ ($q \in I\!R$), and consists of all complex numbers $z = p + iq$ such that $\mathbf{E}\, e^{pX_t} < \infty$ for some $t > 0$. Furthermore, if $z \in \mathbb{C}_0$, we have the *characteristic exponent* of the process X, given by

$$\psi(z) = az + \frac{1}{2}\sigma^2 z^2 + \int_{I\!R} \big(e^{zy} - 1 - zh(y) \big)\Pi(dy) \tag{12}$$

having $\mathbf{E}\, |e^{zX_t}| < \infty$ for all $t \geq 0$, and $\mathbf{E}\, e^{zX_t} = e^{t\psi(z)}$. Formula (12) reduces to formula (9) when $\mathrm{Re}(z) = 0$.
Now we consider a *Lévy market* with two assets: a deterministic savings account $B = \{B_t\}_{t\geq 0}$, with

$$B_t = e^{rt}, \qquad r \geq 0,$$

where we take $B_0 = 1$ for simplicity, and a stock $S = \{S_t\}_{t\geq 0}$, with random evolution modeled by

$$S_t = S_0 e^{X_t}, \qquad S_0 = e^x > 0, \tag{13}$$

where $X = \{X_t\}_{t\geq 0}$ is a Lévy process.
In this model we assume that the stock pays dividends, with constant rate $\delta \geq 0$, and that the given probability measure $\mathbf{P}$ is the choosen equivalent martingale measure. In other words, prices are computed as expectations

with respect to $\mathbf{P}$, and the discounted and reinvested process $\{e^{-(r-\delta)t}S_t\}$ is a $\mathbf{P}$–martingale.
In terms of the characteristic exponent of the process this means that

$$\psi(1) = r - \delta, \tag{14}$$

based on the fact, that $\mathbf{E}\, e^{-(r-\delta)t+X_t} = e^{-t(r-\delta+\psi(1))} = 1$, and condition (14) can also be formulated in terms of the characteristic triplet of the process X as

$$a = r - \delta - \sigma^2/2 - \int_{I\!R} \big(e^y - 1 - h(y)\big)\Pi(dy). \tag{15}$$

In the case, when

$$X_t = \sigma W_t + at \quad (t \geq 0), \tag{16}$$

where $W = \{W_t\}_{t\geq 0}$ is a Wiener process, we obtain the Black–Scholes–Merton (1973) model (see [4],[18]).
In the market model considered we introduce some derivative assets. More precisely, we consider call and put options, of both European and American types.
Let us assume that τ is a stopping time with respect to the given filtration $\mathbf{F}$, that is $\tau\colon \Omega \to [0,\infty]$ belongs to $\mathcal{F}_t$ for all $t \geq 0$; and introduce the notation

$$C(S_0, K, r, \delta, \tau, \psi) = \mathbf{E}\, e^{-r\tau}(S_\tau - K)^+ \tag{17}$$

$$P(S_0, K, r, \delta, \tau, \psi) = \mathbf{E}\, e^{-r\tau}(K - S_\tau)^+ \tag{18}$$

If $\tau = T$, where T is a fixed constant time, then formulas (17) and (18) give the price of the European call and put options respectively.

5.1 Put–Call duality and dual markets

Proposition 1. *Consider a Lévy market with driving process X with characteristic exponent $\psi(z)$, defined in* (12)*, on the set $\mathbb{C}_0$ in* (11)*. Then, for the expectations introduced in* (17) *and* (18) *we have*

$$C(S_0, K, r, \delta, \tau, \psi) = P(K, S_0, \delta, r, \tau, \tilde{\psi}), \tag{19}$$

where

$$\tilde{\psi}(z) = \tilde{a}z + \frac{1}{2}\tilde{\sigma}^2 z^2 + \int_{I\!R} \big(e^{zy} - 1 - zh(y)\big)\tilde{\Pi}(dy) \tag{20}$$

is the characteristic exponent (of a certain Lévy process) that satisfies

$$\tilde{\psi}(z) = \psi(1-z) - \psi(1), \quad \textit{for } 1 - z \in \mathbb{C}_0,$$

and in consequence,

$$\begin{cases} \tilde{a} & = \delta - r - \sigma^2/2 - \int_{I\!R} \big(e^y - 1 - h(y)\big)\tilde{\Pi}(dy), \\ \tilde{\sigma} & = \sigma, \\ \tilde{\Pi}(dy) & = e^{-y}\Pi(-dy). \end{cases} \tag{21}$$

Proof. Consider the martingale $Z = \{Z_t\}_{t\geq 0}$ defined by $Z_t = e^{X_t-(r-\delta)t}$ ($t \geq 0$). Following Shiryaev et al. [25] we introduce the *dual martingale measure* $\tilde{\mathbf{P}}$ given by its restrictions $\tilde{\mathbf{P}}_t$ to $\mathcal{F}_t$ by

$$\frac{d\tilde{\mathbf{P}}_t}{d\mathbf{P}_t} = Z_t,$$

where $\mathbf{P}_t$ is the restriction of $\mathbf{P}$ to $\mathcal{F}_t$. Now

$$\begin{aligned} C(S_0, K, r, \delta, \tau, \psi) = \mathbf{E}\, e^{-r\tau}(S_0 e^{X_\tau} - K)^+ &= \mathbf{E}\, Z_\tau e^{-\delta\tau}(S_0 - K e^{-X_\tau})^+ = \\ &= \tilde{\mathbf{E}} e^{-\delta\tau}(S_0 - K e^{\tilde{X}_\tau})^+. \end{aligned}$$

where $\tilde{\mathbf{E}}$ denotes expectation with respect to $\tilde{\mathbf{P}}$, and the process $\tilde{X} = \{\tilde{X}_t\}_{t\geq 0}$ given by $\tilde{X}_t = -X_t$ ($t \geq 0$) is the *dual process* (see [3]). In order to conclude the proof, that is, in order to verify that

$$\tilde{\mathbf{E}} e^{-\delta\tau}(S_0 - K e^{\tilde{X}_\tau})^+ = P(K, S_0, \delta, r, \tau, \tilde{\psi}),$$

we must verify the dual process $\tilde{X}$ is a Lévy process with characteristic exponent defined by (20) and (21). First, for a complex z such that $1-z \in \mathbb{C}_0$, we have

$$\tilde{\mathbf{E}} e^{z\tilde{X}_t} = \mathbf{E}\, Z_t e^{-zX_t} = \mathbf{E}\, e^{-(r-\delta)t} e^{X_t} e^{-zX_t} = e^{t\big(\psi(1-z)-\psi(1)\big)}.$$

Second, defining $\tilde{\psi}(z) = \psi(1-z) - \psi(1)$, we have

$$\begin{aligned} \tilde{\psi}(z) &= a(1-z) + \sigma^2(1-z)^2/2 + \int_{I\!R} \big(e^{(1-z)y} - 1 - (1-z)h(y)\big)\Pi(dy) \\ &\quad - a - \frac{1}{2}\sigma^2 - \int_{I\!R} \big(e^y - 1 - h(y)\big)\Pi(dy) \\ &= -(a+\sigma^2)z + \frac{1}{2}\sigma^2 z^2 + \int_{I\!R} \big(e^{(1-z)y} - e^y + zh(y)\big)\Pi(dy). \end{aligned}$$

The integral term is transformed as follows:

$$\begin{aligned}\int_{\mathbb{R}} \big(e^{(1-z)y} - e^{y} + zh(y)\big)\Pi(dy) \\ = z\int_{\mathbb{R}}(1-e^{y})h(y)\Pi(dy) + \int_{\mathbb{R}}\big(e^{-zy} - 1 + zh(y)\big)e^{y}\Pi(dy) \\ = z\int_{\mathbb{R}}(1-e^{y})h(y)\Pi(dy) + \int_{\mathbb{R}}\big(e^{zy} - 1 - zh(y)\big)\tilde{\Pi}(dy),\end{aligned}$$

where we introduced the change of variables $y = -u$ in the last integral, and denoted $\tilde{\Pi}(dy) = e^{-y}\Pi(-dy)$. The final calculation, taking into account (15), is

$$\begin{aligned}-\tilde{a} &= a + \sigma^2/2 + \int_{\mathbb{R}}(e^{y}-1)h(y)\Pi(dy) \\ &= r - \delta - \sigma^2/2 - \int_{\mathbb{R}}\big(e^{y} - 1 - h(y)\big)\Pi(dy) + \sigma^2/2 + \int_{\mathbb{R}}(e^{y}-1)h(y)\Pi(dy) \\ &= r - \delta + \sigma^2/2 + \int_{\mathbb{R}}\big(e^{y} - 1 - h(y)\big)\tilde{\Pi}(dy).\end{aligned}$$

This concludes the proof. □

Some remarks are in order. Our Proposition 1 is very similar to Proposition 1 in Schroder (1999). The main difference is that the particular structure of the underlying process (Lévy process are a particular case of the model considered in [23]) allows to completely characterize the distribution of the dual process $\tilde{X}$ under the dual martingale measure $\tilde{\mathbf{P}}$, and to give a simpler proof.

The proof of the proposition motivates us to introduce the following market model. Given a Lévy market with driving process characterized by ψ in (12), consider a market model with two assets, a deterministic savings account $\tilde{B} = \{\tilde{B}_t\}_{t\geq 0}$, given by

$$B_t = e^{\delta t}, \qquad r \geq 0,$$

and a stock $\tilde{S} = \{\tilde{S}_t\}_{t\geq 0}$, modeled by

$$\tilde{S}_t = Ke^{\tilde{X}_t}, \qquad S_0 = e^{x} > 0,$$

where $\tilde{X} = \{\tilde{X}_t\}_{t\geq 0}$ is a Lévy process with characteristic exponent under $\tilde{\mathbf{P}}$ given by $\tilde{\psi}$ in (20). This market is the *auxiliary market* in Detemple

(2001), and we call it *dual market*; accordingly, we call *Put–Call duality* the relation (19). It must be noticed that Peskir and Shiryaev (2000) propose the same denomination for a different relation in [21]. Finally observe, that in the dual market (i.e. with respect to $\tilde{\mathbf{P}}$), the process $\{e^{-(\delta-r)t}\tilde{S}_t\}$ is a martingale. As a consequence, we obtain the Put–Call symmetry in the Black–Scholes–Merton model: In this case $\Pi = 0$, we have no jumps, and the characteristic exponents are

$$\psi(z) = (r-\delta-\sigma^2/2)z + \sigma^2 z^2/2,$$
$$\tilde{\psi}(z) = (\delta-r-\sigma^2/2)z + \sigma^2 z^2/2.$$

and relation (19) is the result known as put–call symmetry.

5.2 Symmetric markets

It is interesting to note, that in a market with no jumps the distribution (or laws) of the discounted (and reinvested) stocks in both the given and dual Lévy markets coincide. It is then natural to define a market to be *symmetric* when this relation hold, i.e. when

$$\mathcal{L}\big(e^{-(r-\delta)t+X_t} \mid P\big) = \mathcal{L}\big(e^{-(\delta-r)t-X_t} \mid \tilde{P}\big), \tag{22}$$

meaning equality in law. In view of (21), and to the fact that the characteristic triplet determines the law of a Lévy processes, we obtain that a necessary and sufficient for condition for (22) to hold is

$$\Pi(dy) = e^{-y}\Pi(-dy). \tag{23}$$

This ensures $\tilde{\Pi} = \Pi$, and from this follows $a-(r-\delta) = \tilde{a}-(\delta-r)$, giving (22), as we always have $\tilde{\sigma} = \sigma$. Condition (23) answers a question raised by Carr and Chesney (1996), see [5].

5.3 Examples and applications

In this section we consider that the Lévy measure of the process has the form

$$\Pi(dy) = e^{\beta y}\Pi_0(dy),$$

where $\Pi_0(dy)$ is a symmetric measure, i.e. $\Pi_0(dy) = \Pi_0(-dy)$. In many cases, the Lévy measure has a Radon-Nykodim density, and we have

$$\Pi(dy) = e^{\beta y}p(y)dy, \tag{24}$$

where $p(x) = p(-x)$, that is, the function $p(x)$ is even.
In this way, we want to model the asymmetry of the market through the parameter β. As a consequence of (23), we obtain that when $\beta = -1/2$ we have a symmetric market. This proposal is similar, in certain sense, to the skewness premium introduced by Bates (1997) in [2]. The idea is to describe numerically the departure from the symmetry, the main difference with Bates (1997) is that the parameter β is a property of the market, independent of the derivative asset considered. It is also interesting to note, that practically all parametric models proposed in the literature, in what concerns Lévy markets, including diffusions with jumps, can be reparametrized in the form (24) (with the exception of Kou (2000), see anyhow Kou and Wang (2001)). Let us consider some examples

5.3.1 Generalized Hyperbolic Model

This model has been proposed by Eberlein and Prause (1998) as they "allow a more realistic description of asset returns than the classical normal distribution" (see [8]). This model has $\sigma = 0$, and a Lévy measure given by (24), with

$$p(y) = \frac{1}{|y|}\Big(\int_0^\infty \frac{\exp\big(-\sqrt{2z+\alpha^2}|y|\big)}{\pi^2 z\big(J^2_{|\lambda|}(\delta\sqrt{2z}) + Y^2_{|\lambda|}(\delta\sqrt{2z})\big)} dz + \mathbf{1}_{\{\lambda\geq 0\}}\lambda e^{-\alpha|y|}\Big),$$

where $\alpha, \beta, \lambda, \mu$ are real parameters that satisfy the conditions $0 \leq |\beta| < \alpha$, and $\delta > 0$; and J_λ, Y_λ are the Bessel functions of the first and second kind (for details see [8]). Particular cases are the hyperbolic distribution, obtained when $\lambda = 1$; and the normal inverse gaussian when $\lambda = -1/2$. The statistical estimation $\beta = -24.91$ is given in [8] for the daily returns of the DAX (German stock index) for the period 15/12/93 to 26/11/97 (The other parameters are also estimated). This indicates the absence of symmetry.

5.3.2 The CGMY market model

This Lévy market model, proposed by Carr et al. (2002) in [6] is characterized by $\sigma = 0$ and Lévy measure given by (24), where the function $p(x)$ is given by

$$p(y) = \frac{C}{|y|^{1+Y}} e^{-\alpha|y|}.$$

The parameters satisfy $C > 0$, $Y < 2$, and $G = \alpha + \beta \geq 0$, $M = \alpha - \beta \geq 0$, where C, G, M, Y are the parameters used in [6].
For studying the presence of a pure diffusion component in the model, condition $\sigma = 0$ is relaxed, and risk neutral distribution are estimated in a five parameters model. Values of $\beta = (G - M)/2$ are given for different assets, and in the general situation, the parameter β is negative, and less than $-1/2$.

5.3.3 Diffusions with jumps

Consider the jump–diffusion model proposed by Merton (1976) in [19]. The driving Lévy process in this model has Lévy measure given by

$$\Pi(dy) = \lambda \frac{1}{\delta\sqrt{2\pi}} e^{-(y-\mu)^2/(2\delta^2)},$$

and is direct to verify that condition (23) holds if and only if $2\mu + \delta^2 = 0$. This result was obtained by Bates (1997) in [2]. The Lévy measure also corresponds to the form in (24), if we take $\beta = \mu/\delta^2$, and

$$p(y) = \lambda \frac{1}{\delta\sqrt{2\pi}} \exp\big(-(x^2 + \mu^2)/(2\delta^2)\big).$$

A recent alternative jump distribution was proposed by Kou and Wang (2001) in [15]. The Lévy measure has the form (24), where

$$p(y) = \lambda e^{-\alpha|y|}.$$

It can be observed that this is a particular case of the CGMY model, when $Y = -1$.

6 Conclusions

In this paper we have extended the results obtained by Gerber and Shiu (1996) for the bidimensional Geometric Brownian Motion to the case of bidimensional Geometric Lévy motion. We have shown that using the Dual market method it is possible to price many derivatives, with payoffs homogenous of any degree, written in terms of two assets driven by geometric Lévy motions, in the European case and for the American perpetual case. Another important fact in this paper is the possibility of having a stochastic discount, this allow us to consider derivatives as quanto derivatives.

Many extensions can be carry on, a natural one would be the extension to the multidimensional case, i.e. to study the pricing problem of derivatives written in terms of many assets. Finally, we derive a put-call relation, that allow us to obtain a call price from a put price of another asset price diffusion, by a change of probability, the first to point this out for the geometric Brownian Motion case were Peskir and Shirjaev (2001), but as we see it is also true for more general processes. Then we derive necessary and sufficient conditions for the denominated put-call symmetry hold.

7 Appendix

How to obtain an EMM ($\mathcal{Q}^{\theta}$)

The procedure introduced in this section is in spirit of Gerber and Shiu (1994). Take the original probability measure P and suppose that relative prices $\{\frac{S^j}{S^1}\}$ are not martingales under P, then we will show how to find EMM.
Let

$$M(z, t; \theta) = \frac{M(z + \theta, t)}{M(\theta, t)}$$

where $M(\theta, t) = \mathsf{E}(e^{\theta \cdot X_t'})$. Now find a vector θ^* such that the probability

$$d\mathcal{Q}_t^{\theta^*} = \frac{e^{\theta^* \cdot X_t'}}{\mathsf{E}(e^{\theta^* \cdot X_t'})}$$

be an EMM. To this end, suppose that $X_t^1 = rt$, as in Gerber and Shiu (1994), then it is enough to prove:

$$S_0^j = \mathsf{E}^*(e^{-rt} S_t^j) \ \ \forall j, \ \forall t$$

where E^* is the expectation under $\mathcal{Q}^{\theta^*}$, take $1_j = (0, \ldots, \underbrace{1}_{j-position}, \ldots, 0)$, then

$$r = \log[M(1_j, 1; \theta^*)] = \log\left[\frac{M(1_j + \theta^*, 1)}{M(\theta^*, 1)}\right]$$

The solution of this equation allow us to construct $\mathcal{Q}^{\theta^*}$. Now to extend the above procedure to our model we need that $\{\frac{S_t^j}{S_t^1}\}$ be a martingale, as $S_0^1 = 1$, then is enough to prove that

$$S_0^j = \mathsf{E}^*(\frac{S_t^j}{S_t^1}) \ \forall j, \ \forall t$$

$$1 = \mathsf{E}^*(e^{X_t^2 - X_t^1})$$

Defining $\bar{1}_j = (-1, 0, \ldots, \underbrace{1}_{j-position}, \ldots, 0)$, we have

$$1 = M(\bar{1}_j, 1; \theta^*)$$

In this way we obtain θ^*, then $\mathcal{Q}^{\theta^*}$.

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