Has the Phillips Curve Flattened?*

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Abstract

We contribute to the recent debate on the instability of the slope of the Phillips curve by offering insights from a flexible time-varying instrumental variable approach robust to weak instruments. Our robust approach focuses directly on the Phillips curve and allows general forms of instability, in contrast to current approaches based either on structural models with time-varying parameters or instrumental variable estimates in ad-hoc sub-samples. We find evidence of a weakening of the slope of the Phillips curve starting around 1980. We also offer novel insights on the Phillips curve during the recent pandemic: The flattening has reverted and the Phillips curve is back.

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1 Introduction

Inflation and unemployment seem to have become disconnected in the last decades. The correlation between inflation and real activity at business cycle frequencies has decreased in the 1990s (e.g. Atkeson and Ohanian, 2001, Stock and Watson, 2007, 2008, 2020), especially during the years of the expansion that followed the recent financial crisis of 2007-2009 - the so-called missing deflation (see, among others, Hall, 2011, and Ball and Mazumder, 2011, 2021). The decrease in the cyclical correlation between inflation and unemployment has been attributed by some to a flattening of the slope of the Phillips curve. The instability of the parameters in macroeconomic models such as the Phillips curve has been emphasized by many researchers, including Boldea and Hall (2013). More recently, the dramatic increase in both inflation and inflation expectations has raised important policy questions: if the Phillips curve indeed flattened, it would imply that more extreme policy measures would be necessary to maintain inflation at its target value. Thus, the question of whether the Phillips curve flattened is of high empirical relevance.

One of the main challenges in the estimation of the Phillips curve is the presence of endogeneity, as inflation and unemployment are jointly determined in equilibrium. There are two main approaches to handling endogeneity: estimating the Phillips curve as part of structural macroeconomic models (either Structural VARs or DSGEs) or focusing on the Phillips curve relationship by relying on instrumental variables methods. Both have advantages and disadvantages. On the one hand, it is well-known how to estimate DSGEs and Structural VARs in the presence of instabilities; however, DSGEs and Structural VARs are full-information estimation procedures, and potential misspecification in any other part of the model might contaminate the estimates of the Phillips curve parameters. This could be a serious problem during the recent financial crisis, when structural models have to confront serious misspecification challenges (Canova, Ferroni and Matthes, 2020; Kuo, Inoue and Rossi, 2020; Den Haan and Drechsel, 2021). On the other hand, limited-information approaches, such as instrumental variables (e.g. Galí and Gertler, 1999, or Galí et al., 2005), are less affected by potential misspecification but methodologies to address time-variation in instrumental variable models were lacking in the literature.

This paper makes two main contributions. The first contribution of this paper is to directly estimate the time-varying structural Phillips curve via limited-information methods in an environment robust to instabilities. To our knowledge, this is the first paper that estimates a general time-varying Phillips curve using instrumental variables. Our approach relies on the novel methodology proposed by Inoue et al. (2022) to estimate local projections and instrumental variables models with time-varying parameters (TVP-IV). Importantly, their approach allows time-variation in the parameters of the first stage regression as well as in the main regression, thus permitting a time-varying relationship between the instruments and the endogenous variables. The presence of weak instruments, however, might invalidate instrumental variable estimation. Given the empirical importance of the weak instrument issue in the estimation of the Phillips curve, our second contribution is to propose a novel methodological TVP-IV estimator robust to weak instruments.

Using the framework above, we shed light on whether and how the Phillips curve changed over time in a way consistent with the seminal, reduced-form approach by Galí and Gertler (1999). We find that the slope of the Phillips curve weakened since the early 1980s: the slope decreased, in absolute value, by 68 percent in the last two decades. However, we also find that it started reverting back in the most recent pandemic period. We also find that the decrease in the correlation between unemployment and inflation cannot be attributed to monetary policy; rather, to the decrease in the slope of the Phillips curve. Our results are complemented by a comparison with the commonly used benchmark IV model with constant parameters and the importance of instabilities is also demonstrated via structural break tests. We establish our main results in a specification that relies on a valid and strong set of instruments. However, we show that our results are robust to using the same set of instruments and specifications in the literature, carefully accounting for the presence of weak instruments. In addition, our findings do not rely on the rational expectations assumption, as our specification uses SPF survey measures of inflation expectations.

It is crucial to note that even small changes (in magnitude) in the slope of the Phillips curve could have important economic consequences. When shocks are big or their effect compounds, even a small flattening (in magnitude) could have important effects on the economy and on the conduct of monetary policy. For example, changes in the inflation target represent significant policy changes, and firms will likely respond by adjusting prices more frequently with higher trend inflation; thus, the slope of the Phillips curve would become steeper - see Ball et al. (1988). In order to stimulate the economy in this environment, central banks would decrease interest rates more than under constant price flexibility, as pointed out by L'Huillier and Schoenle (2022).[1]

[1]The flattening of the Phillips curve is not the only potential explanation for the disconnect between inflation and real variables during the Great Recession; other explanations which we entertain in our analysis include the possibility that inflation or economic slack are mis-measured and that monetary

Relative to the existing literature, our more general approach allows us to confirm the change in the slope of the Phillips curve in the 1990s as a robust result that emerges directly in the classical and appealing instrumental variable approach - a result that was infeasible prior to our work due to lacking methodologies. The decrease in the slope of the Phillips curve, and its recent rise during the pandemic, come to the fore as the reason behind the decrease in the correlation between inflation and unemployment, without requiring auxiliary and restrictive assumptions on other parts of the model describing the economy, nor prior knowledge of the time and pattern of the instability. As we show, our results are very robust to using different measures of slack, different approaches to measure inflation expectations (including using survey's forecasts) and different model specifications.

In contrast to our work, most of the existing literature relies either on reduced-form time-varying parameter approaches (Ball and Mazumder, 2019; Ashley and Verbrugge, 2019) or semi-structural time-varying parameter approaches (Galí and Gambetti, 2019); structural models' estimation in given sub-samples (Del Negro et al., 2020) or via full-information, time-varying parameter VARs (Cogley and Sargent, 2005; Primiceri, 2006; Cogley and Sbordone, 2008); or instrumental variable estimation in either given sub-samples (Barnichon and Mesters, 2020, 2021) or with discrete, estimated breakpoints (Hall, Han and Boldea, 2012), Markov-switching models (Groen and Mumtaz, 2008) and non-linearities (Forbes, Gagnon and Collins, 2021). More in detail, a first strand of the literature uses time-varying parameter methods in reduced-form or semi-structural models. Reduced-form approaches study the correlation between inflation and unemployment without resolving the endogeneity problem. For example, Stock and Watson (2009) survey the literature on the evaluation of inflation forecasts in the United States and suggest that Phillips curve forecasts are better than competing multivariate forecasts, although their performance is episodic relative to a univariate benchmark, pointing to the presence of instabilities. Ball and Mazumder (2019) argue that expected inflation was backward-looking until the late 1990s, but then became strongly anchored at the central bank's target value, which would explain why inflation did not decrease in the high unemployment period around the Great Recession. Differently from Stock and Watson (2009) and Ball and Mazumder (2019), we study not only the correlation but also the structural Phillips curve. Ashley and Verbrugge (2019) show that the slope of the correlation varies across business cycle phases. Using a model where unemployment is decomposed in persistent and

policy is better at stabilizing inflation, thus flattening aggregate demand.

transitory components, they find that inflation responds differently depending on whether fluctuations in the unemployment gap are persistent or transitory. More broadly, a large literature has pointed to the existence of time-variation in inflation (see e.g. Mertens and Nason, 2020). Our findings of time-variation in the slope of the Phillips correlation are in line with theirs; the difference between their approach and ours is that we take a model-free approach to time-variation.

A second strand of the literature focuses on instabilities in semi-structural models. Galí and Gambetti (2019) adopt a semi-structural approach by estimating a time-varying parameter Vector Autoregression (VAR) model to identify economic shocks via sign and long-run restriction, then use such shocks to purge the Phillips curve variables and achieve identification of the Phillips curve parameters. In a similar spirit, Bergholt et al. (2022) estimate structural shocks using sign restrictions in constant-parameter VARs; then, they investigate changes in the Phillips curve over time using inflation and unemployment data purged by the relevant shocks in either sub-samples or rolling windows. Differently from Galí and Gambetti (2019) and Bergholt et al. (2022), we directly estimate the Phillips curve using instrumental variable methods that do not require identifying all the structural shocks in the economy.

A third strand of the literature relies on structural models. For example, Cogley and Sargent (2005) and Primiceri (2006) estimate time-varying parameter structural VAR models for the US economy. Cogley and Sbordone (2008) include a time-varying trend component in inflation, modeled as a random walk; they estimate the time-varying parameter VAR under cross-equation restrictions following the approach in Sbordone (2002, 2006). In related work, Ascari, Bonomolo and Haque (2022) use a piece-wise long-run trend in a time-varying VAR to describe a long-run Phillips curve. Del Negro et al. (2020) investigate whether the flattening of the Phillips curve explains the disconnect between inflation and unemployment by focusing on time-varying parameter structural VARs and DSGE models, accounting for the potential time-variation in the relationship between inflation and real activity by separately estimating their models in two sub-samples, before and after 1989. The break date is determined by an a-priori choice, as a compromise between choosing a date where the economy became more stable (i.e. the Great Moderation, that started in 1984) and the stability of inflation itself, which seems to date back to the mid-1990s. Differently from their work, we rely directly on estimating the structural Phillips curve via limited-information methods, which are more robust to misspecification, and let the instability in inflation dynamics freely emerge within our time-varying instrumental variable estimator.

A fourth strand of the literature focuses on instrumental variables or external information. McLeay and Tenreyro (2019) argue that the fact that inflation follows a seemingly exogenous statistical process, unrelated to the output gap, does not mean that the Phillips curve has disappeared. They show that, in a theoretical model, monetary policy can generate a negative correlation between inflation and the output gap by increasing inflation when output is below potential, thus blurring the identification of the Phillips curve. They find evidence against the disappearance of the Phillips curve using regional data. The identification problem pointed out by McLeay and Tenreyro (2019), however, can be addressed by using instrumental variables, like we do. Barnichon and Mesters (2020, 2021) estimate the Phillips curve and the Phillips multiplier using narrative monetary policy shocks as instruments to address the endogeneity problem. To take into account time-variation, they split the sample at a known break date. The more general framework by Hall, Han and Boldea (2012) allows for multiple discrete shifts at unknown breakpoints in the conditional mean parameters, while the variance is assumed to be constant. Overall, while these papers perform sub-sample analysis (for example, Del Negro et al. (2020) use 1990 as the break point estimate), none of these papers allow for general patterns of time variation, which is instead the main contribution of our paper.

The remainder of the paper is organized as follows. The next section presents the methodology. Section 3 discusses the empirical evidence on the evolution of both the Phillips relation as well as the slope of the structural Phillips curve over time. Section 4 investigates whether the decrease in the correlation between unemployment and inflation is due to monetary policy or to a decrease in the slope of the Phillips curve, and Section 5 discusses the most recent evidence on the Phillips curve, including the recent pandemic. Section 6 concludes.

2 The Phillips Curve Model and the Methodology

Our benchmark Phillips curve is the classic version by Galí and Gertler (1999):

$$\pi_t = c + \gamma_f E_t\left(\pi_{t+1}\right) + \gamma_b \pi_{t-1} + \lambda x_t + u_t,$$

where π_t denotes inflation, x_t denotes the measure of real marginal cost, $E_t\left(.\right)$ denotes conditional expectations at time t and u_t is an unobserved shock. This specification is the same as Galí and Gertler (1999) and Galí et al. (2005): a hybrid New Keynesian Phillips Curve (NKPC) with lagged inflation and the unemployment gap as the forcing variable. We will estimate the NKPC using instrumental variables under the rational expectations assumption.

Our main focus is estimating the slope of the Phillips curve, namely λ, which is the object of a lively debate. On the one hand, several researchers found that the slope of the Phillips curve has flattened or even that the Phillips curve "died," that is, its slope approached zero – see Coibion and Gorodnichenko (2015), Blanchard (2016), Ball and Mazumder (2019), and Stock and Watson (2020), among others. On the other hand, Barnichon and Mesters (2021) and Bergholt et al. (2022), among others, argue that it did not. For example, using a Phillips-multiplier approach, Barnichon and Mesters (2021) argue that the inflation-unemployment trade-off went from being very large before 1990 to being small, but still significant, after 1990, and that the decline in the trade-off is mostly due to the anchoring of inflation expectations. Tenreyro and Twaites (2016) argue that the disconnect between inflation and real activity may not only be due to a flat Phillips curve but also to a flat aggregate demand, such as one where monetary policy strongly responds to inflation. For example, if the central bank achieves perfect inflation stability, the researcher would observe inflation to be uncorrelated with real activity even if the Phillips curve slope were not zero. Del Negro et al. (2020) find that the slope of the Phillips curve substantially weakened over time, and that is the main reason for the disconnect between inflation and unemployment. Bergholt et al. (2022) find that the Phillips curve is "dead" only unconditionally: once it is purged for supply shocks, the Phillips curve is alive and well, and may even have steepened since the financial crisis.[2]

As is well-known in the literature, there are several econometric challenges in estimating the structural Phillips curve. A first challenge is that the forcing variable x_t may be correlated with the structural error term u_t, thus resulting in an endogeneity problem. An additional challenge is that the expected inflation term $E_t(\pi_{t+1})$ is not only endogenous but also unobservable. To address these issues, we consider an Instrumental Variable (IV) approach to identification, as in Galí and Gertler (1999), Galí et al. (2005) and Barnichon and Mesters (2020), among others. Suppose Z_t is a vector of valid instruments such that

$$E[Z_t(\pi_t - \gamma_b \pi_{t-1} - \gamma_f \pi_{t+1} - \lambda x_t)] = 0.$$

Under the rational expectations assumption, we can include as instruments in Z_t any

[2]Bergholt et al. (2022) estimate the Phillips curve using OLS after purging the variables by supply shocks.

predetermined variables, such as lags of the endogenous variables.

2.1 The Time-Varying Instrumental Variable (TVP-IV) Estimator

Our contribution is to estimate the Phillips curve slope allowing the parameters to be time-varying using the time-varying parameter instrumental variable (TVP-IV) approach by Inoue et al. (2022). We consider the following time-varying parameter Phillips curve:

$$\pi_t = c_t + \gamma_{f,t}\pi_{t+1} + \gamma_{b,t}\pi_{t-1} + \lambda_t x_t + u_t,$$

where the variance of the error term is also allowed to be time-varying. The parameters of primary interest are thus $c_t, \gamma_{f,t}, \gamma_{b,t}, \lambda_t$ and the shock volatility – in particular, the slope of the Phillips curve, λ_t.

We estimate the parameters of interest using an IV approach. The TVP-IV regression as follows:[3]

$$\begin{bmatrix} x_t \\ \pi_{t+1} \\ \pi_t \end{bmatrix} = \begin{bmatrix} \beta'_{x,z,t} & c_{x,t} & \beta_{x,\pi_b,t} \\ \beta'_{\pi_f,z,t} & c_{\pi_f,t} & \beta_{\pi_f,\pi_b,t} \\ \lambda_t\beta'_{x,z,t} + \gamma_{f,t}\beta'_{\pi_f,z,t} & \lambda_t c_{x,t} + \gamma_{f,t}c_{\pi_f,t} + c_t & \beta_{\pi,\pi_b,t} \end{bmatrix} \begin{bmatrix} z_t \\ 1 \\ \pi_{t-1} \end{bmatrix} + \begin{bmatrix} v_{x,t} \\ v_{\pi_f,t} \\ v_{\pi,t} \end{bmatrix}, \tag{1}$$

where $(x_t, \pi_{t+1})'$ are endogenous variables, and $Z_t = (z'_t, \pi_{t-1})'$, where z_t are the (excluded) instruments, and $\beta_{\pi,\pi_b,t} = \lambda_t\beta_{x,\pi_b,t} + \gamma_{f,t}\beta_{\pi_f,\pi_b,t} + \gamma_{b,t}$. Note that the estimation directly constrains the parameter $\beta_{\pi,\pi_b,t}$ to be a function of the actual parameters of interest. The parameter path is estimated according to a minimum weighted average risk criterion, as in Müller and Petalas (2010) and, in particular, Inoue et al. (2022).

Let $\theta_t = [\beta'_{x,z,t}, c_{x,t}, \beta_{x,\pi_b,t}, \beta'_{\pi_f,z,t}, c_{\pi_f,t}, \beta_{\pi_f,\pi_b,t}, \lambda_t, c_t, \gamma_{f,t}, \gamma_{b,t}, vech(\Sigma_{v,t})']'$, where $\Sigma_{v,t}$ is the covariance matrix of $[v_{x,t}, v_{\pi_f,t}, v_{\pi,t}]'$. Let θ_t evolve slowly over time according to a deterministic function of time, where $\theta_t = \theta + \delta_t$ and δ_t describes small amounts of time variation (of magnitude $T^{-1/2}$). The risk of estimating the wrong parameter path depends on the true parameter path θ_t and no estimator achieves uniformly low risk over all such paths. As explained below, we estimate the parameter path based on Müller and Petalas (2010) and, in particular, Inoue et al. (2022), by minimizing the

[3]The first two equations below are the first-stage equations, and the third equation is obtained by substituting the first-stage equations into the Phillips curve.

weighted average risk, where the weighting is over alternative true parameter paths. The optimal parameter path estimator will assume a weighting function for δ_t such that θ_t is a multivariate Gaussian random walk with a small variance:

$$\theta_t = \theta_{t-1} + \epsilon_t,$$

where ϵ_t is an i.i.d. disturbance with variance $(1/T)\sigma^2_{\epsilon,t}$, and T is the total sample size. Modeling the time variation in the parameters as a random walk has a long tradition in the literature – see e.g. Canova (1983), Cogley and Sargent (2005) and Primiceri (2006), among others.

Let us highlight that we not only allow the coefficients of the Phillips curve to be time-varying in the main regression but also in the first stage regression, thus allowing a time-varying relationship between the instruments and the endogenous variables.

2.2 The TVP-IV Estimator Robust to Weak Instruments

We estimate the reduced-form parameters in eq. (1) by rewriting it as eq. (2):

$$\begin{bmatrix} x_t \\ \pi_{t+1} \\ \pi_t \end{bmatrix} = \begin{bmatrix} \beta'_{x,z,t} & c_{x,t} & \beta_{x,\pi_b,t} \\ \beta'_{\pi_f,z,t} & c_{\pi_f,t} & \beta_{\pi_f,\pi_b,t} \\ \beta'_{\pi,z,t} & c_{\pi,t} & \beta_{\pi,\pi_b,t} \end{bmatrix} \begin{bmatrix} z_t \\ 1 \\ \pi_{t-1} \end{bmatrix} + \begin{bmatrix} v_{x,t} \\ v_{\pi_f,t} \\ v_{\pi,t} \end{bmatrix}, \quad (2)$$

where $\beta'_{\pi,z,t} = \lambda_t \beta'_{x,z,t} + \gamma_{f,t}\beta'_{\pi_f,z,t}$, $c_{\pi,t} = \lambda_t c_{x,t} + \gamma_{f,t} c_{\pi_f,t} + c_t$, and $\beta_{\pi,\pi_b,t} = \lambda_t \beta_{x,\pi_b,t} + \gamma_{f,t}\beta_{\pi_f,\pi_b,t} + \gamma_{b,t}$. In particular, the reduced-form parameters are $\beta'_{\pi,z,t}$, $c_{\pi,t}$ and $\beta_{\pi,\pi_b,t}$, rather than λ_t, γ_f, and γ_b. We let $\theta_t^{rf} = [\beta'_{x,z,t}, c_{x,t}, \beta_{x,\pi_b,t}, \beta'_{\pi_f,z,t}, c_{\pi_f,t}, \beta_{\pi_f,\pi_b,t}, \beta_{\pi,z,t}, c_{\pi,t}, \beta_{\pi,\pi_b,t}, vech(\Sigma_{v,t})']'$, where $\Sigma_{v,t}$ is the covariance matrix of $[v_{x,t}, v_{\pi_f,t}, v_{\pi,t}]'$. Note that, unlike the previous section, in this section we estimate directly the vector of reduced-form parameters, rather than the structural parameters underlying them. The parameters of primary interest $(\hat{\lambda}_t, \hat{\gamma}_f, \hat{\gamma}_b)$ can then be recovered from the reduced-from parameter estimate $\hat{\theta}_t^{rf}$.

Here below we provide the detailed algorithm we used to estimate the time varying parameters, adapted from Müller and Petalas (2010), as discussed in Inoue et al. (2022).

2.3 Description of the Estimation Algorithm

Let θ_t include all the slope and (co)variance parameters in eq.(2), which can be written as $\{\theta_t\}_{t=1}^T = \{\theta + \delta_t\}_{t=1}^T$, with $\sum_{t=1}^T \delta_t = 0$. The sample information about θ and $\{\delta_t\}_{t=1}^T$ is approximately independent and described by the pseudo model (see Müller and Petalas (2010)):

$$\begin{aligned} \widehat{\theta} &= \theta + T^{-1/2}\widehat{S}\nu_0, \\ \widehat{H}\widehat{V}^{-1}s_t(\widehat{\theta}) &= \widehat{S}^{-1}\delta_t + \nu_t, \quad t = 1, \cdots, T, \end{aligned} \tag{3}$$

with $\nu_t \overset{\text{i.i.d.}}{\sim} \mathcal{N}(0, \widehat{H})$. Here, $\widehat{H}$, $\widehat{V}$, and $\widehat{S}$ are consistent estimators of the counterparts defined below.

Consider the corresponding stable system of eq.(2), where all the parameters are time-invariant, denoted as θ. Let $y_t = (x_t, \pi_{t+1}, \pi_t)'$, $v_t = (v_{x,t}, v_{\pi_f,t}, v_{\pi,t})'$, and $f(y_t|z_t, \pi_{t-1}, \theta)$ denote a family of conditional density functions for y_t under the assumption that $\{v_t\}$ has zero mean and covariance matrix Σ_v.

In particular, let $\sum_{t=1}^T \ell_t(\theta), \ell_t(\theta) = \ln f(y_t|z_t, \pi_{t-1}, \theta)$ denote the (potentially misspecified) likelihood of eq.(2), let $s_t(\theta) = \partial\ell_t(\theta)/\partial\theta, t = 1, \ldots, T$, denote the sequence of $(q \times 1)$ score vectors, let $h_t(\theta) = -\partial s_t(\theta)/\partial\theta', t = 1, \ldots, T$, denote the sequence of $(q \times q)$ Hessians. Then, $\sqrt{T}(\widehat{\theta} - \theta_0) \Rightarrow \mathcal{N}(0, S)$, where the sandwich matrix S is typically estimated as $\widehat{S} = \widehat{H}^{-1}\widehat{V}\widehat{H}^{-1}$ and $\widehat{H} = \frac{1}{T}\sum_{t=1}^T h_t(\widehat{\theta})$. If the score vectors are i.i.d., then $\widehat{V} = \frac{1}{T}\sum_{t=1}^T s_t(\widehat{\theta})s_t(\widehat{\theta})'$; however, in this context, the residuals in eq.(2) might be serially correlated, thus HAC estimators, such as Newey and West (1987), may be used to account for the serial correlation.

Assuming an approximately stationary model and a weighting function for $\{\delta_t\}_{t=1}^T$ that is a demeaned multivariate Gaussian random walk, as in Müller and Petalas (2010), the asymptotically WAR minimizing path estimators $\{\widehat{\theta}_t\}_{t=1}^T$ can be obtained as follows:

1. For $t = 1, \ldots, T$, let $\widetilde{x}_t$ and $\widetilde{y}_t$ be all the elements of $\widehat{H}^{-1}s_t(\widehat{\theta})$ and $\widehat{H}\widehat{V}^{-1}s_t(\widehat{\theta})$, respectively.

2. For $c_i \in C = \{0, c_1, c_2, \ldots, c_{n_G}\}$,[4] $i = 0, 1, \ldots, n_G$, compute

   (a) $r_i = 1 - \frac{c_i}{T}$, $\xi_{i,1} = \widetilde{x}_1$, and $\xi_{i,t} = r_i\xi_{i,t-1} + \widetilde{x}_t - \widetilde{x}_{t-1}, t = 2, \ldots, T$;

[4] For the factor of proportionality $\frac{c^2}{T^2}$, Müller and Petalas (2010) suggest a default choice of minimizing WAR relative to an equal-probability mixture of $c \in \{0, 5, 10, \ldots, 50\}$, which represents the standard deviation of the endpoint of the random walk weighting function and covers a wide range of magnitudes for the time variation.

(b) the residuals $\{\widetilde{\xi}_{i,t}\}_{t=1}^{T}$ of a linear regression of $\{\xi_{i,t}\}_{t=1}^{T}$ on $\{r_i^{t-1} I_q\}_{t=1}^{T}$;

(c) $\overline{\xi}_{i,T} = \widetilde{\xi}_{i,T}$, and $\overline{\xi}_{i,t} = r_i \overline{\xi}_{i,t+1} + \widetilde{\xi}_{i,t} - \widetilde{\xi}_{i,t+1}, t = 1, \ldots, T-1$;

(d) $\{\widehat{\theta}_{i,t}\}_{t=1}^{T} = \{\widehat{\theta} + \widetilde{x}_t - r_i \overline{\xi}_{i,t}\}_{t=1}^{T}$;

(e) $qLL(c_i) = \sum_{t=1}^{T} (r_i \overline{\xi}_{i,t} - \widetilde{x}_t)' \widetilde{y}_t$ where $\widetilde{w}_0 = 1$ and $\widetilde{w}_i = \sqrt{T(1-r_i^2) r_i^{T-1} / ((1-r_i^{2T}))} exp[-\frac{1}{2} qLL(c_i)]$.

3. Compute $w_i = \widetilde{w}_i / \sum_{j=0}^{n_G} \widetilde{w}_j$.

4. The parameter path estimator is given by $\{\widehat{\theta}_t\}_{t=1}^{T} = \{\sum_{i=0}^{n_G} w_i \widehat{\theta}_{i,t}\}_{t=1}^{T}$.

5. With the weighting functions for $\{\delta_t\}_{t=1}^{T}$ and θ interpreted as priors from a Bayesian perspective, the approximate posterior for θ_t is a mixture of multivariate normals $\mathcal{N}\left(\widehat{\theta}_{i,t}, T^{-1} \widehat{S} \kappa_t(c_i)\right), i = 0, \ldots, n_G$ with mixing probabilities w_i where $\widehat{S} = \widehat{H}^{-1} \widehat{V} \widehat{H}^{-1}$, $\kappa_t(c) = \frac{c(1+e^{2c}+e^{2ct/T}+e^{2c(1-t/T)})}{2e^{2c}-2}$, and $\kappa_t(0) = 1$.

In the weak instrument case, we need an extra step to recover the structural parameter estimates $(\hat{\lambda}_t, \hat{\gamma}_{f,t}, \hat{\gamma}_{b,t})$ from the reduced-form parameters θ_t^{rf}. Therefore, in the weak instrument case, we implement the procedure described above in points 1-5, with θ_t^{rf} instead of θ_t, followed by the additional step below:

6. In steps 4-5, we obtain the point estimates of the reduced form parameters $\hat{\theta}_t^{rf}$ as well as their joint distribution, denoted by $F_{\hat{\theta}_t^{rf}}(\cdot)$, for $t = 1, 2, ...$ The estimates $\hat{\lambda}_t, \hat{\gamma}_{f,t}, \hat{\gamma}_{b,t}$ can be recovered from the reduced-from parameter estimate $\hat{\theta}_t^{rf}$. Then, we randomly draw $\mathcal{M}$ times from the joint distribution $F_{\hat{\theta}_t^{rf}}(\cdot)$ and repeatedly recover $\hat{\lambda}_t^i, \hat{\gamma}_{f,t}^i, \hat{\gamma}_{b,t}^i$, $i = 1, 2, ..., \mathcal{M}$. The confidence bands and the median estimate can be obtained by the corresponding quantiles.

This estimation procedure differs from that in Inoue et al. (2022) in step 6; in fact, this estimation procedure is robust to weak instruments since that won't affect the estimation of the reduced form parameters θ_t. The detailed formulas used for the calculations are provided in the Appendix.

2.4 Theoretical Justification of the Methodology

The theoretical justification for the robustness of the proposed methodology to weak instruments is discussed in the next propositions, while the theoretical assumptions are presented in Section D of the Appendix. While Inoue et al. (2022) focus on a local projections approach when the data generating process follows a time-varying parameter vector moving process, this paper considers instrumental variables estimation of time series models, such as the Phillips curve, allowing parameters to vary over time.

The first proposition shows that the time-varying estimates are optimal, in the sense that the expected loss of estimating the parameter path in the misspecified TVP-IV model and the best inference in a corresponding correctly specified model is identical (asymptotically). The second proposition instead states that inference is robust even in the presence of weak instruments.

Proposition 1. Let the assumptions in the Appendix hold and let θ^*_T be the time-varying parameter path that minimizes the expected risk in the correctly specified model in the sense of Müller and Petalas (2010, Condition 1), with parameter path equal to $\theta_t = \theta_0 + T^{-\frac{1}{2}}\delta_0\left(\frac{t}{T}\right)$ relative to the distribution of Δ being a mixture of $N(e\widehat{\theta} + \Sigma_i\widehat{s}, \Sigma_i)$ with mixing probabilities $\tilde{w}_i$ defined in equation (21) in Müller and Petalas (2010), where e is a stacked vector of T identity matrices of the same dimension as θ, $\widehat{s} = [s_1(\widehat{\theta})', s_2(\widehat{\theta})', ..., s_T(\widehat{\theta})']'$, Σ_i is defined in Theorem 4 in Müller and Petalas (2010). Let $\widehat{\Theta}^{r*}_T$ be the reduced-form parameter path that minimizes expected risk of the TVP-IV model in equation (2) relative to the distribution of $\Delta^r \sim N(e\widehat{\theta}^r + \Sigma^r\underline{\widehat{s}}^r, \Sigma^r)$, where $\underline{\widehat{s}}^r$ is the vector of stacked scores $\underline{\widehat{s}}^r_t$, where $\underline{\widehat{s}}^r_t = \widehat{\Xi}^{-1}\widehat{V}^{-1}s_t(\theta_0)$, and $\Sigma^r = K + (I_{Tk} - KD_{\tilde{h}^r})e(e'D_{\tilde{h}^r}e - e'D_{\tilde{h}^r}KD_{\tilde{h}^r}e)^{-1}e'(I_{Tk} - D_{\tilde{h}^r}K)$, where $K = \Sigma_\delta(D_{\tilde{h}^r}\Sigma_\delta + I_{Tk})^{-1}$, $\Sigma_\delta = E_\delta(\delta\delta')$, $e = [I_k \cdot I_k]'$, $D_{\tilde{h}^r} = I \otimes \tilde{h}^r$, $\tilde{h}^r = \widehat{S}^{-1} = \widehat{H}\widehat{V}^{-1}\widehat{H}$. In addition, let $\Theta^*_T(\Delta_T)$ be the reduced-form parameter path that minimizes the expected risk relative to the distribution Δ_T. Assume that the researcher minimizes a loss function $L_T(\cdot)$ defined over θ, δ and the action a^* which is the parameter path estimate) such that $L_T\left(\theta_0, \left[\delta_0\left(\frac{1}{T}\right), ..., \delta_0\left(\frac{T}{T}\right)\right], a^*(\Delta_{1,T})\right) - L_T\left(\theta_0, \left[\delta_0\left(\frac{1}{T}\right), ..., \delta_0\left(\frac{T}{T}\right)\right], a^*(\Delta_{2,T})\right) \rightarrow 0$ whenever the total variation between the two mixtures $\Delta_{1,T}$ and $\Delta_{2,T}$ converges to zero, where Δ_{1T} and Δ_{2T} are two mixtures of n_G normal distributions. Then the difference between $L_T\left(\theta_0, \left[\delta_0\left(\frac{1}{T}\right), ..., \delta_0\left(\frac{T}{T}\right)\right], \Theta^*_T\right)$ in the correctly specified model and $L_T\left(\theta_0, \left[\delta_0\left(\frac{1}{T}\right), ..., \delta_0\left(\frac{T}{T}\right)\right], \Theta^{r*}_T\right)$ in the misspecified TVP-LP model converges to zero in the Prohorov metric.

Proposition 2. Let $C_{\theta,1-\alpha}$ denote the $(1-\alpha)$ credible set for the reduced-form parameter from the quasi posterior distribution defined in step 5 of the Algorithm presented in section 2.3. Let θ^{rf} denote the reduced-form parameter such that $\theta^{rf} = g(\theta)$ for some function $g(\cdot)$, and define $C_{\theta^{rf},1-\alpha}$ as $C_{\theta^{rf},1-\alpha} = \{\theta^{rf} \in \Theta^{rf} : \text{There is } \theta \in \Theta \text{ such that } \theta^{rf} = g(\theta)\}$. Then

$$P(\theta^{rf} \in C_{\theta^{rf},1-\alpha}) \geq 1-\alpha$$

where P is the probability measure implied by the quasi posterior distribution of θ in step 5 of the aforementioned algorithm, and the equality holds if g is one-to-one. [5]

Throughout the paper, we will compare our results to the benchmark model with constant parameters. In particular, we report results based on instability tests and comment on the differences between our time-varying approach and the constant parameter instrumental variable model typically considered in the literature.

3 Has the Phillips Curve Flattened Over Time?

In this section we discuss our main empirical evidence on the three main concepts surrounding the estimation of the Phillips curve in the literature: the Phillips relation, the slope of the Phillips curve and the Phillips multiplier.

The Phillips Relation

We start by empirically investigating the reduced-form relationship between inflation and the labor share over time, following Stock and Watson (2020). We focus on the estimated slope ($\beta_{1,t}$) in the following Phillips relation:

$$E_t \Delta_4 \pi_t^4 = \beta_{0,t} + \beta_{1,t} x_t^4, \tag{4}$$

where x_t^4 is the change in the average value of variable "x" between times t and t-3 and $\Delta_4 = (1 - L^4)$, L denotes the lag operator such that $L x_t = x_{t-1}$. There are several candidate choices for both inflation and real marginal cost measures – see e.g. the literature review in Mavroeidis et al. (2014). In our analysis, inflation (π_t) is measured

[5] The fact that the time variation in the parameters is related to the score is reminiscent of score-driven models, such as Creal et al. (2013) and Harvey (2013). However, the latter consider a predictive framework, and do not consider optimality properties of their estimator, instrumental variables approaches, or weak instruments.

by personal consumption expenditure price index (PCE excluding food and energy, PCExFE) and x_t is a measure of slack.

Figure 1 reports the time-varying estimate of $\beta_{1,t}$ for various measures of slack for the US. The estimate based on our TVP-LP estimator is reported by the dashed line (in red). Each panel in the figure corresponds to a different measure of slack, inspired by Stock and Watson (2020, Table 1). We consider: the unemployment gap, as measured by the Congressional Budget Office (CBO) in panel (a); the GDP gap, also from the CBO, in panel (b); the unemployment gap filtered using a two-sided filter[6] in panel (c); a measure of the short-term unemployment gap[7] in panel (d); the employment-population ratio (again obtained via a two-sided filter) in panel (e); the employment-population ratio focusing on population of age between 25 and 54 year-old (again obtained via a two-sided filter) in panel (f); the capacity utilization rate in panel (g); the unemployment rate measured as a real-time slack in panel (h); and the short-term unemployment rate in panel (i).[8]

The instability in the correlation between inflation and the real marginal cost measure is confirmed by standard parameter instability tests, no matter which measure we use. For example, Elliot and Müller's (2006) qLL test on $\beta_{1,t}$ ranges from -12.50 to -6.55 across the various specifications that we consider. The null hypothesis of no time variation is rejected at the 5% or 10% significance level for all slack measures except the CBO output gap and the short-term unemployment rate measured in real time.

We compare our results with Stock and Watson's (2020), who estimate the Phillips relation in three sub-samples. Their estimate is depicted by the solid (black) line, together with 90 percent confidence bands (black dotted lines).

INSERT FIGURE 1 HERE

As Figure 1 shows, the slope of the Phillips relation substantially flattened over time, and this emerges clearly in the data no matter whether we estimate the relation

[6]The two-sided filter used in this section to obtain the gap measure is the same as Stock and Watson (2020) and it is a band-pass Butterworth filter of degree 6, with lower and upper cutoffs corresponding to periods of 32 and 6 quarters, respectively.

[7]The short-term unemployment gap is obtained from the short-term unemployment rate (those unemployed 26 weeks or less as a fraction of the labor force), i.e. the measure of slack in Ball and Mazumder (2019), using the two-sided filter described in the previous footnote.

[8]All slack measures are standardized, and they have the same mean and standard deviation as the unemployment gap from the CBO. They have also been transformed in order to be positively correlated with the CBO output gap. Appendix A provides more details on the data.

in sub-samples or using our time-varying estimator. Relative to Stock and Watson's (2020) estimates, we find a more gradual decline in the slope magnitude in the 1970s, as well as a flatter curve in the most recent period. Thus, the Phillips relation (i.e. correlation) has disappeared in the data in the most recent period.

But does it mean that the structural Phillips curve has disappeared? Not necessarily, as the Phillips relation measures the (reduced-form) correlation between inflation and unemployment, while the Phillips curve measures the trade-off between inflation and unemployment due to supply shocks. The latter will be considered in the next sub-section.

The Slope of the Phillips Curve

In what follows, we will directly estimate the structural Phillips curve using the TVP-IV-based approach that flexibly allows the parameters to change over time while, at the same time, avoiding the endogeneity problem. In fact, the estimator has the advantage of letting the parameters change over time in a flexible way, including the variance of the error term, as well as the advantage of being robust to endogeneity, as it relies on instrumental variables.[9]

Understanding whether the structural Phillips curve flattened is important, as it implies that more extreme policy measures become necessary to maintain inflation at its target value. It is also an important issue for the design of optimal monetary policy and the desired inflation target. Even small changes (in magnitude) in the slope of the Phillips curve could have important consequences. For example, changes in the inflation target represent significant policy changes, and firms will likely respond by adjusting prices more frequently with higher trend inflation; thus, the slope of the Phillips curve would become steeper - see Ball et al. (1988). In order to stimulate the economy in this environment, central banks should decrease interest rates more than under constant price flexibility, as pointed out by L'Huillier and Schoenle (2022). As a consequence, when shocks are big or their effects compound, as per the mechanisms described in the aforementioned papers, even a small flattening in the slope of the Phillips curve may have important effects on the macroeconomy and on the conduct of monetary policy.

Figure 2 plots the estimates of λ_t using the TVP-IV framework and compares them with the estimates of a benchmark constant parameter model as in Galí and Gertler

[9] Instabilities are empirically relevant. In fact, Elliott and Müller's (2006) qLL test statistics on the reduced form regression, which is included as eq. (2) below, is -904.7377. Thus, the null hypothesis of no time variation is rejected at a 1% significance level, reflecting instabilities in the system.

(1999) and Galí et al. (2005). The main sample ranges from 1970Q1 to 2008Q1. We focus on a model specification where x_t is the unemployment gap estimated by the CBO and expected inflation is the three-quarter-ahead forecast of the mean GDP deflator inflation from the Survey of Professional Forecasters. The set of instruments includes two lags of the unemployment gap (from the CBO) and two lags of the output gap (estimated in real-time using a one-sided quadratically detrending procedure).[10]

INSERT FIGURE 2 HERE

Our results reveal a flattening of the slope of the Phillips curve (λ_t) in the last two decades. The slope decreased, in absolute value, by approximately 68%: it was around -0.12 in the early 1970s and became -0.04 in the most recent sample. In particular, notice how the slope trended downward in the 1990s, becoming effectively indistinguishable from zero.

An important issue in the estimation of the structural Phillips curve via instrumental variables is the presence of a weak instrument problem (Kleibergen and Mavroeidis, 2009; Mavroeidis et al., 2014). In our analysis, the instruments are both valid and strong. In fact, Hansen's J-statistic equals 1.955, with a p-value of 0.3763, indicating that the instruments are valid. Lewis and Mertens's (2022) weak IV test statistic equals 16.0254, and it is greater than the 90% critical value (14.0533), indicating that the instruments are strong. The Ganics et al. (2021) weak-instrument robust confidence interval for the strength of identification also points to strong instruments (the minimum eigenvalue is 1.66, with a confidence interval equal to $(1.15, 5.79)$, which excludes zero). We also report estimates and confidence intervals for the strength of identification over time in Figure A1 in Appendix B. They are based on Ganics et al.'s (2021) weak-instrument robust estimates and confidence intervals for the TVP-IV estimates. The confidence interval excludes zero at all times, thus implying that this set of instruments is strong in this specification even when considering unstable environments.

Previous papers, notably Galí and Gambetti (2019), have estimated the Phillips curve allowing the parameters to be time-varying. In order to address the endogeneity problem in the estimation of the Phillips curve, their approach relies on purging the OLS estimates using shocks identified via time-varying parameter Structural VARs -

[10]The set of instruments is inspired by Galí and Gertler (1999) and Galí et al. (2005); however, we have excluded lagged inflation from the set of instruments because our analysis suggests that it is affected by a weak instrument problem. Figure A2 in Appendix B shows that our main results are robust to using the same set of instrumental variables as Galí and Gertler (1999) and Galí et al. (2005).

hence, it is a semi-structural approach. In particular, their Structural VAR identifies several macroeconomic shocks, using both sign and long-run restrictions. Bergholt et al. (2022) also rely on a semi-structural approach. While the semi-structural approach addresses the endogeneity problem, it requires researchers to separately identify several shocks underlying the economy. Our approach instead does not require researchers to address the challenging task of separately identifying the shocks, as it relies on instrumental variables, in a way that directly parallels the pioneering work of Galí and Gertler (1999).

The Phillips Multiplier

In a recent paper, Barnichon and Mesters (2021) propose the "Phillips multiplier" as a measure of the inflation-unemployment trade-off faced by policymakers, different from the slope of the Phillips curve. Relative to Barnichon and Mesters (2021), our approach in the previous sub-section can directly and flexibly estimate the time-varying trade-off between inflation and unemployment in the classical specification of the Phillips curve.[11] Furthermore, we can use the same instruments and specification as in the seminal contribution by Galí and Gertler (1999), which makes our approach more directly comparable to theirs.

However, the Phillips multiplier is an alternative measure of the trade-off between inflation and unemployment that we can estimate using our time-varying method. By being estimated using instrumental variable methods, the Phillips multiplier also avoids the typical endogeneity problems afflicting the estimation of the Phillips curve. We focus on the same model specification as in Barnichon and Mesters (2021) with time-varying parameters:

$$\sum_{j=0}^{h} \pi_{t+j} = \mathcal{P}_{h,t} \sum_{j=0}^{h} x_{t+j} + \gamma'_{h,t} W_t + e_{t+h}, \tag{5}$$

where x_t is unemployment, W_t denote control variables including four lags of inflation and unemployment following Barnichon and Mesters (2021) and monetary policy shocks (η_t) are used as instruments for $\sum_{j=0}^{h} x_{t+j}$, as monetary policy shocks are uncorrelated with supply shocks. The parameter $\mathcal{P}_{h,t}$ denotes the Phillips multiplier in unstable environments and can be estimated based on the TVP-LP-IV estimator

[11] Under some assumptions, in particular a constant forward-looking behavior of inflation, Barnichon and Mesters (2021) relate the Phillips multiplier to the slope of the Phillips curve. In our approach, we instead let the forward-looking inflation coefficient be freely time-varying.

in Inoue et al. (2022). Therefore, the Phillips multiplier will estimate the effect of an increase in unemployment on inflation conditional on the presence of monetary policy shocks. Additionally, time-varying parameter impulse responses of inflation and unemployment to monetary policy shocks can be obtained by estimating the following TVP-LP:

$$\begin{aligned} \pi_{t+h} &= \beta_{\pi,h,t+h}\eta_t + \gamma'_{\pi,h,t}W_t + e_{\pi,t+h} \\ x_{t+h} &= \beta_{x,h,t+h}\eta_t + \gamma'_{u,h,t}W_t + e_{x,t+h}, \end{aligned} \tag{6}$$

where $\beta_{\pi,h,t+h}$ and $\beta_{u,h,t+h}$ denote the impulse responses of inflation and unemployment to monetary policy shocks in unstable environments and can be estimated based on the TVP-LP estimator in Inoue et al. (2022).

The top two panels in Figure 3 report the impulse responses of inflation and unemployment to a monetary policy shock, whereas the bottom panel depicts the Phillips multiplier. The dashed (red) lines depict our time-varying estimates (each line corresponds to an impulse response estimated at a given point in time), while the continuous (black) line reports the full-sample estimate. The top two panels in the picture show that, conditionally on a contractionary monetary policy shock, inflation decreases and unemployment increases. The time-varying estimates show that the quantitative extent to which inflation and unemployment respond to a monetary policy shock changes significantly over time. As a result, the estimated Phillips multiplier also varies over time, as shown in the bottom panel of Figure 3.

INSERT FIGURE 3 HERE

In order to shed more light on the nature of the time-variation in the Phillips multiplier, Figure 4 depicts the Phillips multiplier over time for a selected horizon ($h = 12$). Figure 4 (a) depicts the Phillips multiplier over time before 1990, using Romer and Romer-s (2004) monetary policy shocks as instruments. Figure 4 (b) depicts the Phillips multiplier over time after 1990, using high-frequency-identified (HFI) monetary surprises as instruments. For each sub-figure, the TVP-IV estimates are reported by dash-dot red lines, while the continuous black lines report the sub-sample estimates of the Phillips multiplier in Barnichon and Mesters (2021). The figure shows that the Phillips multiplier also decreased substantially over time, and the decrease dates back to the 1970s. After 1990, the TVP-IV multiplier estimates are close to 0.

INSERT FIGURE 4 HERE

Comparing our results in Figure 2 to Figures 3 and 4, the flattening of the Phillips curve is a robust result in our data, no matter whether we consider the Phillips (cor)relation, the Phillips multiplier or the slope of the structural Phillips curve.

4 Why Has the Cyclical Correlation Between Inflation and Unemployment Decreased?

Several researchers have made compelling arguments that the reason for the decrease in the cyclical correlation between inflation and unemployment is related to monetary policy actions. According to this explanation, a more responsive monetary policy to inflation and economic conditions would tighten monetary policy more when it perceives inflation to be increasing, in order to keep the latter under control: this causes unemployment to rise, resulting in a positive correlation between inflation and unemployment that biases the slope coefficient of the Phillips curve toward zero. See Haldane and Quah (1999); Roberts (2006); Williams (2006); Mishkin (2007); Carlstrom, Fuerst, and Paustian (2009); and, more recently, McLeay and Tenreyro (2018).

As is well-known, the correlation between inflation and unemployment is the same as the slope of the Phillips curve only in the presence of no endogeneity bias and no measurement error. Thus, the endogeneity problem can be solved using valid and relevant instruments.[12] In the presence of an endogeneity bias due to monetary policy actions, instrumental variable (IV) estimates will still be consistent provided the instruments satisfy the required statistical conditions – that is, the chosen instruments should be both valid and relevant.

The IV approach we discussed in the previous section, that uses lagged macroeconomic variables as instruments, suggests that the slope of the Phillips curve decreased over time in a manner similar to the decrease in the correlation between inflation and unemployment. Our results therefore suggest that the decrease in the correlation between inflation and unemployment is due to a decrease in the slope of the Phillips curve and not to other factors, among which monetary policy.

As mentioned, our analysis in the previous section is based on an instrumental variable method where instruments are both valid and relevant. However, the theoretical

[12]Some researchers have attempted to solve the endogeneity problem by using regional data; however, there are several issues in using cross-sectional data for this purpose, see Canova, 2023). Here we will instead maintain the same framework as in the classical literature on the Phillips curve debate, which focuses on macroeconomic data; however, we shed light on the issue by using both valid and strong instruments, as well as weak identification robust confidence intervals.

validity of the instruments also requires that the residuals of the Phillips curve are not serially correlated, otherwise the IV procedure may not correctly estimate the slope of the Phillips curve. Lurking inside the residual of the Phillips curve there are both cost push shocks as well as measurement error, and both might be correlated with the lagged instruments via a correlation with their own lags. In our analysis, the residuals of the Phillips curve show some evidence of serial correlation.

To be robust to this potential problem, we consider aggregate demand shocks as instruments, as in Barnichon and Mesters (2020). In particular, we consider monetary policy shocks, which are potentially valid instruments, as both the measurement error and unobserved supply shocks (such as labor supply, price or technology shock) would be uncorrelated with such shocks. Although we use the same model specification and the same identification strategy as Barnichon and Mesters (2020), we will use it in our framework to analyze whether the culprit behind the decrease in the correlation between inflation and output is monetary policy, while their approach does not allow for time-varying parameters.

We estimate the following Phillips curve specification:[13]

$$\pi_t = c_t + \lambda_t x_t + \gamma_{f,t}\pi_{t+1} + \gamma_{b,t}\pi_{t-1} + u_t,$$

where the set of instruments is the Almond parameterization of twenty lags of the Romer and Romer (2004) monetary policy shocks and the forcing variable x_t is the unemployment gap. We estimate the equation in the sample starting in 1974:Q1 and ending in 2007:Q4, due to data limitations in the availability of the monetary policy shock data.[14]

Hansen's J-statistic for over-identification is 0.022, with a p-value of 0.8813, indicating that the instruments are valid in our sample. Ganics et al. (2021) weak instrument-robust procedure implies weak instruments, as the minimum eigenvalue is 0.0000, with a confidence interval equal to $(0, 0.0001)$. Lewis and Mertens's (2022) weak IV test statistic equals 4.1991, also indicating that the instruments are weak.[15] Therefore, we develop a novel TVP-IV methodology to obtain estimates and confidence bands that are robust to weak instruments, described in what follows.[16]

[13] This is the same model specification and sample as in Barnichon and Mesters (2020).

[14] The monetary policy shock series that we use ends in 2007. During the zero lower bound period there are fewer monetary policy shocks anyway, which might invalidate the strength of the instrument.

[15] Newey and West's (1994) HAC-robust variance estimates are implemented following Barnichon and Mesters's (2020) choice, which is 5 lags.

[16] Antoine and Renault (2023) and Kleibergen (2023) propose alternative methods to deal with potential misspecification as well as weak identification.

Figure 5 shows the time-varying estimate of the slope of the Phillips curve; dotted lines report the 90 percent confidence bands robust to weak instruments. The figure confirms our result that the slope of the Phillips curve substantially flattened in the 1980s and 1990s.[17] The slope has decreased, in absolute value, by 62% approximately.[18]

INSERT FIGURE 5 HERE

In conclusion, using a variety of different specifications and, in particular, a specification robust to the presence of measurement error and serial correlation in the residuals of the Phillips curve, coupled with weak instrument robust techniques, we find convincing and robust evidence of a decrease in the correlation between inflation and unemployment in the structural Phillips curve. Our result, which is robust to endogeneity, measurement error and correlation, highlights that the decrease in the correlation is due to a flattening of the Phillips curve, rather than to monetary policy.

5 What's Up with the Phillips Curve in the Recent Pandemics?

Finally, we turn to the Phillips curve during the recent financial crisis and, especially, the recent pandemics. Both have contributed to a substantially unstable macroeconomic environment.[19]

Figure 6 plots the time-varying estimates of the Phillips curve parameters using data up to the end of 2021 focusing on the same specification as Galí et al. (2005). The top panel in the picture shows the slope of the Phillips curve (λ_t) together with 90 percent confidence bands robust to weak instruments. It is clear from the figure that, after hovering close to zero (in absolute value) since 1985 and until the end of 1990s, the slope has started to increase again since the beginning of the 2000s. Thus, relative to the literature that attributes the missing disinflation during the recent financial crisis

[17]Barnichon and Mesters (2020) also study the specification considering the output gap as the forcing variable. We report our result of this specification in Figure A3 in Appendix B, which also confirms that the slope of the Phillips curve substantially flattened in the 1980s and 1990s.

[18]Elliot and Müller's (2006) qLL test statistics of the reduced form regression in eq. (2) is -807.83 for this specification. Thus, the null hypothesis of no time variation is rejected at the 1% significance level, thus pointing to time variation in the system.

[19]Elliot and Müller's (2006) qLL test statistics of the reduced form regression in eq. (2) is -4072.1 for this specification, strongly confirming the presence of instabilities in our extended sample.

to the weakening of the Phillips curve, we find evidence that the Phillips curve is again alive and well.

What else happened during the financial and pandemic crises? Panel (b) in Figure 6 shows a steady increase in the degree of forward-looking behavior in inflation. The upward trend, that started during the great moderation, is hovering around 0.6. On the other hand, the degree of backward-looking behavior in inflation has weakened substantially. The downward trend, which started since the 1970s, has brought the parameter close to 0.1, a value that is statistically insignificantly different from zero.

INSERT FIGURE 6 HERE

Overall, our findings suggest that, in setting prices, agents pay more attention to the future and less to the past. The fact that past inflation has lost importance in the agents' price-setting behavior may explain the decrease in the overall serial correlation in inflation and its lack of predictability over time (Stock and Watson, 2007). Our results on the degree of forward- and backward-looking indexation are also consistent with those in Cogley and Sbordone (2008), who, using a very different time-varying parameter VAR-based methodology, similarly find that estimates of the backward-looking indexation parameter are close to zero and that the forward-looking component is instead prominent.

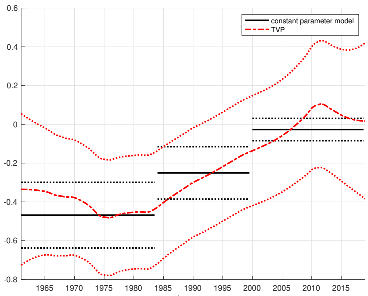
6 Conclusion

We contribute to the debate surrounding the instability of the relation between unemployment and inflation over time by offering insights from a flexible time-varying instrumental variable approach.

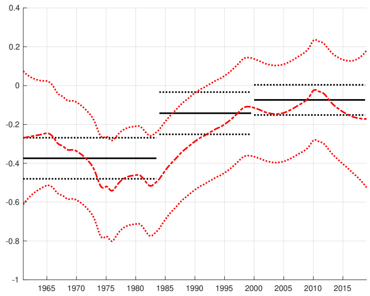
We find that the weakening of the cyclical correlation between inflation and unemployment is due to a flattening in the slope of the Phillips curve, rather than to monetary policy. The slope of the structural Phillips curve has decreased over time since the 1980s. In the most recent period since the Great Recession and during the recent pandemic, the slope of the Phillips curve has increased again.

Our results are based on an approach that has the advantage of avoiding endogeneity while, at the same time, being robust to changes in the economic environment. In addition, by virtue of the approach taken in this paper, our conclusions do not require making auxiliary assumptions on the rest of the economy nor estimating a fully specified model, and hence are more robust to misspecification than existing, full-information approaches. We demonstrate the robustness of our results to various specifications that feature both strong instruments as well as weak-identification robust ones.

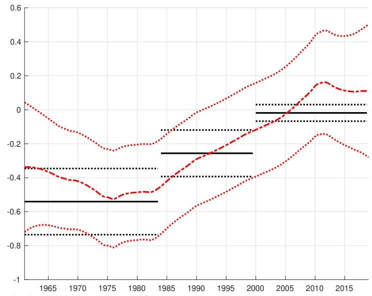
Figures and Tables



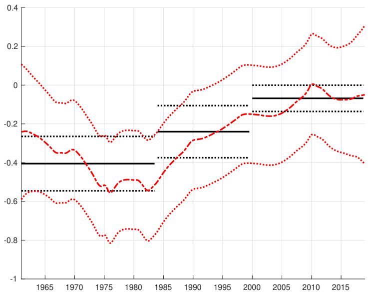
(a) Unemployment gap (CBO)



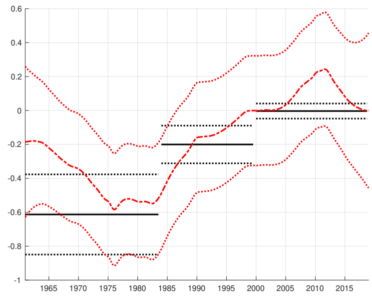
(b) GDP gap (CBO)



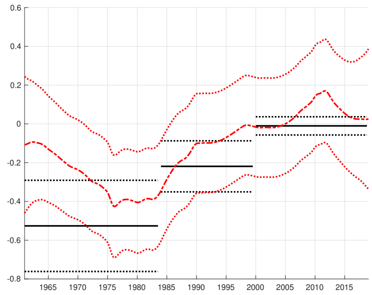
(c) Unemployment gap (two-sided filtered)



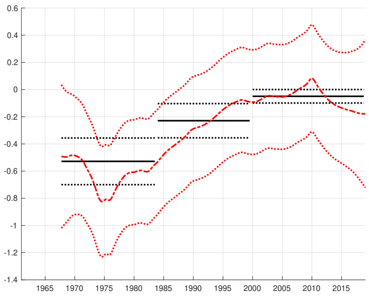
(d) Short-term unemployment gap (two-sided filtered)



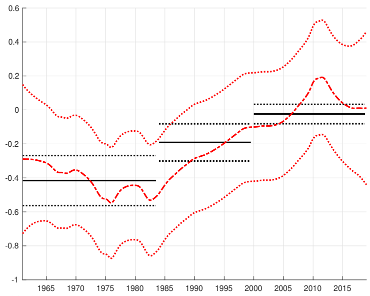
(e) Employment-population ratio (two-sided filtered)



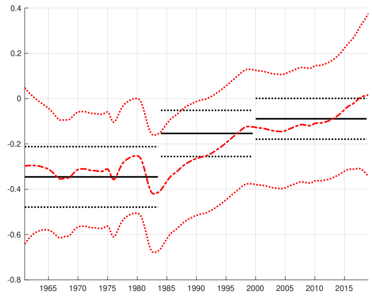
(f) Employment-population ratio ages 25-54 (two-sided filtered)



(g) Capacity utlilization rate (two-sided filtered)



(h) Unemployment rate (Real-time slack)



(i) Short-term unemployment rate (Real-time slack)

Figure 1: **The time-varying Phillips relation**. The figure shows the estimated slope (β_1) in the Phillips relation: $E_t\Delta_4\pi_t^4 = \beta_0 + \beta_1 x_t^4$, where x_t^4 is the change in the average value of variable "x" between times t and t-3 and $\Delta_4 = (1 - L^4)$. Inflation is measured by PCE-xFE and x_t are the various measures of slack for the US (See Stock and Watson's (2020) Table 1). The period is 1961-2019.

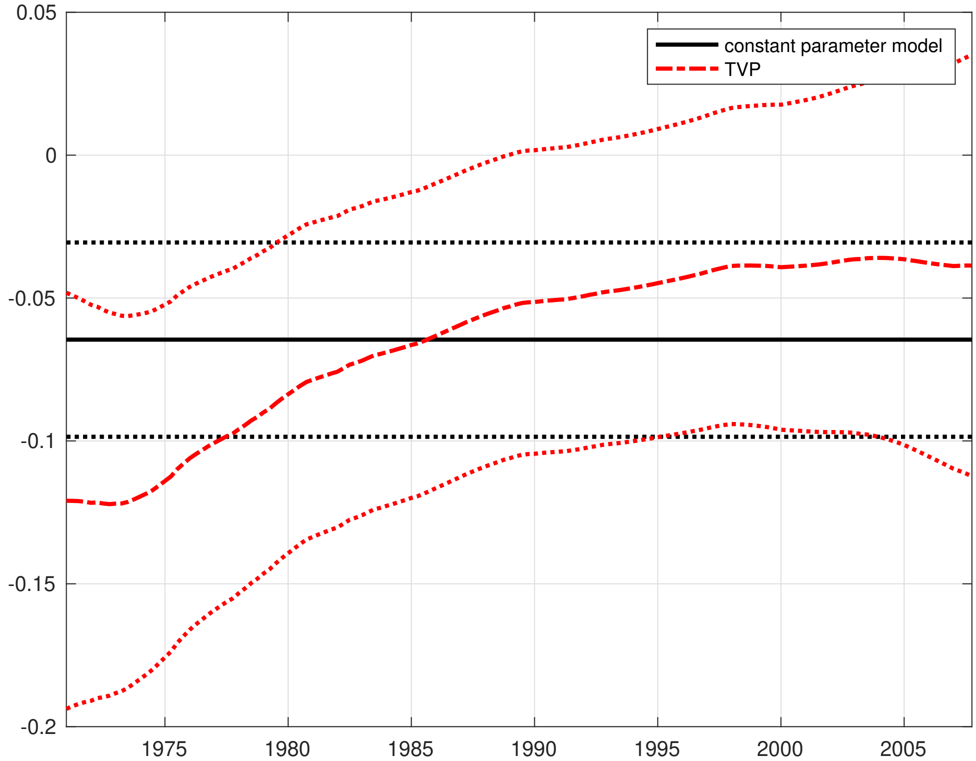
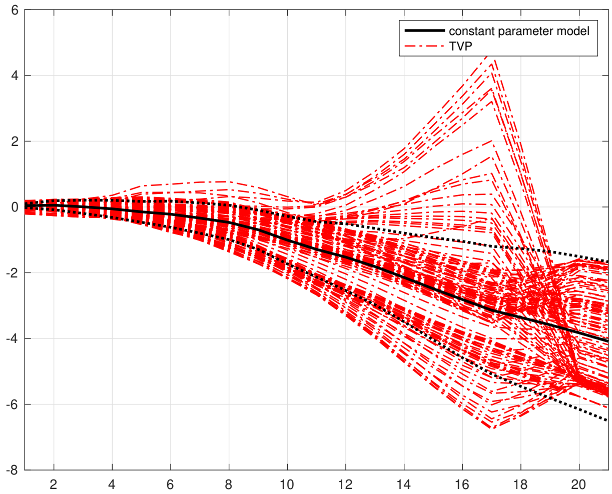
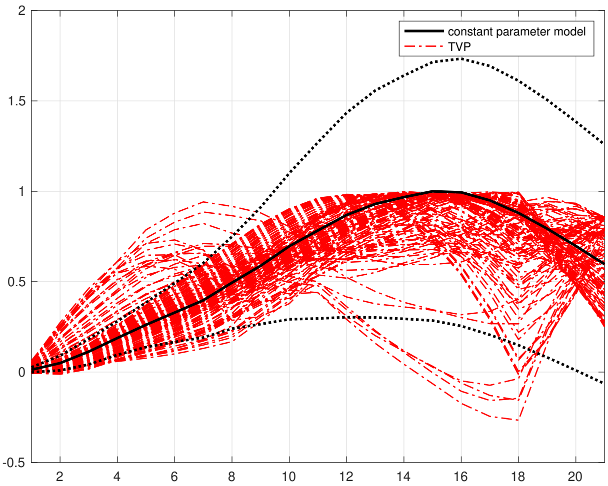


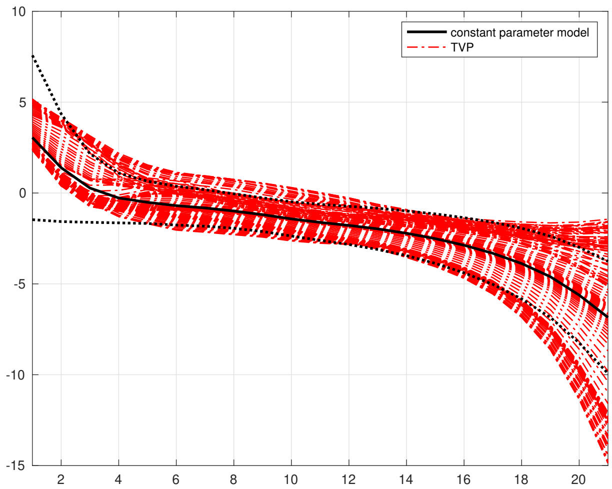
Figure 2: **The time-varying Phillips curve**. The figure shows the coefficient λ_t of the structural Phillips curve estimated using the TVP-IV method (dashed lines) versus the full-sample constant estimate, together with 90% confidence bands. The sample is 1970Q1-2008Q1. HAC-robust variance estimates are implemented with 4 lags, following Mavroeidis et al. (2014).



(a) impulse responses of inflation (TVP-LP estimation)

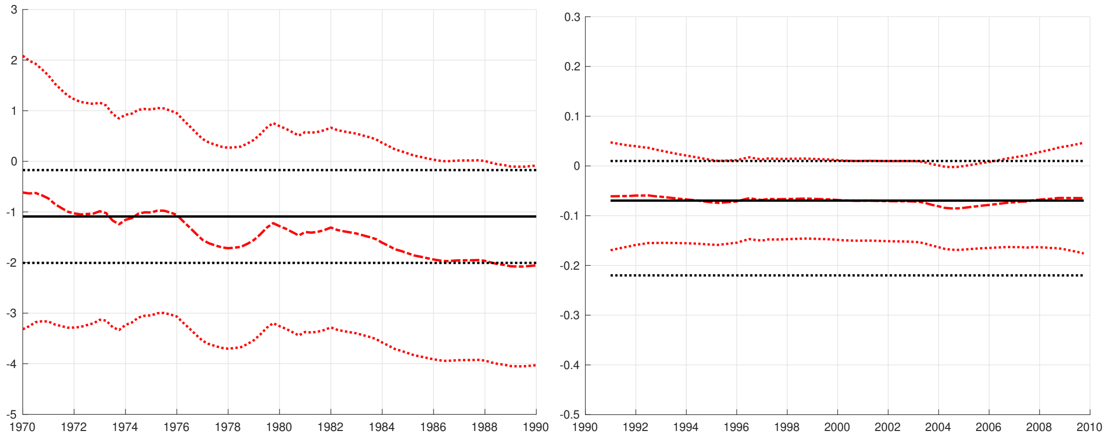


(b) impulse responses of unemployment (TVP-LP estimation)



(c) Phillips multiplier (one-step TVP-LP-IV estimation)

Figure 3: **The time-varying Phillips multiplier**. The figure shows the US Phillips multiplier in the sample 1969q1-2007q4. The black continuous line is Barnichon and Mesters' (2021, Fig. 1) full-sample multiplier and the red dashed lines are the TVP-LP-IV multipliers.



(a) $h = 12$, pre-1990, RR shocks (b) $h = 12$, post-1990, HFI shocks

Figure 4: **The time-varying Phillips multiplier**. The figure shows the TVP-LP estimated US Phillips multiplier over time for selected horizons (dashed red line), using Romer and Romer's shocks as instruments for pre-1990 sub-samples and HFI shocks as instruments for post-1990 sub-samples as to compare with Barnichon and Mesters's (2021) constant multiplier estimates, reported by the black solid line.

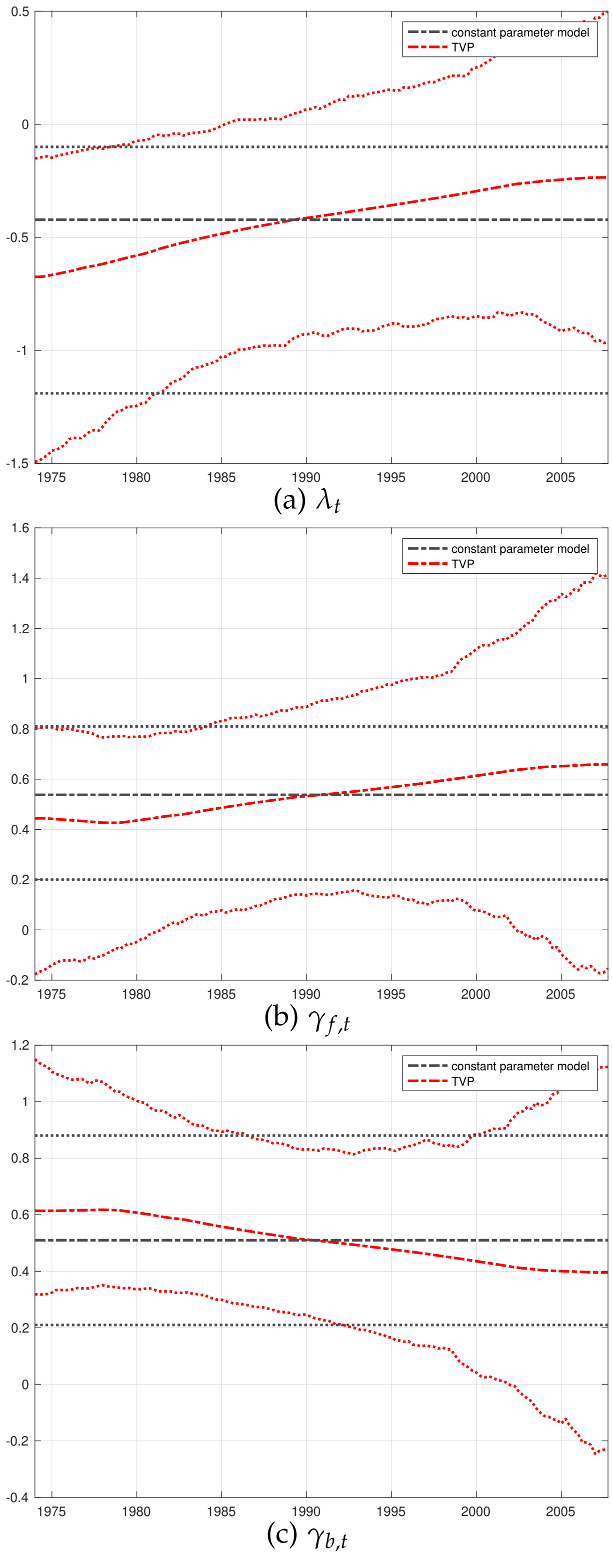
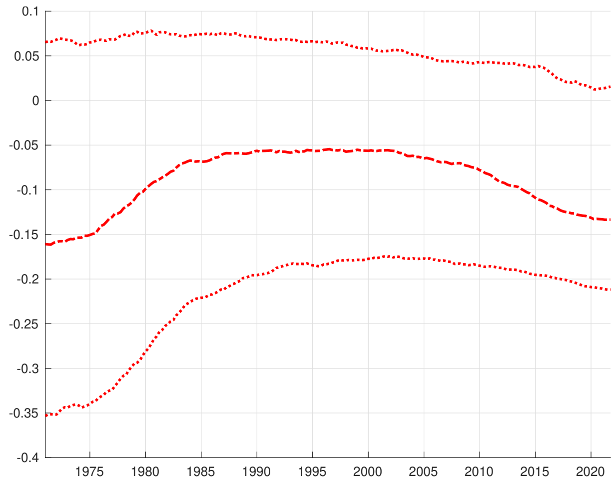
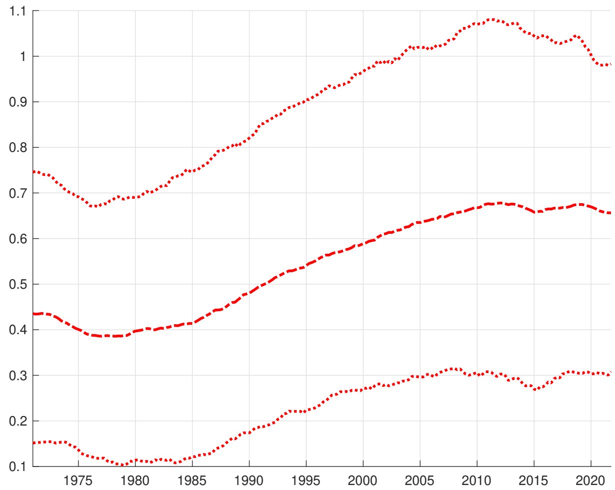


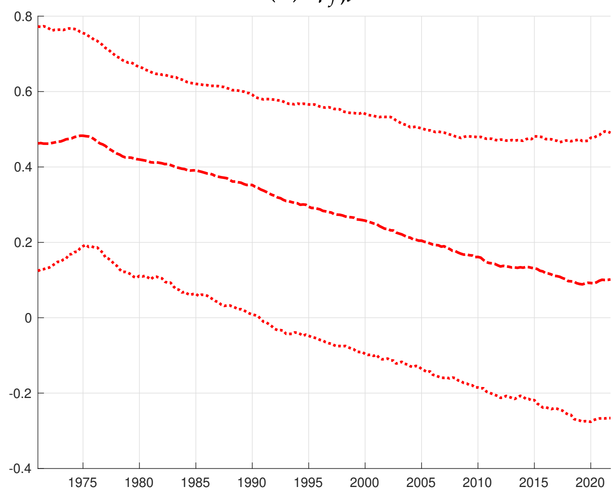
Figure 5: **The time-varying Phillips curve using monetary policy shocks as instruments**. The figure reports the estimated TVP-IV coefficients as well as their full-sample counterparts, together with 90% weak-instrument robust confidence bands. The sample is 1974Q1:2007Q4. The specification is as follows: we use the Almond parameterization of the 20 lags of Romer and Romer monetary policy shocks, and the unemployment gap obtained via the Hodrick-Prescott filter. The choice of the number of lags considered in the Newey and West's (1987) estimator, set equal to five, closely follows Barnichon and Mesters (2020). The bands are smoothed using a seven quarter centered moving average.



(a) λ_t



(b) $\gamma_{f,t}$



(c) $\gamma_{b,t}$

Figure 6: **The time-varying Phillips curve during the pandemic.** The figure shows the estimated TVP-IV coefficients together with 90% weak-instrument robust confidence bands, 1970Q1:2021Q4. The specification is the same as Galí et al. (2005). We estimate the hybrid NKPC with one lag of inflation and the unemployment gap (CBO) as forcing variable using the instrument set: four lags of inflation and two lags of the unemployment gap (CBO), wage inflation, and output gap (CBO). HAC-robust variance estimates are implemented with Lazarus et al.'s (2018) recommendation, which implies to 19 lags. The bands are smoothed using a seven quarter centered moving average.

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Appendix

This Appendix is composed of five sections. Section A contains a detailed description of the data. Section B reports additional empirical results. Section C outlines formulas for the scores and Hessians in the multivariate system. Section D presents high-level assumptions, and Section E contains proofs for the theoretical results.

A Data Description

This Appendix describes the data used in this paper. The data are quarterly and the span of the data is determined by data availability.

Inflation is measured as the "Implicit GDP deflator" (mnemonics "GDPDEF"). The data is transformed as follows: 100 times the log difference of the GDP deflator. For labor share, we use the "Business Sector: Labor Share for All Employed Persons" (mnemonics "PRS84006173"). The data is transformed as: 100*ln(PRS84006173/100). The instruments used follow Galí et al. (2001), including four lags of inflation and two lags of the labor share, wage inflation, and output gap. For wage inflation, we use the "Business Sector: Hourly Compensation for All Employed Persons" (mnemonics "HCOMPBS"). The data is transformed as: 100 times the log difference of the HCOMPBS. The output gap is an economic measure of the difference between the actual output of an economy and its potential output. For the output, we use "Real Gross Domestic Product" (mnemonics "GDPC96") and "Population Level" (mnemonics "CNP16OV"). The data is transformed as: 100*ln(GDPC96/CNP16OV). For the potential output, we use "Real Potential Gross Domestic Product" (mnemonics "GDPPOT"). The data is transformed as: 100*ln(GDPPOT/CNP16OV). All are available from the Federal Reserve Bank of St. Louis's FRED database. "GDPDEF" is available from 1947Q1 - 2022Q1. "PRS84006173" is available from 1947Q1 - 2022Q1. "HCOMPBS" is available from 1947Q1 - 2022Q1. "GDPC96" is available from 1947Q1 - 2017Q2. (This series has been discontinued. It was a duplicate of "GDPC1", which will continue to be updated.) "CNP16OV" is available from 1948Q1 - 2022Q1. "GDPPOT" is available from 1949Q1 - 2031Q4.

The government spending shocks is Ramey's (2011) military news variable. The update series (up to 2015) that we use is from Ramey and Zubairy (2018).

The monetary policy shock is from Romer and Romer (2004) from 1969Q1 to 2007Q4 and the updated data we use is available from Wieland and Yang (2020) at:
https://www.openicpsr.org/openicpsr/project/135741/version/V1/view.[1]

[1]The Romer and Romer (2004) monetary policy shock series is available from 1969Q1 to 2007Q4. The government spending shock series is available from 1989Q1 to 2015Q4. The monetary policy shock series is originally available at the monthly frequency, and we aggregated it at the quarterly frequency by summing the monthly values.

B Additional Empirical Results

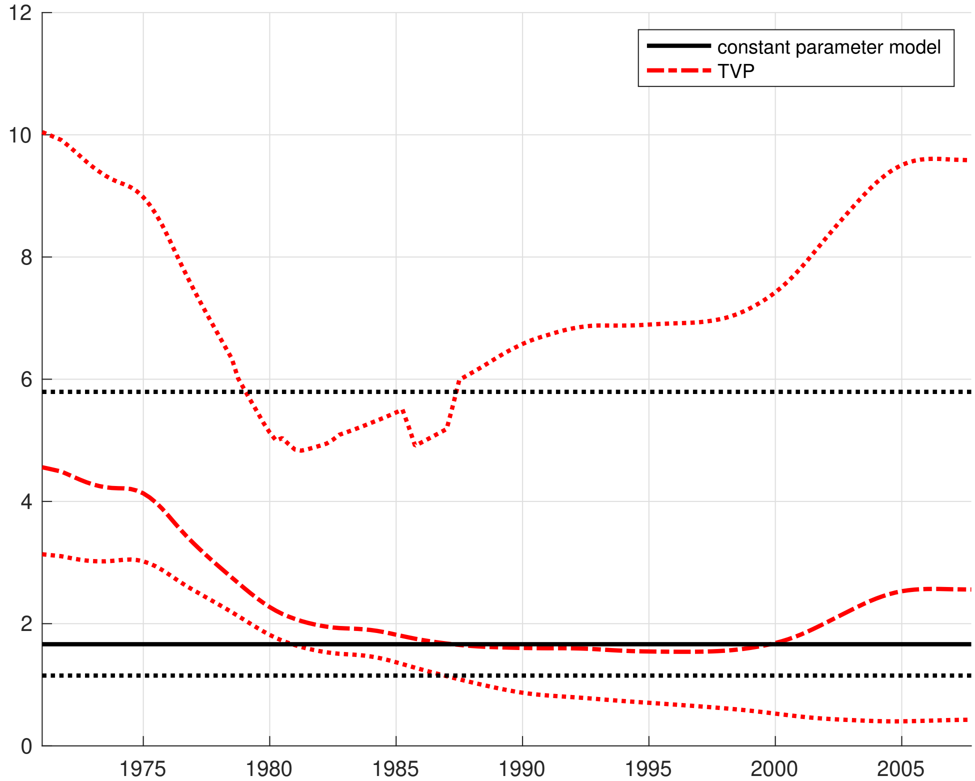
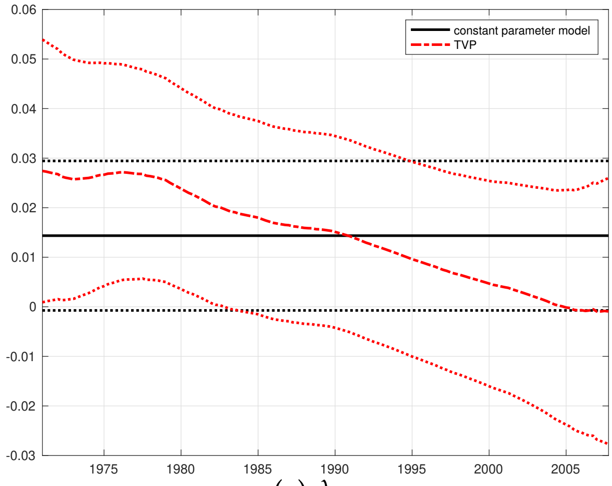
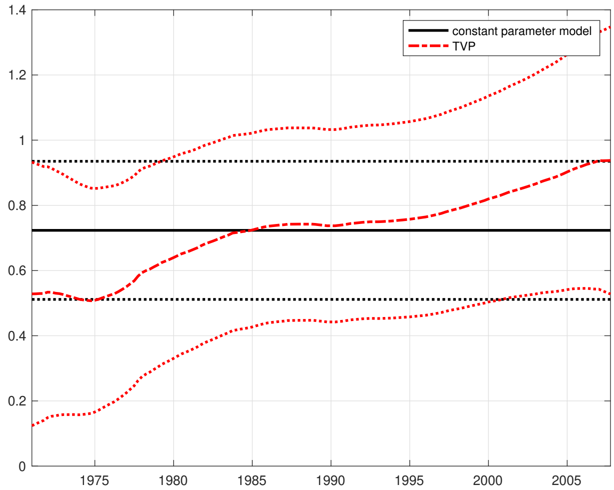


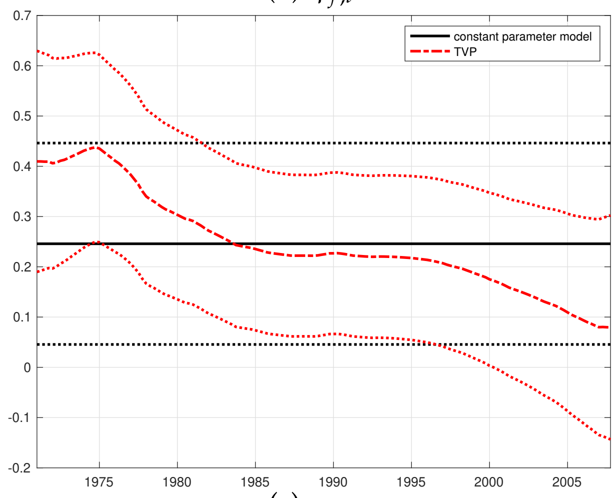
Figure A1: **The time-varying Ganics et al.'s (2021) weak-instrument robust confidence interval for the strength of identification**. The figure shows the Ganics et al.'s (2021) weak-instrument robust estimate for the strength of identification, considering constant parameter model (the black solid line) and considering TVP-IV model (the red dashed line). The dotted lines are the corresponding 90% confidence intervals.



(a) λ_t

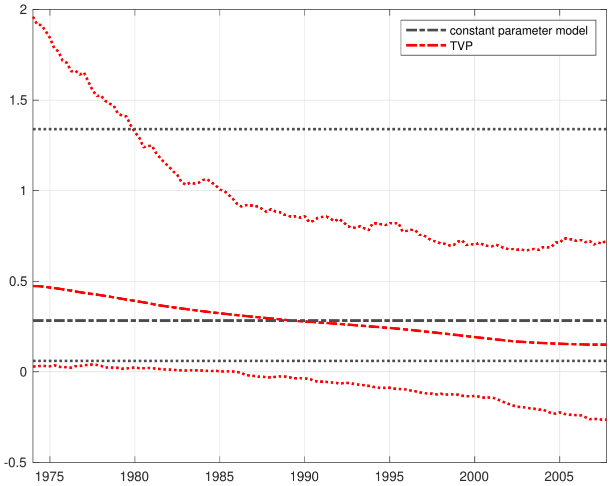


(b) $\gamma_{f,t}$

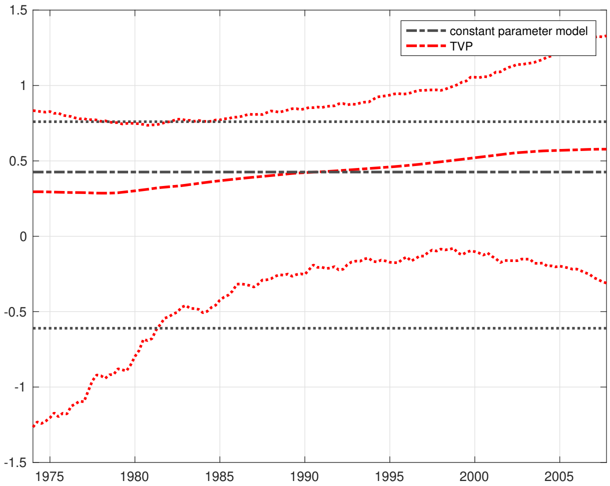


(c) $\gamma_{b,t}$

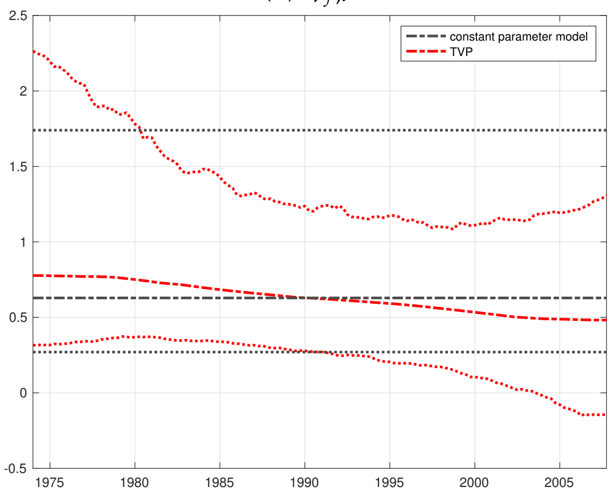
Figure A2: **The time-varying Phillips curve using labor share as the forcing variable.** The figure shows the estimated TVP-IV coefficients as well as the full-sample estimates together with 90% confidence bands. The sample is 1970Q1:2008Q1. The specification is the same as Galí et al. (2005). We estimate the hybrid NKPC with one lag of inflation and the labor share as forcing variable. The Galí et al.'s (2005) instrument set includes four lags of inflation and two lags of the labor share, wage inflation, and quadratically-detrended output. HAC-robust variance estimates are implemented with 4 lags, following Mavroeidis et al. (2014).



(a) λ_t



(b) $\gamma_{f,t}$



(c) $\gamma_{b,t}$

Figure A3: **The time-varying Phillips curve using monetary policy shocks as instruments and output gap as the forcing variable.** The figure shows the TVP-IV estimated coefficients and the full-sample estimates, together with the 90% confidence bands robust to weak instruments. The sample is 1974Q1:2007Q4. The specification uses as instruments the Almond parameterization of 20 lags of the Romer and Romer monetary policy shocks, and the output gap obtained via the Hodrick-Prescott filter as the forcing variable. HAC-robust variance estimates are implemented with 5 lags, following Barnichon and Mesters (2020). The bands are smoothed using a 7-quarter moving average.

C Likelihood, Scores, and Hessians in the Multivariate System

For notational simplicity, we omit the subscript t in all the parameters in what follows.

General framework

Suppose we are interested in the following hybrid Phillips curve:

$$\pi_t = c + \lambda x_t + \gamma_f \pi_{t+1} + \gamma_b \pi_{t-1} + u_t, \tag{7}$$

where instruments are needed for x_t and π_{t+1}. Let z_t denote the instrumental variables excluding π_{t-1}, thus the reduced-form equations for x_t and π_{t+1} are:

$$\begin{aligned} x_t &= c_x + \beta'_{x,z} z_t + \beta_{x,\pi_b} \pi_{t-1} + u_{x,t}, \\ \pi_{t+1} &= c_{\pi_f} + \beta'_{\pi_f,z} z_t + \beta_{\pi_f,\pi_b} \pi_{t-1} + u_{\pi_f,t}. \end{aligned} \tag{8}$$

Combining eqs. (7) and (8), we have

$$\begin{aligned} \pi_t &= c + \lambda \left(c_x + \beta'_{x,z} z_t + \beta_{x,\pi_b} \pi_{t-1} + u_{x,t}\right) + \gamma_f \left(c_{\pi_f} + \beta'_{\pi_f,z} z_t + \beta_{\pi_f,\pi_b} \pi_{t-1} + u_{\pi_f,t}\right) + \gamma_b \pi_{t-1} + u_t \\ &= \left(c + \lambda c_x + \gamma_f c_{\pi_f}\right) + \left(\lambda \beta'_{x,z} + \gamma_f \beta'_{\pi_f,z}\right) z_t + \left(\lambda \beta_{x,\pi_b} + \gamma_f \beta_{\pi_f,\pi_b} + \gamma_b\right) \pi_{t-1} + \left(\lambda u_{x,t} + \gamma_f u_{\pi_f,t} + u_t\right), \end{aligned} \tag{9}$$

and, equivalently, we have the following reduced-form equations:

$$\begin{bmatrix} x_t \\ \pi_{t+1} \\ \pi_t \end{bmatrix} = \begin{bmatrix} \beta'_{x,z} & c_x & \beta_{x,\pi_b} \\ \beta'_{\pi_f,z} & c_{\pi_f} & \beta_{\pi_f,\pi_b} \\ \lambda \beta'_{x,z} + \gamma_f \beta'_{\pi_f,z} & \lambda c_x + \gamma_f c_{\pi_f} + c & \lambda \beta_{x,\pi_b} + \gamma_f \beta_{\pi_f,\pi_b} + \gamma_b \end{bmatrix} \begin{bmatrix} z_t \\ 1 \\ \pi_{t-1} \end{bmatrix} + \begin{bmatrix} v_{x,t} \\ v_{\pi_f,t} \\ v_{\pi,t} \end{bmatrix}, \tag{10}$$

where $(x_t, \pi_{t+1})'$ are (2×1) endogenous variables.

Let the vector of endogenous variables $(x_t, \pi_{t+1})'$ be denoted by $y_{1,t}$ and the LHS variable of interest in eq.7 (π_t) be denoted by $y_{2,t}$. Let $z_{1,t} = (z'_t, 1)'$; $z_{2,t} = \pi_{t-1}$; $\alpha_1 = (\beta'_{x,z}, \beta'_{\pi_f,z})$; and $\alpha_2 = \begin{pmatrix} c_x & \beta_{x,\pi_b} \\ c_{\pi_f} & \beta_{\pi_f,\pi_b} \end{pmatrix}$. Also, let $m = (\lambda, \gamma_f)$, $\mu = (c, \gamma_b)$, $v_{1,t} = (v'_x, v_{\pi_f})'$ and $v_{2,t} = v_\pi$. Then the reduced-form equation above can be rewritten as:

$$\begin{pmatrix} y_{1,t} \\ y_{2,t} \end{pmatrix} = \begin{pmatrix} \alpha_1 & \alpha_2 \\ m\alpha_1 & m\alpha_2 + \mu \end{pmatrix} \begin{pmatrix} z_{1,t} \\ z_{2,t} \end{pmatrix} + \begin{pmatrix} v_{1,t} \\ v_{2,t} \end{pmatrix}. \tag{11}$$

More generally, our structural model can be written as:

$$y_{2,t} = m y_{1,t} + controls + \epsilon_t, \tag{12}$$

where $y_{1,t}$ is $(n_1 \times 1)$ and $y_{2,t}$ is $(n_2 \times 1)$, and our target is to estimate the $(n_1 \times n_2)$ parameter matrix m.

Let Z_t be the $(k \times 1)$ vector of instruments $[z'_{1,t}, z'_{2,t}]'$, where $z_{1,t}$ is $(k_1 \times 1)$, $z_{2,t}$ is $(k_2 \times 1)$, such that $k = k_1 + k_2$. Note that $z_{1,t}$ includes constant and exogenous control variables, and $z_{2,t}$ includes excluded exogenous variables. Also, let $\alpha = [\alpha_1 \;\; \alpha_2]$ is $(n_1 \times k)$, m is $(n_2 \times n_1)$, μ is $(n_2 \times k_2)$. Then eq.(11) can be rewritten as:

$$\begin{pmatrix} y_{1,t} \\ y_{2,t} \end{pmatrix} = \begin{pmatrix} \alpha \\ m\alpha + [0 \;\; \mu] \end{pmatrix} \begin{pmatrix} z_{1,t} \\ z_{2,t} \end{pmatrix} + \begin{pmatrix} v_{1,t} \\ v_{2,t} \end{pmatrix}. \tag{13}$$

The parameters of interest are m, the $(n_1 \times k) + 1 : (n_1 \times k) + 2$-th elements in $\bar{B}$, and π_b, the $(n_1 \times k) + n_1 n_2 + n_2 k_1$-th element in $\bar{B}$, where $\bar{B}$ is the $\bar{q} \times 1$ vector, such that

$$\bar{B} = (\alpha_{1,:} \cdots \alpha_{n_1,:}, m_{1,:} \cdots m_{n_2,:}, \mu_{1,:} \cdots \mu_{n_2,:})', \tag{14}$$

and $\bar{q} = n_1 k + n_2(n_1 + k_1)$.

Log-likelihood

Write (13) as

$$y_t = \begin{bmatrix} \alpha \\ \tilde{\alpha} \end{bmatrix} \begin{bmatrix} z_{2,t} \\ z_{1,t} \end{bmatrix} + \begin{bmatrix} v_{1,t} \\ v_{2,t} \end{bmatrix} = X'_t B + v_t, \tag{15}$$

where $X'_t = (I_n \otimes \begin{bmatrix} z'_{2,t} & z'_{1,t} \end{bmatrix})$ is $(n \times nk)$, v_t has zero mean and covariance matrix Σ, B is $(q \times 1)$, where $q = nk$, stacking all the parameter elements with the following order:

$$B = (\alpha_{1,:} \cdots \alpha_{n_1,:}, \tilde{\alpha}_{1,:} \cdots \tilde{\alpha}_{n_2,:})', \tag{16}$$

$\tilde{\alpha} = m\alpha + \begin{bmatrix} 0 & \mu \end{bmatrix}$, and the subscriptions refer to the elements, e.g., $\alpha_{1,:}$ refers to the first row elements in α.

Once we knowingly incorrectly assume that v_t is an i.i.d. normal random variable, we can write the quasi-log-likelihood based on eq. (15):

$$\log L = \sum_t \ell_t = -\frac{T(n_1 + n_2)}{2} \log 2\pi - \frac{T}{2} \sum_t \log |\Sigma| - \frac{1}{2} \sum_t (y_t - X'_t B)' \Sigma^{-1} (y_t - X'_t B). \tag{17}$$

Scores and Hessians in the strongly identified case

If we need the derivatives w.r.t. $\theta = (\bar{B}', vech(\Sigma)')'$, we now only need to know the derivative of B w.r.t. $\bar{B}$ so that we can apply the chain rule: $\frac{\partial B}{\partial \bar{B}}\frac{\partial l_t}{\partial B} = \frac{\partial l_t}{\partial \bar{B}}$, $\frac{\partial l_t}{\partial B'}\left(\frac{\partial B}{\partial \bar{B}}\right)' = \frac{\partial l_t}{\partial \bar{B}'}$, and

$$\underbrace{\begin{bmatrix} \frac{\partial B_1}{\partial \bar{B}_1} & \cdots & \frac{\partial B_1}{\partial \bar{B}_{\bar{q}}} \\ \vdots & \ddots & \vdots \\ \frac{\partial B_q}{\partial \bar{B}_1} & \cdots & \frac{\partial B_q}{\partial \bar{B}_{\bar{q}}} \end{bmatrix}}_{\left(\frac{\partial B}{\partial \bar{B}}\right)'} = \begin{bmatrix} I_{(n_1k \times n_1k)} & O_{(n_1k \times n_1n_2)} & O_{(n_1k \times n_2k_1)} \\ m \otimes I_{k(n_2k \times n_1k)} & I_{n_2} \otimes \alpha'_{(n_2k \times n_1n_2)} & \begin{bmatrix} O_{(n_2k_2 \times (n_2k_1))} \\ I_{n_2k_1} \end{bmatrix} \end{bmatrix}_{(q \times \bar{q})} . \tag{18}$$

We also need the second order derivatives $\frac{\partial^2 l_t}{\partial \bar{B}_t \partial \bar{B}_t'}$, which is $(\bar{q} \times \bar{q})$:

$$\begin{aligned} \frac{\partial^2 l_t}{\partial \bar{B} \partial \bar{B}'} &= \frac{\partial \frac{\partial l_t}{\partial \bar{B}}}{\partial \bar{B}'} = \frac{\partial \frac{\partial B}{\partial \bar{B}} \frac{\partial l_t}{\partial \bar{B}_t}}{\partial \bar{B}_t'} = \begin{bmatrix} \frac{\partial \frac{\partial B}{\partial \bar{B}} \frac{\partial l_t}{\partial B}}{\partial \bar{B}_1} \cdots \frac{\partial \frac{\partial B}{\partial \bar{B}} \frac{\partial l_t}{\partial B}}{\partial \bar{B}_{\bar{q}}} \end{bmatrix} = \begin{bmatrix} \frac{\partial B}{\partial \bar{B}} \cdot \frac{\partial \frac{\partial l_t}{\partial B}}{\partial \bar{B}_1} + \frac{\partial \frac{\partial B}{\partial \bar{B}}}{\partial \bar{B}_1} \cdot \frac{\partial l_t}{\partial B} \cdots \frac{\partial B}{\partial \bar{B}} \cdot \frac{\partial \frac{\partial l_t}{\partial B}}{\partial \bar{B}_{\bar{q}}} + \frac{\partial \frac{\partial B}{\partial \bar{B}}}{\partial \bar{B}_{\bar{q}}} \cdot \frac{\partial l_t}{\partial B} \end{bmatrix} \\ &= \frac{\partial B}{\partial \bar{B}} \cdot \frac{\partial \frac{\partial l_t}{\partial B}}{\partial \bar{B}'} + \begin{bmatrix} \frac{\partial \frac{\partial B}{\partial \bar{B}}}{\partial \bar{B}_1} \cdot \frac{\partial l_t}{\partial B} \cdots \frac{\partial \frac{\partial B}{\partial \bar{B}}}{\partial \bar{B}_{\bar{q}}} \cdot \frac{\partial l_t}{\partial B} \end{bmatrix} \\ &= \frac{\partial B}{\partial \bar{B}} \cdot \frac{\partial^2 l_t}{\partial B \partial B'} \left(\frac{\partial B}{\partial \bar{B}}\right)' + \underbrace{\begin{bmatrix} \frac{\partial \frac{\partial B}{\partial \bar{B}}}{\partial \bar{B}_1} \cdots \frac{\partial \frac{\partial B}{\partial \bar{B}}}{\partial \bar{B}_{\bar{q}}} \end{bmatrix}}_{\Delta_{(\bar{q} \times q\bar{q})}} \left(I_{\bar{q}} \otimes \frac{\partial l_t}{\partial B}\right). \end{aligned}$$

We derive each submatrix in Δ in turn. For the first n_1k elements in Δ, we have

$$\left(\frac{\partial \frac{\partial B}{\partial \bar{B}}}{\partial \bar{B}_j}\right)' = \begin{bmatrix} O_{(n_1k \times n_1k)} & O_{(n_1k \times n_1n_2)} & O_{(n_1k \times n_2k_1)} \\ O_{(n_2k \times n_1k)} & I_{n_2} \otimes (e'_{\alpha,j})_{(n_2k \times n_1n_2)} & O_{(n_2k \times n_2k_1)} \end{bmatrix}_{(q \times \bar{q})},$$

where $e'_{\alpha,j}$ is a $(k \times n_1)$ matrix with a certain entry being 1 and the remaining entries being 0. For example, $e'_{\alpha,1} = \begin{bmatrix} 1 & 0 \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{bmatrix}$, $e'_{\alpha,2} = \begin{bmatrix} 0 & 0 \cdots & 0 \\ 1 & \ddots & \vdots \\ 0 & \cdots & 0 \end{bmatrix}$, in line with the corresponding element in α'.

For the next n_1n_2 elements, i.e., the $n_1k + 1 : n_1k + n_1n_2$ elements in Δ, we have

$$\left(\frac{\partial \frac{\partial B}{\partial \bar{B}}}{\partial \bar{B}_{n_1k_2+j}}\right)' = \begin{bmatrix} O_{(n_1k \times n_1k)} & O_{(n_1k \times n_1n_2)} & O_{(n_1k \times n_2k_1)} \\ e_{m,j} \otimes I_{k_{(n_2k \times n_1k)}} & O_{(n_2k \times n_1n_2)} & O_{(n_2k \times n_2k_1)} \end{bmatrix}_{(q \times \bar{q})},$$

where $e_{m,j}$ is a $(n_2 \times n_1)$ matrix whose $(1, j)$ entry is one and all other entries are zeros.

For the remaining elements, the derivatives are 0.

Below we summarize the expressions for scores and Hessians.

The score is $s_t = \frac{\partial l_t(\theta)}{\partial \theta} = \begin{bmatrix} \frac{\partial l_t}{\partial \bar{B}} \\ \frac{\partial l_t}{\partial vech(\Sigma)} \end{bmatrix} = \begin{bmatrix} \frac{\partial B}{\partial \bar{B}} \frac{\partial l_t}{\partial B} \\ \frac{\partial l_t}{\partial vech(\Sigma)} \end{bmatrix}$, where

$$\frac{\partial l_t}{\partial B} = X_t \Sigma^{-1}(y_t - X_t' B), \tag{19}$$

$$\frac{\partial l_t}{\partial \Sigma} = -\frac{1}{2}\Sigma^{-1} + \frac{1}{2}\Sigma^{-1}(y_t - X_t'B)(y_t - X_t'B)'\Sigma^{-1}, \tag{20}$$

$$\frac{\partial l_t}{\partial vech(\Sigma)} = D' vec\left(\frac{\partial l_t}{\partial \Sigma}\right), \qquad \frac{\partial l_t}{\partial vec(\Sigma)} = DD' vec\left(\frac{\partial l_t}{\partial \Sigma}\right), \tag{21}$$

where D denote the duplication matrix such that $D \cdot vech(V) = vec(V)$. Besides, denote $D^+ = (D'D)^{-1}D'$ such that $D^+ \cdot vec(V) = vech(V)$.

The Hessian is

$$\begin{aligned} h_t = -\frac{\partial s_t}{\partial \theta'} &= -\begin{bmatrix} \frac{\partial^2 l_t}{\partial \bar{B} \partial \bar{B}'} & \frac{\partial^2 l_t}{\partial \bar{B} \partial vech(\Sigma)'} \\ \frac{\partial^2 l_t}{\partial vech(\Sigma) \partial \bar{B}'} & \frac{\partial^2 l_t}{\partial vech(\Sigma) \partial vech(\Sigma)'} \end{bmatrix} \\ &= -\begin{bmatrix} \frac{\partial B}{\partial \bar{B}} \frac{\partial^2 l_t}{\partial B \partial B'} \left(\frac{\partial B}{\partial \bar{B}}\right)' + \Delta\left(I_{\bar{m}} \otimes \frac{\partial l_t}{\partial B}\right) & \frac{\partial B}{\partial \bar{B}} \frac{\partial^2 l_t}{\partial B \partial vech(\Sigma)'} \\ \frac{\partial^2 l_t}{\partial vech(\Sigma) \partial B'} \left(\frac{\partial B}{\partial \bar{B}}\right)' & \frac{\partial^2 l_t}{\partial vech(\Sigma) \partial vech(\Sigma)'} \end{bmatrix}, \end{aligned} \tag{22}$$

where

$$\begin{aligned} \frac{\partial^2 l_t}{\partial B \partial B'} &= -X_t \Sigma^{-1} X_t', \\ \frac{\partial^2 l_t}{\partial B \partial vech(\Sigma)'} &= \left[-D'\left(\Sigma^{-1}(y_t - X_t'B) \otimes \Sigma^{-1}\right) X_t'\right]', \\ \frac{\partial^2 l_t}{\partial B \partial vec(\Sigma)'} &= \left[-DD'\left(\Sigma^{-1}(y_t - X_t'B) \otimes \Sigma^{-1}\right) X_t'\right]' \end{aligned} \tag{23}$$

$$\begin{aligned} \frac{\partial^2 l_t}{\partial vech(\Sigma) \partial vech(\Sigma)'} &= D'\left(\Sigma^{-1} \otimes \left(\frac{1}{2}\Sigma^{-1} - \Sigma^{-1}(y_t - X_t'B)(y_t - X_t'B)'\Sigma^{-1}\right)\right) D, \\ \frac{\partial^2 l_t}{\partial vec(\Sigma) \partial vec(\Sigma)'} &= DD'\left(\Sigma^{-1} \otimes \left(\frac{1}{2}\Sigma^{-1} - \Sigma^{-1}(y_t - X_t'B)(y_t - X_t'B)'\Sigma^{-1}\right)\right) DD'. \end{aligned} \tag{24}$$

Scores and Hessians in the weakly identified case

When instruments are potentially weak, we estimate eq. (15) with θ replaced by $\theta = (B', vech(\Sigma)')'$. The corresponding scores and Hessians are $s_t = \frac{\partial l_t}{\partial \theta^{rf}} = \begin{bmatrix} \frac{\partial l_t}{\partial B} \\ \frac{\partial l_t}{\partial vech(\Sigma)} \end{bmatrix}$, and

$$h_t = -\frac{\partial s_t}{\partial \theta^{rf\prime}} = -\begin{bmatrix} \frac{\partial^2 l_t}{\partial B \partial B'} & \frac{\partial^2 l_t}{\partial B \partial vech(\Sigma)'} \\ \frac{\partial^2 l_t}{\partial vech(\Sigma) \partial B'} & \frac{\partial^2 l_t}{\partial vech(\Sigma) \partial vech(\Sigma)'} \end{bmatrix}, \tag{25}$$

where each component is defined earlier.

D Assumptions

To use Theorem 5 of Müller and Petalas (2010), we impose the following high-level assumptions:
Assumption 1.

(a) y_t and $Z_t = [z'_{1t}\ z'_{2t}]'$ satisfy the reduced-form equation (15) and have finite fourth moments uniformly in t.

(b) $E(Z_t v_t) = 0$.

(c) Let $\theta_t = [\bar{B}'_t\ \text{vech}(\Sigma_t)']'$ and $\theta_0 = [\bar{B}'_0\ \text{vech}(\Sigma_0)']'$ in the strongly identified case, and let $\theta_t = [B'_t\ \text{vech}(\Sigma_t)']'$, $\theta_0 = [B'_0\ \text{vech}(\Sigma_0)']'$ in the weakly identified case. They satisfy $\theta_t = \theta_0 + T^{-\frac{1}{2}}\delta\left(\frac{t}{T}\right)$ where $\delta(\cdot)$ is piecewise continuous with at most a finite number of discontinuities and left and right limits everywhere.

(d) $T^{-\frac{1}{2}}\sum_{t=1}^{[\cdot T]} s_t(\theta_t) \Rightarrow V^{\frac{1}{2}}\mathcal{W}(\cdot)$, where long-run variance matrix V is positive definite, and $\mathcal{W}(\cdot)$ denotes the standard Brownian motion vector.

(e) There is a negative definite matrix, Ξ, such that $\sup_{s\in[0,1]}\left|\frac{1}{T}\sum_{t=1}^{[sT]} h_t(\theta_0) - s\Xi\right| = o_p(1)$, and $E(Z_t Z'_t)$ is nonsingular.

Assumption 1(a) specifies the reduced-form relationship between endogenous variables y_t and exogenous variables Z_t. Assumption 1(b) imposes that Z_t is exogenous. According to Assumption 1(c), we estimate the structural parameters when they are known to be strongly identified. Otherwise, we estimate the reduced-form parameters. As in Müller and Petalas (2010), we focus on time variations of order $O(T^{-\frac{1}{2}})$. Assumption 1(d) is satisfied if $\text{vec}(X_t\Sigma^{-1}(y_t - X'_t B_t))$ and $\text{vech}\left(\Sigma^{-1}(y_t - X'_t B)(y_t - X'_t B_t)'\Sigma^{-1} - \Sigma^{-1}\right)$ satisfy assumptions for the invariance principle. Assumption 1(e) is a high-level assumption that imposes that the sample Hessian converges to a negative definite matrix.

In the strongly identified case, we assume that the instruments are relevant:

Assumption 2.

α is of full rank.

E Proofs

First, we show that Condition 3 in Müller and Petalas (2010) is satisfied for the weak instruments case. To be self-contained, we present the results as a lemma:

Lemma. *Suppose that Assumption 1 in holds. Then*

(i) $T^{-\frac{1}{2}}\sum_{t=1}^{[\cdot T]}\widehat{s}^r_t \Rightarrow J^r(\cdot) - \iota(\cdot)J^r(1)$, *where* $\widehat{s}^r_t = \widehat{\Xi}\widehat{V}^{-1}\widehat{s}_t - \frac{1}{T}\sum_{j=1}^{T}\widehat{\Xi}\widehat{V}^{-1}\widehat{s}_j$, $J^r(\cdot) = \Gamma^{\frac{1}{2}}\mathcal{W}(\cdot) +$

$\Gamma \int_0^{t(\cdot)} \delta(r)dr$, $\Gamma = \Xi V^{-1}\Xi$, $\Xi = \text{plim}_{T\to\infty}(1/T)\sum_{t=1}^T h_t(\theta_0)$, and V is the long-run variance-covariance matrix of $s_t(\theta_0)$;

(ii) $T^{\frac{1}{2}}(\widehat{\theta}^r - \theta_0) \Rightarrow \Gamma^{-\frac{1}{2}}\mathcal{W}(1) + \int_0^1 \delta_0(r)dr$, *where* $\widehat{\theta}^r = \widehat{\theta} + (\sum_{t=1}^T \widehat{\Xi}\widehat{V}^{-1}\widehat{h}_t)^{-1}\sum_{t=1}^T \widehat{\Xi}\widehat{V}^{-1}\widehat{s}_t$;

(iii) there exist matrices $\widetilde{h}_t^r$ *such that* $\sup_{s\in[0,1]}\left|\frac{1}{T}\sum_{t=1}^{[sT]}\widetilde{h}_t^r - s\Gamma\right| = o_p(1)$.

where $\Xi = \lim_{T\to\infty}\frac{1}{T}\sum_{t=1}^T E[h_t(\theta_0)]$, $\widehat{\Xi} = \frac{1}{T}\sum_{t=1}^T h_t(\theta_0)$, *and* $\widehat{V}$ *is a consistent estimator of* V. [2]

Proof of (ii): Consider a compact set, $\overline{\Theta}_0$, that contains θ_0. It follows from the Chebyshev inequality that

$$\frac{1}{T}\sum_{t=1}^T h_t(\theta) \overset{p}{\to} \lim_{T\to\infty}\frac{1}{T}\sum_{t=1}^T E(h_t(\theta)), \tag{26}$$

for all $\theta \in \overline{\Theta}_0$. Because $\frac{1}{T}\sum_{t=1}^T h_t(\theta)$ satisfies a Lipschitz condition,

$$\sup_{\theta\in\overline{\Theta}_0}\left|\frac{1}{T}\sum_{t=1}^T h_t(\theta) - \lim_{T\to\infty}\frac{1}{T}\sum_{t=1}^T E(h_t(\theta))\right| = o_p(1), \tag{27}$$

(Andrews, 1992). Thus, for any sequence $\overline{\theta}_t \to \theta_0$,

$$\left|\frac{1}{T}\sum_{t=1}^T h_t(\overline{\theta}_t) - \frac{1}{T}\sum_{t=1}^T E(h_t(\overline{\theta}_t))\right| = \left|\frac{1}{T}\sum_{t=1}^T h_t(\overline{\theta}_t) - \Xi\right| = o_p(1). \tag{28}$$

Applying the mean value theorem to the first order condition for the MLE of θ based on (17),

$$\begin{aligned} T^{\frac{1}{2}}(\widehat{\theta} - \theta_0) &= -\left[\frac{1}{T}\sum_{t=1}^T h_t(\overline{\theta})\right]^{-1} T^{-\frac{1}{2}}\sum_{t=1}^T s_t(\theta_0) \\ &= -\left[\frac{1}{T}\sum_{t=1}^T h_t(\overline{\theta})\right]^{-1} T^{-\frac{1}{2}}\sum_{t=1}^T \left[s_t(\theta_t) - h_t(\overline{\theta}_t)T^{-\frac{1}{2}}\delta\left(\frac{t}{T}\right)\right] \\ &= -\left[\frac{1}{T}\sum_{t=1}^T h_t(\overline{\theta})\right]^{-1} T^{-\frac{1}{2}}\sum_{t=1}^T s_t(\theta_t) + \left[\frac{1}{T}\sum_{t=1}^T h_t(\overline{\theta})\right]^{-1}\frac{1}{T}\sum_{t=1}^T h_t(\theta_t)\delta\left(\frac{t}{T}\right), \end{aligned} \tag{29}$$

where $\overline{\theta}$ and $\overline{\theta}_t$ are points between $\widehat{\theta}$ and θ_0 and between θ_t and θ_0, respectively. Using Assumption 1(c) and the Chebyshev inequality,

$$\frac{1}{T}\sum_{t=1}^T h_t(\theta_t)\delta\left(\frac{t}{T}\right) \overset{p}{\to} \Xi\int_0^1 \delta_0(r)dr. \tag{30}$$

[2]In our setup, $\widehat{\Xi}_t$ and $\widehat{V}_t$ in Condition 3 of Müller and Petalas (2010) are $\widehat{\Xi}$ and $\widehat{V}$, respectively.

It follows from (29), (30) and Assumption 1(c) that

$$T^{\frac{1}{2}}(\widehat{\theta}-\theta_0) \ \xrightarrow{d}\ \Xi^{-1}V^{\frac{1}{2}}\mathcal{W}(1)+\int_0^1 \delta_0(r)dr. \tag{31}$$

Since

$$\widehat{\theta}^r \ \equiv\ \widehat{\theta}+\widehat{\Xi}^{-1}\frac{1}{T}\sum_{t=1}^{T}s_t(\widehat{\theta}) \ =\ \widehat{\theta}, \tag{32}$$

it follows that

$$\begin{aligned} T^{\frac{1}{2}}(\widehat{\theta}^r-\theta_0) &= T^{\frac{1}{2}}(\widehat{\theta}-\theta_0)+o_p(1) \\ &\xrightarrow{d} \Gamma^{-\frac{1}{2}}\mathcal{W}(1)+\int_0^1 \delta_0(r)dr. \end{aligned} \tag{33}$$

Because the RHS is $\Gamma^{-1}J^r(1)$ in our setup, condition (ii) is satisfied.

Proof of (i): Next, we show that condition (i) in the lemma is satisfied. It follows that

$$\begin{aligned} T^{-\frac{1}{2}}\sum_{t=1}^{[\cdot T]}\widehat{s}_t^r &\equiv \widehat{\Xi}\widehat{V}^{-1}T^{-\frac{1}{2}}\sum_{t=1}^{[\cdot T]}s_t(\widehat{\theta})-\widehat{\Xi}\widehat{V}^{-1}\frac{1}{T}\sum_{t=1}^{T}s_t(\widehat{\theta}) \\ &= \Xi V^{-1}\left[T^{-\frac{1}{2}}\sum_{t=1}^{[\cdot T]}s_t(\theta_0)-T^{-1}\sum_{t=1}^{[\cdot T]}h_t(\bar{\theta})T^{\frac{1}{2}}(\widehat{\theta}-\theta_0)\right]-\Xi V^{-1}\frac{1}{T}\sum_{t=1}^{T}s_t(\theta_0)+o_p(1) \\ &\Rightarrow \Xi V^{-1}\left(V^{\frac{1}{2}}\mathcal{W}(\cdot)+\Xi\int_0^{\iota(\cdot)}\delta_0(r)dr\right)-\iota(\cdot)\Xi V^{-1}\left(V^{\frac{1}{2}}\mathcal{W}(1)+\Xi\int_0^1\delta_0(r)dr\right) \\ &= \left(\Gamma^{\frac{1}{2}}\mathcal{W}(\cdot)+\Gamma\int_0^{\iota(\cdot)}\delta_0(r)d\lambda\right)-\iota(\cdot)\left(\Gamma^{\frac{1}{2}}\mathcal{W}(1)+\Gamma\int_0^1\delta_0(r)dr\right), \end{aligned} \tag{34}$$

which is $J^r(\cdot)-\iota(\cdot)J^r(1)$, where $\iota(x)=x$. Thus, condition (i) is satisfied.[3]

Proof of (iii): Finally, it follows from Assumption 1(e) that

$$\sup_{\lambda\in[0,1]}\left|\frac{1}{T}\sum_{t=1}^{[\lambda T]}\widetilde{h}_t^r-\lambda\Xi V^{-1}\Xi\right| \ =\ o_p(1), \tag{35}$$

where $\tilde{h}_t^r=\widehat{\Xi}\widehat{V}^{-1}\widehat{\Xi}$, satisfying the third condition.

Because Condition 3 is satisfied, credible sets, C_θ, for the reduced-form parameters, B and Σ, satisfy the conclusion of Theorem 5 of Müller and Petalas (2010). Based on C_θ, a conservative confidence set for m can be constructed as

$$\{m : \exists\theta \text{ such that } \theta\in C_\theta\}, \tag{36}$$

[3]Note that the corresponding Condition 3 in Müller and Petalas' (2010) is stated for $\widehat{\Xi}_t$ and $\widehat{V}_t^{-1}$ instead of $\widehat{\Xi}$ and $\widehat{V}^{-1}$; we let $\widehat{\Xi}_t=\widehat{\Xi}$ and $\widehat{V}_t^{-1}=\widehat{V}^{-1}$ to ensure that the estimate is positive definite in any given sample.

for example.

In the strongly identified case, one can show that Condition 3 is satisfied using analogous arguments as above and Assumption 2.